

INSTRUCTOR'S RESOURCE MANUAL

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SOUTHERN ILLINOIS UNIVERSITY - EDWARDSVILLE

CALCULUS

EIGHTH EDITION

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1.1 Concepts Review

1. rational
2. $\sqrt{2}; \pi$
3. real
4. theorems

Problem Set 1.1

1. $4 - 2(8 - 11) + 6 = 4 - 2(-3) + 6$
 $= 4 + 6 + 6 = 16$
2. $3[2 - 4(7 - 12)] = 3[2 - 4(-5)]$
 $= 3[2 + 20] = 3(22) = 66$
3. $-4[5(-3 + 12 - 4) + 2(13 - 7)]$
 $= -4[5(5) + 2(6)] = -4[25 + 12]$
 $= -4(37) = -148$
4. $5[-1(7 + 12 - 16) + 4] + 2$
 $= 5[-1(3) + 4] + 2 = 5(-3 + 4) + 2$
 $= 5(1) + 2 = 5 + 2 = 7$
5. $\frac{5}{7} - \frac{1}{13} = \frac{65}{91} - \frac{7}{91} = \frac{58}{91}$
6. $\frac{3}{4-7} + \frac{3}{21} - \frac{1}{6} = \frac{3}{-3} + \frac{3}{21} - \frac{1}{6}$
 $= -\frac{42}{42} + \frac{6}{42} - \frac{7}{42} = -\frac{43}{42}$
7. $\frac{1}{3}\left[\frac{1}{2}\left(\frac{1}{4} - \frac{1}{3}\right) + \frac{1}{6}\right] = \frac{1}{3}\left[\frac{1}{2}\left(\frac{3-4}{12}\right) + \frac{1}{6}\right]$
 $= \frac{1}{3}\left[\frac{1}{2}\left(-\frac{1}{12}\right) + \frac{1}{6}\right]$
 $= \frac{1}{3}\left[-\frac{1}{24} + \frac{4}{24}\right]$
 $= \frac{1}{3}\left(\frac{3}{24}\right) = \frac{1}{24}$

$$\begin{aligned} 8. \quad & -\frac{1}{3}\left[\frac{2}{5} - \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)\right] \\ &= -\frac{1}{3\left[\frac{2}{5} - \frac{1}{2}\left(\frac{5-3}{15}\right)\right]} \\ &= -\frac{1}{3\left[\frac{2}{5} - \frac{1}{2}\left(\frac{2}{15}\right)\right]} = -\frac{1}{3\left[\frac{2}{5} - \frac{1}{15}\right]} \\ &= -\frac{1}{3\left(\frac{6}{15} - \frac{1}{15}\right)} = -\frac{1}{3\left(\frac{5}{15}\right)} = -\frac{1}{9} \end{aligned}$$

$$\begin{aligned} 9. \quad & \frac{14}{21}\left(\frac{2}{5-\frac{1}{3}}\right)^2 = \frac{14}{21}\left(\frac{2}{\frac{14}{3}}\right)^2 = \frac{14}{21}\left(\frac{6}{14}\right)^2 \\ &= \frac{14}{21}\left(\frac{3}{7}\right)^2 = \frac{2}{3}\left(\frac{9}{49}\right) = \frac{6}{49} \end{aligned}$$

$$10. \quad \frac{\left(\frac{2}{7} - 5\right)}{\left(1 - \frac{1}{7}\right)} = \frac{\left(\frac{2}{7} - \frac{35}{7}\right)}{\left(\frac{7}{7} - \frac{1}{7}\right)} = \frac{\left(-\frac{33}{7}\right)}{\left(\frac{6}{7}\right)} = -\frac{33}{6} = -\frac{11}{2}$$

$$11. \quad \frac{\frac{11}{7} - \frac{12}{21}}{\frac{11}{7} + \frac{12}{21}} = \frac{\frac{11}{7} - \frac{4}{7}}{\frac{11}{7} + \frac{4}{7}} = \frac{\frac{7}{7}}{\frac{15}{7}} = \frac{7}{15}$$

$$12. \quad \frac{\frac{1}{2} - \frac{3}{4} + \frac{7}{8}}{\frac{1}{2} + \frac{3}{4} - \frac{7}{8}} = \frac{\frac{4}{8} - \frac{6}{8} + \frac{7}{8}}{\frac{4}{8} + \frac{6}{8} - \frac{7}{8}} = \frac{\frac{5}{8}}{\frac{3}{8}} = \frac{5}{3}$$

$$13. \quad 1 - \frac{1}{1+\frac{1}{2}} = 1 - \frac{1}{\frac{3}{2}} = 1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}$$

$$\begin{aligned} 14. \quad & 2 + \frac{3}{1+\frac{5}{2}} = 2 + \frac{3}{\frac{2+5}{2}} = 2 + \frac{3}{\frac{7}{2}} \\ &= 2 + \frac{6}{7} = \frac{14}{7} + \frac{6}{7} = \frac{20}{7} \end{aligned}$$

$$\begin{aligned} 15. \quad & (\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) = (\sqrt{5})^2 - (\sqrt{3})^2 \\ &= 5 - 3 = 2 \end{aligned}$$

$$16. (\sqrt{5} - \sqrt{3})^2 = (\sqrt{5})^2 - 2(\sqrt{5})(\sqrt{3}) + (\sqrt{3})^2 \\ = 5 - 2\sqrt{15} + 3 = 8 - 2\sqrt{15}$$

$$17. 3\sqrt{2}(\sqrt{2} - \sqrt{8}) = 3\sqrt{4} - 3\sqrt{16} \\ = 3 \cdot 2 - 3 \cdot 4 \\ = 6 - 12 = -6$$

$$18. 2\sqrt[3]{4}[\sqrt[3]{2} + \sqrt[3]{16}] = 2\sqrt[3]{8} + 2\sqrt[3]{64} \\ = 2 \cdot 2 + 2 \cdot 4 \\ = 4 + 8 = 12$$

$$19. \left(\frac{7}{4} + \frac{1}{2}\right)^{-2} = \left(\frac{7}{4} + \frac{2}{4}\right)^{-2} = \left(\frac{9}{4}\right)^{-2} \\ = \frac{1}{\left(\frac{9}{4}\right)^2} = \frac{1}{\frac{81}{16}} = \frac{16}{81}$$

$$20. \left(\frac{1}{\sqrt{2}} - \frac{5}{2\sqrt{2}}\right)^{-2} = \left(\frac{2}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}\right)^{-2} \\ = \left(-\frac{3}{2\sqrt{2}}\right)^{-2} \\ = \frac{1}{\left(-\frac{3}{2\sqrt{2}}\right)^2} = \frac{1}{\frac{9}{8}} \\ = \frac{8}{9}$$

$$21. (3x - 4)(x + 1) = 3x^2 + 3x - 4x - 4 \\ = 3x^2 - x - 4$$

$$22. (2x - 3)^2 = (2x - 3)(2x - 3) \\ = 4x^2 - 6x - 6x + 9 \\ = 4x^2 - 12x + 9$$

$$23. (3x - 9)(2x + 1) = 6x^2 + 3x - 18x - 9 \\ = 6x^2 - 15x - 9$$

$$24. (4x - 11)(3x - 7) = 12x^2 - 28x - 33x + 77 \\ = 12x^2 - 61x + 77$$

$$25. (3t^2 - t + 1)^2 \\ = (3t^2 - t + 1)(3t^2 - t + 1) \\ = 9t^4 - 3t^3 + 3t^2 - 3t^3 + t^2 - t + 3t^2 - t + 1 \\ = 9t^4 - 6t^3 + 7t^2 - 2t + 1$$

$$26. (2t + 3)^3 = (2t + 3)(2t + 3)(2t + 3) \\ = (4t^2 + 12t + 9)(2t + 3) \\ = 8t^3 + 12t^2 + 24t^2 + 36t + 18t + 27 \\ = 8t^3 + 36t^2 + 54t + 27$$

$$27. \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$$

$$28. \frac{x^2 - x - 6}{x - 3} = \frac{(x - 3)(x + 2)}{(x - 3)} = x + 2$$

$$29. \frac{t^2 - 4t - 21}{t + 3} = \frac{(t + 3)(t - 7)}{t + 3} = t - 7$$

$$30. \frac{2x - 2x^2}{x^3 - 2x^2 + x} = \frac{2x(1 - x)}{x(x^2 - 2x + 1)} \\ = \frac{-2x(x - 1)}{x(x - 1)(x - 1)} \\ = -\frac{2}{x - 1}$$

$$31. \frac{12}{x^2 + 2x} + \frac{4}{x} + \frac{2}{x + 2} \\ = \frac{12}{x(x + 2)} + \frac{4(x + 2)}{x(x + 2)} + \frac{2x}{x(x + 2)} \\ = \frac{12 + 4x + 8 + 2x}{x(x + 2)} \\ = \frac{6x + 20}{x(x + 2)} \\ = \frac{2(3x + 10)}{x(x + 2)}$$

$$32. \frac{2}{6y - 2} + \frac{y}{9y^2 - 1} - \frac{2y + 1}{1 - 3y} \\ = \frac{2}{2(3y - 1)} + \frac{y}{(3y + 1)(3y - 1)} + \frac{2y + 1}{3y - 1} \\ = \frac{2(3y + 1)}{2(3y + 1)(3y - 1)} + \frac{2y}{2(3y + 1)(3y - 1)} \\ + \frac{(2y + 1)(3y + 1)}{2(3y + 1)(3y - 1)} \\ = \frac{6y + 2 + 2y + 12y^2 + 10y + 2}{2(3y + 1)(3y - 1)}$$

$$\begin{aligned}
 &= \frac{12y^2 + 18y + 4}{2(3y+1)(3y-1)} \\
 &= \frac{2(6y^2 + 9y + 2)}{2(3y+1)(3y-1)} \\
 &= \frac{6y^2 + 9y + 2}{(3y+1)(3y-1)}
 \end{aligned}$$

$$\begin{aligned}
 33. \quad &\frac{t^2 + t - 12}{x^2 - 1} \cdot \frac{x^2 - 6x - 7}{8t - t^2 - 15} \\
 &= \frac{(t+4)(t-3)(x-7)(x+1)}{-(x+1)(x-1)(t-3)(t-5)} \\
 &= -\frac{(t+4)(x-7)}{(x-1)(t-5)}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad &\frac{\frac{x}{x-3} - \frac{2}{x^2 - 4x + 3}}{\frac{5}{x-1} + \frac{5}{x+3}} = \frac{\frac{x}{x-3} - \frac{2}{(x-3)(x-1)}}{\frac{5}{x-1} + \frac{5}{x+3}} \\
 &= \frac{\frac{x(x-1) - 2}{(x-3)(x-1)}}{\frac{5(x-1) + 5(x+3)}{(x-1)(x+3)}} \\
 &= \frac{x(x-1) - 2}{5(x-3) + 5(x-1)} \\
 &= \frac{x^2 - x - 2}{5x - 15 + 5x - 5} \\
 &= \frac{(x-2)(x+1)}{10(x-2)} = \frac{x+1}{10}
 \end{aligned}$$

$$35. \quad \text{a. } 0 \cdot 0 = 0 \quad \text{b. } \frac{0}{0} \text{ is undefined.}$$

$$\text{c. } \frac{0}{17} = 0 \quad \text{d. } \frac{3}{0} \text{ is undefined.}$$

$$\text{e. } 0^5 = 0 \quad \text{f. } 17^0 = 1$$

36. If $\frac{0}{0} = a$, then $0 = 0 \cdot a$, but this is meaningless because a could be any real number. No single value satisfies $\frac{0}{0} = a$.

$$37. \quad \text{a. } -3 < -7; \text{ False} \quad \text{b. } -1 > -17; \text{ True}$$

$$\text{c. } -3 < -\frac{22}{7}; -\frac{21}{7} < -\frac{22}{7}; \text{ False}$$

$$\text{d. } -5 > -\sqrt{26}; -\sqrt{25} > -\sqrt{26}; \text{ True}$$

$$\text{e. } \frac{6}{7} < \frac{34}{39}; \frac{234}{273} < \frac{238}{273}; \text{ True}$$

$$\text{f. } -\frac{5}{7} < -\frac{44}{59}; -\frac{295}{413} < -\frac{308}{413}; \text{ False}$$

$$38. \quad \text{a. } a < b; a^2 < ab \text{ and } ab < b^2, \text{ so } a^2 < b^2$$

$$\text{b. } a < b; \frac{a}{b} < 1; \frac{1}{b} < \frac{1}{a}; \frac{1}{a} > \frac{1}{b}$$

$$39. \quad a < b; 2a < a + b \text{ and } a + b < 2b, \text{ so}$$

$$2a < a + b < 2b; a < \frac{a+b}{2} < b$$

$$40. \quad \text{a. is false if } a < 0$$

$$\text{b. is always true}$$

$$\text{c. is always true}$$

$$\text{d. is false}$$

$$41. \quad \text{a. True; If } x \text{ is positive, then } x^2 \text{ is positive.}$$

$$\text{b. False; Take } x = -2. \text{ Then } x^2 > 0 \text{ but } x < 0.$$

$$\text{c. False; Take } x = \frac{1}{2}. \text{ Then } x^2 = \frac{1}{4} < x.$$

$$\text{d. True; } n = 2 \text{ is even and prime.}$$

$$\text{e. True; Let } x \text{ be any number. Take } y = x^2 + 1. \text{ Then } y > x^2.$$

$$\text{f. False; There is no number larger than every } x^2.$$

$$\text{g. True; } 1/n \text{ can be made arbitrarily close to 0.}$$

$$\text{h. True } 2^{-n} \text{ can be made arbitrarily close to 0.}$$

$$\begin{aligned}
 42. \quad \text{a. If } n \text{ is odd, then there is an integer } k \text{ such that} \\
 n = 2k + 1. \text{ Then} \\
 n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1
 \end{aligned}$$

- b. Prove the contrapositive. Suppose n is even. Then there is an integer k such that $n = 2k$. Then
- $$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$
- Thus n^2 is even.
- c. Parts (a) and (b) prove that n is odd if and only if n^2 is odd.
43. a. $243 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
- b. $127 = 1 \cdot 127$
- c. $5100 = 2 \cdot 2550 = 2 \cdot 2 \cdot 1275 = 2 \cdot 2 \cdot 3 \cdot 425$
 $= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 85 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \cdot 17$
- d. $346 = 2 \cdot 173$
44. Let $A = b \cdot c^2 \cdot d^3$; then $A^2 = b^2 \cdot c^4 \cdot d^6$, so the square of the number is the product of primes which occur an even number of times
45. $\sqrt{2} = \frac{p}{q}$; $2 = \frac{p^2}{q^2}$; $2q^2 = p^2$; Since the prime factors of p^2 must occur an even number of times, $2q^2$ would not be valid and $\frac{p}{q} = \sqrt{2}$ must be irrational.
46. $\sqrt{3} = \frac{p}{q}$; $3 = \frac{p^2}{q^2}$; $3q^2 = p^2$; Since the prime factors of p^2 must occur an even number of times, $3q^2$ would not be valid and $\frac{p}{q} = \sqrt{3}$ must be irrational.
47. Let a , b , p , and q be natural numbers, so $\frac{a}{b}$ and $\frac{p}{q}$ are rational. $\frac{a}{b} + \frac{p}{q} = \frac{aq + bp}{bq}$ This sum is the quotient of natural numbers, so it is also rational.
48. Let a be an irrational number and p and q be natural numbers. $a \cdot \frac{p}{q} = \frac{ap}{q}$. Since the numerator is not a natural number, the product is irrational.
49. a. $-\sqrt{9} = -3$; rational
- b. $0.375 = \frac{3}{8}$; rational
- c. $1 - \sqrt{2}$; irrational
- d. $(1 + \sqrt{3})^2 = 1 + 2\sqrt{3} + 3 = 4 + 2\sqrt{3}$; irrational
- e. $(3\sqrt{2})(5\sqrt{2}) = 15\sqrt{4} = 30$; rational
- e. $5\sqrt{2}$; irrational
50. The sum of two irrational numbers is not always irrational. If the numbers are additive inverses, the sum is 0, which is rational.
51. If m were a perfect square, its prime factors would occur even numbers of times. If m is not a perfect square, some factors will occur an odd number of times and $\sqrt{m}$ will be irrational.
52. $\sqrt{6} + \sqrt{3} \approx 4.18154$ is irrational.
53. $\sqrt{2} - \sqrt{3} + \sqrt{6} \approx 2.13165$ is irrational.
54. $\log_{10} 5 \approx 0.69897$ is irrational
55. a. Converse: If I get an A in this course, then I do all of the homework.
 Contrapositive: If I don't get an A in this course, then I don't do all of the homework.
- b. Converse: If x is an integer, then x is a real number.
 Contrapositive: If x is not an integer, then x is not a real number.
- c. Converse: If $\triangle ABC$ is an isosceles triangle, then $\triangle ABC$ is an equilateral triangle.
 Contrapositive: If $\triangle ABC$ is not an isosceles triangle, then $\triangle ABC$ is not an equilateral triangle.

1.2 Concepts Review

1. 0.333..... (3s repeat); 0.200... (0s repeat);
3.14159...
2. rational
3. rational; irrational
4. real

Problem Set 1.2

1.
$$\begin{array}{r} .0833 \\ 12 \overline{)1.0000} \\ \underline{96} \\ 40 \\ \underline{36} \\ 40 \end{array}$$
2.
$$\begin{array}{r} .285714 \\ 7 \overline{)2.000000} \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 30 \\ \underline{28} \\ 2 \end{array}$$

3.
$$\begin{array}{r} .846153 \\ 13 \overline{)11.000000} \\ \underline{104} \\ 60 \\ \underline{52} \\ 80 \\ \underline{78} \\ 20 \\ \underline{13} \\ 70 \\ \underline{65} \\ 50 \\ \underline{39} \\ 11 \end{array}$$

4.
$$\begin{array}{r} .294117 \\ 17 \overline{)5.000000} \\ \underline{34} \\ 160 \\ \underline{153} \\ 70 \\ \underline{68} \\ 20 \\ \underline{17} \\ 30 \\ \underline{17} \\ 130 \\ \underline{119} \\ 11 \end{array}$$

5.
$$\begin{array}{r} 3.66 \\ 3 \overline{)11.00} \\ \underline{9} \\ 20 \\ \underline{18} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

$$\begin{array}{r}
 6. \quad \begin{array}{r} .846153 \\ 13 \overline{) 11.000000} \\ \underline{10 \ 4} \\ 60 \\ \underline{52} \\ 80 \\ \underline{78} \\ 20 \\ \underline{13} \\ 70 \\ \underline{65} \\ 50 \\ \underline{39} \\ 11 \end{array}
 \end{array}$$

$$\begin{array}{l}
 7. \quad x = 0.123123123... \\
 1000x = 123.123123... \\
 \hline
 x = 0.123123... \\
 999x = 123 \\
 \hline
 x = \frac{123}{999} = \frac{41}{333}
 \end{array}$$

$$\begin{array}{l}
 8. \quad x = 0.217171717... \\
 1000x = 217.171717... \\
 10x = 2.171717... \\
 \hline
 990x = 215 \\
 \hline
 x = \frac{215}{990} = \frac{43}{198}
 \end{array}$$

$$\begin{array}{l}
 9. \quad x = 2.56565656... \\
 100x = 256.565656... \\
 \hline
 x = 2.565656... \\
 99x = 254 \\
 \hline
 x = \frac{254}{99}
 \end{array}$$

$$\begin{array}{l}
 10. \quad x = 3.929292... \\
 100x = 392.929292... \\
 \hline
 x = 3.929292... \\
 99x = 389 \\
 \hline
 x = \frac{389}{99}
 \end{array}$$

$$\begin{array}{l}
 11. \quad x = 0.199999... \\
 100x = 19.99999... \\
 \hline
 10x = 1.99999... \\
 \hline
 90x = 18 \\
 \hline
 x = \frac{18}{90} = \frac{1}{5}
 \end{array}$$

$$\begin{array}{l}
 12. \quad x = 0.399999... \\
 100x = 39.99999... \\
 \hline
 10x = 3.99999... \\
 \hline
 90x = 36 \\
 \hline
 x = \frac{36}{90} = \frac{2}{5}
 \end{array}$$

13. Those rational numbers that can be expressed by a terminating decimal followed by zeros.

14. $\frac{p}{q} = p \left(\frac{1}{q} \right)$, so we only need to look at $\frac{1}{q}$. If $q = 2^n \cdot 5^m$, then $\frac{1}{q} = \left(\frac{1}{2} \right)^n \cdot \left(\frac{1}{5} \right)^m = (0.5)^n (0.2)^m$. The product of any number of terminating decimals is also a terminating decimal, so $(0.5)^n$ and $(0.2)^m$, and hence their product, $\frac{1}{q}$, is a terminating decimal. Thus $\frac{p}{q}$ has a terminating decimal expansion.

15. Answers will vary. Possible answer: 0.000001 , $\frac{1}{\pi^{12}} \approx 0.0000010819...$

16. Smallest positive integer: 1; There is no smallest positive rational or irrational number.

17. Answers will vary. Possible answer: 3.14159101001...

18. There is no real number between 0.9999... (repeating 9's) and 1. 0.9999... and 1 represent the *same* real number.

19. Irrational

20. Answers will vary. Possible answers: $-\pi$ and π , $-\sqrt{2}$ and $\sqrt{2}$

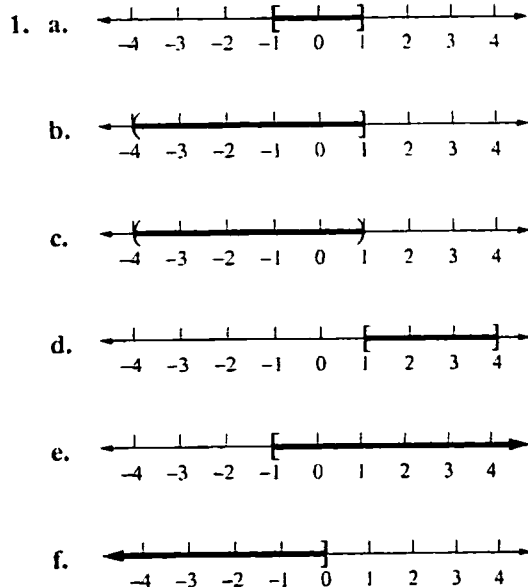
21. $(\sqrt{3} + 1)^3 \approx 20.39230485$

22. $(\sqrt{2} - \sqrt{3})^4 \approx 0.0102051443$
23. $\sqrt[4]{1.123} - \sqrt[3]{1.09} \approx 0.00028307388$
24. $(3.1415)^{-1/2} \approx 0.5641979034$
25. $\frac{\sqrt{130} - \sqrt{5}}{3^{1.2} - 3} \approx 12.43322783$
26. $\frac{(0.00121)(5.23 \times 10^{-3})}{6.16 \times 10^{-4}} \approx 0.0102732143$
27. $\sqrt{8.9\pi^2 + 1} - 3\pi \approx 0.000691744752$
28. $\sqrt[4]{(6\pi^2 - 2)\pi} \approx 3.661591807$
29. Let a and b be real numbers with $a < b$. Let n be a natural number that satisfies $1/n < b - a$. Let $S = \{k : k/n > b\}$. Since a nonempty set of integers that is bounded below contains a least element, there is a $k_0 \in S$ such that $k_0/n > b$ but $(k_0 - 1)/n \leq b$. Then $\frac{k_0 - 1}{n} = \frac{k_0}{n} - \frac{1}{n} > b - \frac{1}{n} > a$. Thus, $a < \frac{k_0 - 1}{n} \leq b$. If $\frac{k_0 - 1}{n} < b$, then choose $r = \frac{k_0 - 1}{n}$. Otherwise, choose $r = \frac{k_0 - 2}{n}$.
30. Answers will vary. Possible answer: $\approx 120 \text{ in}^3$
31. $r = 4000 \text{ mi} \times 5280 \frac{\text{ft}}{\text{mi}} = 21,120,000 \text{ ft}$
equator $= 2\pi r = 2\pi(21,120,000) \approx 132,700,874 \text{ ft}$
32. Answers will vary. Possible answer:
 $70 \frac{\text{beats}}{\text{min}} \times 60 \frac{\text{min}}{\text{hr}} \times 24 \frac{\text{hr}}{\text{day}} \times 365 \frac{\text{day}}{\text{year}} \times 20 \text{ yr}$
 $= 735,840,000 \text{ beats}$
33. $V = \pi r^2 h = \pi \left(\frac{16}{2} \cdot 12 \right)^2 (270 \cdot 12)$
 $\approx 93,807,453.98 \text{ in.}^3$
volume of one board foot (in inches):
 $1 \cdot 3 \cdot 12 \cdot 3 \cdot 12 = 144 \text{ in.}^3$
number of board feet:
 $\frac{93,807,453.98}{144} \approx 651,441 \text{ board ft}$
34. $V = \pi(8.004)^2(270) - \pi(8)^2(270) \approx 54.3 \text{ in.}^3$
35. a. At $x = 2\pi$: 286.866542
b. At $x = 2.15$: 9.16925
c. At $x = 2.71828$: 16.34874967
d. At $x = 1.1$: 4.292
36. $x^4 - 3x^3 + 4x^2 + 6x - 10$
 $= (x^3 - 3x^2 + 4x + 6)x - 10$
 $= [(x^2 - 3x + 4)x + 6]x - 10$
 $= [((x - 3)x + 4)x + 6]x - 10$
a. At $x = 1$: -2
b. At $x = \pi$: 52.71823452
c. At $x = 14.2$: 32,950.5856
d. At $x = 1.2157$: 0.0000269681
37. a. -2 b. -2
c. $x = 2.4444\dots$;
 $10x = 24.4444\dots$
 $\frac{x = 2.4444\dots}{9x = 22}$
 $x = \frac{22}{9}$
d. 1
e. $n = 1: x = 0, n = 2: x = \frac{3}{2}, n = 3: x = -\frac{2}{3},$
 $n = 4: x = \frac{5}{4}$
The upper bound is $\frac{3}{2}$.
f. $\sqrt{2}$
38. a. Answers will vary. Possible answer: An example is $S = \{x : x^2 < 5, x \text{ a rational number}\}$. Here the least upper bound is $\sqrt{5}$, which is real but irrational.
b. True

1.3 Concepts Review

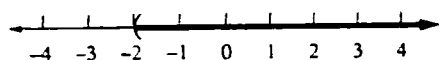
- interval; intervals
- $[-1, 5); (-\infty, 2]$
- $b > 0; b < 0$
- $-5, -4, 3$

Problem Set 1.3

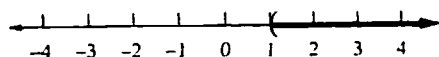


2. a. $(2, 7)$ b. $[-3, 4)$
 c. $(-\infty, 2]$ d. $[-1, 3]$

3. $x - 7 < 2x - 5$
 $-2 < x; (-2, \infty)$



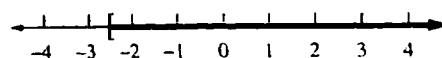
4. $3x - 5 < 4x - 6$
 $1 < x; (1, \infty)$



5. $7x - 2 \leq 9x + 3$

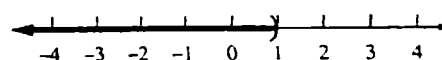
$$-5 \leq 2x$$

$$x \geq -\frac{5}{2}; \left[-\frac{5}{2}, \infty\right)$$



6. $5x - 3 > 6x - 4$

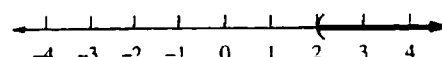
$$1 > x; (-\infty, 1)$$



7. $10x + 1 > 8x + 5$

$$2x > 4$$

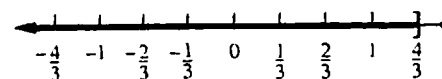
$$x > 2; (2, \infty)$$



8. $-2x + 5 \geq 4x - 3$

$$8 \geq 6x$$

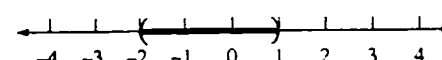
$$x \leq \frac{4}{3}; \left(-\infty, \frac{4}{3}\right]$$



9. $-4 < 3x + 2 < 5$

$$-6 < 3x < 3$$

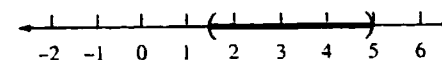
$$-2 < x < 1; (-2, 1)$$



10. $-3 < 4x - 9 < 11$

$$6 < 4x < 20$$

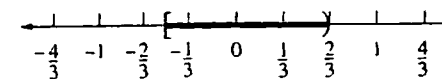
$$\frac{3}{2} < x < 5; \left(\frac{3}{2}, 5\right)$$



11. $-3 < 1 - 6x \leq 4$

$$-4 < -6x \leq 3$$

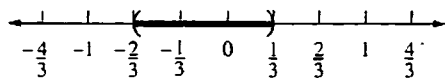
$$\frac{2}{3} > x \geq -\frac{1}{2}; \left[-\frac{1}{2}, \frac{2}{3}\right)$$



12. $4 < 5 - 3x < 7$

$-1 < -3x < 2$

$\frac{1}{3} > x > -\frac{2}{3}; \left(-\frac{2}{3}, \frac{1}{3}\right)$

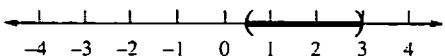


13. $2 + 3x < 5x + 1 < 16$

$2 + 3x < 5x + 1$ and $5x + 1 < 16$

$1 < 2x$ and $5x < 15$

$x > \frac{1}{2}$ and $x < 3; \left(\frac{1}{2}, 3\right)$

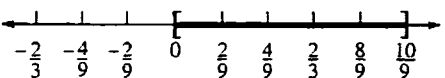


14. $2x - 4 \leq 6 - 7x \leq x + 6$

$2x - 4 \leq -7x$ and $6 - 7x \leq x + 6$

$9x \leq 10$ and $10x \geq 0$

$x \leq \frac{10}{9}$ and $x \geq 0; \left[0, \frac{10}{9}\right]$



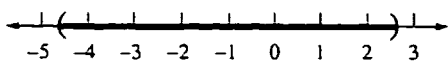
15. $x^2 + 2x - 12 < 0;$

$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-12)}}{2(1)} = \frac{-2 \pm \sqrt{52}}{2}$

$= -1 \pm \sqrt{13}$

$[x - (-1 + \sqrt{13})][x - (-1 - \sqrt{13})] < 0;$

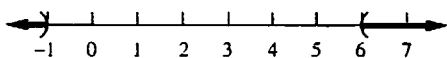
$(-1 - \sqrt{13}, -1 + \sqrt{13})$



16. $x^2 - 5x - 6 > 0$

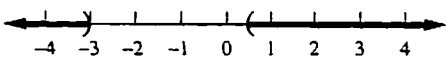
$(x + 1)(x - 6) > 0;$

$(-\infty, -1) \cup (6, \infty)$



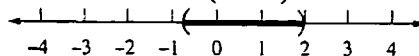
17. $2x^2 + 5x - 3 > 0; (2x - 1)(x + 3) > 0;$

$(-\infty, -3) \cup \left(\frac{1}{2}, \infty\right)$

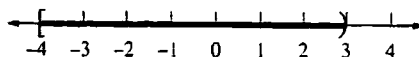


18. $4x^2 - 5x - 6 < 0$

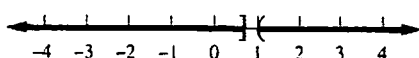
$(4x + 3)(x - 2) < 0; \left(-\frac{3}{4}, 2\right)$



19. $\frac{x+4}{x-3} \leq 0; [-4, 3)$



20. $\frac{3x-2}{x-1} \geq 0; \left(-\infty, \frac{2}{3}\right] \cup (1, \infty)$

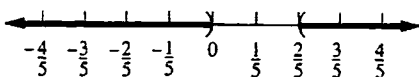


21. $\frac{2}{x} < 5$

$\frac{2}{x} - 5 < 0$

$\frac{2-5x}{x} < 0;$

$(-\infty, 0) \cup \left(\frac{2}{5}, \infty\right)$

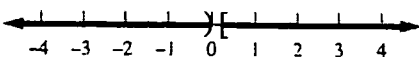


22. $\frac{7}{4x} \leq 7$

$\frac{7}{4x} - 7 \leq 0$

$\frac{7-28x}{4x} \leq 0;$

$(-\infty, 0) \cup \left[\frac{1}{4}, \infty\right)$

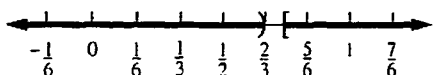


$$23. \quad \frac{1}{3x-2} \leq 4$$

$$\frac{1}{3x-2} - 4 \leq 0$$

$$\frac{1-4(3x-2)}{3x-2} \leq 0$$

$$\frac{9-12x}{3x-2} \leq 0; \left(-\infty, \frac{2}{3}\right) \cup \left[\frac{3}{4}, \infty\right)$$

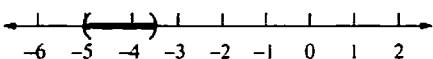


$$24. \quad \frac{3}{x+5} > 2$$

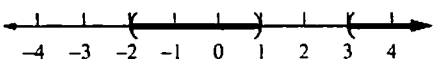
$$\frac{3}{x+5} - 2 > 0$$

$$\frac{3-2(x+5)}{x+5} > 0$$

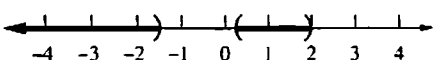
$$\frac{-2x-7}{x+5} > 0; \left(-5, -\frac{7}{2}\right)$$



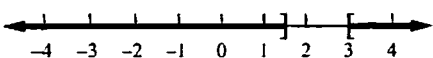
$$25. (x+2)(x-1)(x-3) > 0; (-2, 1) \cup (3, 8)$$



$$26. (2x+3)(3x-1)(x-2) < 0; \left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{1}{3}, 2\right)$$

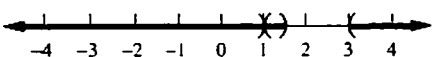


$$27. (2x-3)(x-1)^2(x-3) \geq 0; \left(-\infty, \frac{3}{2}\right] \cup [3, \infty)$$



$$28. (2x-3)(x-1)^2(x-3) > 0;$$

$$\left(-\infty, 1\right) \cup \left(1, \frac{3}{2}\right) \cup (3, \infty)$$

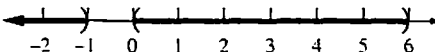


$$29. \quad x^3 - 5x^2 - 6x < 0$$

$$x(x^2 - 5x - 6) < 0$$

$$x(x+1)(x-6) < 0;$$

$$(-\infty, -1) \cup (0, 6)$$

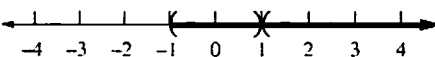


$$30. \quad x^3 - x^2 - x + 1 > 0$$

$$(x^2 - 1)(x - 1) > 0$$

$$(x+1)(x-1)^2 > 0;$$

$$(-1, 1) \cup (1, \infty)$$



$$31. \quad \text{a. } 3x + 7 > 1 \text{ and } 2x + 1 < 3$$

$$3x > -6 \text{ and } 2x < 2$$

$$x > -2 \text{ and } x < 1; (-2, 1)$$

$$\text{b. } 3x + 7 > 1 \text{ and } 2x + 1 > -4$$

$$3x > -6 \text{ and } 2x > -5$$

$$x > -2 \text{ and } x > -\frac{5}{2}; (-2, \infty)$$

$$\text{c. } 3x + 7 > 1 \text{ and } 2x + 1 < -4$$

$$x > -2 \text{ and } x < -\frac{5}{2}; \emptyset$$

$$32. \quad \text{a. } 2x - 7 > 1$$

$$\{4 < x\} \text{ or } 2x + 1 < 3$$

$$2x > 8 \text{ or } 2x < 2; x > 4 \text{ or } x < 1;$$

$$(-\infty, 1) \cup (4, \infty)$$

$$\text{b. } 2x - 7 \leq 1$$

$$\{x \leq 4\} \text{ or } 2x + 1 < 3$$

$$x \leq 4 \text{ or } x < 1; (-\infty, 4]$$

$$\text{c. } 2x - 7 \leq 1$$

$$\{x \leq 4\} \text{ or } 2x + 1 > 3$$

$$x \leq 4 \text{ or } x > 1; (-\infty, \infty)$$

$$33. \quad \text{a. } (x+1)(x^2 + 2x - 7) \geq x^2 - 1$$

$$x^3 + 3x^2 - 5x - 7 \geq x^2 - 1$$

$$x^3 + 2x^2 - 5x - 6 \geq 0$$

$$(x+3)(x+1)(x-2) \geq 0$$

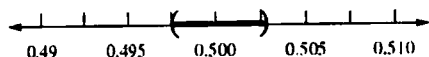
$$[-3, -1] \cup [2, \infty)$$

b. $x^4 - 2x^2 \geq 8$
 $x^4 - 2x^2 - 8 \geq 0$
 $(x^2 - 4)(x^2 + 2) \geq 0$
 $(x^2 + 2)(x + 2)(x - 2) \geq 0$
 $(-\infty, -2] \cup [2, \infty)$

c. $(x^2 + 1)^2 - 7(x^2 + 1) + 10 < 0$
 $[(x^2 + 1) - 5][(x^2 + 1) - 2] < 0$
 $(x^2 - 4)(x^2 - 1) < 0$
 $(x + 2)(x + 1)(x - 1)(x - 2) < 0$
 $(-2, -1) \cup (1, 2)$

34. Suppose $x > 0$. If we divide both sides of the inequality $1 > 0$ by x , we obtain $1/x > 0$. To prove the converse, divide both sides of the equation $1 > 0$ by $1/x$. This gives $\frac{1}{1/x} > \frac{0}{1/x}$, which is equivalent to $x > 0$.

35. a. $1.99 < \frac{1}{x} < 2.01$
 $1.99x < 1 < 2.01x$
 $1.99x < 1$ and $1 < 2.01x$
 $x < \frac{1}{1.99}$ and $x > \frac{1}{2.01}$
 $\frac{1}{2.01} < x < \frac{1}{1.99}$
 $\left(\frac{1}{2.01}, \frac{1}{1.99}\right)$

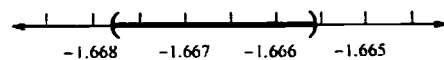


b. $2.99 < \frac{1}{x+2} < 3.01$
 $2.99(x+2) < 1 < 3.01(x+2)$
 $2.99x + 5.98 < 1$ and
 $1 < 3.01x + 6.02$

$$x < \frac{-4.98}{2.99} \text{ and } x > \frac{-5.02}{3.01}$$

$$-\frac{5.02}{3.01} < x < -\frac{4.98}{2.99}$$

$$\left(-\frac{5.02}{3.01}, -\frac{4.98}{2.99}\right)$$



c. $3 - \varepsilon < \frac{1}{x+2} < 3 + \varepsilon$
 $(3 - \varepsilon)(x + 2) < 1 < (3 + \varepsilon)(x + 2)$
 $(3 - \varepsilon)x + (3 - \varepsilon)2 < 1 < (3 + \varepsilon)x + (3 + \varepsilon)2$

$$x < \frac{1 - 2(3 - \varepsilon)}{3 - \varepsilon} \text{ and } x > \frac{1 - 2(3 + \varepsilon)}{3 + \varepsilon}$$

$$\left(\frac{1 - 2(3 + \varepsilon)}{3 + \varepsilon}, \frac{1 - 2(3 - \varepsilon)}{3 - \varepsilon}\right)$$

36. $1 + x + x^2 + x^3 + \dots + x^{99} \leq 0$;
 $(-\infty, -1)$

37. $\frac{1}{R} \leq \frac{1}{10} + \frac{1}{20} + \frac{1}{30}$
 $\frac{1}{R} \leq \frac{6+3+2}{60}$

$$\frac{1}{R} \leq \frac{11}{60}$$

$$R \geq \frac{60}{11}$$

$$\frac{1}{R} \geq \frac{1}{20} + \frac{1}{30} + \frac{1}{40}$$

$$\frac{1}{R} \geq \frac{6+4+3}{120}$$

$$\frac{1}{R} \geq \frac{13}{120}$$

$$R \leq \frac{120}{13}$$

$$\frac{60}{11} \leq R \leq \frac{120}{13}$$

1.4 Concepts Review

1. $-1; 5$
2. $|a+b| \leq |a|+|b|$
3. b, c
4. $\frac{0.2}{5} = 0.04$

Problem Set 1.4

1. $|x+2| < 1$;
 $-1 < x+2 < 1$
 $-3 < x < -1$
 $(-3, -1)$
2. $|x-2| \geq 5$;
 $x-2 \leq -5$ or $x-2 \geq 5$
 $x \leq -3$ or $x \geq 7$
 $(-\infty, -3] \cup [7, \infty)$
3. $|2x-1| > 2$;
 $2x-1 < -2$ or $2x-1 > 2$
 $2x < -1$ or $2x > 3$;
 $x < -\frac{1}{2}$ or $x > \frac{3}{2}$; $(-\infty, -\frac{1}{2}) \cup (\frac{3}{2}, \infty)$
4. $|4x+5| \leq 10$;
 $-10 \leq 4x+5 \leq 10$
 $-15 \leq 4x \leq 5$
 $-\frac{15}{4} \leq x \leq \frac{5}{4}$; $[-\frac{15}{4}, \frac{5}{4}]$
5. $|\frac{x}{4}+1| < 1$
 $-1 < \frac{x}{4}+1 < 1$
 $-2 < \frac{x}{4} < 0$;
 $-8 < x < 0$; $(-8, 0)$
6. $|\frac{2x}{7}-5| \geq 7$
 $\frac{2x}{7}-5 \leq -7$ or $\frac{2x}{7}-5 \geq 7$
 $\frac{2x}{7} \leq -2$ or $\frac{2x}{7} \geq 12$
 $x \leq -7$ or $x \geq 42$;
 $(-\infty, -7] \cup [42, \infty)$
7. $|2x-7| > 3$;
 $2x-7 < -3$ or $2x-7 > 3$
 $2x < 4$ or $2x > 10$
 $x < 2$ or $x > 5$; $(-\infty, 2) \cup (5, \infty)$
8. $|5x-6| > 1$;
 $5x-6 < -1$ or $5x-6 > 1$
 $5x < 5$ or $5x > 7$
 $x < 1$ or $x > \frac{7}{5}$; $(-\infty, 1) \cup (\frac{7}{5}, \infty)$
9. $|4x+2| \geq 10$;
 $4x+2 \leq -10$ or $4x+2 \geq 10$
 $4x \leq -12$ or $4x \geq 8$
 $x \leq -3$ or $x \geq 2$
 $(-\infty, -3] \cup [2, \infty)$
10. $|\frac{x}{2}+7| \geq 2$;
 $\frac{x}{2}+7 \leq -2$ or $\frac{x}{2}+7 \geq 2$
 $\frac{x}{2} \leq -9$ or $\frac{x}{2} \geq -5$
 $x \leq -18$ or $x \geq -10$
 $(-\infty, -18] \cup [-10, \infty)$
11. $|2+\frac{5}{x}| > 1$;
 $2+\frac{5}{x} < -1$ or $2+\frac{5}{x} > 1$
 $3+\frac{5}{x} < 0$ or $1+\frac{5}{x} > 0$
 $\frac{3x+5}{x} < 0$ or $\frac{x+5}{x} > 0$;
 $(-\infty, -5) \cup (-\frac{5}{3}, 0) \cup (0, \infty)$
12. $|\frac{1}{x}-3| > 6$;
 $\frac{1}{x}-3 < -6$ or $\frac{1}{x}-3 > 6$
 $\frac{1}{x}+3 < 0$ or $\frac{1}{x}-9 > 0$
 $\frac{1+3x}{x} < 0$ or $\frac{1-9x}{x} > 0$;
 $(-\frac{1}{3}, 0) \cup (0, \frac{1}{9})$

$$13. x^2 - 3x - 4 \geq 0;$$

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)} = \frac{3 \pm 5}{2} = -1, 4$$

$$(x+1)(x-4) = 0; (-\infty, -1] \cup [4, \infty)$$

$$14. x^2 - 4x + 4 \leq 0; x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)} = 2$$

$$(x-2)(x-2) \leq 0; x = 2$$

$$15. 3x^2 + 17x - 6 > 0;$$

$$x = \frac{-17 \pm \sqrt{(17)^2 - 4(3)(-6)}}{2(3)} = \frac{-17 \pm 19}{6} = -6, \frac{1}{3}$$

$$(3x-1)(x+6) > 0; (-\infty, -6) \cup \left(\frac{1}{3}, \infty\right)$$

$$16. 14x^2 + 11x - 15 \leq 0;$$

$$x = \frac{-11 \pm \sqrt{(11)^2 - 4(14)(-15)}}{2(14)} = \frac{-11 \pm 31}{28}$$

$$x = -\frac{3}{2}, \frac{5}{7}$$

$$\left(x + \frac{3}{2}\right)\left(x - \frac{5}{7}\right) \leq 0; \left[-\frac{3}{2}, \frac{5}{7}\right]$$

$$17. |x-3| < 0.5 \Leftrightarrow 5|x-3| < 5(0.5) \Leftrightarrow |5x-15| < 2.5$$

$$18. |x+2| < 0.3 \Leftrightarrow 4|x+2| < 4(0.3) \Leftrightarrow |4x+8| < 1.2$$

$$19. |x-2| < \frac{\varepsilon}{6} \Leftrightarrow 6|x-2| < \varepsilon \Leftrightarrow |6x-12| < \varepsilon$$

$$20. |x+4| < \frac{\varepsilon}{2} \Leftrightarrow 2|x+4| < \varepsilon \Leftrightarrow |2x+8| < \varepsilon$$

$$21. |3x-15| < \varepsilon \Leftrightarrow |3(x-5)| < \varepsilon$$

$$\Leftrightarrow 3|x-5| < \varepsilon$$

$$\Leftrightarrow |x-5| < \frac{\varepsilon}{3}; \delta = \frac{\varepsilon}{3}$$

$$22. |4x-8| < \varepsilon \Leftrightarrow |4(x-2)| < \varepsilon$$

$$\Leftrightarrow 4|x-2| < \varepsilon$$

$$\Leftrightarrow |x-2| < \frac{\varepsilon}{4}; \delta = \frac{\varepsilon}{4}$$

$$23. |6x+36| < \varepsilon \Leftrightarrow |6(x+6)| < \varepsilon$$

$$\Leftrightarrow 6|x+6| < \varepsilon$$

$$\Leftrightarrow |x+6| < \frac{\varepsilon}{6}; \delta = \frac{\varepsilon}{6}$$

$$24. |5x+25| < \varepsilon \Leftrightarrow |5(x+5)| < \varepsilon$$

$$\Leftrightarrow 5|x+5| < \varepsilon$$

$$\Leftrightarrow |x+5| < \frac{\varepsilon}{5}; \delta = \frac{\varepsilon}{5}$$

$$25. C = \pi d$$

$$|C-10| \leq 0.02$$

$$|\pi d - 10| \leq 0.02$$

$$\left|\pi\left(d - \frac{10}{\pi}\right)\right| \leq 0.02$$

$$\left|d - \frac{10}{\pi}\right| \leq \frac{0.02}{\pi} \approx 0.0064$$

We must measure the diameter to an accuracy of 0.0064 in.

$$26. |C-50| \leq 1.5, \left|\frac{5}{9}(F-32)-50\right| \leq 1.5;$$

$$\frac{5}{9}|(F-32)-90| \leq 1.5$$

$$|F-122| \leq 2.7$$

We are allowed an error of 2.7° F .

$$27. |x-1| < 2|x-3|$$

$$|x-1| < |2x-6|$$

$$(x-1)^2 < (2x-6)^2$$

$$x^2 - 2x + 1 < 4x^2 - 24x + 36$$

$$3x^2 - 22x + 35 > 0$$

$$(3x-7)(x-5) > 0;$$

$$\left(-\infty, \frac{7}{3}\right) \cup (5, \infty)$$

$$28. |2x-1| \geq |x+1|$$

$$(2x-1)^2 \geq (x+1)^2$$

$$4x^2 - 4x + 1 \geq x^2 + 2x + 1$$

$$3x^2 - 6x \geq 0$$

$$3x(x-2) \geq 0$$

$$(-\infty, 0] \cup [2, \infty)$$

$$\begin{aligned}
 29. \quad & 2|2x-3| < |x+10| \\
 & |4x-6| < |x+10| \\
 & (4x-6)^2 < (x+10)^2 \\
 & 16x^2 - 48x + 36 < x^2 + 20x + 100 \\
 & 15x^2 - 68x - 64 < 0 \\
 & (5x+4)(3x-16) < 0: \\
 & \left(-\frac{4}{5}, \frac{16}{3}\right)
 \end{aligned}$$

$$\begin{aligned}
 30. \quad & |3x-1| < 2|x+6| \\
 & |3x-1| < |2x+12| \\
 & (3x-1)^2 < (2x+12)^2 \\
 & 9x^2 - 6x + 1 < 4x^2 + 48x + 144 \\
 & 5x^2 - 54x - 143 < 0 \\
 & (5x+11)(x-13) < 0; \\
 & \left(-\frac{11}{5}, 13\right)
 \end{aligned}$$

31. $|x| < |y| \Rightarrow |x||x| \leq |x||y|$ and $|x||y| < |y||y|$ Order property: $x < y \Leftrightarrow xz < yz$ when z is positive.

$$\begin{aligned}
 & \Rightarrow |x|^2 < |y|^2 \\
 & \Rightarrow x^2 < y^2
 \end{aligned}$$

Transitivity
 $(|x|^2 = x^2)$

Conversely,

$$\begin{aligned}
 x^2 < y^2 & \Rightarrow |x|^2 < |y|^2 & (x^2 = |x|^2) \\
 & \Rightarrow |x|^2 - |y|^2 < 0 & \text{Subtract } |y|^2 \text{ from each side.} \\
 & \Rightarrow (|x| - |y|)(|x| + |y|) < 0 & \text{Factor the difference of two squares.} \\
 & \Rightarrow |x| - |y| < 0 & \text{This is the only factor that can be negative.} \\
 & \Rightarrow |x| < |y| & \text{Add } |y| \text{ to each side.}
 \end{aligned}$$

$$\begin{aligned}
 32. \quad & 0 < a < b \Rightarrow a = (\sqrt{a})^2 \text{ and } b = (\sqrt{b})^2, \text{ so} \\
 & (\sqrt{a})^2 < (\sqrt{b})^2, \text{ and, by Problem 31,} \\
 & |\sqrt{a}| < |\sqrt{b}| \Rightarrow \sqrt{a} < \sqrt{b}
 \end{aligned}$$

$$33. \text{ a. } |a-b| = |a+(-b)| \leq |a| + |-b| = |a| + |b|$$

$$\text{b. } |a-b| \geq ||a| - |b|| \geq |a| - |b| \text{ Use Property 4 of absolute values.}$$

$$\text{c. } |a+b+c| = |(a+b)+c| \leq |a+b| + |c| \leq |a| + |b| + |c|$$

$$\begin{aligned}
 34. \quad & \left| \frac{1}{x^2+3} - \frac{1}{|x|+2} \right| = \left| \frac{1}{x^2+3} + \left(-\frac{1}{|x|+2} \right) \right| \\
 & \leq \left| \frac{1}{x^2+3} \right| + \left| -\frac{1}{|x|+2} \right| \\
 & = \left| \frac{1}{x^2+3} \right| + \left| \frac{1}{|x|+2} \right| \\
 & = \frac{1}{x^2+3} + \frac{1}{|x|+2}
 \end{aligned}$$

by the Triangular Inequality, and since

$$x^2+3 > 0, |x|+2 > 0 \Rightarrow \frac{1}{x^2+3} > 0, \frac{1}{|x|+2} > 0.$$

$$x^2+3 \geq 3 \text{ and } |x|+2 \geq 2, \text{ so}$$

$$\frac{1}{x^2+3} \leq \frac{1}{3} \text{ and } \frac{1}{|x|+2} \leq \frac{1}{2}, \text{ thus,}$$

$$\frac{1}{x^2+3} + \frac{1}{|x|+2} \leq \frac{1}{3} + \frac{1}{2}$$

$$35. \left| \frac{x-2}{x^2+9} \right| = \left| \frac{x+(-2)}{x^2+9} \right|$$

$$\left| \frac{x-2}{x^2+9} \right| \leq \left| \frac{x}{x^2+9} \right| + \left| \frac{-2}{x^2+9} \right|$$

$$\left| \frac{x-2}{x^2+9} \right| \leq \frac{|x|}{x^2+9} + \frac{2}{x^2+9} = \frac{|x|+2}{x^2+9}$$

$$\text{Since } x^2+9 \geq 9, \frac{1}{x^2+9} \leq \frac{1}{9}$$

$$\frac{|x|+2}{x^2+9} \leq \frac{|x|+2}{9}$$

$$\left| \frac{x-2}{x^2+9} \right| \leq \frac{|x|+2}{9}$$

$$36. |x| \leq 2 \Rightarrow |x^2+2x+7| \leq |x^2| + |2x| + 7$$

$$\leq 4+4+7=15$$

$$\text{and } |x^2+1| \geq 1 \text{ so } \frac{1}{x^2+1} \leq 1.$$

$$\text{Thus, } \left| \frac{x^2+2x+7}{x^2+1} \right| = |x^2+2x+7| \left| \frac{1}{x^2+1} \right|$$

$$\leq 15 \cdot 1 = 15$$

$$37. \left| x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 + \frac{1}{8}x + \frac{1}{16} \right|$$

$$\leq |x^4| + \frac{1}{2}|x^3| + \frac{1}{4}|x^2| + \frac{1}{8}|x| + \frac{1}{16}$$

$$\leq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \quad \text{since } |x| \leq 1.$$

$$\text{So } \left| x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 + \frac{1}{8}x + \frac{1}{16} \right| \leq 1.9375 < 2.$$

$$38. \text{ a. } x < x^2$$

$$x - x^2 < 0$$

$$x(1-x) < 0$$

$$x < 0 \text{ or } x > 1$$

$$\text{ b. } x^2 < x$$

$$x^2 - x < 0$$

$$x(x-1) < 0$$

$$0 < x < 1$$

$$39. a \neq 0 \Rightarrow$$

$$0 \leq \left(a - \frac{1}{a} \right)^2 = a^2 - 2 + \frac{1}{a^2}$$

$$\text{so, } 2 \leq a^2 + \frac{1}{a^2} \text{ or } a^2 + \frac{1}{a^2} \geq 2$$

$$40. a < b$$

$$a+a < a+b \text{ and } a+b < b+b$$

$$2a < a+b < 2b$$

$$a < \frac{a+b}{2} < b$$

$$41. 0 < a < b$$

$$a^2 < ab \text{ and } ab < b^2$$

$$a^2 < ab < b^2$$

$$a < \sqrt{ab} < b$$

$$42. \sqrt{ab} \leq \frac{1}{2}(a+b) \Leftrightarrow ab \leq \frac{1}{4}(a^2+2ab+b^2)$$

$$\Leftrightarrow 0 \leq \frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2 = \frac{1}{4}(a^2-2ab+b^2)$$

$$\Leftrightarrow 0 \leq \frac{1}{4}(a-b)^2 \text{ which is always true.}$$

$$43. \text{ For a rectangle the area is } ab, \text{ while for a square}$$

$$\text{the area is } a^2 = \left(\frac{a+b}{2} \right)^2. \text{ From Problem 42,}$$

$$\sqrt{ab} \leq \frac{1}{2}(a+b) \Leftrightarrow ab \leq \left(\frac{a+b}{2} \right)^2 \text{ so the square}$$

$$\text{has the largest area.}$$

$$44. A = \pi r^2; A = 4\pi(10)^2 = 400\pi$$

$$|4\pi r^2 - 400\pi| < 0.01$$

$$4\pi|r^2 - 100| < 0.01$$

$$|r^2 - 100| < \frac{0.01}{4\pi}$$

$$-\frac{0.01}{4\pi} < r^2 - 100 < \frac{0.01}{4\pi}$$

$$\sqrt{100 - \frac{0.01}{4\pi}} < r < \sqrt{100 + \frac{0.01}{4\pi}}$$

$$\delta \approx 0.00004 \text{ in}$$

1.5 Concepts Review

1. II; IV

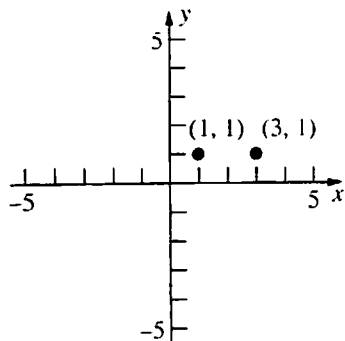
2. $\sqrt{(x+2)^2 + (y-3)^2}$

3. $(x+4)^2 + (y-2)^2 = 25$

4. $\left(\frac{-2+5}{2}, \frac{3+7}{2}\right) = (1.5, 5)$

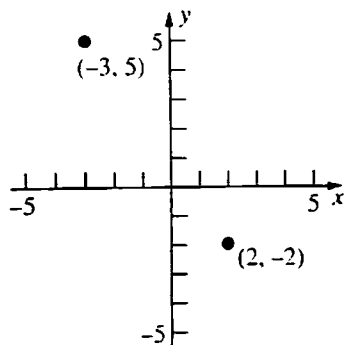
Problem Set 1.5

1.



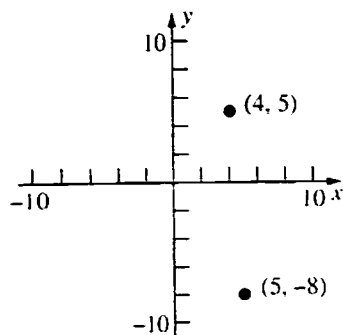
$$d = \sqrt{(3-1)^2 + (1-1)^2} = \sqrt{4} = 2$$

2.



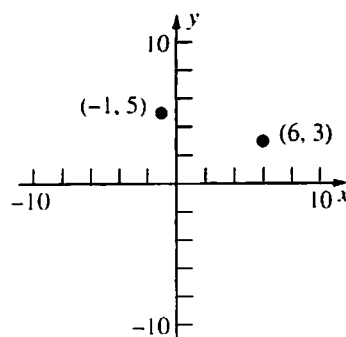
$$d = \sqrt{(-3-2)^2 + (5+2)^2} = \sqrt{74} \approx 8.60$$

3.



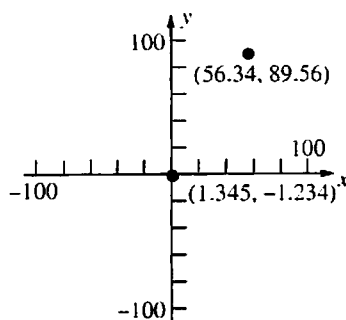
$$d = \sqrt{(4-5)^2 + (5+8)^2} = \sqrt{170} \approx 13.04$$

4.



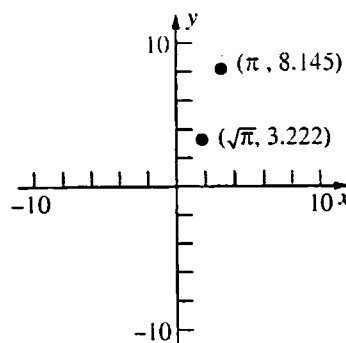
$$d = \sqrt{(-1-6)^2 + (5-3)^2} = \sqrt{49+4} = \sqrt{53} \approx 7.28$$

5.



$$d = \sqrt{(1.345-56.34)^2 + (-1.234-89.56)^2} \approx 106.151$$

6.



$$d = \sqrt{(\sqrt{\pi} - \pi)^2 + (3.222-8.145)^2} \approx 5.110$$

7. $d_1 = \sqrt{(5+2)^2 + (3-4)^2} = \sqrt{49+1} = \sqrt{50}$

$$d_2 = \sqrt{(5-10)^2 + (3-8)^2} = \sqrt{25+25} = \sqrt{50}$$

$$d_3 = \sqrt{(-2-10)^2 + (4-8)^2} = \sqrt{144+16} = \sqrt{160}$$

$d_1 = d_2$ so the triangle is isosceles.

8. $a = \sqrt{(2-4)^2 + (-4-0)^2} = \sqrt{4+16} = \sqrt{20}$

$$b = \sqrt{(4-8)^2 + (0+2)^2} = \sqrt{16+4} = \sqrt{20}$$

$$c = \sqrt{(2-8)^2 + (-4+2)^2} = \sqrt{36+4} = \sqrt{40}$$

$a^2 + b^2 = c^2$, so the triangle is a right triangle.

9. $(-1, -1), (-1, 3); (7, -1), (7, 3); (1, 1), (5, 1)$

10. $\sqrt{(x-3)^2 + (0-1)^2} = \sqrt{(x-6)^2 + (0-4)^2};$
 $x^2 - 6x + 10 = x^2 - 12x + 52$
 $6x = 42$
 $x = 7 \Rightarrow (7, 0)$

11. $\left(\frac{-2+4}{2}, \frac{-2+3}{2}\right) = \left(1, \frac{1}{2}\right);$
 $d = \sqrt{(1+2)^2 + \left(\frac{1}{2}-3\right)^2} = \sqrt{9 + \frac{25}{4}} \approx 3.91$

12. midpoint of $AB = \left(\frac{1+2}{2}, \frac{3+6}{2}\right) = \left(\frac{3}{2}, \frac{9}{2}\right)$
midpoint of $CD = \left(\frac{4+3}{2}, \frac{7+4}{2}\right) = \left(\frac{7}{2}, \frac{11}{2}\right)$
 $d = \sqrt{\left(\frac{3}{2}-\frac{7}{2}\right)^2 + \left(\frac{9}{2}-\frac{11}{2}\right)^2}$
 $= \sqrt{4+1} = \sqrt{5} \approx 2.24$

13. $(x-1)^2 + (y-1)^2 = 1$

14. $(x+2)^2 + (y-3)^2 = 4^2$
 $(x+2)^2 + (y-3)^2 = 16$

15. $(x-2)^2 + (y+1)^2 = r^2$
 $(5-2)^2 + (3+1)^2 = r^2$
 $r^2 = 9+16 = 25$
 $(x-2)^2 + (y+1)^2 = 25$

16. $(x-4)^2 + (y-3)^2 = r^2$
 $(6-4)^2 + (2-3)^2 = r^2$
 $r^2 = 4+1 = 5$
 $(x-4)^2 + (y-3)^2 = 5$

17. center = $\left(\frac{1+3}{2}, \frac{3+7}{2}\right) = (2, 5)$
radius = $\frac{1}{2}\sqrt{(1-3)^2 + (3-7)^2} = \frac{1}{2}\sqrt{4+16}$
 $= \frac{1}{2}\sqrt{20} = \sqrt{5}$
 $(x-2)^2 + (y-5)^2 = 5$

18. Since the circle is tangent to the x -axis, $r = 4$.

$$(x-3)^2 + (y-4)^2 = 16$$

19. Substitute $x = \frac{1}{4}$ into the equation and solve for y .

$$\left(-\frac{3}{4}\right)^2 + (y-1)^2 = 1$$

$$(y-1)^2 = \frac{7}{16}$$

$$y-1 = \pm \frac{\sqrt{7}}{4}$$

$$y = 1 \pm \frac{\sqrt{7}}{4}$$

20. Substitute $y = 1$ into the equation and solve for x .

$$(x-1)^2 + (0)^2 = 1$$

$$x-1 = \pm 1$$

$$x = 0, 2$$

21. $x^2 + 2x + 10 + y^2 - 6y - 10 = 0$

$$x^2 + 2x + y^2 - 6y = 0$$

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = 1 + 9$$

$$(x+1)^2 + (y-3)^2 = 10$$

$$\text{center} = (-1, 3); \text{radius} = \sqrt{10}$$

22. $x^2 + y^2 - 6y = 36$

$$x^2 + (y^2 - 6y + 9) = 16 + 9$$

$$x^2 + (y-3)^2 = 25$$

$$\text{center} = (0, 3); \text{radius} = \sqrt{5}$$

23. $x^2 + y^2 - 12x + 35 = 0$

$$x^2 - 12x + y^2 = -35$$

$$(x^2 - 12x + 36) + y^2 = -35 + 36$$

$$(x-6)^2 + y^2 = 1$$

$$\text{center} = (6, 0); \text{radius} = 1$$

24. $x^2 + y^2 - 10x + 10y = 0$

$$(x^2 - 10x + 25) + (y^2 + 10y + 25) = 25 + 25$$

$$(x-5)^2 + (y+5)^2 = 50$$

$$\text{center} = (5, -5); \text{radius} = \sqrt{50} = 5\sqrt{2}$$

25. $4x^2 + 16x + 15 + 4y^2 + 6y = 0$

$$4(x^2 + 4x + 4) + 4\left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = -15 + 16 + \frac{9}{4}$$

$$4(x+2)^2 + 4\left(y + \frac{3}{4}\right)^2 = \frac{13}{4}$$

$$(x+2)^2 + \left(y + \frac{3}{4}\right)^2 = \frac{13}{16}$$

$$\text{center} = \left(-2, -\frac{3}{4}\right); \text{radius} = \frac{\sqrt{13}}{4}$$

26. $4x^2 + 16x + \frac{105}{16} + 4y^2 + 3y = 0$

$$4\left(x^2 + 4x + 4\right) + 4\left(y^2 + \frac{3}{4}y + \frac{9}{64}\right)$$

$$= -\frac{105}{16} + 16 + \frac{9}{16}$$

$$4(x+2)^2 + 4\left(y + \frac{3}{8}\right)^2 = 10$$

$$(x+2)^2 + \left(y + \frac{3}{8}\right)^2 = \frac{5}{2}$$

$$\text{center} = \left(-2, -\frac{3}{8}\right); \text{radius} = \sqrt{\frac{5}{2}} = \frac{\sqrt{10}}{2}$$

27. center: $\left(\frac{2+6}{2}, \frac{-1+3}{2}\right) = (4, 1)$

$$\text{midpoint} = \left(\frac{2+6}{2}, \frac{3+3}{2}\right) = (4, 3)$$

$$\text{inscribed circle: radius} = \sqrt{(4-4)^2 + (1-3)^2}$$

$$= \sqrt{4} = 2$$

$$(x-4)^2 + (y-1)^2 = 4$$

circumscribed circle:

$$\text{radius} = \sqrt{(4-2)^2 + (1-3)^2} = \sqrt{8}$$

$$(x-4)^2 + (y-1)^2 = 8$$

28. The radius of each circle is $\sqrt{16} = 4$. The centers are $(1, -2)$ and $(-9, 10)$. The length of the belt is the sum of half the circumference of the first circle, half the circumference of the second circle, and twice the distance between their centers.

$$L = \frac{1}{2} \cdot 2\pi(4) + \frac{1}{2} \cdot 2\pi(4) + 2\sqrt{(1+9)^2 + (-2-10)^2}$$

$$= 8\pi + 2\sqrt{100+144}$$

$$\approx 56.37$$

29. $AC = \sqrt{AB^2 + BC^2} = \sqrt{(214)^2 + (179)^2}$

$$= \sqrt{77,837} \approx 278.99$$

$$\text{Cost by truck} = 3.71(214 + 179) = \$1458.03$$

Cost by plane = $4.82(279) = \$1344.78$; cheaper by plane.

30. Distance running = 6 mi

$$\text{Distance swimming} = \sqrt{(10-6)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{16 + \frac{1}{4}} = \sqrt{16.25}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Rate}} = \frac{6}{8} + \frac{\sqrt{16.25}}{3} \approx 2.09 \text{ hr}$$

31. Put the vertex of the right angle at the origin with the other vertices at $(a, 0)$ and $(0, b)$. The midpoint of the hypotenuse is $\left(\frac{a}{2}, \frac{b}{2}\right)$. The

distances from the vertices are

$$\sqrt{\left(a - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2},$$

$$\sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(b - \frac{b}{2}\right)^2} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2}, \text{ and}$$

$$\sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2},$$

which are all the same.

32. From Problem 31, the midpoint of the hypotenuse, $(4, 3)$, is equidistant from the vertices. This is the center of the circle. The radius is $\sqrt{16+9} = 5$. The equation of the circle is

$$(x-4)^2 + (y-3)^2 = 25.$$

33. $x^2 + y^2 - 4x - 2y - 11 = 0$

$$(x^2 - 4x + 4) + (y^2 - 2y + 1) = 11 + 4 + 1$$

$$(x-2)^2 + (y-1)^2 = 16$$

$$x^2 + y^2 + 20x - 12y + 72 = 0$$

$$(x^2 + 20x + 100) + (y^2 - 12y + 36)$$

$$= -72 + 100 + 36$$

$$(x+10)^2 + (y-6)^2 = 64$$

center of first circle: $(2, 1)$

center of second circle: $(-10, 6)$

$$d = \sqrt{(2+10)^2 + (1-6)^2} = \sqrt{144+25}$$

$$= \sqrt{169} = 13$$

However, the radii only sum to $4 + 8 = 12$, so the circles must not intersect if the distance between their centers is 13.

34. $x^2 + ax + y^2 + by + c = 0$

$$\left(x^2 + ax + \frac{a^2}{4}\right) + \left(y^2 + by + \frac{b^2}{4}\right)$$

$$= -c + \frac{a^2}{4} + \frac{b^2}{4}$$

$$\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \frac{a^2 + b^2 - 4c}{4}$$

$$\frac{a^2 + b^2 - 4c}{4} > 0 \Rightarrow a^2 + b^2 > 4c$$

35. Label the points C , P , Q , and R as shown in the figure below. Let $d = |OP|$, $h = |OQ|$, and $a = |PR|$. Triangles $\triangle OPR$ and $\triangle CQR$ are similar because each contains a right angle and they share angle $\angle QRC$. For an angle of 30° , $d/h = \sqrt{3}/2$. Thus, using a property of similar triangles,

$$|QC|/|RC| = \sqrt{3}/2$$

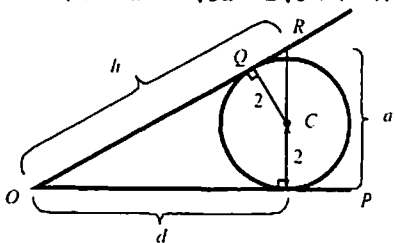
$$\frac{2}{a-2} = \frac{\sqrt{3}}{2}$$

$$a = 2 + 4/\sqrt{3}$$

$$\text{Thus, } h = 2a = 2(2 + 4/\sqrt{3}) = 4(1 + 2/\sqrt{3})$$

By the Pythagorean Theorem, we have

$$d = \sqrt{h^2 - a^2} = \sqrt{3}a = 2\sqrt{3} + 4 \approx 7.464$$



36. The equations of the two circles are

$$(x - R)^2 + (y - R)^2 = R^2$$

$$(x - r)^2 + (y - r)^2 = r^2$$

Let (a, a) denote the point where the two circles touch. This point must satisfy

$$(a - R)^2 + (a - R)^2 = R^2$$

$$(a - R)^2 = \frac{R^2}{2}$$

$$a = \left(1 \pm \frac{\sqrt{2}}{2}\right)R$$

$$\text{Since } a < R, a = \left(1 - \frac{\sqrt{2}}{2}\right)R.$$

At the same time, the point where the two circles touch must satisfy

$$(a - r)^2 + (a - r)^2 = r^2$$

$$a = \left(1 \pm \frac{\sqrt{2}}{2}\right)r$$

$$\text{Since } a > r, a = \left(1 + \frac{\sqrt{2}}{2}\right)r.$$

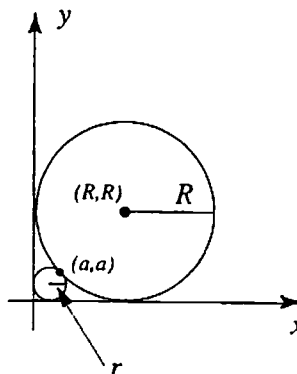
Equating the two expressions for a yields

$$\left(1 - \frac{\sqrt{2}}{2}\right)R = \left(1 + \frac{\sqrt{2}}{2}\right)r$$

$$r = \frac{1 - \frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}}R = \frac{\left(1 - \frac{\sqrt{2}}{2}\right)^2}{\left(1 + \frac{\sqrt{2}}{2}\right)\left(1 - \frac{\sqrt{2}}{2}\right)}R$$

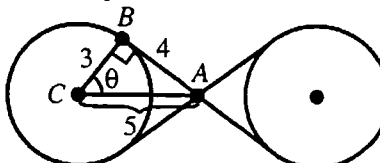
$$r = \frac{1 - \sqrt{2} + \frac{1}{2}}{1 - \frac{1}{2}}R$$

$$r = (3 - 2\sqrt{2})R \approx 0.1716R$$



37. The centers of the circles are

$\sqrt{(10-2)^2 + (8-2)^2} = \sqrt{100} = 10$ units apart, so the belts cross at a point 5 units from each center. The belt makes a right angle with the radius at point B , as shown in the figure.



From the Pythagorean Theorem, the length of

the belt from A to B is $\sqrt{5^2 - 3^2} = \sqrt{16} = 4$.

$\sin \theta = \frac{4}{5} \Rightarrow \theta \approx 0.93$ radians. The situation on

the lower half of the wheel is identical, and the two wheels are identical, so the length of the belt around each wheel is

$3(2\pi - 1.86) \approx 13.3$ units. The length of the belt is $2(13.3) + 4(4) \approx 42.6$ units.

$$38. \quad 2\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x-3)^2 + (y-4)^2}$$

$$4(x^2 - 2x + 1 + y^2 - 2y + 1)$$

$$= x^2 - 6x + 9 + y^2 - 8y + 16$$

$$3x^2 - 2x + 3y^2 = 9 + 16 - 4 - 4;$$

$$3x^2 - 2x + 3y^2 = 17; x^2 - \frac{2}{3}x + y^2 = \frac{17}{3};$$

$$\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + y^2 = \frac{17}{3} + \frac{1}{9}$$

$$\left(x - \frac{1}{3}\right)^2 + y^2 = \frac{52}{9}$$

$$\text{center: } \left(\frac{1}{3}, 0\right); \text{ radius: } \left(\frac{\sqrt{52}}{3}\right)$$

39. Let a , b , and c be the lengths of the sides of the right triangle, with c the length of the hypotenuse. Then the Pythagorean Theorem

says that $a^2 + b^2 = c^2$

$$\text{Thus, } \frac{\pi a^2}{8} + \frac{\pi b^2}{8} = \frac{\pi c^2}{8} \text{ or}$$

$$\frac{1}{2}\pi\left(\frac{a}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 = \frac{1}{2}\pi\left(\frac{c}{2}\right)^2$$

$$\frac{1}{2}\pi\left(\frac{x}{2}\right)^2 \text{ is the area of a semicircle with}$$

diameter x , so the circles on the legs of the triangle have total area equal to the area of the semicircle on the hypotenuse.

From $a^2 + b^2 = c^2$,

$$\frac{\sqrt{3}}{4}a^2 + \frac{\sqrt{3}}{4}b^2 = \frac{\sqrt{3}}{4}c^2$$

$$\frac{\sqrt{3}}{4}x^2 \text{ is the area of an equilateral triangle with}$$

sides of length x , so the equilateral triangles on the legs of the right triangle have total area equal to the area of the equilateral triangle on the hypotenuse of the right triangle.

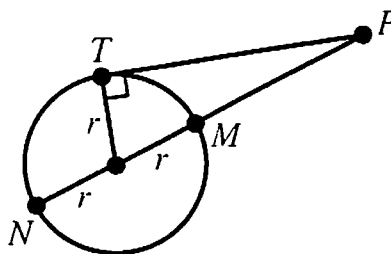
40. See the figure below. The angle at T is a right angle, so the Pythagorean Theorem gives

$$(PM + r)^2 = (PT)^2 + r^2$$

$$\Leftrightarrow (PM)^2 + 2rPM + r^2 = (PT)^2 + r^2$$

$$\Leftrightarrow PM(PM + 2r) = (PT)^2$$

$$PM + 2r = PN \text{ so this gives } (PM)(PN) = (PT)^2$$



41. The lengths A , B , and C are the same as the corresponding distances between the centers of the circles:

$$A = \sqrt{(-2)^2 + (8)^2} = \sqrt{68} \approx 8.2$$

$$B = \sqrt{(6)^2 + (8)^2} = \sqrt{100} = 10$$

$$C = \sqrt{(8)^2 + (0)^2} = \sqrt{64} = 8$$

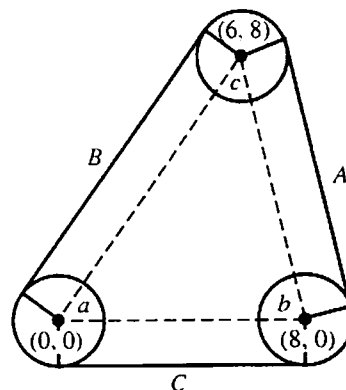
Each circle has radius 2, so the part of the belt around the wheels is

$$2(2\pi - a - \pi) + 2(2\pi - b - \pi) + 2(2\pi - c - \pi)$$

$$= 2[3\pi - (a + b + c)] = 2(2\pi) = 4\pi$$

Since $a + b + c = \pi$, the sum of the angles of a triangle.

The length of the belt is $\approx 8.2 + 10 + 8 + 4\pi$
 ≈ 38.8 units.



42. In Problems 28 and 41, the curved portions of the belt have total length $2\pi r$. The lengths of the straight portions will be the same as the lengths of the sides. The belt will have length $2\pi r + d_1 + d_2 + \dots + d_n$.

1.6 Concepts Review

1. $\frac{(d-b)}{(c-a)}$
2. 0
3. $y = mx + b; x = k$
4. $Ax + By + C = 0$

Problem Set 1.6

1. $\frac{2-1}{2-1} = 1$
2. $\frac{7-5}{4-3} = 1$
3. $\frac{-6-3}{-5-2} = \frac{9}{7}$
4. $\frac{-6+4}{0-2} = 1$
5. $\frac{5-0}{0-3} = -\frac{5}{3}$
6. $\frac{6-0}{0+6} = 1$
7. $\frac{-3.456 - 5.678}{7.654 + 1.234} \approx -1.028$
8. $\frac{\sqrt{2} - \sqrt{3}}{1.642 - \pi} \approx 0.212$
9. $y - 2 = -1(x - 2)$
 $y - 2 = -x + 2$
 $x + y - 4 = 0$
10. $y - 4 = -1(x - 3)$
 $y - 4 = -x + 3$
 $x + y - 7 = 0$
11. $y = 2x + 3$
 $2x - y + 3 = 0$
12. $y = 0x + 5$
 $0x + y - 5 = 0$
13. $m = \frac{8-3}{4-2} = \frac{5}{2};$
 $y - 3 = \frac{5}{2}(x - 2)$
 $2y - 6 = 5x - 10$
 $5x - 2y - 4 = 0$
14. $m = \frac{2-1}{8-4} = \frac{1}{4};$
 $y - 1 = \frac{1}{4}(x - 4)$
 $4y - 4 = x - 4$
 $x - 4y + 0 = 0$
15. $m = \frac{5+3}{2-2}$: undefined; $x + 0y - 2 = 0$
16. $x = -5; x + 0y + 5 = 0$
17. $3y = -2x + 1; y = -\frac{2}{3}x + \frac{1}{3}$; slope $= -\frac{2}{3}$;
 y -intercept $= \frac{1}{3}$
18. $-4y = 5x - 6$
 $y = -\frac{5}{4}x + \frac{3}{2}$
slope $= -\frac{5}{4}$; y -intercept $= \frac{3}{2}$
19. $6 - 2y = 10x - 2$
 $-2y = 10x - 8$
 $y = -5x + 4$;
slope $= -5$; y -intercept $= 4$
20. $4x + 5y = -20$
 $5y = -4x - 20$
 $y = -\frac{4}{5}x - 4$
slope $= -\frac{4}{5}$; y -intercept $= -4$
21. a. $m = 2$;
 $y + 3 = 2(x - 3)$
 $y = 2x - 9$
- b. $m = -\frac{1}{2}$;
 $y + 3 = -\frac{1}{2}(x - 3)$
 $y = -\frac{1}{2}x - \frac{3}{2}$
- c. $2x + 3y = 6$
 $3y = -2x + 6$
 $y = -\frac{2}{3}x + 2$;
 $m = -\frac{2}{3}$;
 $y + 3 = -\frac{2}{3}(x - 3)$
 $y = -\frac{2}{3}x - 1$

d. $m = \frac{3}{2};$

$$y + 3 = \frac{3}{2}(x - 3)$$

$$y = \frac{3}{2}x - \frac{15}{2}$$

e. $m = \frac{-1-2}{3+1} = -\frac{3}{4};$

$$y + 3 = -\frac{3}{4}(x - 3)$$

$$y = -\frac{3}{4}x - \frac{3}{4}$$

f. $x = 3$ g. $y = -3$

22. a. $3x + cy = 5$
 $3(3) + c(1) = 5$
 $c = -4$

b. $c = 0$

c. $2x + y = -1$
 $y = -2x - 1$

$$m = -2;$$

$$3x + cy = 5$$

$$cy = -3x + 5$$

$$y = -\frac{3}{c}x + \frac{5}{c}$$

$$-2 = -\frac{3}{c}$$

$$c = \frac{3}{2}$$

d. c must be the same as the coefficient of x ,
so $c = 3$.

e. $y - 2 = 3(x + 3);$
perpendicular slope $= -\frac{1}{3};$

$$-\frac{1}{3} = -\frac{3}{c}$$

$$c = 9$$

23. $m = \frac{3}{2};$

$$y + 1 = \frac{3}{2}(x + 2)$$

$$y = \frac{3}{2}x + 2$$

24. a. $m = 2;$

$$kx - 3y = 10$$

$$-3y = -kx + 10$$

$$y = \frac{k}{3}x - \frac{10}{3}$$

$$\frac{k}{3} = 2; k = 6$$

b. $m = -\frac{1}{2};$

$$\frac{k}{3} = -\frac{1}{2}$$

$$k = -\frac{3}{2}$$

c. $2x + 3y = 6$

$$3y = -2x + 6$$

$$y = -\frac{2}{3}x + 2;$$

$$m = \frac{3}{2}; \frac{k}{3} = \frac{3}{2}; k = \frac{9}{2}$$

25. $y = 3(3) - 1 = 8; (3, 9)$ is above the line.

26. $(a, 0), (0, b); m = \frac{b-0}{0-a} = -\frac{b}{a}$

$$y = -\frac{b}{a}x + b; \frac{bx}{a} + y = b; \frac{x}{a} + \frac{y}{b} = 1$$

27. $2x + 3y = 4$

$$-3x + y = 5$$

$$2x + 3y = 4$$

$$9x - 3y = -15$$

$$\frac{11x}{11} = -11$$

$$x = -1$$

$$-3(-1) + y = 5$$

$$y = 2$$

Point of intersection: $(-1, 2)$

$$3y = -2x + 4$$

$$y = -\frac{2}{3}x + \frac{4}{3}$$

$$m = \frac{3}{2}$$

$$y - 2 = \frac{3}{2}(x + 1)$$

$$y = \frac{3}{2}x + \frac{7}{2}$$

$$28. \begin{aligned} 4x - 5y &= 8 \\ 2x + y &= -10 \end{aligned}$$

$$\begin{aligned} 4x - 5y &= 8 \\ -4x - 2y &= 20 \\ \hline -7y &= 28 \end{aligned}$$

$$\begin{aligned} y &= -4 \\ 4x - 5(-4) &= 8 \\ 4x &= -12 \\ x &= -3 \end{aligned}$$

Point of intersection: $(-3, -4)$;

$$\begin{aligned} 4x - 5y &= 8 \\ -5y &= -4x + 8 \\ y &= \frac{4}{5}x - \frac{8}{5} \end{aligned}$$

$$m = -\frac{5}{4}$$

$$\begin{aligned} y + 4 &= -\frac{5}{4}(x + 3) \\ y &= -\frac{5}{4}x - \frac{31}{4} \end{aligned}$$

$$29. \begin{aligned} 3x - 4y &= 5 \\ 2x + 3y &= 9 \end{aligned}$$

$$\begin{aligned} 9x - 12y &= 15 \\ 8x + 12y &= 36 \\ \hline 17x &= 51 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} 3(3) - 4y &= 5 \\ -4y &= -4 \\ y &= 1 \end{aligned}$$

Point of intersection: $(3, 1)$; $3x - 4y = 5$;

$$\begin{aligned} -4y &= -3x + 5 \\ y &= \frac{3}{4}x - \frac{5}{4} \end{aligned}$$

$$m = -\frac{4}{3}$$

$$\begin{aligned} y - 1 &= -\frac{4}{3}(x - 3) \\ y &= -\frac{4}{3}x + 5 \end{aligned}$$

$$30. \begin{aligned} 5x - 2y &= 5 \\ 2x + 3y &= 6 \end{aligned}$$

$$\begin{aligned} 15x - 6y &= 15 \\ 4x + 6y &= 12 \\ \hline 19x &= 27 \end{aligned}$$

$$x = \frac{27}{19}$$

$$2\left(\frac{27}{19}\right) + 3y = 6$$

$$3y = \frac{60}{19}$$

$$y = \frac{20}{19}$$

Point of intersection: $\left(\frac{27}{19}, \frac{20}{19}\right)$;

$$\begin{aligned} 5x - 2y &= 5 \\ -2y &= -5x + 5 \end{aligned}$$

$$y = \frac{5}{2}x - \frac{5}{2}$$

$$m = -\frac{2}{5}$$

$$\begin{aligned} y - \frac{20}{19} &= -\frac{2}{5}\left(x - \frac{27}{19}\right) \\ y &= -\frac{2}{5}x + \frac{54}{95} + \frac{20}{19} \\ y &= -\frac{2}{5}x + \frac{154}{95} \end{aligned}$$

$$31. \begin{aligned} A &= 3, B = 4, C = -6 \\ d &= \frac{|3(-3) + 4(2) + (-6)|}{\sqrt{(3)^2 + (4)^2}} = \frac{7}{5} \end{aligned}$$

$$32. d = \frac{|2(4) - 2(-1) + 4|}{\sqrt{(2)^2 + (2)^2}} = \frac{14}{\sqrt{8}} = \frac{7\sqrt{2}}{2}$$

$$33. \begin{aligned} A &= 12, B = -5, C = 1 \\ d &= \frac{|12(-2) - 5(-1) + 1|}{\sqrt{(12)^2 + (-5)^2}} = \frac{18}{13} \end{aligned}$$

$$34. \begin{aligned} A &= 2, B = -1, C = -5 \\ d &= \frac{|2(3) - 1(-1) - 5|}{\sqrt{(2)^2 + (-1)^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \end{aligned}$$

35. $2x + 4(0) = 5$

$$x = \frac{5}{2}$$

$$d = \frac{\left| 2\left(\frac{5}{2}\right) + 4(0) - 7 \right|}{\sqrt{(2)^2 + (4)^2}} = \frac{2}{\sqrt{20}} = \frac{\sqrt{5}}{5}$$

36. $7(0) - 5y = -1$

$$y = \frac{1}{5}$$

$$d = \frac{\left| 7(0) - 5\left(\frac{1}{5}\right) - 6 \right|}{\sqrt{(7)^2 + (-5)^2}} = \frac{7}{\sqrt{74}} = \frac{7\sqrt{74}}{74}$$

37. $120,000(0.08) = 9600$; $V = 120,000 - 9600t$

38. Slope = -9600 ; The bulldozer depreciates at \$9600 per year.

39. $(0, 700,000)$, $(10, 820,000)$

$$m = \frac{820,000 - 700,000}{10 - 0} = 12,000$$

$$N = 12,000n + 700,000$$

$$\text{At } n = 25: N = 12,000(25) + 700,000 = 1,000,000$$

40. $(0, 80,000)$, $(20, 2000)$

$$m = \frac{2000 - 80,000}{20 - 0} = -3900$$

$$V = -3900n + 80,000$$

41. a. When $x = 0$, $P = -2000$, which indicates that the company loses money if no items are sold.

b. Slope = 450; this is the amount of profit gained with the sale of each item.

42. a. Slope = 0.75; this is the amount of money added to the cost with each item produced.

b. When $x = 0$, $C = 200$. This is the fixed cost, that is, the cost to produce zero units..

43. If (x_0, y_0) is on both lines, then

$$2x_0 - y_0 + 4 = 0 \text{ and } x_0 + 3y_0 - 6 = 0 \text{ so}$$

$$2x_0 - y_0 + 4 + k(x_0 + 3y_0 - 6) = 0 + 0 \cdot k = 0,$$

which means that (x_0, y_0) is on the line

$$2x - y + 4 + k(x + 3y - 6) = 0 \text{ regardless of the value of } k.$$

44. $\frac{x}{a} + \frac{y}{a} = 1$

$$\frac{2}{a} + \frac{3}{a} = 1$$

$$\frac{5}{a} = 1$$

$$a = 5$$

$$\frac{x}{5} + \frac{y}{5} = 1$$

$$x + y - 5 = 0$$

45. $m = \frac{-2 - 3}{1 + 2} = -\frac{5}{3}$; $m = \frac{3}{5}$; passes through

$$\left(\frac{-2 + 1}{2}, \frac{3 - 2}{2} \right) = \left(-\frac{1}{2}, \frac{1}{2} \right)$$

$$y - \frac{1}{2} = \frac{3}{5} \left(x + \frac{1}{2} \right)$$

$$y = \frac{3}{5}x + \frac{4}{5}$$

46. $m = \frac{0 - 4}{2 - 0} = -2$; $m = \frac{1}{2}$; passes through

$$\left(\frac{0 + 2}{2}, \frac{4 + 0}{2} \right) = (1, 2)$$

$$y - 2 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{3}{2}$$

$$m = \frac{6 - 0}{4 - 2} = 3$$
; $m = -\frac{1}{3}$; passes through

$$\left(\frac{2 + 4}{2}, \frac{0 + 6}{2} \right) = (3, 3)$$

$$y - 3 = -\frac{1}{3}(x - 3)$$

$$y = -\frac{1}{3}x + 4$$

$$\frac{1}{2}x + \frac{3}{2} = -\frac{1}{3}x + 4$$

$$\frac{5}{6}x = \frac{5}{2}$$

$$x = 3$$

$$y = \frac{1}{2}(3) + \frac{3}{2} = 3$$

center = $(3, 3)$

47. Let the origin be at the vertex as shown in the figure below. The center of the circle is then

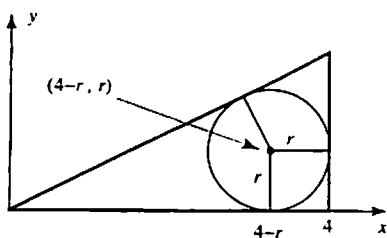
$(4 - r, r)$, so it has equation

$$(x - (4 - r))^2 + (y - r)^2 = r^2. \text{ Along the side of}$$

length 5, the y -coordinate is always $\frac{3}{4}$ times the x -coordinate. Thus, we need to find the value of r for which there is exactly one x -solution to $(x-4+r)^2 + (y-r)^2 = r^2$. Solving for x in this equation gives

$$x = \frac{4}{25} \left(16 - r \pm \sqrt{24(-r^2 + 7r - 6)} \right).$$
 There is

exactly one solution when $-r^2 + 7r - 6 = 0$, that is, when $r = 1$ or $r = 6$. The root $r = 6$ is extraneous. Thus, the largest circle that can be inscribed in this triangle has radius $r = 1$.



48. The line tangent to the circle at (a, b) will be perpendicular to the line through (a, b) and the center of the circle, which is $(0, 0)$. The line through (a, b) and $(0, 0)$ has slope
- $$m = \frac{0-b}{0-a} = \frac{b}{a}; ax + by = r^2 \Rightarrow y = -\frac{a}{b}x + \frac{r^2}{b}$$
- so $ax + by = r^2$ has slope $-\frac{a}{b}$ and is perpendicular to the line through (a, b) and $(0, 0)$, so it is tangent to the circle at (a, b) .

49. $12a + 0b = 36$
 $a = 3$
 $3^2 + b^2 = 36$
 $b = \pm 3\sqrt{3}$
 $3x - 3\sqrt{3}y = 36$
 $x - \sqrt{3}y = 12$
 $3x + 3\sqrt{3}y = 36$
 $x + \sqrt{3}y = 12$

50. $A = m, B = -1, C = B - b; (0, 0)$

$$d = \frac{|m(0) - 1(0) + B - b|}{\sqrt{m^2 + (-1)^2}} = \frac{|B - b|}{\sqrt{m^2 + 1}}$$

- The midpoint of the side from $(0, 0)$ to $(a, 0)$ is
51. $\left(\frac{0+a}{2}, \frac{0+0}{2} \right) = \left(\frac{a}{2}, 0 \right)$

The midpoint of the side from $(0, 0)$ to (b, c) is

$$\left(\frac{0+b}{2}, \frac{0+c}{2} \right) = \left(\frac{b}{2}, \frac{c}{2} \right)$$

$$m_1 = \frac{c-0}{b-a} = \frac{c}{b-a}$$

$$m_2 = \frac{\frac{c}{2}-0}{\frac{b}{2}-\frac{a}{2}} = \frac{c}{b-a}; m_1 = m_2$$

52. See the figure below. The midpoints of the sides are

$$P \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right), Q \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2} \right),$$

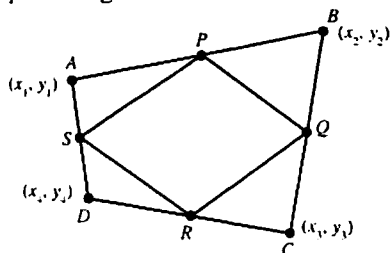
$$R \left(\frac{x_3 + x_4}{2}, \frac{y_3 + y_4}{2} \right), \text{ and}$$

$$S \left(\frac{x_1 + x_4}{2}, \frac{y_1 + y_4}{2} \right). \text{ The slope of } PS \text{ is}$$

$$\frac{\frac{1}{2}[y_1 + y_4 - (y_1 + y_2)]}{\frac{1}{2}[x_1 + x_4 - (x_1 + x_2)]} = \frac{y_4 - y_2}{x_4 - x_2}. \text{ The slope of}$$

$$QR \text{ is } \frac{\frac{1}{2}[y_3 + y_4 - (y_2 + y_3)]}{\frac{1}{2}[x_3 + x_4 - (x_2 + x_3)]} = \frac{y_4 - y_2}{x_4 - x_2}. \text{ Thus}$$

PS and QR are parallel. The slopes of SR and PQ are both $\frac{y_3 - y_1}{x_3 - x_1}$, so $PQRS$ is a parallelogram.



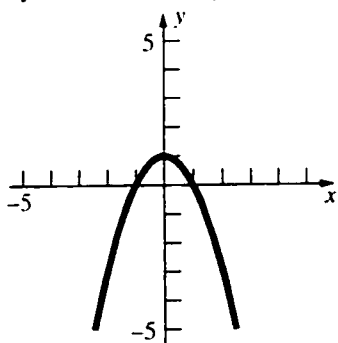
53. $x^2 + (y-6)^2 = 25$; passes through $(3, 2)$
 tangent line: $3x - 4y = 1$
 The dirt hits the wall at $y = 8$.

1.7 Concepts Review

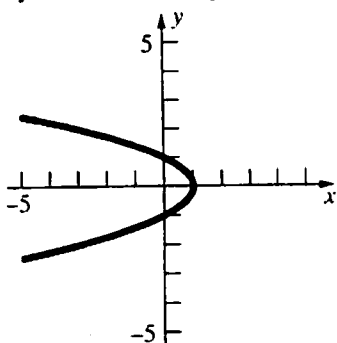
1. y -axis
2. $(4, -2)$
3. 8; -2, 1, 4
4. line; parabola

Problem Set 1.7

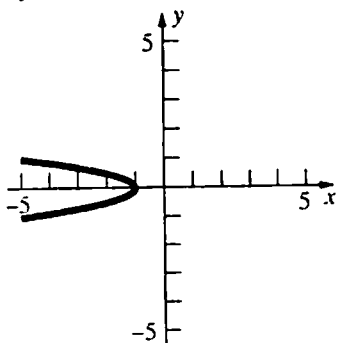
1. $y = -x^2 + 1$; y -intercept = 1; $y = (1+x)(1-x)$;
 x -intercepts = -1, 1
 Symmetric with respect to the y -axis



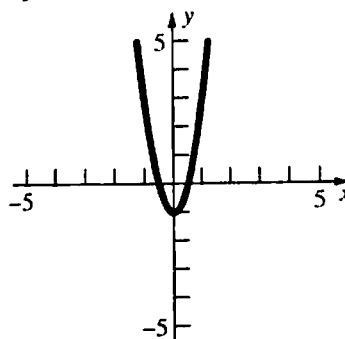
2. $x = -y^2 + 1$; y -intercepts = -1, 1;
 x -intercept = 1
 Symmetric with respect to the x -axis.



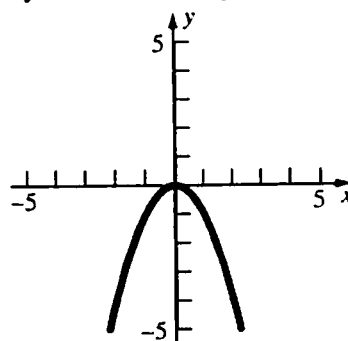
3. $x = -4y^2 - 1$; x -intercept = -1
 Symmetric with respect to the x -axis



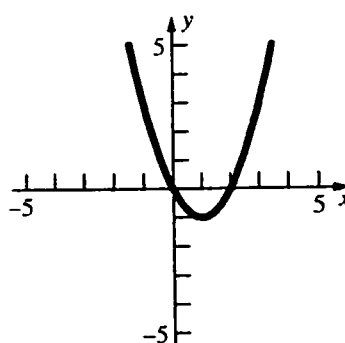
4. $y = 4x^2 - 1$; y -intercept = -1
 $y = (2x+1)(2x-1)$; x -intercepts = $-\frac{1}{2}, \frac{1}{2}$
 Symmetric with respect to the y -axis.



5. $x^2 + y = 0$; $y = -x^2$
 x -intercept = 0, y -intercept = 0
 Symmetric with respect to the y -axis



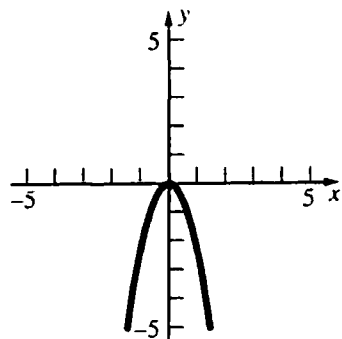
6. $y = x^2 - 2x$; y -intercept = 0
 $y = x(2-x)$; x -intercepts = 0, 2



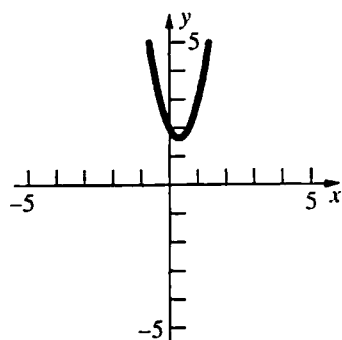
7. $7x^2 + 3y = 0$; $3y = -7x^2$; $y = -\frac{7}{3}x^2$

x -intercept = 0, y -intercept = 0

Symmetric with respect to the y -axis



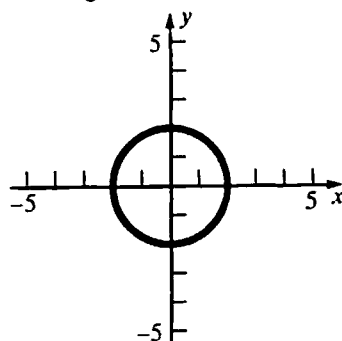
8. $y = 3x^2 - 2x + 2$; y -intercept = 2



9. $x^2 + y^2 = 4$

x -intercepts = -2, 2; y -intercepts = -2, 2

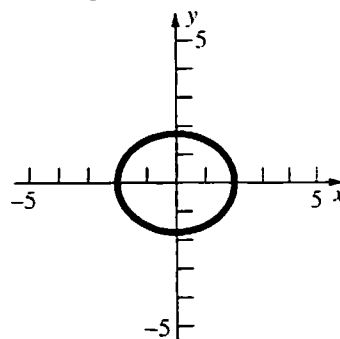
Symmetric with respect to the x -axis, y -axis, and origin



10. $3x^2 + 4y^2 = 12$; y -intercepts = $-\sqrt{3}, \sqrt{3}$

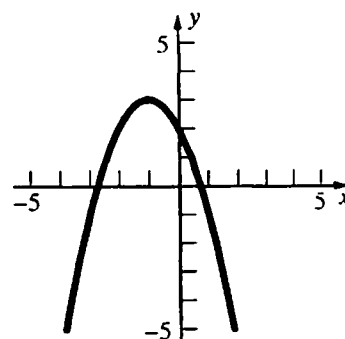
x -intercepts = -2, 2

Symmetric with respect to the x -axis, y -axis, and origin



11. $y = -x^2 - 2x + 2$; y -intercept = 2

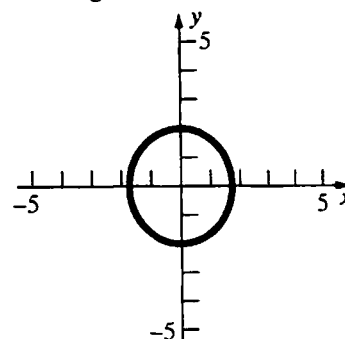
$$x\text{-intercepts} = \frac{2 \pm \sqrt{4+8}}{-2} = \frac{2 \pm 2\sqrt{3}}{-2} = -1 \pm \sqrt{3}$$



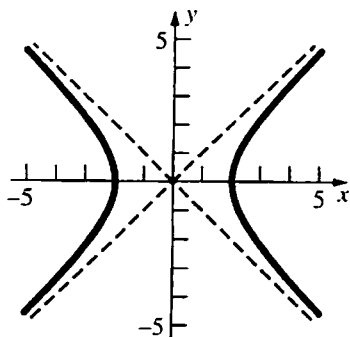
12. $4x^2 + 3y^2 = 12$; y -intercepts = -2, 2

x -intercepts = $-\sqrt{3}, \sqrt{3}$

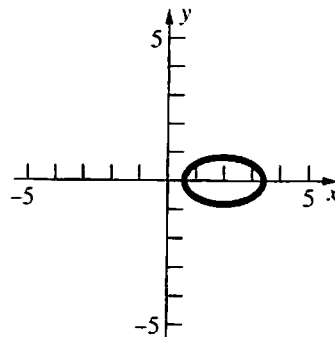
Symmetric with respect to the x -axis, y -axis, and origin



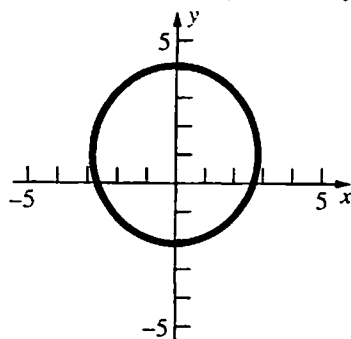
13. $x^2 - y^2 = 4$
 x -intercept = -2, 2
 Symmetric with respect to the x -axis, y -axis,
 and origin



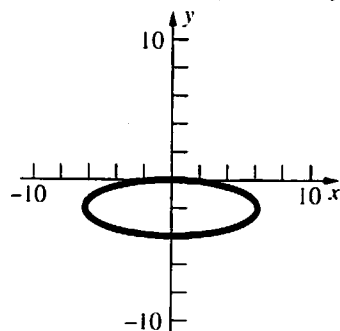
16. $x^2 - 4x + 3y^2 = -2$
 x -intercepts = $2 \pm \sqrt{2}$
 Symmetric with respect to the x -axis



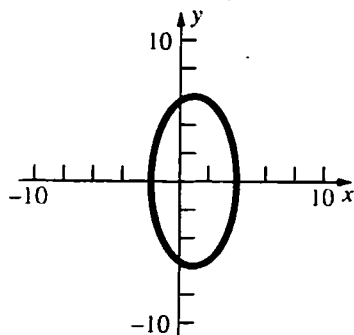
14. $x^2 + (y - 1)^2 = 9$; y -intercepts = -2, 4
 x -intercepts = $-2\sqrt{2}, 2\sqrt{2}$
 Symmetric with respect to the y -axis



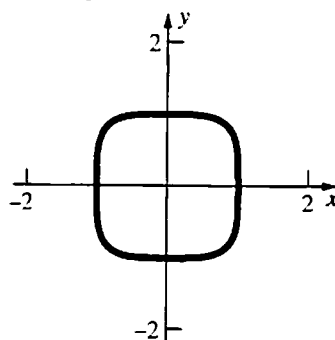
17. $x^2 + 9(y + 2)^2 = 36$; y -intercepts = -4, 0
 x -intercept = 0
 Symmetric with respect to the y -axis



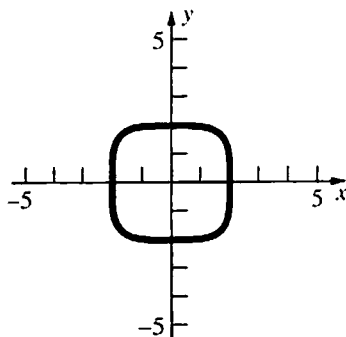
15. $4(x - 1)^2 + y^2 = 36$;
 y -intercepts = $\pm\sqrt{32} = \pm4\sqrt{2}$
 x -intercepts = -2, 4
 Symmetric with respect to the x -axis



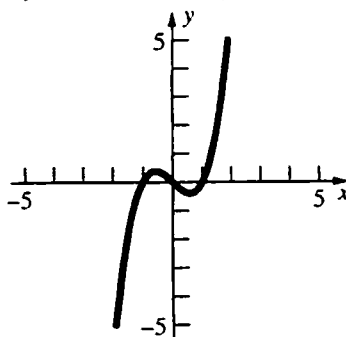
18. $x^4 + y^4 = 1$; y -intercepts = -1, 1
 x -intercepts = -1, 1
 Symmetric with respect to the x -axis, y -axis,
 and origin



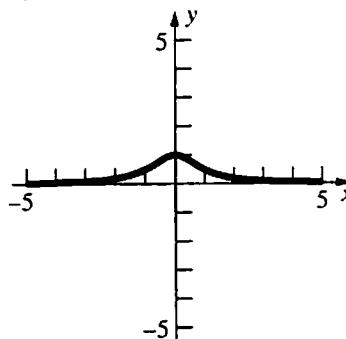
19. $x^4 + y^4 = 16$; y -intercepts = $-2, 2$
 x -intercepts = $-2, 2$
 Symmetric with respect to the y -axis, x -axis and origin



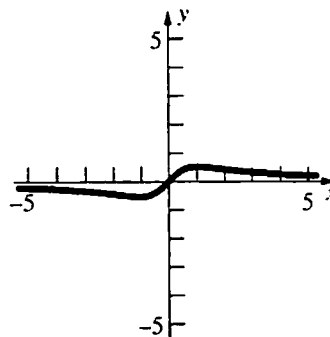
20. $y = x^3 - x$; y -intercepts = 0 ;
 $y = x(x^2 - 1) = x(x + 1)(x - 1)$;
 x -intercepts = $-1, 0, 1$
 Symmetric with respect to the origin



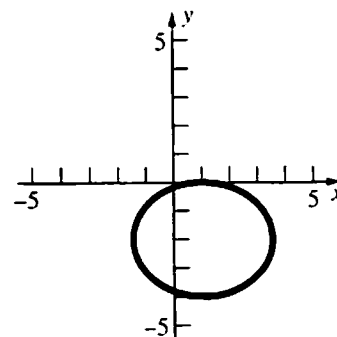
21. $y = \frac{1}{x^2 + 1}$; y -intercept = 1
 Symmetric with respect to the y -axis



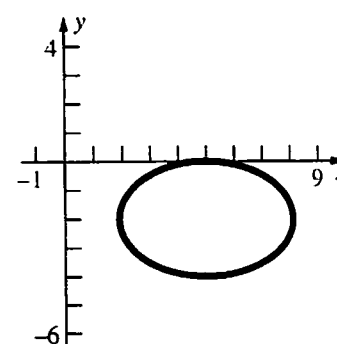
22. $y = \frac{x}{x^2 + 1}$; y -intercept = 0
 x -intercept = 0
 Symmetric with respect to the origin



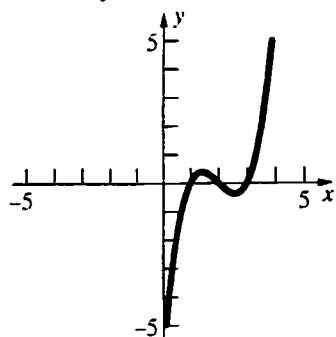
23. $2x^2 - 4x + 3y^2 + 12y = -2$
 $2(x^2 - 2x + 1) + 3(y^2 + 4y + 4) = -2 + 2 + 12$
 $2(x - 1)^2 + 3(y + 2)^2 = 12$
 y -intercepts = $-2 \pm \frac{\sqrt{30}}{3}$
 x -intercept = 1



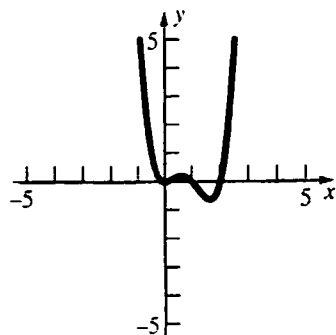
24. $4(x - 5)^2 + 9(y + 2)^2 = 36$; x -intercept = 5



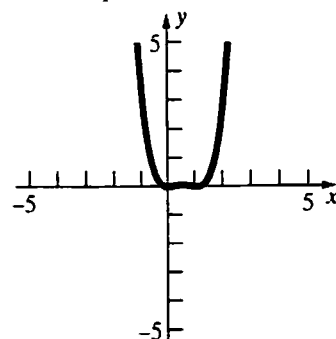
25. $y = (x-1)(x-2)(x-3)$; y -intercept = -6
 x -intercepts = $1, 2, 3$



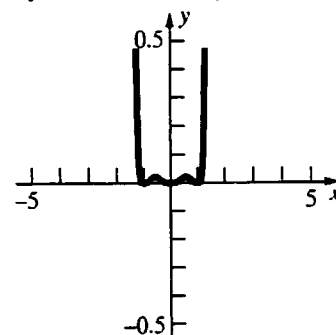
26. $y = x^2(x-1)(x-2)$; y -intercept = 0
 x -intercepts = $0, 1, 2$



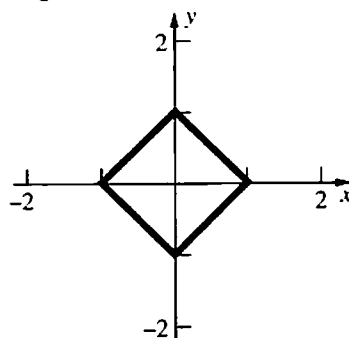
27. $y = x^2(x-1)^2$; y -intercept = 0
 x -intercepts = $0, 1$



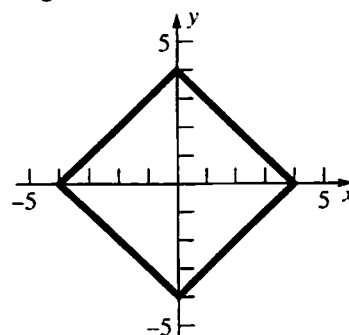
28. $y = x^4(x-1)^4(x+1)^4$; y -intercept = 0
 x -intercepts = $-1, 0, 1$
 Symmetric with respect to the y -axis



29. $|x| + |y| = 1$; y -intercepts = $-1, 1$;
 x -intercepts = $-1, 1$
 Symmetric with respect to the x -axis, y -axis and origin



30. $|x| + |y| = 4$; y -intercepts = $-4, 4$;
 x -intercepts = $-4, 4$
 Symmetric with respect to the x -axis, y -axis and origin



$$-x + 1 = (x+1)^2$$

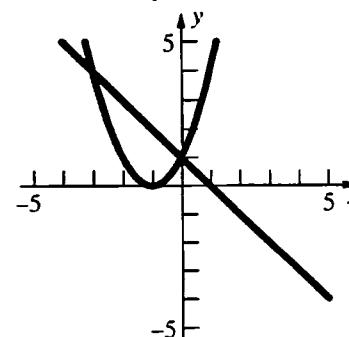
$$-x + 1 = x^2 + 2x + 1$$

31. $x^2 + 3x = 0$

$$x(x+3) = 0$$

$$x = 0, -3$$

Intersection points: $(0, 1)$ and $(-3, 4)$

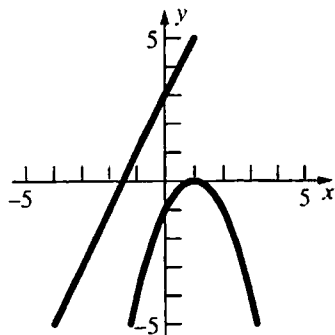


32. $2x+3=-(x-1)^2$

$$2x-3=-x^2+2x-1$$

$$x^2+4=0$$

No points of intersection



33. $-2x+3=-2(x-4)^2$

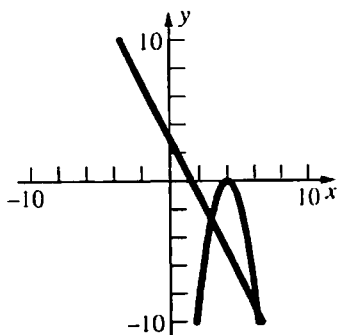
$$-2x+3=-2x^2+16x-32$$

$$2x^2-18x+35=0$$

$$x = \frac{18 \pm \sqrt{324 - 280}}{4} = \frac{18 \pm 2\sqrt{11}}{4} = \frac{9 \pm \sqrt{11}}{2};$$

Intersection points: $\left(\frac{9-\sqrt{11}}{2}, -6+\sqrt{11}\right),$

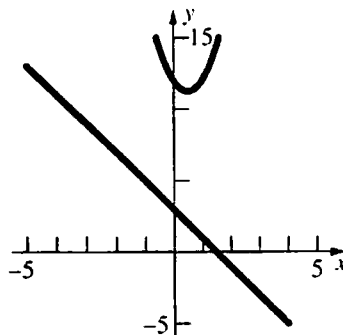
$$\left(\frac{9+\sqrt{11}}{2}, -6-\sqrt{11}\right)$$



34. $-2x+3=-3x^2-3x+12$

$$3x^2-x+9=0$$

No points of intersection

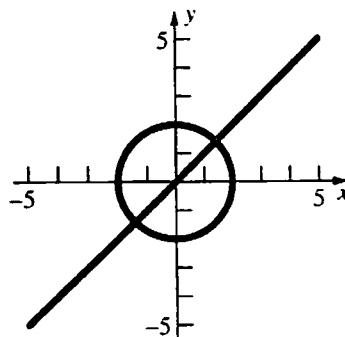


$$x^2+x^2=4$$

35. $x^2=2$

$$x = \pm\sqrt{2}$$

Intersection points: $(-\sqrt{2}, -\sqrt{2}), (\sqrt{2}, \sqrt{2})$



36. $2x^2+3(x-1)^2=12$

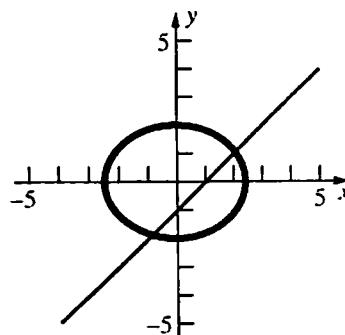
$$2x^2+3x^2-6x+3=12$$

$$5x^2-6x-9=0$$

$$x = \frac{6 \pm \sqrt{36+180}}{10} = \frac{6 \pm 6\sqrt{6}}{10} = \frac{3 \pm 3\sqrt{6}}{5}$$

Intersection points:

$$\left(\frac{3-3\sqrt{6}}{5}, \frac{-2-3\sqrt{6}}{5}\right), \left(\frac{3+3\sqrt{6}}{5}, \frac{-2+3\sqrt{6}}{5}\right)$$



37.

$$y = 3x + 1$$

$$x^2 + 2x + (3x + 1)^2 = 15$$

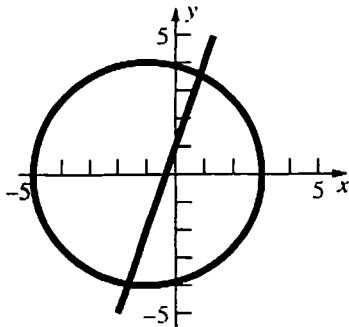
$$x^2 + 2x + 9x^2 + 6x + 1 = 15$$

$$10x^2 + 8x - 14 = 0$$

$$2(5x^2 + 4x - 7) = 0$$

$$x \approx -1.65, 0.85$$

Intersection points: $(-1.65, -3.95)$ and $(0.85, 3.55)$



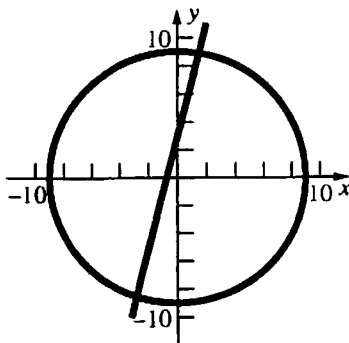
$$38. \quad x^2 + (4x + 3)^2 = 81$$

$$x^2 + 16x^2 + 24x + 9 = 81$$

$$17x^2 + 24x - 72 = 0$$

$$x \approx -2.88, 1.47$$

Intersection points: $(-2.88, -8.52)$, $(1.47, 8.88)$



$$39. \quad a. \quad y = x^2; (2)$$

$$b. \quad ax^3 + bx^2 + cx + d, \text{ with } a > 0: (1)$$

$$c. \quad ax^3 + bx^2 + cx + d, \text{ with } a < 0: (3)$$

$$d. \quad y = ax^3, \text{ with } a > 0: (4)$$

$$40. \quad x^2 + y^2 = 13; (-2, -3), (-2, 3), (2, -3), (2, 3)$$

$$d_1 = \sqrt{(2+2)^2 + (-3+3)^2} = 4$$

$$d_2 = \sqrt{(2+2)^2 + (-3-3)^2} = \sqrt{52} = 2\sqrt{13}$$

$$d_3 = \sqrt{(2-2)^2 + (3+3)^2} = 6$$

Three such distances

$$41. \quad x^2 + 2x + y^2 - 2y = 20; (-2, 1 + \sqrt{21}),$$

$$(-2, 1 - \sqrt{21}), (2, 1 + \sqrt{13}), (2, 1 - \sqrt{13})$$

$$d_1 = \sqrt{(-2-2)^2 + [1 + \sqrt{21} - (1 + \sqrt{13})]^2}$$

$$= \sqrt{16 + (\sqrt{21} - \sqrt{13})^2}$$

$$= \sqrt{50 - 2\sqrt{273}} \approx 4.12$$

$$d_2 = \sqrt{(-2-2)^2 + [1 + \sqrt{21} - (1 - \sqrt{13})]^2}$$

$$= \sqrt{16 + (\sqrt{21} + \sqrt{13})^2}$$

$$= \sqrt{50 + 2\sqrt{273}} \approx 9.11$$

$$d_3 = \sqrt{(-2+2)^2 + [1 + \sqrt{21} - (1 - \sqrt{21})]^2}$$

$$= \sqrt{0 + (\sqrt{21} + \sqrt{21})^2} = \sqrt{(2\sqrt{21})^2} = 2\sqrt{21} \approx 9.17$$

$$d_4 = \sqrt{(-2-2)^2 + [1 - \sqrt{21} - (1 + \sqrt{13})]^2}$$

$$= \sqrt{16 + (-\sqrt{21} - \sqrt{13})^2} = \sqrt{50 + 2\sqrt{273}} \approx 9.11$$

$$d_5 = \sqrt{(-2-2)^2 + [1 - \sqrt{21} - (1 - \sqrt{13})]^2}$$

$$= \sqrt{16 + (\sqrt{13} - \sqrt{21})^2} = \sqrt{50 - 2\sqrt{273}} \approx 4.12$$

$$d_6 = \sqrt{(2-2)^2 + [1 + \sqrt{13} - (1 - \sqrt{13})]^2}$$

$$= \sqrt{0 + (\sqrt{13} + \sqrt{13})^2} = \sqrt{(2\sqrt{13})^2} = 2\sqrt{13} \approx 7.21$$

Four such distances ($d_2 = d_4$ and $d_1 = d_5$).

1.8 Chapter Review

Concepts Test

1. False: p and q must be integers.
2. True: $\frac{p_1}{q_1} - \frac{p_2}{q_2} = \frac{p_1q_2 - p_2q_1}{q_1q_2}$; since p_1, q_1, p_2 , and q_2 are integers, so are $p_1q_2 - p_2q_1$ and q_1q_2 .
3. False: If the numbers are opposites ($-\pi$ and π) then the sum is 0, which is rational.
4. True: Between any two distinct real numbers there are both a rational and an irrational number.
5. False: $0.999\dots$ is equal to 1.
6. True: $(a^m)^n = (a^n)^m = a^{mn}$
7. False: $(a \cdot b) \cdot c = a^{bc}$; $a \cdot (b \cdot c) = a^{b^c}$
8. True: Since $x \leq y \leq z$ and $x \geq z$, $x = y = z$
9. True: If x was not 0, then $\varepsilon = \frac{|x|}{2}$ would be a positive number less than $|x|$.
10. True: $y - x = -(x - y)$ so
 $(x - y)(y - x) = (x - y)(-1)(x - y)$
 $= (-1)(x - y)^2$.
 $(x - y)^2 \geq 0$ for all x and y , so
 $-(x - y)^2 \leq 0$.
11. True: $a < b < 0$; $a < b$; $\frac{a}{b} > 1$; $\frac{1}{b} < \frac{1}{a}$
12. True: $[a, b]$ and $[b, c]$ share point b in common.
13. True: If (a, b) and (c, d) share a point then $c < b$ so they share the infinitely many points between b and c .
14. True: $\sqrt{x^2} = |x| = -x$ if $x < 0$.
15. False: If $x = 0$, the number has no sign.
16. False: This only holds if x and y are greater than 0.
17. True: $|x| < |y| \Leftrightarrow |x|^4 < |y|^4$
 $|x|^4 = x^4$ and $|y|^4 = y^4$, so $x^4 < y^4$.
18. True: $|x + y| = -(x + y)$
 $= -x + (-y) = |x| + |y|$
19. True: If $r = 0$, then
 $\frac{1}{1+|r|} = \frac{1}{1-r} = \frac{1}{1-|r|} = 1$.
For any r , $1+|r| \geq 1-|r|$.
Since $|r| < 1$, $1-|r| > 0$ so
 $\frac{1}{1+|r|} \leq \frac{1}{1-|r|}$; also, $-1 < r < 1$.
If $-1 < r < 0$, then $|r| = -r$ and
 $1-r = 1+|r|$, so
 $\frac{1}{1+|r|} = \frac{1}{1-r} \leq \frac{1}{1-|r|}$.
If $0 < r < 1$, then $|r| = r$ and
 $1-r = 1-|r|$, so
 $\frac{1}{1+|r|} \leq \frac{1}{1-r} = \frac{1}{1-|r|}$.
20. True: If $|r| > 1$, then $1-|r| < 0$. Thus, since
 $1+|r| \geq 1-|r|$, $\frac{1}{1-|r|} \leq \frac{1}{1+|r|}$. If
 $r > 1$, $|r| = r$, and $1-r = 1-|r|$, so
 $\frac{1}{1-|r|} = \frac{1}{1-r} \leq \frac{1}{1+|r|}$.
If $r < -1$, $|r| = -r$ and $1-r = 1+|r|$,
so $\frac{1}{1-|r|} \leq \frac{1}{1-r} = \frac{1}{1+|r|}$.
21. True: If x and y are the same sign, then
 $\|x| - |y|| = |x - y|$. $|x - y| \leq |x + y|$
when x and y are the same sign, so
 $\|x| - |y|| \leq |x + y|$. If x and y
have opposite signs then either
 $\|x| - |y|| = |x - (-y)| = |x + y|$
($x > 0, y < 0$) or
 $\|x| - |y|| = |-x - y| = |x + y|$
($x < 0, y > 0$). In either case

- $\|x\| - \|y\| = \|x + y\|$.
If either $x = 0$ or $y = 0$, the inequality is easily seen to be true.
22. True: If y is positive,
then $x = \sqrt{y}$ satisfies
 $x^2 = (\sqrt{y})^2 = y$.
23. True: For every real number y , whether it is positive, zero, or negative, the cube root $x = \sqrt[3]{y}$ satisfies $x^3 = (\sqrt[3]{y})^3 = y$.
24. True: For example $x^2 \leq 0$ has solution $\{0\}$.
25. True: $x^2 + ax + y^2 + y = 0$
 $x^2 + ax + \frac{a^2}{4} + y^2 + y + \frac{1}{4} = \frac{a^2}{4} + \frac{1}{4}$
 $\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{a^2 + 1}{4}$
is a circle for all values of a .
26. False: If $x = b = 0$ and $c < 0$, the equation does not represent a circle.
27. True; $y - b = \frac{3}{4}(x - a)$
 $y = \frac{3}{4}x - \frac{3a}{4} + b$;
If $x = a + 4$:
 $y = \frac{3}{4}(a + 4) - \frac{3a}{4} + b$
 $= \frac{3a}{4} + 3 - \frac{3a}{4} + b = b + 3$
28. True: If the points are on the same line, they have equal slope. Then the reciprocals of the slopes are also equal.
29. True: If $ab > 0$, a and b have the same sign, so (a, b) is in either the first or third quadrant.
30. True: Let $x = \varepsilon/2$. If $\varepsilon > 0$, then $x > 0$ and $x < \varepsilon$.
31. True: If $ab = 0$, a or b is 0, so (a, b) lies on the x -axis or the y -axis. If $a = b = 0$, (a, b) is the origin.
32. True: $y_1 = y_2$, so (x_1, y_1) and (x_2, y_2) are on the same horizontal line.
33. True: $d = \sqrt{[(a+b) - (a-b)]^2 + (a-a)^2}$
 $= \sqrt{(2b)^2} = |2b|$
34. False: The equation of a vertical line cannot be written in point-slope form.
35. True: This is the general linear equation.
36. True: Two non-vertical lines are parallel if and only if they have the same slope.
37. False: The slopes of perpendicular lines are negative reciprocals.
38. True: If a and b are rational and $(a, 0), (0, b)$ are the intercepts, the slope is $-\frac{b}{a}$ which is rational.
39. False: $ax + y = c \Rightarrow y = -ax + c$
 $ax - y = c \Rightarrow y = ax - c$
 $(a)(-a) \neq -1$.
40. True: The equation is
 $(3 + 2m)x + (6m - 2)y + 4 - 2m = 0$
which is the equation of a straight line unless $3 + 2m$ and $6m - 2$ are both 0, and there is no real number m such that
 $3 + 2m = 0$ and $6m - 2 = 0$.

Sample Test Problems

1. a. $\left(n + \frac{1}{n}\right)^n; \left(1 + \frac{1}{1}\right)^1 = 2; \left(2 + \frac{1}{2}\right)^2 = \frac{25}{4};$
 $\left(-2 + \frac{1}{-2}\right)^{-2} = \frac{4}{25}$
- b. $(n^2 - n + 1)^2; [(1)^2 - (1) + 1]^2 = 1;$
 $[(2)^2 - (2) + 1]^2 = 9;$
 $[(-2)^2 - (-2) + 1]^2 = 49$
- c. $4^{3/n}; 4^{3/1} = 64; 4^{3/2} = 8; 4^{-3/2} = \frac{1}{8}$

$$\begin{aligned} \text{d. } \sqrt[n]{\left|\frac{1}{n}\right|}; \sqrt[n]{\left|\frac{1}{1}\right|} &= 1; \sqrt[2]{\left|\frac{1}{2}\right|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}; \\ -2\sqrt[3]{\left|\frac{1}{-2}\right|} &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{2. a. } \left(1 + \frac{1}{m} + \frac{1}{n}\right) \left(1 - \frac{1}{m} + \frac{1}{n}\right)^{-1} &= \frac{1 + \frac{1}{m} + \frac{1}{n}}{1 - \frac{1}{m} + \frac{1}{n}} \\ &= \frac{mn + n + m}{mn - n + m} \end{aligned}$$

$$\begin{aligned} \text{b. } \frac{\frac{2}{x+1} - \frac{x}{x^2-x-2}}{\frac{3}{x+1} - \frac{2}{x-2}} &= \frac{\frac{2}{x+1} - \frac{x}{(x-2)(x+1)}}{\frac{3}{x+1} - \frac{2}{x-2}} \\ &= \frac{\frac{2(x-2) - x}{(x+1)(x-2)}}{\frac{3(x-2) - 2(x+1)}{(x+1)(x-2)}} \\ &= \frac{x-4}{x-8} \end{aligned}$$

$$\text{c. } \frac{(t^3-1)}{t-1} = \frac{(t-1)(t^2+t+1)}{t-1} = t^2+t+1$$

3. Let a , b , c , and d be integers.

$$\frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{a}{2b} + \frac{c}{2d} = \frac{ad+bc}{2bd} \text{ which is rational.}$$

4. $x = 4.1282828\dots$

$$1000x = 4128.282828\dots$$

$$\underline{10x = 41.282828\dots}$$

$$990x = 4807$$

$$x = \frac{4807}{990}$$

5. Answers will vary. Possible answer:

$$\sqrt{\frac{13}{50}} \approx 0.50990\dots$$

$$\text{6. } \frac{\left(\sqrt[3]{8.15 \times 10^4} - 1.32\right)^2}{3.24} \approx 545.39$$

$$\text{7. } (\pi - \sqrt{2.0})^{2.5} - \sqrt[3]{2.0} \approx 2.66$$

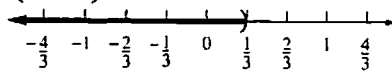
$$\text{8. } \sin^2(2.45) + \cos^2(2.40) - 1.00 \approx -0.0495$$

$$\text{9. } 1 - 3x > 0$$

$$3x < 1$$

$$x < \frac{1}{3}$$

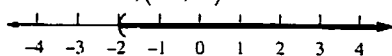
$$\left(-\infty, \frac{1}{3}\right)$$



$$\text{10. } 6x + 3 > 2x - 5$$

$$4x > -8$$

$$x > -2; (-2, \infty)$$

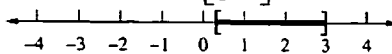


$$\text{11. } 3 - 2x \leq 4x + 1 \leq 2x + 7$$

$$3 - 2x \leq 4x + 1 \text{ and } 4x + 1 \leq 2x + 7$$

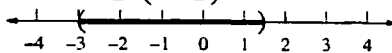
$$6x \geq 2 \text{ and } 2x \geq 6$$

$$x \geq \frac{1}{3} \text{ and } x \leq 3; \left[\frac{1}{3}, 3\right]$$



$$\text{12. } 2x^2 + 5x - 3 < 0; (2x-1)(x+3) < 0;$$

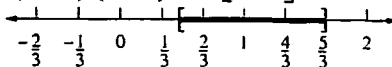
$$-3 < x < \frac{1}{2}; \left(-3, \frac{1}{2}\right)$$



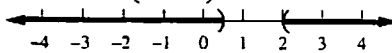
$$\text{13. } 21t^2 - 44t + 12 \leq -3; 21t^2 - 44t + 15 \leq 0;$$

$$t = \frac{44 \pm \sqrt{44^2 - 4(21)(15)}}{2(21)} = \frac{44 \pm 26}{42} = \frac{3}{7}, \frac{5}{3}$$

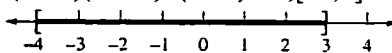
$$\left(t - \frac{3}{7}\right)\left(t - \frac{5}{3}\right) \leq 0; \left[\frac{3}{7}, \frac{5}{3}\right]$$



$$\text{14. } \frac{2x-1}{x-2} > 0; \left(-\infty, \frac{1}{2}\right) \cup (2, \infty)$$

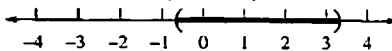


$$\text{15. } (x+4)(2x-1)^2(x-3) \leq 0; [-4, 3]$$



$$\text{16. } |3x-4| < 6; -6 < 3x-4 < 6; -2 < 3x < 10;$$

$$-\frac{2}{3} < x < \frac{10}{3}; \left(-\frac{2}{3}, \frac{10}{3}\right)$$



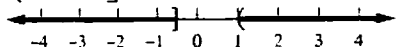
17. $\frac{3}{1-x} \leq 2$

$$\frac{3}{1-x} - 2 \leq 0$$

$$\frac{3-2(1-x)}{1-x} \leq 0$$

$$\frac{2x+1}{1-x} \leq 0;$$

$$\left(-\infty, -\frac{1}{2}\right] \cup (1, \infty)$$



18. $|12-3x| \geq |x|$

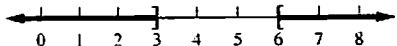
$$(12-3x)^2 \geq x^2$$

$$144-72x+9x^2 \geq x^2$$

$$8x^2-72x+144 \geq 0$$

$$8(x-3)(x-6) \geq 0$$

$$(-\infty, 3] \cup [6, \infty)$$



19. For example, if $x = -2$, $| -(-2) | = 2 \neq -2$

$$|-x| \neq x \text{ for any } x < 0$$

20. If $|-x| = x$, then $|x| = x$.

$$x \geq 0$$

21. $|t-5| = |-(5-t)| = |5-t|$

$$\text{If } |5-t| = 5-t, \text{ then } 5-t \geq 0.$$

$$t \leq 5$$

22. $|t-a| = |-(a-t)| = |a-t|$

$$\text{If } |a-t| = a-t, \text{ then } a-t \geq 0.$$

$$t \leq a$$

23. If $|x| \leq 2$, then

$$0 \leq |2x^2+3x+2| \leq |2x^2| + |3x| + 2 \leq 8+6+2=16$$

$$\text{also } |x^2+2| \geq 2 \text{ so } \frac{1}{|x^2+2|} \leq \frac{1}{2}. \text{ Thus}$$

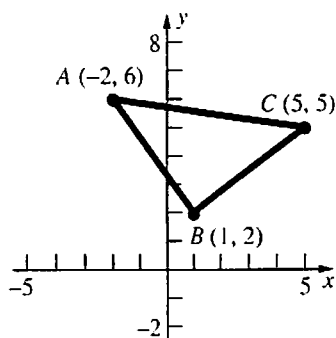
$$\left| \frac{2x^2+3x+2}{x^2+2} \right| = |2x^2+3x+2| \left| \frac{1}{x^2+2} \right| \leq 16 \left(\frac{1}{2} \right) = 8$$

24. a. The distance between x and 5 is 3.

b. The distance between x and -1 is less than or equal to 2.

c. The distance between x and a is greater than b .

25.



$$d(A, B) = \sqrt{(1+2)^2 + (2-6)^2} = \sqrt{9+16} = 5$$

$$d(B, C) = \sqrt{(5-1)^2 + (5-2)^2} = \sqrt{16+9} = 5$$

$$d(A, C) = \sqrt{(5+2)^2 + (5-6)^2} = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$$

$$(AB)^2 + (BC)^2 = (AC)^2, \text{ so } \triangle ABC \text{ is a right triangle.}$$

26. midpoint: $\left(\frac{1+7}{2}, \frac{2+8}{2}\right) = (4, 5)$

$$d = \sqrt{(4-3)^2 + (5+6)^2} = \sqrt{1+121} = \sqrt{122}$$

27. center = $\left(\frac{2+10}{2}, \frac{0+4}{2}\right) = (6, 2)$

$$\text{radius} = \frac{1}{2} \sqrt{(10-2)^2 + (4-0)^2} = \frac{1}{2} \sqrt{64+16} = 2\sqrt{5}$$

$$\text{circle: } (x-6)^2 + (y-2)^2 = 20$$

28. $x^2 + y^2 - 8x + 6y = 0$

$$x^2 - 8x + 16 + y^2 + 6y + 9 = 16 + 9$$

$$(x-4)^2 + (y+3)^2 = 25;$$

$$\text{center} = (4, -3), \text{ radius} = 5$$

$$x^2 - 2x + y^2 + 2y = 2$$

29. $x^2 - 2x + 1 + y^2 + 2y + 1 = 2 + 1 + 1$

$$(x-1)^2 + (y+1)^2 = 4$$

$$\text{center} = (1, -1)$$

$$x^2 + 6x + y^2 - 4y = -7$$

$$x^2 + 6x + 9 + y^2 - 4y + 4 = -7 + 9 + 4$$

$$(x+3)^2 + (y-2)^2 = 6$$

$$\text{center} = (-3, 2)$$

$$d = \sqrt{(-3-1)^2 + (2+1)^2} = \sqrt{16+9} = 5$$

30. a. $3x + 2y = 6$

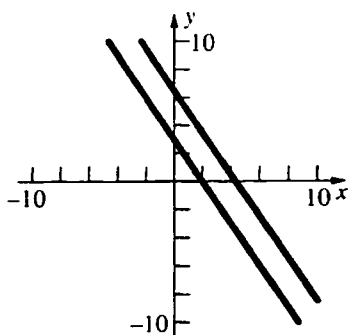
$$2y = -3x + 6$$

$$y = -\frac{3}{2}x + 3$$

$$m = -\frac{3}{2}$$

$$y - 2 = -\frac{3}{2}(x - 3)$$

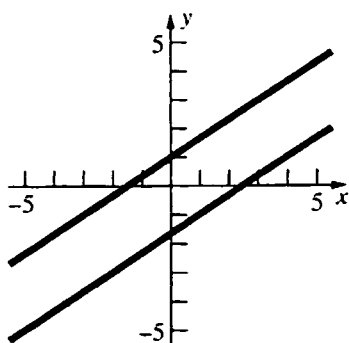
$$y = -\frac{3}{2}x + \frac{13}{2}$$



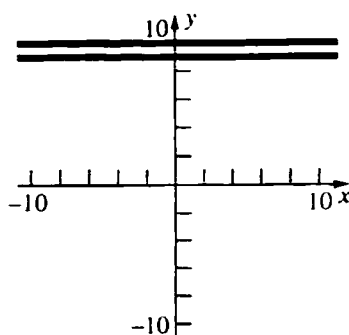
b. $m = \frac{2}{3}$;

$$y + 1 = \frac{2}{3}(x - 1)$$

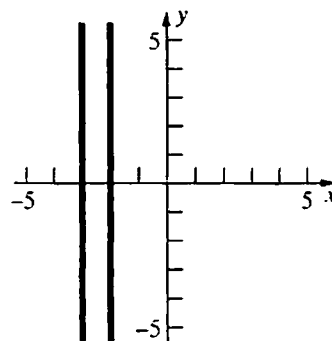
$$y = \frac{2}{3}x - \frac{5}{3}$$



c. $y = 9$



d. $x = -3$



31. a. $m = \frac{3-1}{7+2} = \frac{2}{9}$;

$$y - 1 = \frac{2}{9}(x + 2)$$

$$y = \frac{2}{9}x + \frac{13}{9}$$

b. $3x - 2y = 5$

$$-2y = -3x + 5$$

$$y = \frac{3}{2}x - \frac{5}{2}$$

$$m = \frac{3}{2}$$

$$y - 1 = \frac{3}{2}(x + 2)$$

$$y = \frac{3}{2}x + 4$$

c. $3x + 4y = 9$

$$4y = -3x + 9$$

$$y = -\frac{3}{4}x + \frac{9}{4}; m = -\frac{3}{4}$$

$$y - 1 = -\frac{3}{4}(x + 2)$$

$$y = -\frac{3}{4}x + \frac{1}{2}$$

d. $x = -2$

e. contains $(-2, 1)$ and $(0, 3)$; $m = \frac{3-1}{0+2} = 1$;

$$y = x + 3$$

32. $m_1 = \frac{3+1}{5-2} = \frac{4}{3}$; $m_2 = \frac{11-3}{11-5} = \frac{8}{6} = \frac{4}{3}$;

$$m_3 = \frac{11+1}{11-2} = \frac{12}{9} = \frac{4}{3}$$

$m_1 = m_2 = m_3$, so the points lie on the same line.

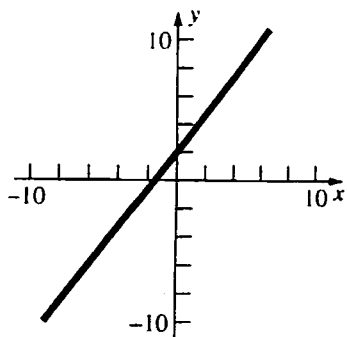
33. The figure is a cubic with respect to y .

The equation is (b) $x = y^3$.

34. The figure is a quadratic, opening downward, with a negative y -intercept. The equation is (c)

$y = ax^2 + bx + c$, with $a < 0$, $b > 0$, and $c < 0$.

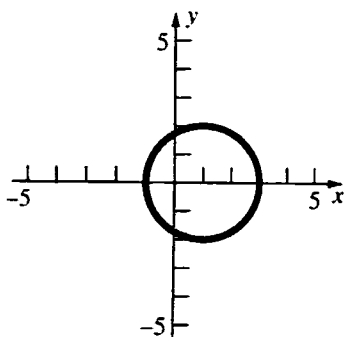
35.



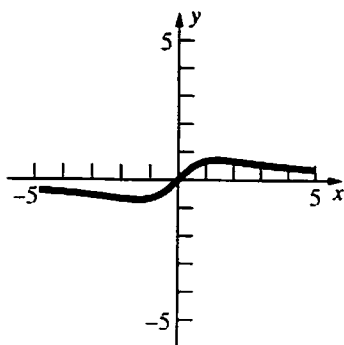
36. $x^2 - 2x + y^2 = 3$

$x^2 - 2x + 1 + y^2 = 4$

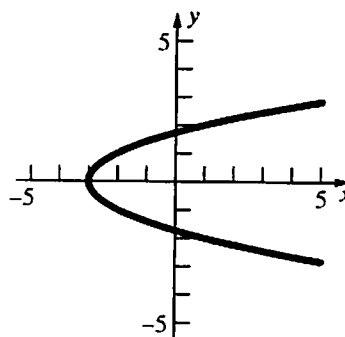
$(x-1)^2 + y^2 = 4$



37.



38.



39. $y = x^2 - 2x + 4$ and $y - x = 4$;

$x + 4 = x^2 - 2x + 4$

$x^2 - 3x = 0$

$x(x-3) = 0$

points of intersection: (0, 4) and (3, 7)

40. $4x - y = 2$

$y = 4x - 2$;

$m = -\frac{1}{4}$

contains $(a, 0)$, $(0, b)$;

$\frac{ab}{2} = 8$

$ab = 16$

$b = \frac{16}{a}$

$\frac{b-0}{0-a} = -\frac{b}{a} = -\frac{1}{4}$;

$a = 4b$

$a = 4\left(\frac{16}{a}\right)$

$a^2 = 64$

$a = 8$

$b = \frac{16}{8} = 2$; $y = -\frac{1}{4}x + 2$

2.1 Concepts Review

1. domain; range
2. $f(2u) = 3(2u)^2 = 12u^2$; $f(x+h) = 3(x+h)^2$
3. asymptote
4. even; odd; y-axis; origin

Problem Set 2.1

1. a. $f(1) = 1 - 1^2 = 0$
 b. $f(-2) = 1 - (-2)^2 = -3$
 c. $f(0) = 1 - 0^2 = 1$
 d. $f(k) = 1 - k^2$
 e. $f(-5) = 1 - (-5)^2 = -24$
 f. $f\left(\frac{1}{4}\right) = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$
 g. $f(3t) = 1 - (3t)^2 = 1 - 9t^2$
 h. $f(2x) = 1 - (2x)^2 = 1 - 4x^2$
 i. $f\left(\frac{1}{t}\right) = 1 - \left(\frac{1}{t}\right)^2 = 1 - \frac{1}{t^2} = \frac{t^2 - 1}{t^2}$
2. a. $F(1) = 1^3 + 3 \cdot 1 = 4$
 b. $F(\sqrt{2}) = (\sqrt{2})^3 + 3(\sqrt{2}) = 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$
 c. $F\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^3 + 3\left(\frac{1}{4}\right) = \frac{1}{64} + \frac{3}{4} = \frac{49}{64}$
 d. $F(\pi) = \pi^3 + 3\pi$

$$e. F\left(\frac{1}{t}\right) = \left(\frac{1}{t}\right)^3 + 3\left(\frac{1}{t}\right) = \frac{1}{t^3} + \frac{3}{t} = \frac{1 + 3t^2}{t^3}$$

$$f. F(2.718) = (2.718)^3 + 3(2.718) \approx 28.233$$

$$3. a. G(0) = \frac{1}{0-1} = -1$$

$$b. G(0.999) = \frac{1}{0.999-1} = -1000$$

$$c. G(1.01) = \frac{1}{1.01-1} = 100$$

$$d. G(y^2) = \frac{1}{y^2-1}$$

$$e. G(-x) = \frac{1}{-x-1} = -\frac{1}{x+1}$$

$$f. G\left(\frac{1}{x^2}\right) = \frac{1}{\frac{1}{x^2}-1} = \frac{x^2}{1-x^2}$$

$$4. a. \Phi(1) = \frac{1+1^2}{\sqrt{1}} = 2$$

$$b. \Phi(-t) = \frac{-t + (-t)^2}{\sqrt{-t}} = \frac{t^2 - t}{\sqrt{-t}}$$

$$c. \Phi\left(\frac{1}{2}\right) = \frac{\frac{1}{2} + \left(\frac{1}{2}\right)^2}{\sqrt{\frac{1}{2}}} = \frac{\frac{3}{4}}{\sqrt{\frac{1}{2}}} \approx 1.06$$

$$d. \Phi(u+1) = \frac{(u+1) + (u+1)^2}{\sqrt{u+1}} = \frac{u^2 + 3u + 2}{\sqrt{u+1}}$$

$$e. \Phi(x^2) = \frac{(x^2) + (x^2)^2}{\sqrt{x^2}} = \frac{x^2 + x^4}{|x|}$$

$$\begin{aligned} \text{f. } \Phi(x^2 + x) &= \frac{(x^2 + x) + (x^2 + x)^2}{\sqrt{x^2 + x}} \\ &= \frac{x^4 + 2x^3 + 2x^2 + x}{\sqrt{x^2 + x}} \end{aligned}$$

5. a. $f(0.25) = \frac{1}{\sqrt{0.25-3}} = \frac{1}{\sqrt{-2.75}}$ is not defined

b. $f(x) = \frac{1}{\sqrt{\pi-3}} \approx 2.658$

c. $f(3+\sqrt{2}) = \frac{1}{\sqrt{3+\sqrt{2}-3}} = \frac{1}{\sqrt{\sqrt{2}}} = 2^{-0.25} \approx 0.841$

6. a. $f(0.79) = \frac{\sqrt{(0.79)^2 + 9}}{0.79 - \sqrt{3}} \approx -3.293$

b. $f(12.26) = \frac{\sqrt{(12.26)^2 + 9}}{12.26 - \sqrt{3}} \approx 1.199$

c. $f(\sqrt{3}) = \frac{\sqrt{(\sqrt{3})^2 + 9}}{\sqrt{3} - \sqrt{3}}$; undefined

7. a. $x^2 + y^2 = 1$
 $y^2 = 1 - x^2$
 $y = \pm\sqrt{1-x^2}$; not a function

b. $xy + y + x = 1$
 $y(x+1) = 1-x$
 $y = \frac{1-x}{x+1}$; $f(x) = \frac{1-x}{x+1}$

c. $x = \sqrt{2y+1}$
 $x^2 = 2y+1$
 $y = \frac{x^2-1}{2}$; $f(x) = \frac{x^2-1}{2}$

d. $x = \frac{y}{y+1}$
 $xy + x = y$
 $x = y - xy$
 $x = y(1-x)$
 $y = \frac{x}{1-x}$; $f(x) = \frac{x}{1-x}$

8. The graphs on the left are not graphs of functions, the graphs on the right are graphs of functions.

9. $\frac{f(a+h) - f(a)}{h} = \frac{[2(a+h)^2 - 1] - (2a^2 - 1)}{h}$
 $= \frac{4ah + 2h^2}{h} = 4a + 2h$

10. $\frac{F(a+h) - F(a)}{h} = \frac{4(a+h)^3 - 4a^3}{h}$
 $= \frac{4a^3 + 12a^2h + 12ah^2 + 4h^3 - 4a^3}{h}$
 $= \frac{12a^2h + 12ah^2 + 4h^3}{h}$
 $= 12a^2 + 12ah + 4h^2$

11. $\frac{g(x+h) - g(x)}{h} = \frac{\frac{3}{x+h-2} - \frac{3}{x-2}}{h}$
 $= \frac{\frac{3x-6-3x-3h+6}{(x+h-2)(x-2)}}{h} = \frac{-3h}{h(x^2-4x+hx-2h+4)}$
 $= -\frac{3}{x^2-4x+hx-2h+4}$

12. $\frac{G(a+h) - G(a)}{h} = \frac{\frac{a+h}{a+h+4} - \frac{a}{a+4}}{h}$
 $= \frac{\frac{a^2+4a+ah+4h-a^2-ah-4a}{(a+h+4)(a+4)}}{h} = \frac{4h}{h(a^2+8a+ah+4h+16)}$
 $= \frac{4}{a^2+8a+ah+4h+16}$

13. a. $F(z) = \sqrt{2z+3}$
 $2z+3 \geq 0$; $z \geq -\frac{3}{2}$
Domain: $\left\{z \in \mathbb{R} : z \geq -\frac{3}{2}\right\}$

b. $g(v) = \frac{1}{4v-1}$
 $4v-1 = 0$; $v = \frac{1}{4}$
Domain: $\left\{v \in \mathbb{R} : v \neq \frac{1}{4}\right\}$

c. $\psi(x) = \sqrt{x^2 - 9}$
 $x^2 - 9 \geq 0; x^2 \geq 9; |x| \geq 3$
Domain: $\{x \in \mathbb{R} : |x| \geq 3\}$

d. $H(y) = -\sqrt{625 - y^4}$
 $625 - y^4 \geq 0; 625 \geq y^4; |y| \leq 5$
Domain: $\{y \in \mathbb{R} : |y| \leq 5\}$

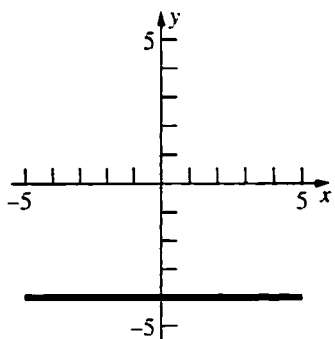
14. a. $f(x) = \frac{4 - x^2}{x^2 - x - 6} = \frac{4 - x^2}{(x - 3)(x + 2)}$
Domain: $\{x \in \mathbb{R} : x \neq -2, 3\}$

b. $G(y) = \sqrt{(y + 1)^{-1}}$
 $\frac{1}{y + 1} \geq 0; y > -1$
Domain: $\{y \in \mathbb{R} : y > -1\}$

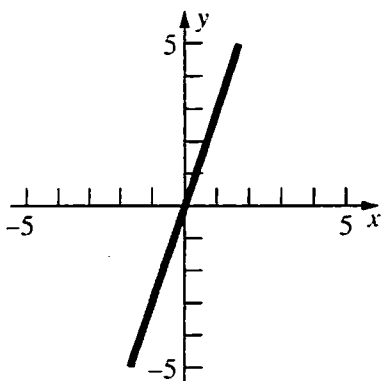
c. $\phi(u) = |2u + 3|$
Domain: $\mathbb{R}$

d. $F(t) = t^{2/3} - 4$
Domain: $\mathbb{R}$

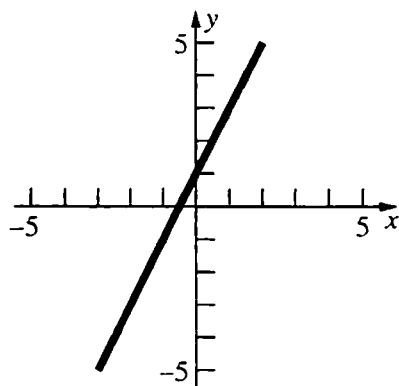
15. $f(x) = -4; f(-x) = -4$; even function



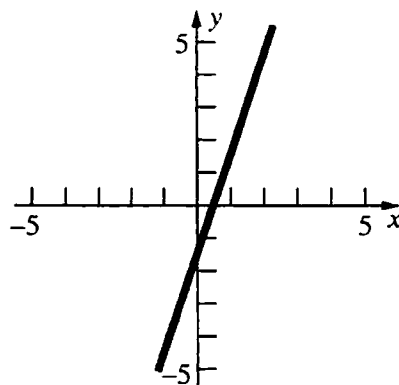
16. $f(x) = 3x; f(-x) = -3x$; odd function



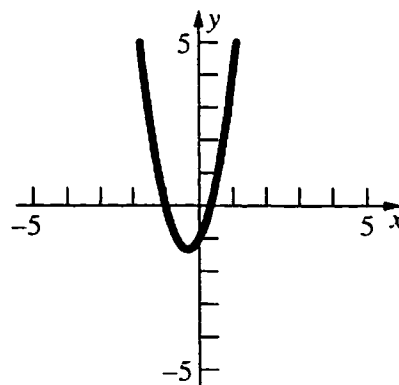
17. $F(x) = 2x + 1; F(-x) = -2x + 1$; neither



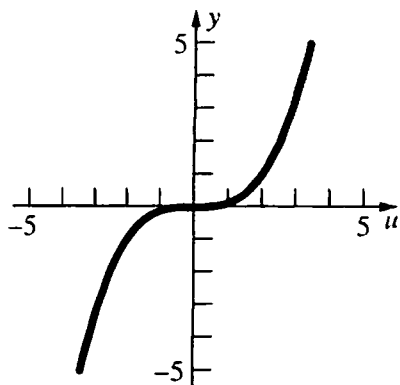
18. $F(x) = 3x - \sqrt{2}; F(-x) = -3x - \sqrt{2}$; neither



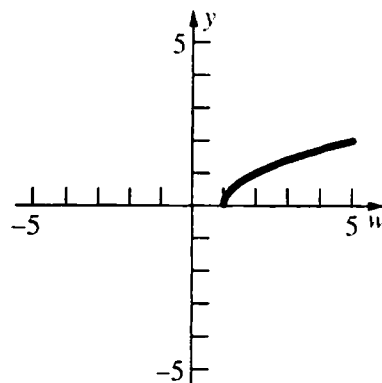
19. $g(x) = 3x^2 + 2x - 1; g(-x) = 3x^2 - 2x - 1$; neither



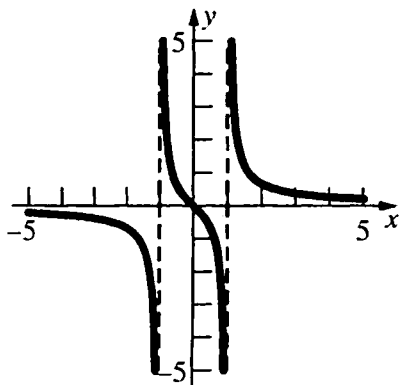
20. $g(u) = \frac{u^3}{8}$; $g(-u) = -\frac{u^3}{8}$; odd function



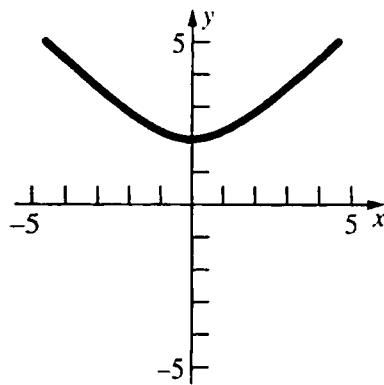
23. $f(w) = \sqrt{w-1}$; $f(-w) = \sqrt{-w-1}$; neither



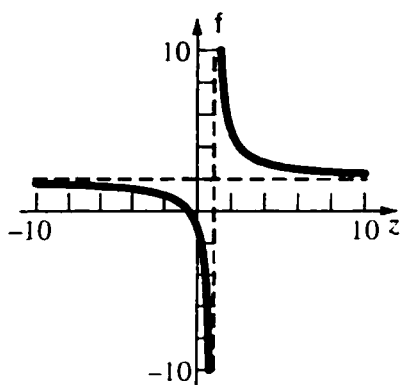
21. $g(x) = \frac{x}{x^2-1}$; $g(-x) = \frac{-x}{x^2-1}$; odd



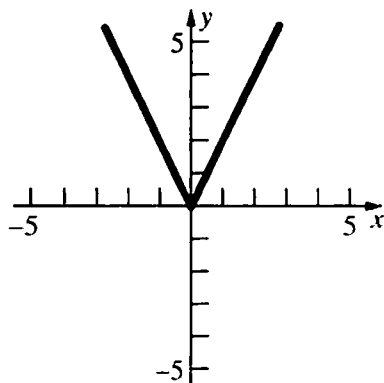
24. $h(x) = \sqrt{x^2+4}$; $h(-x) = \sqrt{x^2+4}$; even function



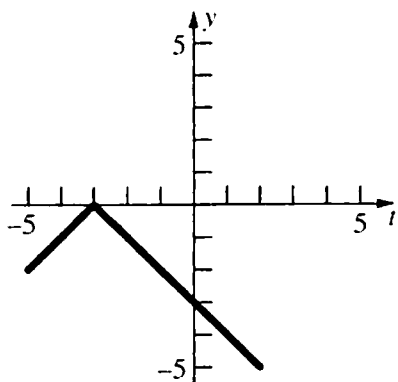
22. $\phi(z) = \frac{2z+1}{z-1}$; $\phi(-z) = \frac{-2z+1}{-z-1}$; neither



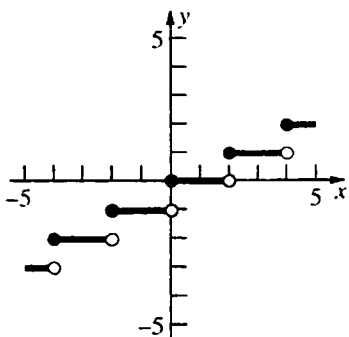
25. $f(x) = |2x|$; $f(-x) = |-2x| = |2x|$; even function



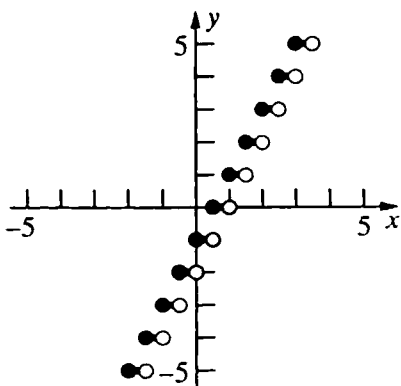
26. $F(t) = -|t+3|$; $F(-t) = -|-t+3|$; neither



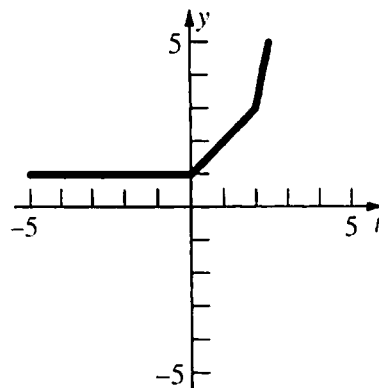
27. $g(x) = \left\lfloor \frac{x}{2} \right\rfloor$; $g(-x) = \left\lfloor -\frac{x}{2} \right\rfloor$; neither



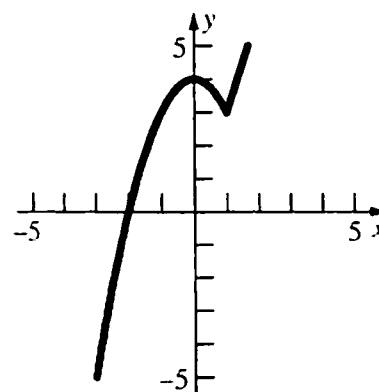
28. $G(x) = \lceil 2x-1 \rceil$; $G(-x) = \lceil -2x+1 \rceil$; neither



29. $g(t) = \begin{cases} 1 & \text{if } t \leq 0 \\ t+1 & \text{if } 0 < t < 2 \\ t^2-1 & \text{if } t \geq 2 \end{cases}$ neither



30. $h(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 1 \\ 3x & \text{if } x > 1 \end{cases}$ neither



31. $T(x) = 5000 + 805x$
Domain: $\{x \in \text{integers: } 0 \leq x \leq 100\}$

$$u(x) = \frac{T(x)}{x} = \frac{5000}{x} + 805$$

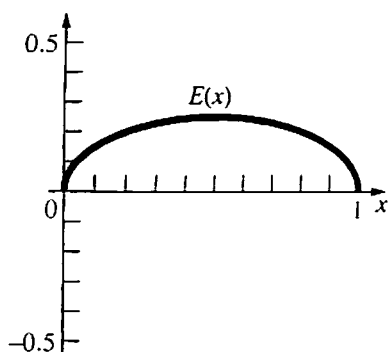
Domain: $\{x \in \text{integers: } 0 < x \leq 100\}$

32. a. $P(x) = 6x - (400 + 5\sqrt{x(x-4)})$
 $= 6x - 400 - 5\sqrt{x(x-4)}$

b. $P(200) \approx -190$; $P(1000) = 610$

c. ABC breaks even when $P(x) = 0$;
 $6x - 400 - 5\sqrt{x(x-4)} = 0$; $x \approx 390$

33. $E(x) = x - x^2$



$\frac{1}{2}$ exceeds its square by the maximum amount.

34. Each side has length $\frac{p}{3}$. The height of the

triangle is $\frac{\sqrt{3}p}{6}$.

$$A(p) = \frac{1}{2} \left(\frac{p}{3} \right) \left(\frac{\sqrt{3}p}{6} \right) = \frac{\sqrt{3}p^2}{36}$$

35. a. $E(x) = 24 + 0.40x$

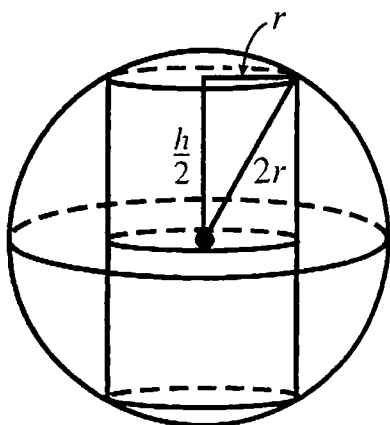
b. $120 = 24 + 0.40x$
 $0.40x = 96; x = 240$ mi

36. The volume of the cylinder is $\pi r^2 h$, where h is the height of the cylinder. From the figure,

$$r^2 + \left(\frac{h}{2} \right)^2 = (2r)^2; \quad \frac{h^2}{4} = 3r^2;$$

$$h = \sqrt{12r^2} = 2r\sqrt{3}.$$

$$V(r) = \pi r^2 (2r\sqrt{3}) = 2\pi r^3 \sqrt{3}$$



37. The area of the two semicircular ends is $\frac{\pi d^2}{4}$.

The length of each parallel side is $\frac{1 - \pi d}{2}$.

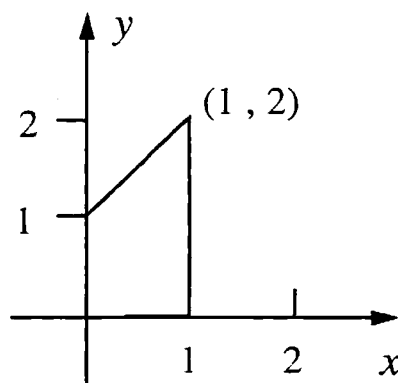
$$A(d) = \frac{\pi d^2}{4} + d \left(\frac{1 - \pi d}{2} \right) = \frac{\pi d^2}{4} + \frac{d - \pi d^2}{2}$$

$$= \frac{2d - \pi d^2}{4}$$

Since the track is one mile long, $\pi d < 1$, so

$$d < \frac{1}{\pi}. \text{ Domain: } \left\{ d \in \mathbb{R} : 0 < d < \frac{1}{\pi} \right\}$$

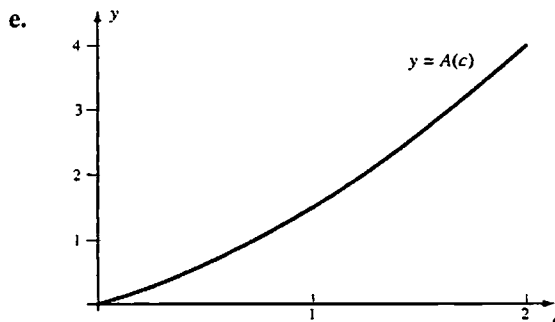
38. a. $A(1) = 1(1) + \frac{1}{2}(1)(2 - 1) = \frac{3}{2}$



b. $A(2) = 2(1) + \frac{1}{2}(2)(3 - 1) = 4$

c. $A(0) = 0$

d. $A(c) = c(1) + \frac{1}{2}(c)(c + 1 - 1) = \frac{1}{2}c^2 + c$



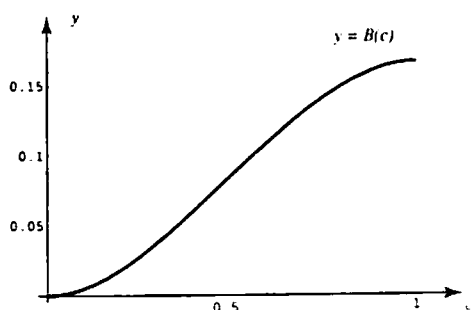
f. Domain: $\{c \in \mathbb{R} : c \geq 0\}$

Range: $\{y \in \mathbb{R} : y \geq 0\}$

39. a. $B(0) = 0$

b. $B\left(\frac{1}{2}\right) = \frac{1}{2}B(1) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$

c.



40. a. $f(x+y) = 2(x+y) = 2x+2y = f(x)+f(y)$

b. $f(x+y) = (x+y)^2 = x^2 + 2xy + y^2$
 $\neq f(x) + f(y)$

c. $f(x+y) = 2(x+y) + 1 = 2x + 2y + 1$
 $\neq f(x) + f(y)$

d. $f(x+y) = -3(x+y) = -3x - 3y = f(x) + f(y)$

41. For any x , $x+0 = x$, so

$$f(x) = f(x+0) = f(x) + f(0), \text{ hence } f(0) = 0.$$

Let m be the value of $f(1)$. For p in $\mathbb{N}$,

$$p = p \cdot 1 = 1 + 1 + \dots + 1, \text{ so}$$

$$f(p) = f(1 + 1 + \dots + 1) = f(1) + f(1) + \dots + f(1) \\ = pf(1) = pm.$$

$$1 = p\left(\frac{1}{p}\right) = \frac{1}{p} + \frac{1}{p} + \dots + \frac{1}{p}, \text{ so}$$

$$m = f(1) = f\left(\frac{1}{p} + \frac{1}{p} + \dots + \frac{1}{p}\right)$$

$$= f\left(\frac{1}{p}\right) + f\left(\frac{1}{p}\right) + \dots + f\left(\frac{1}{p}\right) = pf\left(\frac{1}{p}\right), \text{ hence}$$

$$f\left(\frac{1}{p}\right) = \frac{m}{p}. \text{ Any rational number can be written}$$

as $\frac{p}{q}$ with p, q in $\mathbb{N}$.

$$\frac{p}{q} = p\left(\frac{1}{q}\right) = \frac{1}{q} + \frac{1}{q} + \dots + \frac{1}{q},$$

$$\text{so } f\left(\frac{p}{q}\right) = f\left(\frac{1}{q} + \frac{1}{q} + \dots + \frac{1}{q}\right)$$

$$= f\left(\frac{1}{q}\right) + f\left(\frac{1}{q}\right) + \dots + f\left(\frac{1}{q}\right)$$

$$= pf\left(\frac{1}{q}\right) = p\left(\frac{m}{q}\right) = m\left(\frac{p}{q}\right)$$

42. The player has run $10t$ feet after t seconds. He reaches first base when $t = 9$, second base when $t = 18$, third base when $t = 27$, and home plate when $t = 36$. The player is $10t - 90$ feet from first base when $9 \leq t \leq 18$, hence $\sqrt{90^2 + (10t - 90)^2}$ feet from home plate. The player is $10t - 180$ feet from second base when $18 \leq t \leq 27$, thus he is $90 - (10t - 180) = 270 - 10t$ feet from third base and $\sqrt{90^2 + (270 - 10t)^2}$ feet from home plate. The player is $10t - 270$ feet from third base when $27 \leq t \leq 36$, thus he is $90 - (10t - 270) = 360 - 10t$ feet from home plate.

$$\text{a. } s = \begin{cases} 10t & \text{if } 0 \leq t \leq 9 \\ \sqrt{90^2 + (10t - 90)^2} & \text{if } 9 < t \leq 18 \\ \sqrt{90^2 + (270 - 10t)^2} & \text{if } 18 < t \leq 27 \\ 360 - 10t & \text{if } 27 < t \leq 36 \end{cases}$$

$$\text{b. } s = \begin{cases} 180 - |180 - 10t| & \text{if } 0 \leq t \leq 9 \\ & \text{or } 27 < t \leq 36 \\ \sqrt{90^2 + (10t - 90)^2} & \text{if } 9 < t \leq 18 \\ \sqrt{90^2 + (270 - 10t)^2} & \text{if } 18 < t \leq 27 \end{cases}$$

43. a. $f(1.38) \approx 0.2994$
 $f(4.12) \approx 3.6852$

b.

x	$f(x)$
-4	-4.05
-3	-3.1538
-2	-2.375
-1	-1.8
0	-1.25
1	-0.2
2	1.125
3	2.3846
4	3.55

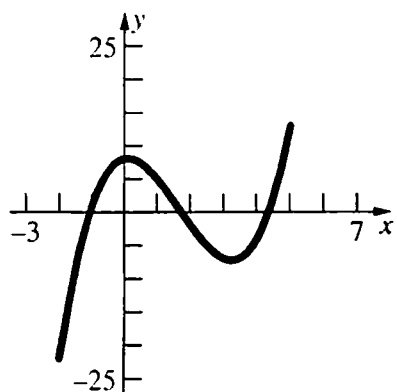
44. a. $f(1.38) \approx -76.8204$
 $f(4.12) \approx 6.7508$

b.

x	$f(x)$
-4	-6.1902
-3	0.4118
-2	13.7651
-1	9.9579

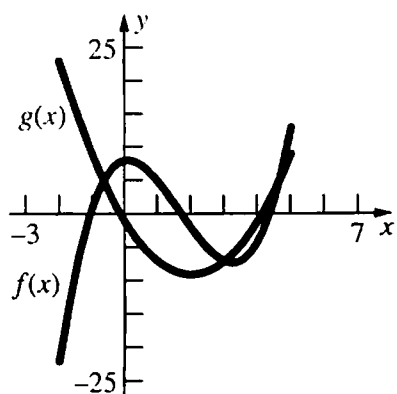
0	0
1	-7.3369
2	-17.7388
3	-0.4521
4	4.4378

45.



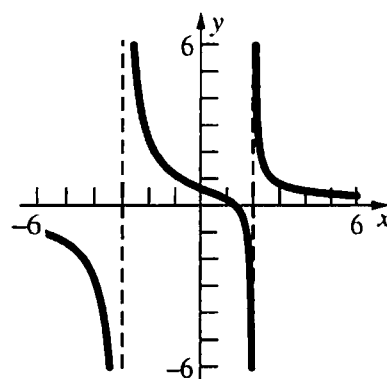
- a. Range: $\{y \in \mathbb{R} : -22 \leq y \leq 13\}$
b. $f(x) = 0$ when $x \approx -1.1, 1.7, 4.3$
 $f(x) \geq 0$ on $[-1.1, 1.7] \cup [4.3, 5]$

46.



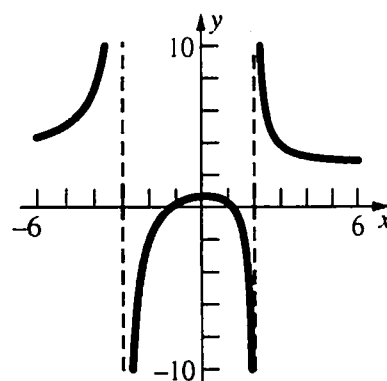
- a. $f(x) = g(x)$ at $x \approx -0.6, 3.0, 4.6$
b. $f(x) \geq g(x)$ on $[-0.6, 3.0] \cup [4.6, 5]$
c. $|f(x) - g(x)|$
 $= |x^3 - 5x^2 + x + 8 - 2x^2 + 8x + 1|$
 $= |x^3 - 7x^2 + 9x + 9|$
Largest value $|f(-2) - g(-2)| = 45$

47.



- a. x -intercept: $3x - 4 = 0; x = \frac{4}{3}$
 y -intercept: $\frac{3 \cdot 0 - 4}{0^2 + 0 - 6} = \frac{2}{3}$
b. $\mathbb{R}$
c. $x^2 + x - 6 = 0; (x + 3)(x - 2) = 0$
Vertical asymptotes at $x = -3, x = 2$
d. Horizontal asymptote at $y = 0$

48.



- a. x -intercepts: $3x^2 - 4 = 0; x = \pm \sqrt{\frac{4}{3}} = \pm \frac{2\sqrt{3}}{3}$
 y -intercept: $\frac{2}{3}$
b. $\mathbb{R}$
c. $x^2 + x - 6 = 0; (x + 3)(x - 2) = 0$
Vertical asymptotes at $x = -3, x = 2$
d. Horizontal asymptote at $y = 3$

2.2 Concepts Review

- $(x^2 + 1)^3$
- $f(g(x))$
- 2; left
- a quotient of two polynomial functions

Problem Set 2.2

- $(f + g)(2) = (2 + 3) + 2^2 = 9$
 - $(f \cdot g)(0) = (0 + 3)(0^2) = 0$
 - $(g/f)(3) = \frac{3^2}{3+3} = \frac{9}{6} = \frac{3}{2}$
 - $(f \circ g)(1) = f(1^2) = 1 + 3 = 4$
 - $(g \circ f)(1) = g(1 + 3) = 4^2 = 16$
 - $(g \circ f)(-8) = g(-8 + 3) = (-5)^2 = 25$
- $(f - g)(2) = (2^2 + 2) - \frac{2}{2+3} = 6 - \frac{2}{5} = \frac{28}{5}$
 - $(f/g)(1) = \frac{1^2 + 1}{\frac{2}{1+3}} = \frac{2}{\frac{2}{4}} = 4$
 - $g^2(3) = \left[\frac{2}{3+3} \right]^2 = \left(\frac{1}{3} \right)^2 = \frac{1}{9}$
 - $(f \circ g)(1) = f\left(\frac{2}{1+3} \right) = \left(\frac{1}{2} \right)^2 + \frac{1}{2} = \frac{3}{4}$
 - $(g \circ f)(1) = g(1^2 + 1) = \frac{2}{2+3} = \frac{2}{5}$
 - $(g \circ g)(3) = g\left(\frac{2}{3+3} \right) = \frac{2}{\frac{1}{3}+3} = \frac{2}{\frac{10}{3}} = \frac{3}{5}$
- $(\Phi + \Psi)(t) = t^3 + 1 + \frac{1}{t}$
 - $(\Phi \circ \Psi)(r) = \Phi\left(\frac{1}{r} \right) = \left(\frac{1}{r} \right)^3 + 1 = \frac{1}{r^3} + 1$
 - $(\Psi \circ \Phi)(r) = \Psi(r^3 + 1) = \frac{1}{r^3 + 1}$

$$d. \quad \Phi^3(z) = (z^3 + 1)^3$$

$$e. \quad (\Phi - \Psi)(5t) = [(5t)^3 + 1] - \frac{1}{5t} \\ = 125t^3 + 1 - \frac{1}{5t}$$

$$f. \quad ((\Phi - \Psi) \circ \Psi)(t) = (\Phi - \Psi)\left(\frac{1}{t} \right) \\ = \left(\frac{1}{t} \right)^3 + 1 - \frac{1}{\frac{1}{t}} = \frac{1}{t^3} + 1 - t$$

$$4. \quad a. \quad (f \cdot g)(x) = \frac{2\sqrt{x^2 - 1}}{x} \\ \text{Domain: } (-\infty, -1] \cup [1, \infty)$$

$$b. \quad f^4(x) + g^4(x) = \left(\sqrt{x^2 - 1} \right)^4 + \left(\frac{2}{x} \right)^4 \\ = (x^2 - 1)^2 + \frac{16}{x^4} \\ \text{Domain: } (-\infty, 0) \cup (0, \infty)$$

$$c. \quad (f \circ g)(x) = f\left(\frac{2}{x} \right) = \sqrt{\left(\frac{2}{x} \right)^2 - 1} = \sqrt{\frac{4}{x^2} - 1} \\ \text{Domain: } [-2, 0) \cup (0, 2]$$

$$d. \quad (g \circ f)(x) = g\left(\sqrt{x^2 - 1} \right) = \frac{2}{\sqrt{x^2 - 1}} \\ \text{Domain: } (-\infty, -1) \cup (1, \infty)$$

$$5. \quad (f \circ g)(x) = f(|1 + x|) = \sqrt{|1 + x|^2 - 4} \\ = \sqrt{x^2 + 2x - 3} \\ (g \circ f)(x) = g\left(\sqrt{x^2 - 4} \right) = |1 + \sqrt{x^2 - 4}| \\ = 1 + \sqrt{x^2 - 4}$$

$$6. \quad g^3(x) = (x^2 + 1)^3 = (x^4 + 2x^2 + 1)(x^2 + 1) \\ = x^6 + 3x^4 + 3x^2 + 1 \\ (g \circ g \circ g)(x) = (g \circ g)(x^2 + 1) \\ = g[(x^2 + 1)^2 + 1] = g(x^4 + 2x^2 + 2) \\ = (x^4 + 2x^2 + 2)^2 + 1 \\ = x^8 + 4x^6 + 8x^4 + 8x^2 + 5$$

$$7. \quad g(3.141) \approx 1.188$$

$$8. \quad g(2.03) \approx 0.000205$$

$$9. \quad [g^2(\pi) - g(\pi)]^{1/3} = [(11 - 7\pi)^2 - |11 - 7\pi|]^{1/3} \\ \approx 4.789$$

10. $[g^3(\pi) - g(\pi)]^{1/3} = [(6\pi - 11)^3 - (6\pi - 11)]^{1/3} \approx 7.807$

11. a. $g(x) = \sqrt{x}, f(x) = x + 7$

b. $g(x) = x^{15}, f(x) = x^2 + x$

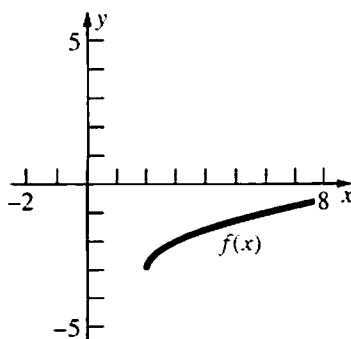
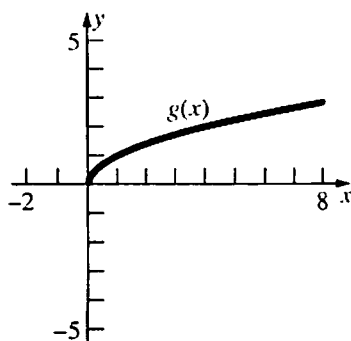
12. a. $f(x) = \frac{2}{x^3}, g(x) = x^2 + x + 1$

b. $f(x) = \log x, g(x) = x^3 + 3x$

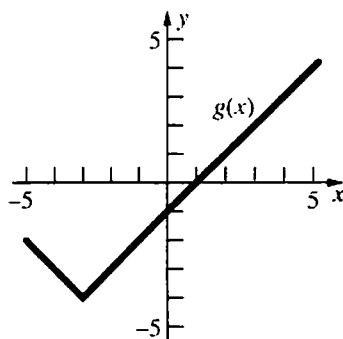
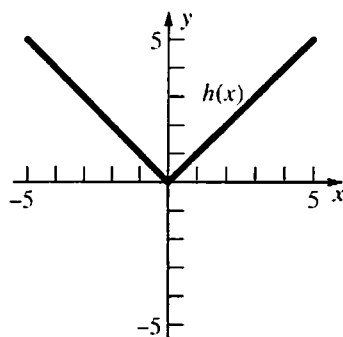
13. $p = f \circ g \circ h$ if $f(x) = \log x, g(x) = \sqrt{x}, h(x) = x^2 + 1$
 $p = f \circ g \circ h$ if $f(x) = \log \sqrt{x}, g(x) = x + 1, h(x) = x^2$

14. $p = f \circ g \circ h \circ l$ if $f(x) = \log x, g(x) = \sqrt{x}, h(x) = x + 1, l(x) = x^2$

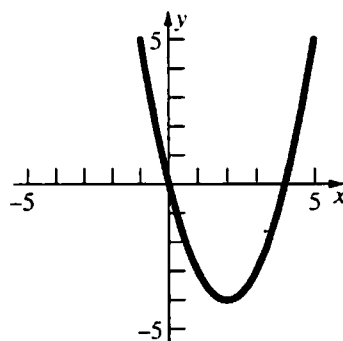
15. Translate the graph of $g(x) = \sqrt{x}$ to the right 2 units and down 3 units.



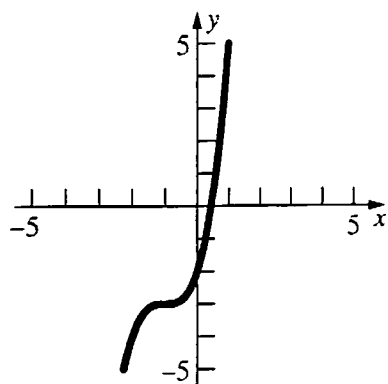
16. Translate the graph of $h(x) = |x|$ to the left 3 units and down 4 units.



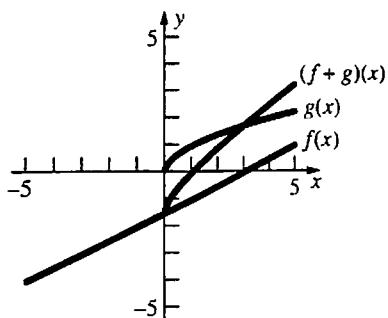
17. Translate the graph of $y = x^2$ to the right 2 units and down 4 units.



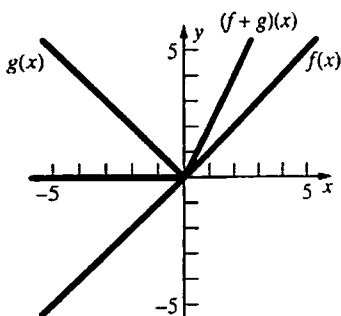
18. Translate the graph of $y = x^3$ to the left 1 unit and down 3 units.



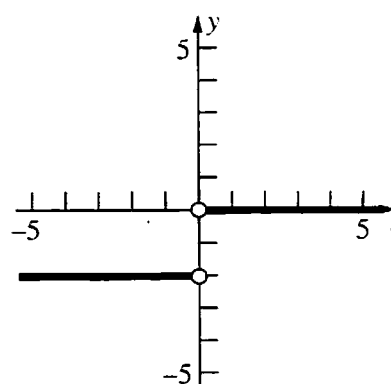
19. $(f + g)(x) = \frac{x-3}{2} + \sqrt{x}$



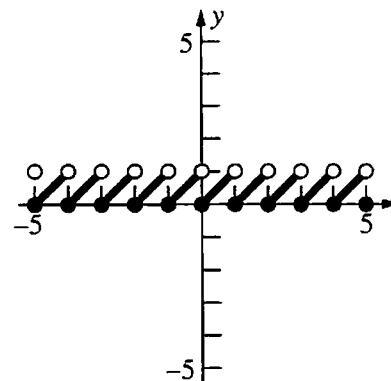
20. $(f + g)(x) = x + |x|$



21. $F(t) = \frac{|t| - t}{t}$



22. $G(t) = t - \lceil t \rceil$



23. a. Even; $(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$ if f and g are both even functions.
- b. Odd; $(f + g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f + g)(x)$ if f and g are both odd functions.
- c. Even;
 $(f \cdot g)(-x) = [f(-x)][g(-x)] = [f(x)][g(x)] = (f \cdot g)(x)$ if f and g are both even functions.
- d. Even; $(f \cdot g)(-x) = [f(-x)][g(-x)] = [-f(x)][-g(x)] = [f(x)][g(x)] = (f \cdot g)(x)$ if f and g are both odd functions.
- e. Odd; $(f \cdot g)(-x) = [f(-x)][g(-x)] = [f(x)][-g(x)] = -[f(x)][g(x)] = -(f \cdot g)(x)$ if f is an even function and g is an odd function.

24. a. $F(x) - F(-x)$ is odd because
 $F(-x) - F(x) = -[F(x) - F(-x)]$

b. $F(x) + F(-x)$ is even because
 $F(-x) + F(-(-x)) = F(-x) + F(x)$
 $= F(x) + F(-x)$

c. $\frac{F(x) - F(-x)}{2}$ is odd and $\frac{F(x) + F(-x)}{2}$ is even.
 $\frac{F(x) - F(-x)}{2} + \frac{F(x) + F(-x)}{2} = \frac{2F(x)}{2} = F(x)$

25. Not every polynomial of even degree is an even function. For example $f(x) = x^2 + x$ is neither even nor odd. Not every polynomial of odd degree is an odd function. For example $g(x) = x^3 + x^2$ is neither even nor odd.

26. a. Neither

b. PF

c. RF

d. PF

e. RF

f. Neither

27. a. $P = \sqrt{29 - 3(2 + \sqrt{t}) + (2 + \sqrt{t})^2}$
 $= \sqrt{t + \sqrt{t} + 27}$

b. When $t = 15$, $P = \sqrt{15 + \sqrt{15} + 27} \approx 7$

28. $R(t) = (120 + 2t + 3t^2)(6000 + 700t)$
 $= 2100t^3 + 19,400t^2 + 96,000t + 720,000$

29. $D(t) = \begin{cases} 400t & \text{if } 0 \leq t \leq 1 \\ \sqrt{(400t)^2 + [300(t-1)]^2} & \text{if } t > 1 \end{cases}$

$D(t) = \begin{cases} 400t & \text{if } 0 \leq t \leq 1 \\ \sqrt{250,000t^2 - 180,000t + 90,000} & \text{if } t > 1 \end{cases}$

30. $D(2.5) \approx 1097$ mi

31. $f(f(x)) = f\left(\frac{ax+b}{cx-a}\right) = \frac{a\left(\frac{ax+b}{cx-a}\right) + b}{c\left(\frac{ax+b}{cx-a}\right) - a}$
 $= \frac{a^2x + ab + bcx - ab}{acx + bc - acx + a^2} = \frac{x(a^2 + bc)}{a^2 + bc} = x$

If $a^2 + bc = 0$, $f(f(x))$ is undefined, while if $x = \frac{a}{c}$, $f(x)$ is undefined.

32. $f(f(f(x))) = f\left(f\left(\frac{x-3}{x+1}\right)\right) = f\left(\frac{\frac{x-3}{x+1} - 3}{\frac{x-3}{x+1} + 1}\right)$
 $= f\left(\frac{x-3-3x-3}{x-3+x+1}\right) = f\left(\frac{-2x-6}{2x-2}\right) = f\left(\frac{-x-3}{x-1}\right)$
 $= \frac{\frac{-x-3}{x-1} - 3}{\frac{-x-3}{x-1} + 1} = \frac{-x-3-3x+3}{-x-3+x-1} = \frac{-4x}{-4} = x$

If $x = -1$, $f(x)$ is undefined, while if $x = 1$, $f(f(x))$ is undefined.

33. a. $f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}}{\frac{1}{x} - 1} = \frac{1}{1-x}$

b. $f(f(x)) = f\left(\frac{x}{x-1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1}$
 $= \frac{x}{x-x+1} = x$

c. $f\left(\frac{1}{f(x)}\right) = f\left(\frac{x-1}{x}\right) = \frac{\frac{x-1}{x}}{\frac{x-1}{x} - 1} = \frac{x-1}{x-1-x}$
 $= 1-x$

34. a. $f(1/x) = \frac{1/x}{\sqrt{1/x} - 1} = \frac{1}{\sqrt{x} - x}$

b. $f(f(x)) = f(x/(\sqrt{x}-1)) = \frac{x/(\sqrt{x}-1)}{\sqrt{\frac{x}{(\sqrt{x}-1)}} - 1}$
 $= \frac{x}{\sqrt{x(\sqrt{x}-1)} + 1 - \sqrt{x}}$

35. $f_1(f_1(x)) = x$;

$f_1(f_2(x)) = \frac{1}{x}$;

$f_1(f_3(x)) = 1-x$;

$f_1(f_4(x)) = \frac{1}{1-x}$;

$$f_1(f_5(x)) = \frac{x-1}{x};$$

$$f_1(f_6(x)) = \frac{x}{x-1};$$

$$f_2(f_1(x)) = \frac{1}{x};$$

$$f_2(f_2(x)) = \frac{1}{\frac{1}{x}} = x;$$

$$f_2(f_3(x)) = \frac{1}{1-x};$$

$$f_2(f_4(x)) = \frac{1}{\frac{1}{1-x}} = 1-x;$$

$$f_2(f_5(x)) = \frac{1}{\frac{x-1}{x}} = \frac{x}{x-1};$$

$$f_2(f_6(x)) = \frac{1}{\frac{x}{x-1}} = \frac{x-1}{x};$$

$$f_3(f_1(x)) = 1-x;$$

$$f_3(f_2(x)) = 1 - \frac{1}{x} = \frac{x-1}{x};$$

$$f_3(f_3(x)) = 1 - (1-x) = x;$$

$$f_3(f_4(x)) = 1 - \frac{1}{1-x} = \frac{x}{x-1};$$

$$f_3(f_5(x)) = 1 - \frac{x-1}{x} = \frac{1}{x};$$

$$f_3(f_6(x)) = 1 - \frac{x}{x-1} = \frac{1}{1-x};$$

$$f_4(f_1(x)) = \frac{1}{1-x};$$

$$f_4(f_2(x)) = \frac{1}{1 - \frac{1}{x}} = \frac{x}{x-1};$$

$$f_4(f_3(x)) = \frac{1}{1 - (1-x)} = \frac{1}{x};$$

$$f_4(f_4(x)) = \frac{1}{1 - \frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{x-1}{x};$$

$$f_4(f_5(x)) = \frac{1}{1 - \frac{x-1}{x}} = \frac{x}{x - (x-1)} = x;$$

$$f_4(f_6(x)) = \frac{1}{1 - \frac{x}{x-1}} = \frac{x-1}{x-1-x} = 1-x;$$

$$f_5(f_1(x)) = \frac{x-1}{x};$$

$$f_5(f_2(x)) = \frac{\frac{1}{x} - 1}{\frac{1}{x}} = 1-x;$$

$$f_5(f_3(x)) = \frac{1-x-1}{1-x} = \frac{x}{x-1};$$

$$f_5(f_4(x)) = \frac{\frac{1}{1-x} - 1}{\frac{1}{1-x}} = \frac{1 - (1-x)}{1} = x;$$

$$f_5(f_5(x)) = \frac{\frac{x-1}{x} - 1}{\frac{x-1}{x}} = \frac{x-1-x}{x-1} = \frac{1}{1-x};$$

$$f_5(f_6(x)) = \frac{\frac{x}{x-1} - 1}{\frac{x}{x-1}} = \frac{x - (x-1)}{x} = \frac{1}{x};$$

$$f_6(f_1(x)) = \frac{x}{x-1};$$

$$f_6(f_2(x)) = \frac{\frac{1}{x}}{\frac{1}{x} - 1} = \frac{1}{1-x};$$

$$f_6(f_3(x)) = \frac{1-x}{1-x-1} = \frac{x-1}{x};$$

$$f_6(f_4(x)) = \frac{\frac{1}{1-x}}{\frac{1}{1-x} - 1} = \frac{1}{1 - (1-x)} = \frac{1}{x};$$

$$f_6(f_5(x)) = \frac{\frac{x-1}{x}}{\frac{x-1}{x} - 1} = \frac{x-1}{x-1-x} = 1-x;$$

$$f_6(f_6(x)) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{x}{x - (x-1)} = x$$

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_4	f_3	f_6	f_5
f_3	f_3	f_5	f_1	f_6	f_2	f_4
f_4	f_4	f_6	f_2	f_5	f_1	f_3
f_5	f_5	f_3	f_6	f_1	f_4	f_2
f_6	f_6	f_4	f_5	f_2	f_3	f_1

a. $f_3 \circ f_3 \circ f_3 \circ f_3 \circ f_3$
 $= (((f_3 \circ f_3) \circ f_3) \circ f_3) \circ f_3$
 $= (((f_1 \circ f_3) \circ f_3) \circ f_3)$
 $= ((f_3 \circ f_3) \circ f_3)$
 $= f_1 \circ f_3 = f_3$

b. $f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5 \circ f_6$
 $= (((((f_1 \circ f_2) \circ f_3) \circ f_4) \circ f_5) \circ f_6)$
 $= (((f_2 \circ f_3) \circ f_4) \circ f_5) \circ f_6$
 $= (f_4 \circ f_4) \circ (f_5 \circ f_6)$
 $= f_5 \circ f_2 = f_3$

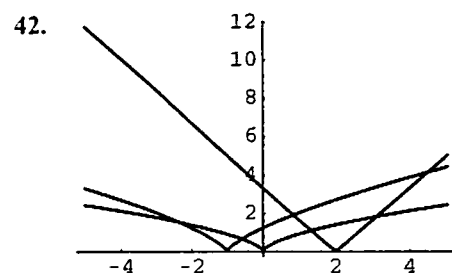
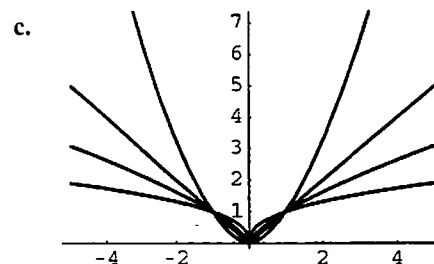
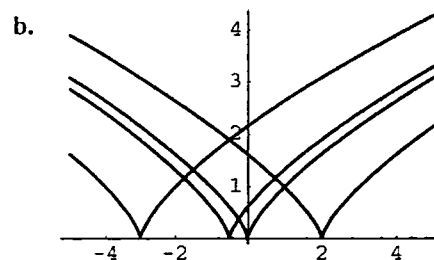
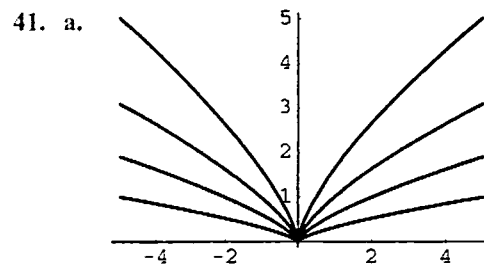
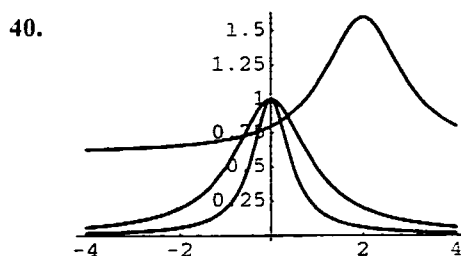
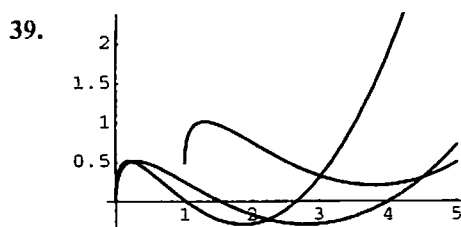
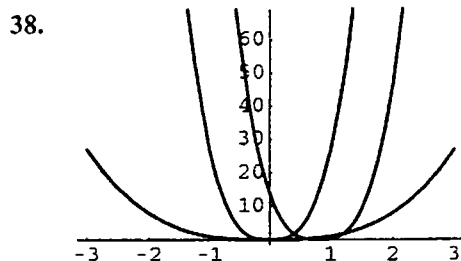
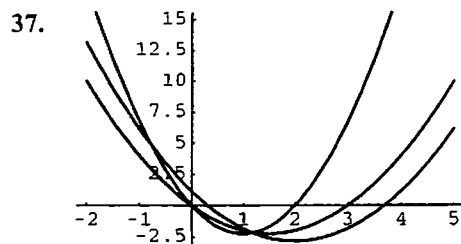
c. If $F \circ f_6 = f_1$, then $F = f_6$.

d. If $G \circ f_3 \circ f_6 = f_1$, then $G \circ f_4 = f_1$ so $G = f_5$.

e. If $f_2 \circ f_3 \circ H = f_5$, then $f_6 \circ H = f_5$ so $H = f_3$.

$$36. (f_1 \circ (f_2 \circ f_3))(x) = f_1((f_2 \circ f_3)(x)) \\ = f_1(f_2(f_3(x)))$$

$$((f_1 \circ f_2) \circ f_3)(x) = (f_1 \circ f_2)(f_3(x)) \\ = f_1(f_2(f_3(x))) \\ = (f_1 \circ (f_2 \circ f_3))(x)$$



2.3 Concepts Review

1. $(-\infty, \infty); [-1, 1]$

2. $2\pi; 2\pi; \pi$

3. odd; even

4. $r = \sqrt{(-4)^2 + 3^2} = 5; \cos \theta = \frac{x}{r} = -\frac{4}{5}$

Problem Set 2.3

1. a. $30\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$

b. $45\left(\frac{\pi}{180}\right) = \frac{\pi}{4}$

c. $-60\left(\frac{\pi}{180}\right) = -\frac{\pi}{3}$

d. $240\left(\frac{\pi}{180}\right) = \frac{4\pi}{3}$

e. $-370\left(\frac{\pi}{180}\right) = -\frac{37\pi}{18}$

f. $10\left(\frac{\pi}{180}\right) = \frac{\pi}{18}$

g. $22\frac{1}{2}\left(\frac{\pi}{180}\right) = \frac{\pi}{8}$

h. $600\left(\frac{\pi}{180}\right) = \frac{10\pi}{3}$

i. $-120\left(\frac{\pi}{180}\right) = -\frac{2\pi}{3}$

2. a. $\frac{7}{6}\pi\left(\frac{180}{\pi}\right) = 210^\circ$

b. $\frac{3}{4}\pi\left(\frac{180}{\pi}\right) = 135^\circ$

c. $-\frac{1}{3}\pi\left(\frac{180}{\pi}\right) = -60^\circ$

d. $\frac{4}{3}\pi\left(\frac{180}{\pi}\right) = 240^\circ$

e. $-\frac{35}{18}\pi\left(\frac{180}{\pi}\right) = -350^\circ$

f. $\frac{3}{18}\pi\left(\frac{180}{\pi}\right) = 30^\circ$

g. $\frac{9}{8}\pi\left(\frac{180}{\pi}\right) = 202.5^\circ$

h. $\frac{10}{3}\pi\left(\frac{180}{\pi}\right) = 600^\circ$

i. $-\frac{4}{3}\pi\left(\frac{180}{\pi}\right) = -240^\circ$

3. a. $33.3\left(\frac{\pi}{180}\right) \approx 0.5812$

b. $46\left(\frac{\pi}{180}\right) \approx 0.8029$

c. $-66.6\left(\frac{\pi}{180}\right) \approx -1.1624$

d. $240.11\left(\frac{\pi}{180}\right) \approx 4.1907$

e. $-369\left(\frac{\pi}{180}\right) \approx -6.4403$

f. $11\left(\frac{\pi}{180}\right) \approx 0.1920$

g. $22.5\left(\frac{\pi}{180}\right) \approx 0.3927$

h. $359\left(\frac{\pi}{180}\right) \approx 6.2657$

i. $-121.35\left(\frac{\pi}{180}\right) \approx -2.1180$

4. a. $3.141\left(\frac{180}{\pi}\right) \approx 180^\circ$

b. $6.28\left(\frac{180}{\pi}\right) \approx 359.8^\circ$

c. $5.00\left(\frac{180}{\pi}\right) \approx 286.5^\circ$

d. $0.001\left(\frac{180}{\pi}\right) \approx 0.057^\circ$

e. $-0.1\left(\frac{180}{\pi}\right) \approx -5.73^\circ$

f. $36.0 \left(\frac{180}{\pi} \right) \approx 2062.6^\circ$

g. $-2.00 \left(\frac{180}{\pi} \right) \approx -114.6^\circ$

h. $1.234 \left(\frac{180}{\pi} \right) \approx 70.70^\circ$

i. $-10.0 \left(\frac{180}{\pi} \right) \approx -573^\circ$

5. a. $\frac{56.4 \tan 34.2^\circ}{\sin 34.1^\circ} \approx 68.37$

b. $\frac{5.34 \tan 21.3^\circ}{\sin 3.1^\circ + \cot 23.5^\circ} \approx 0.8845$

c. $\tan(0.452) \approx 0.4855$

d. $\sin(-0.361) \approx -0.3532$

e. $\cos(-0.361) \approx 0.9355$

f. $\tan(-0.361) \approx -0.3775$

6. a. $\frac{234.1 \sin(1.56)}{\cos(0.34)} \approx 248.3$

b. $\sin^2(2.51) + \sqrt{\cos(0.51)} \approx 1.2828$

7. a. $\frac{56.3 \tan 34.2^\circ}{\sin 56.1^\circ} \approx 46.097$

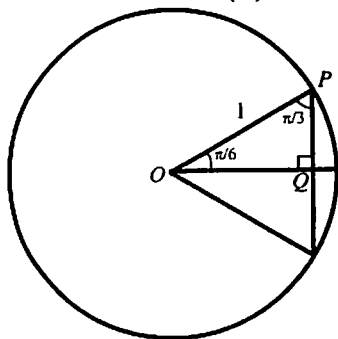
b. $\left(\frac{\sin 35^\circ}{\sin 26^\circ + \cos 26^\circ} \right)^3 \approx 0.0789$

8. Referring to Figure 2, it is clear that $\sin 0 = 0$ and $\cos 0 = 1$. If the angle is $\pi/6$, then the triangle in the figure below is equilateral. Thus,

$$|PQ| = \frac{1}{2}|OP| = \frac{1}{2}. \text{ This implies that } \sin \frac{\pi}{6} = \frac{1}{2}.$$

By the Pythagorean Identity,

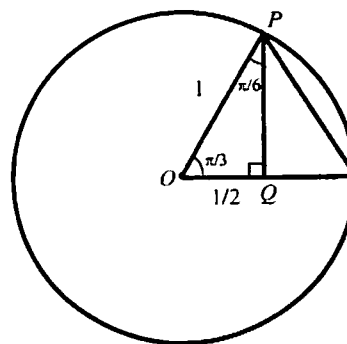
$$\cos^2 \frac{\pi}{6} = 1 - \sin^2 \frac{\pi}{6} = 1 - \left(\frac{1}{2} \right)^2 = \frac{3}{4}. \text{ Thus}$$



$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}. \text{ The results}$$

$\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ were derived in the text. If the angle is $\pi/6$ then the triangle in the figure below is equilateral. Thus $\cos \frac{\pi}{6} = \frac{1}{2}$ and by the

Pythagorean Identity, $\sin \frac{\pi}{6} = \frac{\sqrt{3}}{2}$.



Referring to Figure 2, it is clear that $\sin \frac{\pi}{2} = 1$

and $\cos \frac{\pi}{2} = 0$. The rest of the values are obtained using the same kind of reasoning in the second quadrant.

9. a. $\tan \left(\frac{\pi}{6} \right) = \frac{\sin \left(\frac{\pi}{6} \right)}{\cos \left(\frac{\pi}{6} \right)} = \frac{\sqrt{3}}{3}$

b. $\sec(\pi) = \frac{1}{\cos(\pi)} = -1$

c. $\sec \left(\frac{3\pi}{4} \right) = \frac{1}{\cos \left(\frac{3\pi}{4} \right)} = -\sqrt{2}$

d. $\csc \left(\frac{\pi}{2} \right) = \frac{1}{\sin \left(\frac{\pi}{2} \right)} = 1$

e. $\cot \left(\frac{\pi}{4} \right) = \frac{\cos \left(\frac{\pi}{4} \right)}{\sin \left(\frac{\pi}{4} \right)} = 1$

f. $\tan \left(-\frac{\pi}{4} \right) = \frac{\sin \left(-\frac{\pi}{4} \right)}{\cos \left(-\frac{\pi}{4} \right)} = -1$

$$10. \text{ a. } \tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \sqrt{3}$$

$$\text{ b. } \sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = 2$$

$$\text{ c. } \cot\left(\frac{\pi}{3}\right) = \frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{\sqrt{3}}{3}$$

$$\text{ d. } \csc\left(\frac{\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$$

$$\text{ e. } \tan\left(-\frac{\pi}{6}\right) = \frac{\sin\left(-\frac{\pi}{6}\right)}{\cos\left(-\frac{\pi}{6}\right)} = -\frac{\sqrt{3}}{3}$$

$$\text{ f. } \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

$$11. \text{ a. } (1 + \sin z)(1 - \sin z) = 1 - \sin^2 z \\ = \cos^2 z = \frac{1}{\sec^2 z}$$

$$\text{ b. } (\sec t - 1)(\sec t + 1) = \sec^2 t - 1 = \tan^2 t$$

$$\text{ c. } \sec t - \sin t \tan t = \frac{1}{\cos t} - \frac{\sin^2 t}{\cos t} \\ = \frac{1 - \sin^2 t}{\cos t} = \frac{\cos^2 t}{\cos t} = \cos t$$

$$\text{ d. } \frac{\sec^2 t - 1}{\sec^2 t} = \frac{\tan^2 t}{\sec^2 t} = \frac{\frac{\sin^2 t}{\cos^2 t}}{\frac{1}{\cos^2 t}} = \sin^2 t$$

$$12. \text{ a. } \sin^2 v + \frac{1}{\sec^2 v} = \sin^2 v + \cos^2 v = 1$$

$$\text{ b. } \cos 3t = \cos(2t + t) = \cos 2t \cos t - \sin 2t \sin t \\ = (2\cos^2 t - 1)\cos t - 2\sin^2 t \cos t \\ = 2\cos^3 t - \cos t - 2(1 - \cos^2 t)\cos t \\ = 2\cos^3 t - \cos t - 2\cos t + 2\cos^3 t \\ = 4\cos^3 t - 3\cos t$$

$$\text{ c. } \sin 4x = \sin[2(2x)] = 2\sin 2x \cos 2x \\ = 2(2\sin x \cos x)(2\cos^2 x - 1) \\ = 2(4\sin x \cos^3 x - 2\sin x \cos x) \\ = 8\sin x \cos^3 x - 4\sin x \cos x$$

$$\text{ d. } (1 + \cos \theta)(1 - \cos \theta) = 1 - \cos^2 \theta = \sin^2 \theta$$

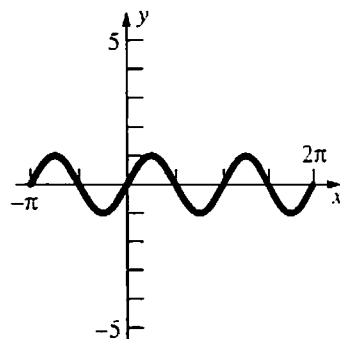
$$13. \text{ a. } \frac{\sin u}{\csc u} + \frac{\cos u}{\sec u} = \sin^2 u + \cos^2 u = 1$$

$$\text{ b. } (1 - \cos^2 x)(1 + \cot^2 x) = (\sin^2 x)(\csc^2 x) \\ = \sin^2 x \left(\frac{1}{\sin^2 x} \right) = 1$$

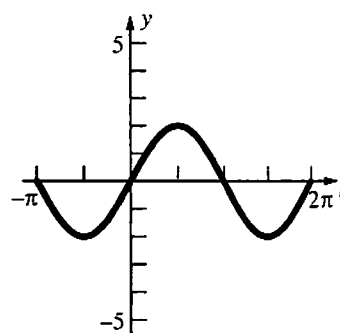
$$\text{ c. } \sin t(\csc t - \sin t) = \sin t \left(\frac{1}{\sin t} - \sin t \right) \\ = 1 - \sin^2 t = \cos^2 t$$

$$\text{ d. } \frac{1 - \csc^2 t}{\csc^2 t} = -\frac{\cot^2 t}{\csc^2 t} = -\frac{\frac{\cos^2 t}{\sin^2 t}}{\frac{1}{\sin^2 t}} \\ = -\cos^2 t = -\frac{1}{\sec^2 t}$$

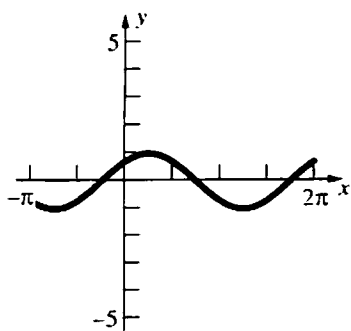
$$14. \text{ a. } y = \sin 2x$$



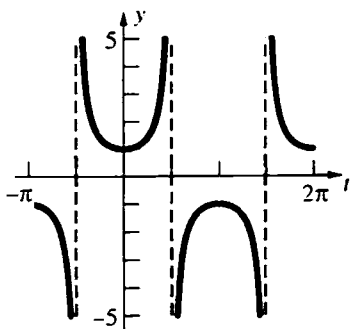
$$\text{ b. } y = 2 \sin t$$



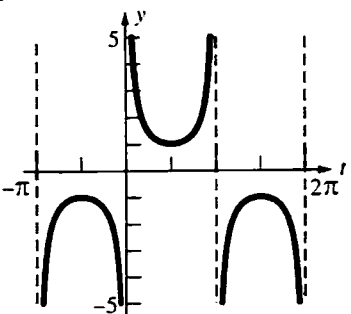
c. $y = \cos\left(x - \frac{\pi}{4}\right)$



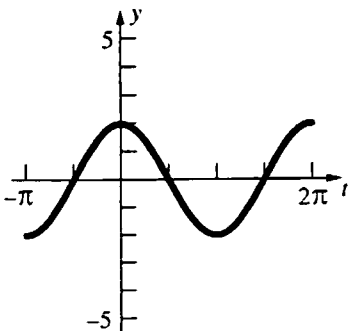
d. $y = \sec t$



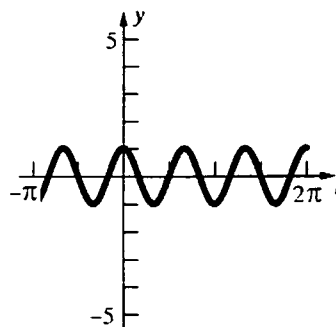
15. a. $y = \csc t$



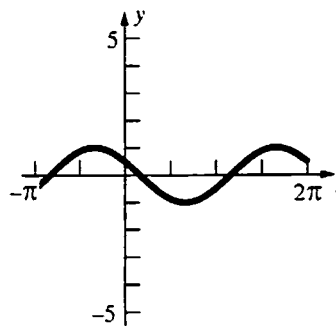
b. $y = 2 \cos t$



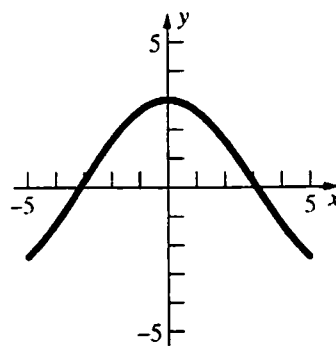
c. $y = \cos 3t$



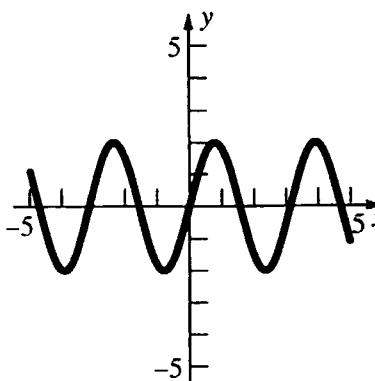
d. $y = \cos\left(t + \frac{\pi}{3}\right)$



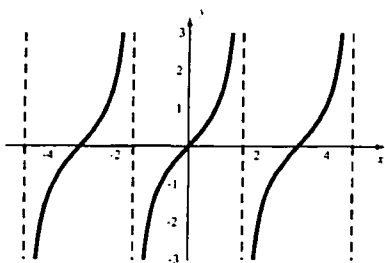
16. $y = 3 \cos \frac{x}{2}$
Period = 4π , amplitude = 3



17. $y = 2 \sin 2x$
Period = π , amplitude = 2

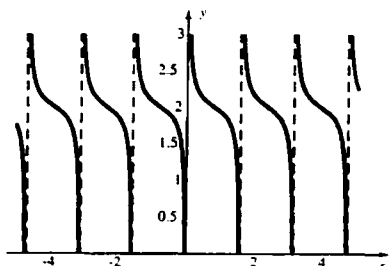


18. $y = \tan x$
Period = π



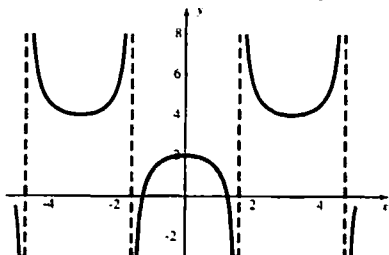
19. $y = 2 + \frac{1}{6} \cot(2x)$

Period = $\frac{\pi}{2}$, shift: 2 units up



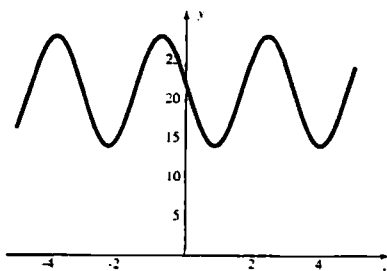
20. $y = 3 + \sec(x - \pi)$

Period = 2π , shift: 3 units up, π units right



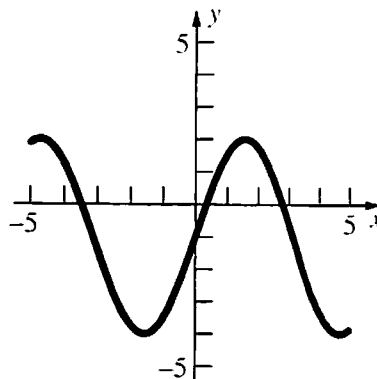
21. $y = 21 + 7 \sin(2x + 3)$

Period = π , amplitude = 7, shift: 21 units up, $\frac{3}{2}$ units left



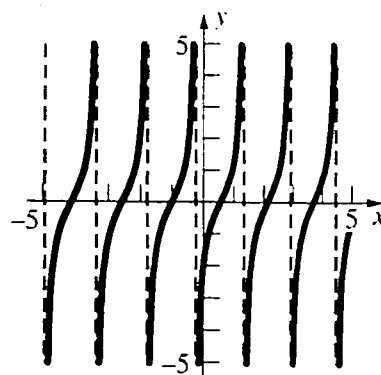
22. $y = 3 \cos\left(x - \frac{\pi}{2}\right) - 1$

Period = 2π , amplitude = 3, shifts: $\frac{\pi}{2}$ units right and 1 unit down.



23. $y = \tan\left(2x - \frac{\pi}{3}\right)$

Period = $\frac{\pi}{2}$, shift: $\frac{\pi}{6}$ units right



24. a. and g.: $y = \sin\left(x + \frac{\pi}{2}\right) = \cos x = -\cos(\pi - x)$

b. and e.: $y = \cos\left(x + \frac{\pi}{2}\right) = \sin(x + \pi)$
 $= -\sin(\pi - x)$

c. and f.: $y = \cos\left(x - \frac{\pi}{2}\right) = \sin x$
 $= -\sin(x + \pi)$

d. and h.:
 $y = \sin\left(x - \frac{\pi}{2}\right) = \cos(x + \pi) = \cos(x - \pi)$

25. a. $-t \sin(-t) = t \sin t$; even

b. $\sin^2(-t) = \sin^2 t$; even

c. $\csc(-t) = \frac{1}{\sin(-t)} = -\csc t$; odd

d. $|\sin(-t)| = |-\sin t| = |\sin t|$; even

e. $\sin(\cos(-t)) = \sin(\cos t)$; even

f. $-x + \sin(-x) = -x - \sin x = -(x + \sin x)$; odd

26. a. $\cot(-t) + \sin(-t) = -\cot t - \sin t$
 $= -(\cot t + \sin t)$; odd

b. $\sin^3(-t) = -\sin^3 t$; odd

c. $\sec(-t) = \frac{1}{\cos(-t)} = \sec t$; even

d. $\sqrt{\sin^4(-t)} = \sqrt{\sin^4 t}$; even

e. $\cos(\sin(-t)) = \cos(-\sin t) = \cos(\sin t)$; even

f. $(-x)^2 + \sin(-x) = x^2 - \sin x$; neither

27. $\cos^2 \frac{\pi}{3} = \frac{1 + \cos 2\left(\frac{\pi}{3}\right)}{2} = \frac{1 + \cos \frac{2\pi}{3}}{2} = \frac{1 - \frac{1}{2}}{2} = \frac{1}{4}$

28. $\sin^2 \frac{\pi}{6} = \frac{1 - \cos 2\left(\frac{\pi}{6}\right)}{2} = \frac{1 - \cos \frac{\pi}{3}}{2} = \frac{1 - \frac{1}{2}}{2} = \frac{1}{4}$

29. $\sin^3 \frac{\pi}{6} = \left(\sin^2 \frac{\pi}{6}\right)\left(\sin \frac{\pi}{6}\right) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

30. $\cos^2 \frac{\pi}{12} = \frac{1 + \cos 2\left(\frac{\pi}{12}\right)}{2} = \frac{1 + \cos \frac{\pi}{6}}{2} = \frac{1 + \frac{\sqrt{3}}{2}}{2}$
 $= \frac{2 + \sqrt{3}}{4}$

31. $\sin^2 \frac{\pi}{8} = \frac{1 - \cos 2\left(\frac{\pi}{8}\right)}{2} = \frac{1 - \cos \frac{\pi}{4}}{2} = \frac{1 - \frac{\sqrt{2}}{2}}{2}$
 $= \frac{2 - \sqrt{2}}{4}$

32. a. $\sin(x - y) = \sin x \cos(-y) + \cos x \sin(-y)$
 $= \sin x \cos y - \cos x \sin y$

b. $\cos(x - y) = \cos x \cos(-y) - \sin x \sin(-y)$
 $= \cos x \cos y + \sin x \sin y$

c. $\tan(x - y) = \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)}$
 $= \frac{\tan x - \tan y}{1 + \tan x \tan y}$

33. $\tan(t + \pi) = \frac{\tan t + \tan \pi}{1 - \tan t \tan \pi} = \frac{\tan t + 0}{1 - (\tan t)(0)}$
 $= \tan t$

34. $\cos(x - \pi) = \cos x \cos(-\pi) - \sin x \sin(-\pi)$
 $= -\cos x - 0 \cdot \sin x = -\cos x$

35. $s = rt = (2.5 \text{ ft})(2\pi \text{ rad}) = 5\pi \text{ ft}$, so the tire goes
 5π feet per revolution, or $\frac{1}{5\pi}$ revolutions per
foot.

$$\left(\frac{1 \text{ rev}}{5\pi \text{ ft}}\right)\left(60 \frac{\text{mi}}{\text{hr}}\right)\left(\frac{1 \text{ hr}}{60 \text{ min}}\right)\left(5280 \frac{\text{ft}}{\text{mi}}\right)$$

$$\approx 336 \text{ rev/min}$$

36. $s = rt = (2 \text{ ft})(150 \text{ rev})(2\pi \text{ rad/rev}) \approx 1885 \text{ ft}$

37. $r_1 t_1 = r_2 t_2$; $6(2\pi)t_1 = 8(2\pi)(21)$
 $t_1 = 28 \text{ rev/sec}$

38. $\Delta y = \sin \alpha$ and $\Delta x = \cos \alpha$

$$m = \frac{\Delta y}{\Delta x} = \frac{\sin \alpha}{\cos \alpha} = \tan \alpha$$

39. a. $\tan \alpha = \sqrt{3}$
 $\alpha = \frac{\pi}{3}$

b. $\sqrt{3}x + 3y = 6$
 $3y = -\sqrt{3}x + 6$
 $y = -\frac{\sqrt{3}}{3}x + 2$; $m = -\frac{\sqrt{3}}{3}$
 $\tan \alpha = -\frac{\sqrt{3}}{3}$
 $\alpha = -\frac{\pi}{6} = \frac{5\pi}{6}$

40. $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$

$$\tan \theta = \tan(\theta_2 - \theta_1) = \frac{\tan \theta_2 + \tan(-\theta_1)}{1 - \tan \theta_2 \tan(-\theta_1)}$$

$$= \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

41. a. $\tan \theta = \frac{3 - 2}{1 + 3(2)} = \frac{1}{7}$
 $\theta \approx 0.1419$

b. $\tan \theta = \frac{-1 - \frac{1}{2}}{1 + \left(\frac{1}{2}\right)(-1)} = -3$
 $\theta \approx 1.8925$

$$\begin{aligned}
 \text{c. } 2x - 6y &= 12 & 2x + y &= 0 \\
 -6y &= -2x + 12 & y &= -2x \\
 y &= \frac{1}{3}x - 2 \\
 m_1 &= \frac{1}{3}, \quad m_2 = -2 \\
 \tan \theta &= \frac{-2 - \frac{1}{3}}{1 + (\frac{1}{3})(-2)} = -7; \theta \approx 1.7127
 \end{aligned}$$

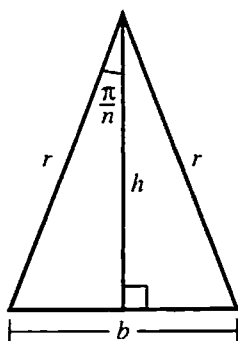
42. Recall that the area of the circle is πr^2 . The measure of the vertex angle of the circle is 2π . Observe that the ratios of the vertex angles must equal the ratios of the areas. Thus, $\frac{t}{2\pi} = \frac{A}{\pi r^2}$, so

$$A = \frac{1}{2} r^2 t.$$

43. $A = \frac{1}{2} (2)(5)^2 = 25\text{cm}^2$

44. Divide the polygon into n isosceles triangles by drawing lines from the center of the circle to the corners of the polygon. If the base of each triangle is on the perimeter of the polygon, then the angle opposite each base has measure $\frac{2\pi}{n}$.

Bisect this angle to divide the triangle into two right triangles (See figure).



$$\sin \frac{\pi}{n} = \frac{b}{2r} \text{ so } b = 2r \sin \frac{\pi}{n} \text{ and } \cos \frac{\pi}{n} = \frac{h}{r} \text{ so}$$

$$h = r \cos \frac{\pi}{n}.$$

$$P = nb = 2rn \sin \frac{\pi}{n}$$

$$A = n \left(\frac{1}{2} bh \right) = nr^2 \cos \frac{\pi}{n} \sin \frac{\pi}{n}$$

45. The base of the triangle is the side opposite the angle t . Then the base has length $2r \sin \frac{t}{2}$ (similar to Problem 45). The radius of the semicircle is $r \sin \frac{t}{2}$ and the height of the triangle is $r \cos \frac{t}{2}$.

$$\begin{aligned}
 A &= \frac{1}{2} \left(2r \sin \frac{t}{2} \right) \left(r \cos \frac{t}{2} \right) + \frac{\pi}{2} \left(r \sin \frac{t}{2} \right)^2 \\
 &= r^2 \sin \frac{t}{2} \cos \frac{t}{2} + \frac{\pi r^2}{2} \sin^2 \frac{t}{2}
 \end{aligned}$$

46.

$$\begin{aligned}
 &\cos \frac{x}{2} \cos \frac{x}{4} \cos \frac{x}{8} \cos \frac{x}{16} \\
 &= \frac{1}{2} \left[\cos \frac{3}{4}x + \cos \frac{1}{4}x \right] \frac{1}{2} \left[\cos \frac{3}{16}x + \cos \frac{1}{16}x \right] \\
 &= \frac{1}{4} \left[\cos \frac{3}{4}x + \cos \frac{1}{4}x \right] \left[\cos \frac{3}{16}x + \cos \frac{1}{16}x \right] \\
 &= \frac{1}{4} \left[\cos \frac{3}{4}x \cos \frac{3}{16}x + \cos \frac{3}{4}x \cos \frac{1}{16}x \right. \\
 &\quad \left. + \cos \frac{1}{4}x \cos \frac{3}{16}x + \cos \frac{1}{4}x \cos \frac{1}{16}x \right] \\
 &= \frac{1}{4} \left[\frac{1}{2} \left(\cos \frac{15}{16}x + \cos \frac{9}{16}x \right) + \frac{1}{2} \left(\cos \frac{13}{16}x + \cos \frac{11}{16}x \right) \right. \\
 &\quad \left. + \frac{1}{2} \left(\cos \frac{7}{16}x + \cos \frac{1}{16}x \right) + \frac{1}{2} \left(\cos \frac{5}{16}x + \cos \frac{3}{16}x \right) \right] \\
 &= \frac{1}{8} \left[\cos \frac{15}{16}x + \cos \frac{13}{16}x + \cos \frac{11}{16}x + \cos \frac{9}{16}x \right. \\
 &\quad \left. + \cos \frac{7}{16}x + \cos \frac{5}{16}x + \cos \frac{3}{16}x + \cos \frac{1}{16}x \right]
 \end{aligned}$$

47.
$$\begin{aligned}
 S_1(n) &= 1 + 2 + 3 + \dots + n \\
 + \quad S_1(n) &= n + (n-1) + (n-2) + \dots + 1 \\
 \hline
 2 \cdot S_1(n) &= (n+1) + (n+1) + (n+1) + \dots + (n+1)
 \end{aligned}$$

Thus, $2 \cdot S_1(n) = n(n+1)$, so $S_1(n) = \frac{n(n+1)}{2}$

48. a. Add up $(x+1)^3 - x^3$ for $x = 0, 1, 2, \dots, n$ to obtain

$$\begin{aligned}
 &1^3 + 0^3 + 2^3 - 1^3 + 3^2 - 2^3 + \dots + (n+1)^3 - n^3 \\
 &= (n+1)^3
 \end{aligned}$$

Now add up $3x^2 + 3x + 1$ for $x = 0, 1, 2, \dots, n$ to obtain

$$\begin{aligned}
 &3 \cdot 0^2 + 3 \cdot 0 + 1 + 3 \cdot 1^2 + 3 \cdot 1 + 1 + 3 \cdot 2^2 + 3 \cdot 2 \\
 &+ 1 + \dots + 3 \cdot n^2 + 3 \cdot n + 1
 \end{aligned}$$

$$\begin{aligned}
 &= 3(0^2 + 1^2 + 2^2 + \dots + n^2) \\
 &+ 3(0 + 1 + 2 + \dots + n) + 1 + 1 + 1 + \dots + 1 \\
 &= 3S_2(n) + 3S_1(n) + n + 1 \\
 \text{Thus, } (n+1)^3 &= 3S_2(n) + 3S_1(n) + n + 1
 \end{aligned}$$

- b. Solve the equation in part a. for $S_2(n)$, and use the formula for $S_1(n)$, to get:

$$\begin{aligned}
 S_2(n) &= \frac{1}{3}[(n+1)^3 - 3S_1(n) - (n+1)] \\
 &= \frac{1}{3}\left[(n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)\right] \\
 &= \frac{1}{3}(n+1)\left[(n+1)^2 - \frac{3n}{2} - 1\right] \\
 &= \frac{1}{6}(n+1)[2(n+1)^2 - 3n - 2] \\
 &= \frac{1}{6}(n+1)[2n^2 + 4n + 2 - 3n - 2] \\
 &= \frac{1}{6}(n+1)(2n^2 + n) = \frac{1}{6}n(n+1)(2n+1)
 \end{aligned}$$

49. $(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$
Add up $(x+1)^4 - x^4$ for $x = 0, 1, 2, \dots, n$ to obtain

$$\begin{aligned}
 &1^4 - 0^4 + 2^4 - 1^4 + 3^4 - 2^4 + \dots \\
 &+ (n+1)^4 - n^4 = (n+1)^4
 \end{aligned}$$

Now add up $4x^3 + 6x^2 + 4x + 1$ for $x = 0, 1, 2, \dots, n$ to obtain

$$\begin{aligned}
 &4(0^3 + 1^3 + 2^3 + \dots + n^3) \\
 &+ 6(0^2 + 1^2 + 2^2 + \dots + n^2) \\
 &+ 4(0 + 1 + 2 + \dots + n) + 1 + 1 + 1 + \dots + 1 \\
 &= 4S_3(n) + 6S_2(n) + 4S_1(n) + n + 1
 \end{aligned}$$

Thus, since $(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$,
 $(n+1)^4 = 4S_3(n) + 6S_2(n) + 4S_1(n) + n + 1$

Solve this equation for $S_3(n)$:

$$\begin{aligned}
 S_3(n) &= \frac{1}{4}[(n+1)^4 - 6S_2(n) - 4S_1(n) - (n+1)] \\
 &= \frac{1}{4}[(n+1)^4 - 6 \cdot \frac{1}{6}n(n+1)(2n+1) \\
 &\quad - 4 \cdot \frac{1}{2}n(n+1) - (n+1)] \\
 &= \frac{1}{4}(n+1)[(n+1)^3 - n(2n+1) - 2n - 1] \\
 &= \frac{1}{4}(n+1)(n^3 + 3n^2 + 3n + 1 - 2n^2 - n - 2n - 1) \\
 &= \frac{1}{4}(n+1)(n^3 + n^2) = \frac{1}{4}n^2(n+1)^2
 \end{aligned}$$

50. $(x+1)^5 - x^5 = 5x^4 + 10x^3 + 10x^2 + 5x + 1$

Add up $(x+1)^5 - x^5$ for $x = 0, 1, 2, \dots, n$ to obtain

$$\begin{aligned}
 &1^5 - 0^5 + 2^5 - 1^5 + 3^5 - 2^5 + \dots + (n+1)^5 - n^5 \\
 &= (n+1)^5
 \end{aligned}$$

Now add up $5x^4 + 10x^3 + 10x^2 + 5x + 1$ for $x = 0, 1, 2, \dots, n$ to obtain

$$\begin{aligned}
 &5(0^4 + 1^4 + 2^4 + \dots + n^4) \\
 &+ 10(0^3 + 1^3 + 2^3 + \dots + n^3) \\
 &+ 10(0^2 + 1^2 + 2^2 + \dots + n^2) \\
 &+ 5(0 + 1 + 2 + \dots + n) + 1 + 1 + 1 + \dots + 1 \\
 &= 5S_4(n) + 10S_3(n) + 10S_2(n) + 5S_1(n) + n + 1
 \end{aligned}$$

Thus, since

$$(x+1)^5 - x^5 = 5x^4 + 10x^3 + 10x^2 + 5x + 1,$$

$$(n+1)^5 = 5S_4(n) + 10S_3(n) + 10S_2(n) + 5S_1(n) + n + 1$$

Solve this equation for $S_4(n)$:

$$\begin{aligned}
 S_4(n) &= \frac{1}{5}[(n+1)^5 - 10S_3(n) - 10S_2(n) - 5S_1(n) - (n+1)] \\
 &= \frac{1}{5}\left[(n+1)^5 - 10 \cdot \frac{1}{4}n^2(n+1)^2 - 10 \cdot \frac{1}{6}n(n+1)(2n+1) \right. \\
 &\quad \left. - 5 \cdot \frac{1}{2}n(n+1) - (n+1)\right] \\
 &= \frac{1}{5}(n+1)\left[(n+1)^4 - \frac{5}{2}n^2(n+1) - \frac{5}{3}n(2n+1) - \frac{5}{2}n - 1\right] \\
 &= \frac{1}{30}(n+1)[6(n+1)^4 - 15n^2(n+1) - 10n(2n+1) - 15n - 6] \\
 &= \frac{1}{30}(n+1)(6n^4 + 24n^3 + 36n^2 + 24n + 6 - 15n^3 \\
 &\quad - 15n^2 - 20n^2 - 10n - 15n - 6) \\
 &= \frac{1}{30}(n+1)(6n^4 + 9n^3 + n^2 - n) \\
 &= \frac{1}{30}n(n+1)(6n^3 + 9n^2 + n - 1)
 \end{aligned}$$

51. The temperature function is

$$T(t) = 80 + 25 \sin\left(\frac{2\pi}{12}\left(t - \frac{7}{2}\right)\right).$$

The normal high temperature for November 15th is then $T(10.5) = 67.5^\circ\text{F}$.

52. The water level function is

$$F(t) = 8.5 + 3.5 \sin\left(\frac{2\pi}{12}(t - 9)\right).$$

The water level at 5:30 P.M. is then

$$F(17.5) \approx 5.12 \text{ ft.}$$

53. As t increases, the point on the rim of the wheel will move around the circle of radius 2.

a. $x(2) \approx 1.902$

$y(2) \approx 0.618$

$$x(6) \approx -1.176$$

$$y(6) \approx -1.618$$

$$x(10) = 0$$

$$y(10) = 2$$

$$x(0) = 0$$

$$y(0) = 2$$

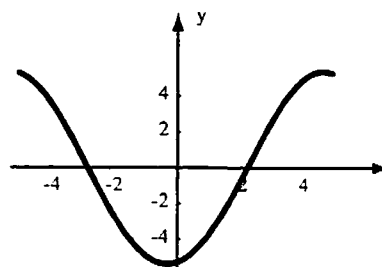
b. $x(t) = -2 \sin\left(\frac{\pi}{5}t\right), y(t) = 2 \cos\left(\frac{\pi}{5}t\right)$

c. The point is at $(2, 0)$ when $\frac{\pi}{5}t = \frac{\pi}{2}$; that is,

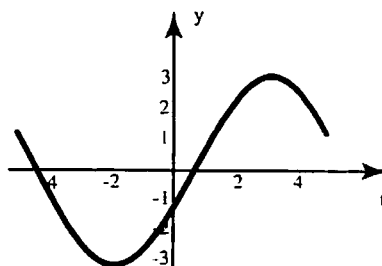
$$\text{when } t = \frac{5}{2}$$

54. Both functions have frequency $\frac{2\pi}{10}$. When you add functions that have the same frequency, the sum has the same frequency.

a. $y(t) = 3 \sin(\pi t / 5) - 5 \cos(\pi t / 5) + 2 \sin((\pi t / 5) - 3)$



b. $y(t) = 3 \cos(\pi / 5 - 2) + \cos(\pi / 5) + \cos((\pi / 5) - 3)$



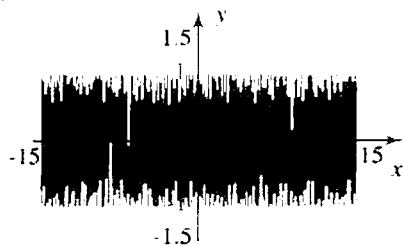
55. a. $C \sin(\omega t + \phi) = (C \sin \omega t) \cos \phi + (C \cos \omega t) \sin \phi$. Thus $A = C \cos \omega t$ and $B = C \sin \omega t$.

b.
$$\begin{aligned} A^2 + B^2 &= (C \cos \omega t)^2 + (C \sin \omega t)^2 \\ &= C^2 (\cos^2 \omega t) + C^2 (\sin^2 \omega t) = C^2 \end{aligned}$$

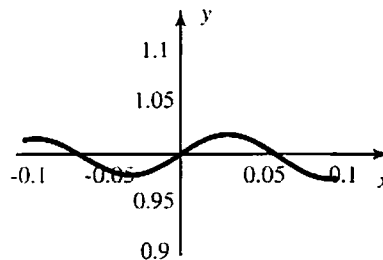
c.
$$\begin{aligned} &A_1 \sin(\omega t + \phi_1) + A_2 \sin(\omega t + \phi_2) + A_3 (\sin \omega t + \phi_3) \\ &= A_1 (\sin \omega t \cos \phi_1 + \cos \omega t \sin \phi_1) \\ &\quad + A_2 (\sin \omega t \cos \phi_2 + \cos \omega t \sin \phi_2) \\ &\quad + A_3 (\sin \omega t \cos \phi_3 + \cos \omega t \sin \phi_3) \\ &= (A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3) \sin \omega t \\ &\quad + (A_1 \sin \phi_1 + A_2 \sin \phi_2 + A_3 \sin \phi_3) \cos \omega t \end{aligned}$$

d. Written response. Answers will vary.

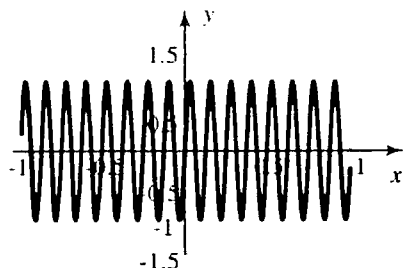
56. (a.), (b.), and (c.) all look similar to this:



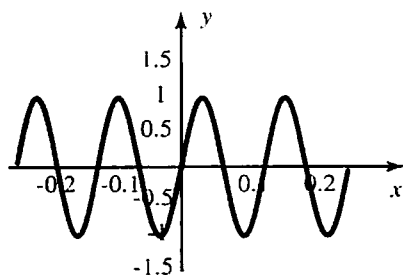
c.



d.



e.



The windows in (a)-(c) are not helpful because the function oscillates too much over the domain plotted. Plots in (d) or (e) show the behavior of the function.

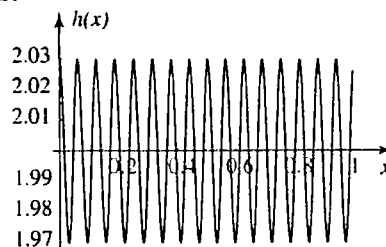
The plot in (a) shows the long term behavior of the function, but not the short term behavior, whereas the plot in (c) shows the short term behavior, but not the long term behavior. The plot in (b) shows a little of each.

58. a.

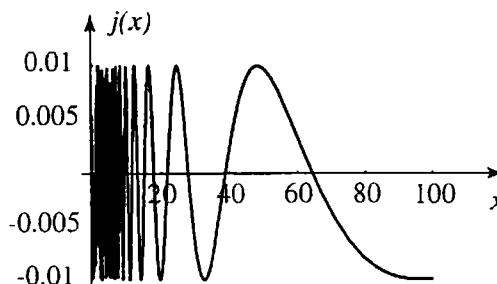
$$h(x) = (f \circ g)(x) = \frac{\frac{3}{100} \cos(100x) + 2}{\left(\frac{1}{100}\right)^2 \cos^2(100x) + 1}$$

$$j(x) = (g \circ f)(x) = \frac{1}{100} \cos\left(100 \frac{3x+2}{x^2+1}\right)$$

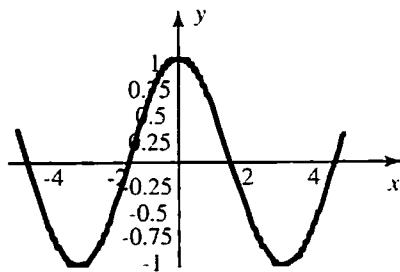
b.



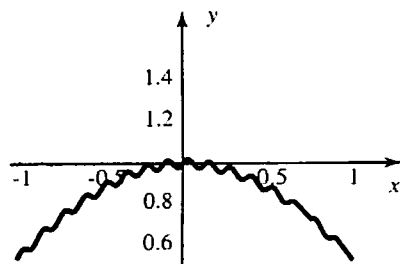
c.



57. a.



b.



2.4 Concepts Review

1. $L; c$
2. 6
3. L ; right
4. $\lim_{x \rightarrow c} f(x) = M$

Problem Set 2.4

1. $\lim_{x \rightarrow 3} (x - 5) = -2$
2. $\lim_{t \rightarrow -1} (1 - 2t) = 3$
3. $\lim_{x \rightarrow -2} (x^2 + 2x - 1) = (-2)^2 + 2(-2) - 1 = -1$
4. $\lim_{x \rightarrow -2} (x^2 + 2t - 1) = (2)^2 + 2t - 1 = 3 + 2t$
5. $\lim_{t \rightarrow -1} \frac{1 - 2t}{\sqrt{3t + 21}} = \frac{1 - 2(-1)}{\sqrt{3(-1) + 21}} = \frac{3}{\sqrt{18}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
6. $\lim_{t \rightarrow -1} \frac{\sqrt{1 - 2t}}{(3t + 2)^3} = \frac{\sqrt{1 - 2(-1)}}{[3(-1) + 2]^3} = \frac{\sqrt{3}}{(-1)^3} = -\sqrt{3}$
7. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 2 + 2 = 4$
8. $\lim_{t \rightarrow -7} \frac{t^2 + 4t - 21}{t + 7} = \lim_{t \rightarrow -7} \frac{(t + 7)(t - 3)}{t + 7} = \lim_{t \rightarrow -7} (t - 3) = -7 - 3 = -10$
9. $\lim_{x \rightarrow -1} \frac{x^3 - 4x^2 + x + 6}{x + 1} = \lim_{x \rightarrow -1} \frac{(x + 1)(x^2 - 5x + 6)}{x + 1} = \lim_{x \rightarrow -1} (x^2 - 5x + 6) = (-1)^2 - 5(-1) + 6 = 12$
10. $\lim_{x \rightarrow 0} \frac{x^4 + 2x^3 - x^2}{x^2} = \lim_{x \rightarrow 0} (x^2 + 2x - 1) = -1$
11. $\lim_{x \rightarrow -t} \frac{x^2 - t^2}{x + t} = \lim_{x \rightarrow -t} \frac{(x + t)(x - t)}{x + t} = \lim_{x \rightarrow -t} (x - t) = -t - t = -2t$
12. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3} (x + 3) = 3 + 3 = 6$

$$13. \lim_{t \rightarrow 2} \frac{\sqrt{(t+4)(t-2)^4}}{(3t-6)^2} = \lim_{t \rightarrow 2} \frac{(t-2)^2 \sqrt{t+4}}{9(t-2)^2} = \lim_{t \rightarrow 2} \frac{\sqrt{t+4}}{9} = \frac{\sqrt{2+4}}{9} = \frac{\sqrt{6}}{9}$$

$$14. \lim_{t \rightarrow 7} \frac{\sqrt{(t-7)^3}}{t-7} = \lim_{t \rightarrow 7} \frac{(t-7)\sqrt{t-7}}{t-7} = \lim_{t \rightarrow 7} \sqrt{t-7} = \sqrt{7-7} = 0$$

$$15. \lim_{x \rightarrow 3} \frac{x^4 - 18x^2 + 81}{(x-3)^2} = \lim_{x \rightarrow 3} \frac{(x^2 - 9)^2}{(x-3)^2} = \lim_{x \rightarrow 3} \frac{(x-3)^2(x+3)^2}{(x-3)^2} = \lim_{x \rightarrow 3} (x+3)^2 = (3+3)^2 = 36$$

$$16. \lim_{u \rightarrow 1} \frac{(3u+4)(2u-2)^3}{(u-1)^2} = \lim_{u \rightarrow 1} \frac{8(3u+4)(u-1)^3}{(u-1)^2} = \lim_{u \rightarrow 1} 8(3u+4)(u-1) = 8[3(1)+4](1-1) = 0$$

$$17. \lim_{h \rightarrow 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 4h}{h} = \lim_{h \rightarrow 0} (h + 4) = 4$$

$$18. \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 2xh}{h} = \lim_{h \rightarrow 0} (h + 2x) = 2x$$

19.

x	$\frac{\sin x}{2x}$
1.	0.420735
0.1	0.499167
0.01	0.499992
0.001	0.5
-1.	0.420735
-0.1	0.499167
-0.01	0.499992
-0.001	0.5

$\lim_{x \rightarrow 0} \frac{\sin x}{2x} = 0.5$

20.

t	$\frac{1-\cos t}{2t}$
1.	0.229849
0.1	0.0249792
0.01	0.00249998
0.001	0.00025
-1.	-0.229849
-0.1	-0.0249792
-0.01	-0.00249998
-0.001	-0.00025

$$\lim_{t \rightarrow 0} \frac{1 - \cos t}{2t} = 0$$

21.

x	$(x - \sin x)^2 / x^2$
1.	0.0251314
0.1	2.775×10^{-6}
0.01	2.7775×10^{-10}
0.001	2.7778×10^{-14}
-1.	0.0251314
-0.1	2.775×10^{-6}
-0.01	2.7775×10^{-10}
-0.001	2.7778×10^{-14}

$$\lim_{x \rightarrow 0} \frac{(x - \sin x)^2}{x^2} = 0$$

22.

x	$(1 - \cos x)^2 / x^2$
1.	0.211322
0.1	0.00249584
0.01	0.0000249996
0.001	2.5×10^{-7}
-1.	0.211322
-0.1	0.00249584
-0.01	0.0000249996
-0.001	2.5×10^{-7}

$$\lim_{x \rightarrow 0} \frac{(1 - \cos x)^2}{x^2} = 0$$

23.

t	$(t^2 - 1) / (\sin(t - 1))$
2.	3.56519
1.1	2.1035
1.01	2.01003
1.001	2.001
0	1.1884
0.9	1.90317
0.99	1.99003
0.999	1.999

$$\lim_{t \rightarrow 1} \frac{t^2 - 1}{\sin(t - 1)} = 2$$

24.

x	$\frac{x - \sin(x - 3) - 3}{x - 3}$
4.	0.158529
3.1	0.00166583
3.01	0.0000166666
3.001	1.66667×10^{-7}
2.	0.158529
2.9	0.00166583
2.99	0.0000166666
2.999	1.66667×10^{-7}

$$\lim_{x \rightarrow 3} \frac{x - \sin(x - 3) - 3}{x - 3} = 0$$

25.

x	$(1 + \sin(x - 3\pi/2)) / (x - \pi)$
$1 + \pi$	0.4597
$0.1 + \pi$	0.0500
$0.01 + \pi$	0.0050
$0.001 + \pi$	0.0005
$-1 + \pi$	-0.4597
$-0.1 + \pi$	-0.0500
$-0.01 + \pi$	-0.0050
$-0.001 + \pi$	-0.0005

$$\lim_{x \rightarrow \pi} \frac{1 + \sin\left(x - \frac{3\pi}{2}\right)}{x - \pi} = 0$$

26.

t	$(1 - \cot t)/(1/t)$
1.	0.357907
0.1	-0.896664
0.01	-0.989967
0.001	-0.999
-1.	-1.64209
-0.1	-1.09666
-0.01	-1.00997
-0.001	-1.001

$$\lim_{t \rightarrow 0} \frac{1 - \cot t}{\frac{1}{t}} = -1$$

27.

x	$(x - \pi/4)^2 / (\tan x - 1)^2$
$1. + \frac{\pi}{4}$	0.0320244
$0.1 + \frac{\pi}{4}$	0.201002
$0.01 + \frac{\pi}{4}$	0.245009
$0.001 + \frac{\pi}{4}$	0.2495
$-1. + \frac{\pi}{4}$	0.674117
$-0.1 + \frac{\pi}{4}$	0.300668
$-0.01 + \frac{\pi}{4}$	0.255008
$-0.001 + \frac{\pi}{4}$	0.2505

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{(x - \frac{\pi}{4})^2}{(\tan x - 1)^2} = 0.25$$

28.

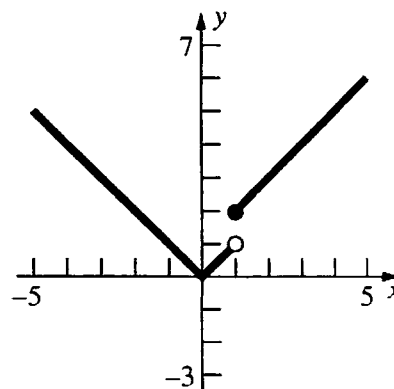
u	$(2 - 2 \sin u)/3u$
$1. + \frac{\pi}{2}$	0.11921
$0.1 + \frac{\pi}{2}$	0.00199339
$0.01 + \frac{\pi}{2}$	0.0000210862
$0.001 + \frac{\pi}{2}$	2.12072×10^{-7}
$-1. + \frac{\pi}{2}$	0.536908
$-0.1 + \frac{\pi}{2}$	0.00226446
$-0.01 + \frac{\pi}{2}$	0.0000213564
$-0.001 + \frac{\pi}{2}$	2.12342×10^{-7}

$$\lim_{u \rightarrow \frac{\pi}{2}} \frac{2 - 2 \sin u}{3u} = 0$$

29. a. $\lim_{x \rightarrow -3} f(x) = 2$
 b. $f(-3) = 1$
 c. $f(-1)$ does not exist.
 d. $\lim_{x \rightarrow -1} f(x) = \frac{5}{2}$
 e. $f(1) = 2$
 f. $\lim_{x \rightarrow 1} f(x)$ does not exist.
 g. $\lim_{x \rightarrow 1^-} f(x) = 2$
 h. $\lim_{x \rightarrow 1^+} f(x) = 1$

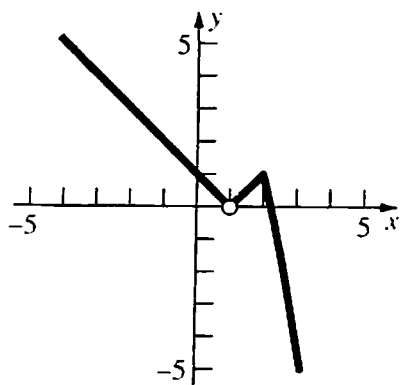
30. a. $\lim_{x \rightarrow -3} f(x)$ does not exist.
 b. $f(-3) = 1$
 c. $f(-1) = 1$
 d. $\lim_{x \rightarrow -1} f(x) = 2$
 e. $f(1) = 1$
 f. $\lim_{x \rightarrow 1} f(x)$ does not exist.
 g. $\lim_{x \rightarrow 1^-} f(x) = 1$
 h. $\lim_{x \rightarrow 1^+} f(x)$ does not exist.

31.



- a. $\lim_{x \rightarrow 0} f(x) = 0$
 b. $\lim_{x \rightarrow 1} f(x)$ does not exist.
 c. $f(1) = 2$
 d. $\lim_{x \rightarrow 1^+} f(x) = 2$

32.



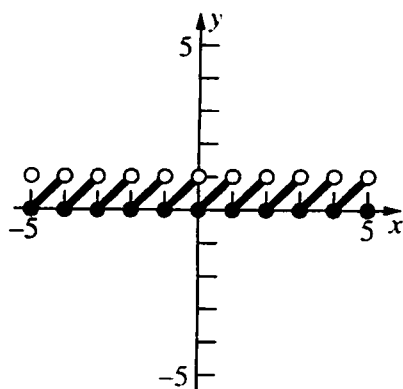
a. $\lim_{x \rightarrow 1} g(x) = 0$

b. $g(1)$ does not exist.

c. $\lim_{x \rightarrow 2} g(x) = 1$

d. $\lim_{x \rightarrow 2^+} g(x) = 1$

33. $f(x) = x - \lfloor x \rfloor$



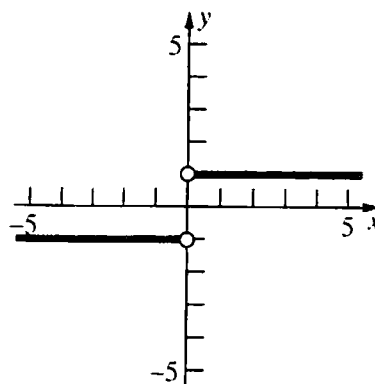
a. $f(0) = 0$

b. $\lim_{x \rightarrow 0} f(x)$ does not exist.

c. $\lim_{x \rightarrow 0^-} f(x) = 1$

d. $\lim_{x \rightarrow \frac{1}{2}} f(x) = \frac{1}{2}$

34. $f(x) = \frac{x}{|x|}$

a. $f(0)$ does not exist.b. $\lim_{x \rightarrow 0} f(x)$ does not exist.

c. $\lim_{x \rightarrow 0^-} f(x) = -1$

d. $\lim_{x \rightarrow \frac{1}{2}} f(x) = 1$

35. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{|x - 1|}$ does not exist.

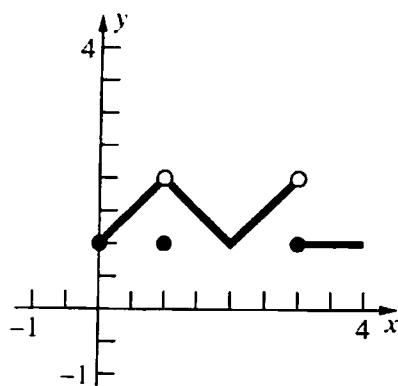
$$\lim_{x \rightarrow 1^-} \frac{x^2 - 1}{|x - 1|} = -2 \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{|x - 1|} = 2$$

$$\begin{aligned} 36. \quad \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{x+2} - \sqrt{2})(\sqrt{x+2} + \sqrt{2})}{x(\sqrt{x+2} + \sqrt{2})} \\ &= \lim_{x \rightarrow 0} \frac{x+2-2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+2} + \sqrt{2})} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{\sqrt{0+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \end{aligned}$$

37. a. $\lim_{x \rightarrow 1} f(x)$ does not exist.

b. $\lim_{x \rightarrow 0} f(x) = 0$

38.

39. $\lim_{x \rightarrow a} f(x)$ exists for $a = -1, 0, 1$.

40. The changed values should not change $\lim_{x \rightarrow a} f(x)$ at any a as long as the changed points are not all together. As x approaches a , the limit should still be a^2 .

41. a. $\lim_{x \rightarrow 1} \frac{|x-1|}{x-1}$ does not exist.
 $\lim_{x \rightarrow 1^-} \frac{|x-1|}{x-1} = -1$ and $\lim_{x \rightarrow 1^+} \frac{|x-1|}{x-1} = 1$

b. $\lim_{x \rightarrow 1^-} \frac{|x-1|}{x-1} = -1$

c. $\lim_{x \rightarrow 1^-} \frac{x^2 - |x-1| - 1}{|x-1|} = -3$

d. $\lim_{x \rightarrow 1^-} \left[\frac{1}{x-1} - \frac{1}{|x-1|} \right]$ does not exist.

42. a. $\lim_{x \rightarrow 1^+} \sqrt{x - \lceil x \rceil} = 0$

b. $\lim_{x \rightarrow 0^+} \frac{1}{x}$ does not exist.

c. $\lim_{x \rightarrow 0^+} x(-1)^{\lceil 1/x \rceil}$ does not exist.

d. $\lim_{x \rightarrow 0^+} \lceil x \rceil (-1)^{\lceil 1/x \rceil} = 0$

e. $\lim_{x \rightarrow 0^+} x \left(\frac{1}{x} \right) = 1$

f. $\lim_{x \rightarrow 0^+} x^2 \left(\frac{1}{x} \right) = 0$

43. $\lim_{x \rightarrow 0} \sqrt{x}$ does not exist since $\sqrt{x}$ is not defined for $x < 0$.

44. $\lim_{x \rightarrow 0^+} x^x = 1$

45. $\lim_{x \rightarrow 0} \sqrt{|x|} = 0$

46. $\lim_{x \rightarrow 0} |x|^x = 1$

47. $\lim_{x \rightarrow 0} \frac{\sin 2x}{4x} = \frac{1}{2}$

48. $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x} = \frac{5}{3}$

49. $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$ does not exist.

50. $\lim_{x \rightarrow 0} x \cos\left(\frac{1}{x}\right) = 0$

51. $\lim_{x \rightarrow 1} \frac{x^3 - 1}{\sqrt{2x+2} - 2} = 6$

52. $\lim_{x \rightarrow 0} \frac{x \sin 2x}{\sin(x^2)} = 2$

53. $\lim_{x \rightarrow 2^-} \frac{x^2 - x - 2}{|x - 2|} = -3$

54. $\lim_{x \rightarrow 1^+} \frac{2}{1 + 2^{1/(x-1)}} = 0$

55. $\lim_{x \rightarrow 0} \sqrt{x}$; The computer gives a value of 0, but $\lim_{x \rightarrow 0^-} \sqrt{x}$ does not exist.

2.5 Concepts Review

1. $L - \varepsilon; L + \varepsilon$
2. $0 < |x - a| < \delta; |f(x) - L| < \varepsilon$
3. $\frac{\varepsilon}{3}$
4. $ma + b$

Problem Set 2.5

1. $0 < |t - a| < \delta \Rightarrow |f(t) - M| < \varepsilon$
2. $0 < |u - b| < \delta \Rightarrow |g(u) - L| < \varepsilon$
3. $0 < |z - d| < \delta \Rightarrow |h(z) - P| < \varepsilon$
4. $0 < |y - e| < \delta \Rightarrow |\phi(y) - B| < \varepsilon$
5. $0 < c - x < \delta \Rightarrow |f(x) - L| < \varepsilon$
6. $0 < t - a < \delta \Rightarrow |g(t) - D| < \varepsilon$
7. $0 < |x - 0| < \delta \Rightarrow |(2x - 1) - (-1)| < \varepsilon$
 $|2x - 1 + 1| < \varepsilon \Leftrightarrow |2x| < \varepsilon$
 $\Leftrightarrow 2|x| < \varepsilon$
 $\Leftrightarrow |x| < \frac{\varepsilon}{2}$
 $\delta = \frac{\varepsilon}{2}; 0 < |x - 0| < \delta$
 $|(2x - 1) - (-1)| = |2x| = 2|x| < 2\delta = \varepsilon$
8. $0 < |x + 21| < \delta \Rightarrow |(3x - 1) - (-64)| < \varepsilon$
 $|3x - 1 + 64| < \varepsilon \Leftrightarrow |3x + 63| < \varepsilon$
 $\Leftrightarrow |3(x + 21)| < \varepsilon$
 $\Leftrightarrow 3|x + 21| < \varepsilon$
 $\Leftrightarrow |x + 21| < \frac{\varepsilon}{3}$
 $\delta = \frac{\varepsilon}{3}; 0 < |x + 21| < \delta$
 $|(3x - 1) - (-64)| = |3x + 63| = 3|x + 21| < 3\delta = \varepsilon$

$$9. 0 < |x - 5| < \delta \Rightarrow \left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon$$

$$\left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon \Leftrightarrow \left| \frac{(x - 5)(x + 5)}{x - 5} - 10 \right| < \varepsilon$$

$$\Leftrightarrow |x + 5 - 10| < \varepsilon$$

$$\Leftrightarrow |x - 5| < \varepsilon$$

$$\delta = \varepsilon; 0 < |x - 5| < \delta$$

$$\left| \frac{x^2 - 25}{x - 5} - 10 \right| = \left| \frac{(x - 5)(x + 5)}{x - 5} - 10 \right| = |x + 5 - 10|$$

$$= |x - 5| < \delta = \varepsilon$$

$$10. 0 < |x - 0| < \delta \Rightarrow \left| \frac{2x^2 - x}{x} - (-1) \right| < \varepsilon$$

$$\left| \frac{2x^2 - x}{x} + 1 \right| < \varepsilon \Leftrightarrow \left| \frac{x(2x - 1)}{x} + 1 \right| < \varepsilon$$

$$\Leftrightarrow |2x - 1 + 1| < \varepsilon$$

$$\Leftrightarrow |2x| < \varepsilon$$

$$\Leftrightarrow 2|x| < \varepsilon$$

$$\Leftrightarrow |x| < \frac{\varepsilon}{2}$$

$$\delta = \frac{\varepsilon}{2}; 0 < |x - 0| < \delta$$

$$\left| \frac{2x^2 - x}{x} - (-1) \right| = \left| \frac{x(2x - 1)}{x} + 1 \right| = |2x - 1 + 1|$$

$$= |2x| = 2|x| < 2\delta = \varepsilon$$

$$11. 0 < |x - 5| < \delta \Rightarrow \left| \frac{2x^2 - 11x + 5}{x - 5} - 9 \right| < \varepsilon$$

$$\left| \frac{2x^2 - 11x + 5}{x - 5} - 9 \right| < \varepsilon \Leftrightarrow \left| \frac{(2x - 1)(x - 5)}{x - 5} - 9 \right| < \varepsilon$$

$$\Leftrightarrow |2x - 1 - 9| < \varepsilon$$

$$\Leftrightarrow |2(x - 5)| < \varepsilon$$

$$\Leftrightarrow |x - 5| < \frac{\varepsilon}{2}$$

$$\delta = \frac{\varepsilon}{2}; 0 < |x - 5| < \delta$$

$$\left| \frac{2x^2 - 11x + 5}{x - 5} - 9 \right| = \left| \frac{(2x - 1)(x - 5)}{x - 5} - 9 \right|$$

$$= |2x - 1 - 9| = |2(x - 5)| = 2|x - 5| < 2\delta = \varepsilon$$

$$12. 0 < |x-1| < \delta \Rightarrow |\sqrt{2x}-\sqrt{2}| < \varepsilon$$

$$|\sqrt{2x}-\sqrt{2}| < \varepsilon$$

$$\Leftrightarrow \left| \frac{(\sqrt{2x}-\sqrt{2})(\sqrt{2x}+\sqrt{2})}{\sqrt{2x}+\sqrt{2}} \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{2x-2}{\sqrt{2x}+\sqrt{2}} \right| < \varepsilon$$

$$\Leftrightarrow 2 \left| \frac{x-1}{\sqrt{2x}+\sqrt{2}} \right| < \varepsilon$$

$$\delta = \frac{\sqrt{2}\varepsilon}{2}; 0 < |x-1| < \delta$$

$$|\sqrt{2x}-\sqrt{2}| = \left| \frac{(\sqrt{2x}-\sqrt{2})(\sqrt{2x}+\sqrt{2})}{\sqrt{2x}+\sqrt{2}} \right|$$

$$= \left| \frac{2x-2}{\sqrt{2x}+\sqrt{2}} \right|$$

$$\frac{2|x-1|}{\sqrt{2x}+\sqrt{2}} \leq \frac{2|x-1|}{\sqrt{2}} < \frac{2\delta}{\sqrt{2}} = \varepsilon$$

$$13. 0 < |x-4| < \delta \Rightarrow \left| \frac{\sqrt{2x-1}}{\sqrt{x-3}} - \sqrt{7} \right| < \varepsilon$$

$$\left| \frac{\sqrt{2x-1}}{\sqrt{x-3}} - \sqrt{7} \right| < \varepsilon \Leftrightarrow \left| \frac{\sqrt{2x-1}-\sqrt{7(x-3)}}{\sqrt{x-3}} \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{(\sqrt{2x-1}-\sqrt{7(x-3)})(\sqrt{2x-1}+\sqrt{7(x-3)})}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{2x-1-(7x-21)}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{-5(x-4)}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} \right| < \varepsilon$$

$$\Leftrightarrow |x-4| \left| \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} \right| < \varepsilon$$

To bound $\frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}$, agree

that $\delta \leq \frac{1}{2}$. If $\delta \leq \frac{1}{2}$, then $\frac{7}{2} < x < \frac{9}{2}$, so

$$0.65 < \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < 1.65 \text{ and}$$

$$\text{hence } |x-4| \left| \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} \right| < \varepsilon$$

$$\Leftrightarrow |x-4| < \frac{\varepsilon}{1.65}$$

For whatever ε is chosen, let δ be the smaller of $\frac{1}{2}$ and $\frac{\varepsilon}{1.65}$.

$$\delta = \min \left\{ \frac{1}{2}, \frac{\varepsilon}{1.65} \right\}, 0 < |x-4| < \delta$$

$$\left| \frac{\sqrt{2x-1}}{\sqrt{x-3}} - \sqrt{7} \right| = |x-4| \left| \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} \right|$$

$$< |x-4|(1.65) < 1.65\delta \leq \varepsilon$$

since $\delta = \frac{1}{2}$ only when $\frac{1}{2} \leq \frac{\varepsilon}{1.65}$ so $1.65\delta \leq \varepsilon$.

$$14. 0 < |x-1| < \delta \Rightarrow \left| \frac{14x^2-20x+6}{x-1} - 8 \right| < \varepsilon$$

$$\left| \frac{14x^2-20x+6}{x-1} - 8 \right| < \varepsilon \Leftrightarrow \left| \frac{2(7x-3)(x-1)}{x-1} - 8 \right| < \varepsilon$$

$$\Leftrightarrow |2(7x-3)-8| < \varepsilon$$

$$\Leftrightarrow |14(x-1)| < \varepsilon$$

$$\Leftrightarrow 14|x-1| < \varepsilon$$

$$\Leftrightarrow |x-1| < \frac{\varepsilon}{14}$$

$$\delta = \frac{\varepsilon}{14}; 0 < |x-1| < \delta$$

$$\left| \frac{14x^2-20x+6}{x-1} - 8 \right| = \left| \frac{2(7x-3)(x-1)}{x-1} - 8 \right|$$

$$= |2(7x-3)-8|$$

$$= |14(x-1)| = 14|x-1| < 14\delta = \varepsilon$$

$$15. 0 < |x-1| < \delta \Rightarrow \left| \frac{10x^3-26x^2+22x-6}{(x-1)^2} - 4 \right| < \varepsilon$$

$$\left| \frac{10x^3-26x^2+22x-6}{(x-1)^2} - 4 \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{(10x-6)(x-1)^2}{(x-1)^2} - 4 \right| < \varepsilon$$

$$\Leftrightarrow |10x-6-4| < \varepsilon$$

$$\Leftrightarrow |10(x-1)| < \varepsilon$$

$$\Leftrightarrow 10|x-1| < \varepsilon$$

$$\Leftrightarrow |x-1| < \frac{\varepsilon}{10}$$

$$\delta = \frac{\varepsilon}{10}; 0 < |x-1| < \delta$$

$$\begin{aligned} & \left| \frac{10x^3 - 26x^2 + 22x - 6}{(x-1)^2} - 4 \right| \\ &= \left| \frac{(10x-6)(x-1)^2}{(x-1)^2} - 4 \right| \\ &= |10x - 6 - 4| = |10(x-1)| \\ &= 10|x-1| < 10\delta = \varepsilon \end{aligned}$$

$$16. \quad 0 < |x-1| < \delta \Rightarrow |(2x^2+1)-3| < \varepsilon$$

$$|2x^2+1-3| = |2x^2-2| = 2|x+1||x-1|$$

To bound $|2x+2|$, agree that $\delta \leq 1$.

$|x-1| < \delta$ implies

$$|2x+2| = |2x-2+4|$$

$$\leq |2x-2| + |4|$$

$$< 2 + 4 = 6$$

$$\delta \leq \frac{\varepsilon}{6}; \delta = \min\left\{1, \frac{\varepsilon}{6}\right\}; 0 < |x-1| < \delta$$

$$|(2x^2+1)-3| = |2x^2-2|$$

$$= |2x+2||x-1| < 6 \cdot \left(\frac{\varepsilon}{6}\right) = \varepsilon$$

$$17. \quad 0 < |x+1| < \delta \Rightarrow |(x^2-2x-1)-2| < \varepsilon$$

$$|x^2-2x-1-2| = |x^2-2x-3| = |x+1||x-3|$$

To bound $|x-3|$, agree that $\delta \leq 1$.

$|x+1| < \delta$ implies

$$|x-3| = |x+1-4|$$

$$\leq |x+1| + |-4|$$

$$< 1 + 4 = 5$$

$$\delta \leq \frac{\varepsilon}{5}; \delta = \min\left\{1, \frac{\varepsilon}{5}\right\}; 0 < |x+1| < \delta$$

$$|(x^2-2x-1)-2| = |x^2-2x-3|$$

$$= |x+1||x-3| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$$

$$18. \quad 0 < |x| < \delta \Rightarrow |x^4-0| = |x^4| < \varepsilon$$

$|x^4| = |x||x^3|$. To bound $|x^3|$, agree that

$\delta \leq 1$. $|x| < \delta \leq 1$ implies $|x^3| = |x|^3 \leq 1$ so

$\delta \leq \varepsilon$.

$$\delta = \min\{1, \varepsilon\}; 0 < |x| < \delta \Rightarrow |x^4| = |x||x^3| < \varepsilon \cdot 1$$

$$= \varepsilon$$

19. Choose $\varepsilon > 0$. Then since $\lim_{x \rightarrow c} f(x) = L$, there is

some $\delta_1 > 0$ such that

$$0 < |x-c| < \delta_1 \Rightarrow |f(x)-L| < \varepsilon.$$

Since $\lim_{x \rightarrow c} f(x) = M$, there is some $\delta_2 > 0$ such

that $0 < |x-c| < \delta_2 \Rightarrow |f(x)-L| < \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and choose x_0 such that $0 < |x_0-c| < \delta$.

$$\text{Thus, } |f(x_0)-L| < \varepsilon \Rightarrow -\varepsilon < f(x_0)-L < \varepsilon$$

$$\Rightarrow -f(x_0)-\varepsilon < -L < -f(x_0)+\varepsilon$$

$$\Rightarrow f(x_0)-\varepsilon < L < f(x_0)+\varepsilon.$$

Similarly,

$$f(x_0)-\varepsilon < M < f(x_0)+\varepsilon.$$

Thus,

$-2\varepsilon < L-M < 2\varepsilon$. As $\varepsilon \rightarrow 0$, $L-M \rightarrow 0$, so $L=M$.

20. Since $\lim_{x \rightarrow c} G(x) = 0$, then given any $\varepsilon > 0$, we

can find $\delta > 0$ such that whenever

$$|x-c| < \delta, |G(x)| < \varepsilon.$$

Take any $\varepsilon > 0$ and the corresponding δ that

works for $G(x)$, then $|x-c| < \delta$ implies

$$|F(x)-0| = |F(x)| \leq |G(x)| < \varepsilon \text{ since}$$

$$\lim_{x \rightarrow c} G(x) = 0.$$

Thus, $\lim_{x \rightarrow c} F(x) = 0$.

21. For all $x \neq 0$, $0 \leq \sin^2\left(\frac{1}{x}\right) \leq 1$ so

$$x^4 \sin^2\left(\frac{1}{x}\right) \leq x^4 \text{ for all } x \neq 0. \text{ By Problem 18,}$$

$\lim_{x \rightarrow 0} x^4 = 0$, so, by Problem 20,

$$\lim_{x \rightarrow 0} x^4 \sin^2\left(\frac{1}{x}\right) = 0.$$

22. $0 < x < \delta \Rightarrow |\sqrt{x} - 0| = |\sqrt{x}| = \sqrt{x} < \varepsilon$
 For $x > 0$, $(\sqrt{x})^2 = x$.
 $\sqrt{x} < \varepsilon \Leftrightarrow (\sqrt{x})^2 = x < \varepsilon^2$
 $\delta = \varepsilon^2$; $0 < x < \delta \Rightarrow \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$

23. $\lim_{x \rightarrow 0^+} |x| : 0 < x < \delta \Rightarrow |x| - 0| < \varepsilon$
 For $x \geq 0$, $|x| = x$.
 $\delta = \varepsilon$; $0 < x < \delta \Rightarrow |x| - 0| = |x| = x < \delta = \varepsilon$
 Thus, $\lim_{x \rightarrow 0^+} |x| = 0$.
 $\lim_{x \rightarrow 0^-} |x| : 0 < 0 - x < \delta \Rightarrow ||x| - 0| < \varepsilon$
 For $x < 0$, $|x| = -x$: note also that $||x|| = |x|$
 since $|x| \geq 0$.
 $\delta = \varepsilon$; $0 < -x < \delta \Rightarrow ||x|| = |x| = -x < \delta = \varepsilon$
 Thus, $\lim_{x \rightarrow 0^-} |x| = 0$.
 since $\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^-} |x| = 0$, $\lim_{x \rightarrow 0} |x| = 0$.

24. Choose $\varepsilon > 0$. Since $\lim_{x \rightarrow a} g(x) = 0$ there is some $\delta_1 > 0$ such that
 $0 < |x - a| < \delta_1 \Rightarrow |g(x) - 0| < \frac{\varepsilon}{B}$.

Let $\delta = \min\{1, \delta_1\}$, then $|f(x)| < B$ for
 $|x - a| < \delta$ or $|x - a| < \delta \Rightarrow |f(x)| < B$. Thus,
 $|x - a| < \delta \Rightarrow |f(x)g(x) - 0| = |f(x)g(x)|$
 $= |f(x)||g(x)| < B \cdot \frac{\varepsilon}{B} = \varepsilon$ so $\lim_{x \rightarrow a} f(x)g(x) = 0$.

25. Choose $\varepsilon > 0$. Since $\lim_{x \rightarrow a} f(x) = L$, there is a $\delta > 0$ such that for $0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$.
 That is, for
 $a - \delta < x < a$ or $a < x < a + \delta$,
 $L - \varepsilon < f(x) < L + \varepsilon$.

Let $f(a) = A$.
 $M = \max\{|L - \varepsilon|, |L + \varepsilon|, |A|\}$, $c = a - \delta$,
 $d = a + \delta$. Then for x in (c, d) , $|f(x)| \leq M$, since
 either $x = a$, in which case
 $|f(x)| = |f(a)| = |A| \leq M$ or $0 < |x - a| < \delta$ so
 $L - \varepsilon < f(x) < L + \varepsilon$ and $|f(x)| < M$.

26. Suppose that $L > M$. Then $L - M = \alpha > 0$. Now
 take $\varepsilon < \frac{\alpha}{2}$ and $\delta = \min\{\delta_1, \delta_2\}$ where
 $0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$ and
 $0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon$.
 Thus, for $0 < |x - a| < \delta$,
 $L - \varepsilon < f(x) < L + \varepsilon$ and $M - \varepsilon < g(x) < M + \varepsilon$.
 Combine the inequalities and use the fact
 that $f(x) \leq g(x)$ to get
 $L - \varepsilon < f(x) \leq g(x) < M + \varepsilon$ which leads to
 $L - \varepsilon < M + \varepsilon$ or $L - M < 2\varepsilon$.
 However,
 $L - M = \alpha > 2\varepsilon$
 which is a contradiction.
 Thus $L \leq M$.

27. (b) and (c) are equivalent to the definition of limit.

28. For every $\varepsilon > 0$ and $\delta > 0$ there is some x with
 $0 < |x - c| < \delta$ such that $|f(x) - L| > \varepsilon$.

29. a. $g(x) = \frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}$

b. No, because $\frac{x+6}{x^4 - 4x^3 + x^2 + x + 6} + 1$ has
 an asymptote at $x \approx 3.49$.

c. If $\delta \leq \frac{1}{4}$, then $2.75 < x < 3$
 or $3 < x < 3.25$ and by graphing
 $y = |g(x)| = \left| \frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6} \right|$
 on the interval $[2.75, 3.25]$, we see that
 $0 < \left| \frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6} \right| < 3$
 so m must be at least three.

2.6 Concepts Review

1. 48
2. 4
3. $-8; -4 + 5c$
4. $0; L; 2L$

Problem Set 2.6

- $$\begin{aligned} 1. \quad & \lim_{x \rightarrow 1} (2x + 1) && 4 \\ &= \lim_{x \rightarrow 1} 2x + \lim_{x \rightarrow 1} 1 && 3 \\ &= 2 \lim_{x \rightarrow 1} x + \lim_{x \rightarrow 1} 1 && 2, 1 \\ &= 2(1) + 1 = 3 \end{aligned}$$
- $$\begin{aligned} 2. \quad & \lim_{x \rightarrow -1} (3x^2 - 1) && 5 \\ &= \lim_{x \rightarrow -1} 3x^2 - \lim_{x \rightarrow -1} 1 && 3 \\ &= 3 \lim_{x \rightarrow -1} x^2 - \lim_{x \rightarrow -1} 1 && 8 \\ &= 3 \left(\lim_{x \rightarrow -1} x \right)^2 - \lim_{x \rightarrow -1} 1 && 2, 1 \\ &= 3(-1)^2 - 1 = 2 \end{aligned}$$
- $$\begin{aligned} 3. \quad & \lim_{x \rightarrow 0} [(2x + 1)(x - 3)] && 6 \\ &= \lim_{x \rightarrow 0} (2x + 1) \cdot \lim_{x \rightarrow 0} (x - 3) && 4, 5 \\ &= \left(\lim_{x \rightarrow 0} 2x + \lim_{x \rightarrow 0} 1 \right) \cdot \left(\lim_{x \rightarrow 0} x - \lim_{x \rightarrow 0} 3 \right) && 3 \\ &= \left(2 \lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} 1 \right) \cdot \left(\lim_{x \rightarrow 0} x - \lim_{x \rightarrow 0} 3 \right) && 2, 1 \\ &= [2(0) + 1](0 - 3) = -3 \end{aligned}$$
- $$\begin{aligned} 4. \quad & \lim_{x \rightarrow \sqrt{2}} [(2x^2 + 1)(7x^2 + 13)] && 6 \\ &= \lim_{x \rightarrow \sqrt{2}} (2x^2 + 1) \cdot \lim_{x \rightarrow \sqrt{2}} (7x^2 + 13) && 4, 3 \\ &= \left(2 \lim_{x \rightarrow \sqrt{2}} x^2 + \lim_{x \rightarrow \sqrt{2}} 1 \right) \cdot \left(7 \lim_{x \rightarrow \sqrt{2}} x^2 + \lim_{x \rightarrow \sqrt{2}} 13 \right) && 8, 1 \\ &= \left[2 \left(\lim_{x \rightarrow \sqrt{2}} x \right)^2 + 1 \right] \left[7 \left(\lim_{x \rightarrow \sqrt{2}} x \right)^2 + 13 \right] && 2 \\ &= [2(\sqrt{2})^2 + 1][7(\sqrt{2})^2 + 13] = 135 \end{aligned}$$

- $$\begin{aligned} 5. \quad & \lim_{x \rightarrow 2} \frac{2x + 1}{5 - 3x} && 7 \\ &= \frac{\lim_{x \rightarrow 2} (2x + 1)}{\lim_{x \rightarrow 2} (5 - 3x)} && 4, 5 \\ &= \frac{\lim_{x \rightarrow 2} 2x + \lim_{x \rightarrow 2} 1}{\lim_{x \rightarrow 2} 5 - \lim_{x \rightarrow 2} 3x} && 3, 1 \\ &= \frac{2 \lim_{x \rightarrow 2} x + 1}{5 - 3 \lim_{x \rightarrow 2} x} && 2 \\ &= \frac{2(2) + 1}{5 - 3(2)} = -5 \end{aligned}$$
- $$\begin{aligned} 6. \quad & \lim_{x \rightarrow -3} \frac{4x^3 + 1}{7 - 2x^2} && 7 \\ &= \frac{\lim_{x \rightarrow -3} (4x^3 + 1)}{\lim_{x \rightarrow -3} (7 - 2x^2)} && 4, 5 \\ &= \frac{\lim_{x \rightarrow -3} 4x^3 + \lim_{x \rightarrow -3} 1}{\lim_{x \rightarrow -3} 7 - \lim_{x \rightarrow -3} 2x^2} && 3, 1 \\ &= \frac{4 \lim_{x \rightarrow -3} x^3 + 1}{7 - 2 \lim_{x \rightarrow -3} x^2} && 8 \\ &= \frac{4 \left(\lim_{x \rightarrow -3} x \right)^3 + 1}{7 - 2 \left(\lim_{x \rightarrow -3} x \right)^2} && 2 \\ &= \frac{4(-3)^3 + 1}{7 - 2(-3)^2} = \frac{107}{11} \end{aligned}$$
- $$\begin{aligned} 7. \quad & \lim_{x \rightarrow 3} \sqrt{3x - 5} && 9 \\ &= \sqrt{\lim_{x \rightarrow 3} (3x - 5)} && 5, 3 \\ &= \sqrt{3 \lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 5} && 2, 1 \\ &= \sqrt{3(3) - 5} = 2 \end{aligned}$$
- $$\begin{aligned} 8. \quad & \lim_{x \rightarrow -3} \sqrt{5x^2 + 2x} && 9 \\ &= \sqrt{\lim_{x \rightarrow -3} (5x^2 + 2x)} && 4, 3 \\ &= \sqrt{5 \lim_{x \rightarrow -3} x^2 + 2 \lim_{x \rightarrow -3} x} && 8 \\ &= \sqrt{5 \left(\lim_{x \rightarrow -3} x \right)^2 + 2 \lim_{x \rightarrow -3} x} && 2 \\ &= \sqrt{5(-3)^2 + 2(-3)} = \sqrt{39} \end{aligned}$$

$$\begin{aligned}
 9. \quad & \lim_{t \rightarrow -2} (2t^3 + 15)^{13} && 8 \\
 & = \left[\lim_{t \rightarrow -2} (2t^3 + 15) \right]^{13} && 4, 3 \\
 & = \left[2 \lim_{t \rightarrow -2} t^3 + \lim_{t \rightarrow -2} 15 \right]^{13} && 8 \\
 & = \left[2 \left(\lim_{t \rightarrow -2} t \right)^3 + \lim_{t \rightarrow -2} 15 \right]^{13} && 2, 1 \\
 & = [2(-2)^3 + 15]^{13} = -1
 \end{aligned}$$

$$\begin{aligned}
 10. \quad & \lim_{w \rightarrow -2} \sqrt{-3w^3 + 7w^2} && 9 \\
 & = \sqrt{\lim_{w \rightarrow -2} (-3w^3 + 7w^2)} && 4, 3 \\
 & = \sqrt{-3 \lim_{w \rightarrow -2} w^3 + 7 \lim_{w \rightarrow -2} w^2} && 8 \\
 & = \sqrt{-3 \left(\lim_{w \rightarrow -2} w \right)^3 + 7 \left(\lim_{w \rightarrow -2} w \right)^2} && 2 \\
 & = \sqrt{-3(-2)^3 + 7(-2)^2} = 2\sqrt{13}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad & \lim_{y \rightarrow 2} \left(\frac{4y^3 + 8y}{y + 4} \right)^{1/3} && 9 \\
 & = \left(\lim_{y \rightarrow 2} \frac{4y^3 + 8y}{y + 4} \right)^{1/3} && 7 \\
 & = \left[\frac{\lim_{y \rightarrow 2} (4y^3 + 8y)}{\lim_{y \rightarrow 2} (y + 4)} \right]^{1/3} && 4, 3 \\
 & = \left(\frac{4 \lim_{y \rightarrow 2} y^3 + 8 \lim_{y \rightarrow 2} y}{\lim_{y \rightarrow 2} y + \lim_{y \rightarrow 2} 4} \right)^{1/3} && 8, 1 \\
 & = \left[\frac{4 \left(\lim_{y \rightarrow 2} y \right)^3 + 8 \lim_{y \rightarrow 2} y}{\lim_{y \rightarrow 2} y + 4} \right]^{1/3} && 2 \\
 & = \left[\frac{4(2)^3 + 8(2)}{2 + 4} \right]^{1/3} = 2
 \end{aligned}$$

$$\begin{aligned}
 12. \quad & \lim_{w \rightarrow 5} (2w^4 - 9w^3 + 19)^{-1/2} \\
 & = \lim_{w \rightarrow 5} \frac{1}{\sqrt{2w^4 - 9w^3 + 19}} && 7 \\
 & = \frac{\lim_{w \rightarrow 5} 1}{\lim_{w \rightarrow 5} \sqrt{2w^4 - 9w^3 + 19}} && 1 \\
 & = \frac{1}{\sqrt{\lim_{w \rightarrow 5} (2w^4 - 9w^3 + 19)}} && 4.5 \\
 & = \frac{1}{\sqrt{\lim_{w \rightarrow 5} 2w^4 - \lim_{w \rightarrow 5} 9w^3 + \lim_{w \rightarrow 5} 19}} && 1, 3 \\
 & = \frac{1}{\sqrt{2 \lim_{w \rightarrow 5} w^4 - 9 \lim_{w \rightarrow 5} w^3 + 19}} && 8 \\
 & = \frac{1}{\sqrt{2 \left(\lim_{w \rightarrow 5} w \right)^4 - 9 \left(\lim_{w \rightarrow 5} w \right)^3 + 19}} && 2 \\
 & = \frac{1}{\sqrt{2(5)^4 - 9(5)^3 + 19}} \\
 & = \frac{1}{\sqrt{144}} = \frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 13. \quad & \lim_{x \rightarrow -1} \frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x+7)} = \lim_{x \rightarrow -1} \frac{x-3}{x+7} \\
 & = \frac{-1-3}{-1+7} = -\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & \lim_{x \rightarrow 2} \frac{x^2 + 7x + 10}{x + 2} = \lim_{x \rightarrow 2} \frac{(x+2)(x+5)}{x+2} \\
 & = \lim_{x \rightarrow 2} (x+5) = 7
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x+2)(x-1)}{(x+1)(x-1)} \\
 & = \lim_{x \rightarrow 1} \frac{x+2}{x+1} = \frac{1+2}{1+1} = \frac{3}{2}
 \end{aligned}$$

$$16. \lim_{x \rightarrow -3} \frac{x^2 - 14x - 51}{x^2 - 4x - 21} = \lim_{x \rightarrow -3} \frac{(x+3)(x-17)}{(x+3)(x-7)} \\ = \lim_{x \rightarrow -3} \frac{x-17}{x-7} = \frac{-3-17}{-3-7} = 2$$

$$17. \lim_{u \rightarrow -2} \frac{u^2 - ux + 2u - 2x}{u^2 - u - 6} = \lim_{u \rightarrow -2} \frac{(u+2)(u-x)}{(u+2)(u-3)} \\ = \lim_{u \rightarrow -2} \frac{u-x}{u-3} = \frac{-2+2}{-2-3} = \frac{0}{-5} = 0$$

$$18. \lim_{x \rightarrow 1} \frac{x^2 + ux - x - u}{x^2 + 2x - 3} = \lim_{x \rightarrow 1} \frac{(x-1)(x+u)}{(x-1)(x+3)} \\ = \lim_{x \rightarrow 1} \frac{x+u}{x+3} = \frac{1+u}{1+3} = \frac{u+1}{4}$$

$$19. \lim_{x \rightarrow \pi} \frac{2x^2 - 6x\pi + 4\pi^2}{x^2 - \pi^2} = \lim_{x \rightarrow \pi} \frac{2(x-\pi)(x-2\pi)}{(x-\pi)(x+\pi)} \\ = \lim_{x \rightarrow \pi} \frac{2(x-2\pi)}{x+\pi} = \frac{2(\pi-2\pi)}{\pi+\pi} = -1$$

$$20. \lim_{w \rightarrow -2} \frac{(w+2)(w^2 - w - 6)}{w^2 + 4w + 4} \\ = \lim_{w \rightarrow -2} \frac{(w+2)^2(w-3)}{(w+2)^2} = \lim_{w \rightarrow -2} (w-3) \\ = -2-3 = -5$$

$$21. \lim_{x \rightarrow a} \sqrt{f^2(x) + g^2(x)} \\ = \sqrt{\lim_{x \rightarrow a} f^2(x) + \lim_{x \rightarrow a} g^2(x)} \\ = \sqrt{\left(\lim_{x \rightarrow a} f(x)\right)^2 + \left(\lim_{x \rightarrow a} g(x)\right)^2} \\ = \sqrt{(3)^2 + (-1)^2} = \sqrt{10}$$

$$22. \lim_{x \rightarrow a} \frac{2f(x) - 3g(x)}{f(x) + g(x)} = \frac{\lim_{x \rightarrow a} [2f(x) - 3g(x)]}{\lim_{x \rightarrow a} [f(x) + g(x)]} \\ = \frac{2 \lim_{x \rightarrow a} f(x) - 3 \lim_{x \rightarrow a} g(x)}{\lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)} = \frac{2(3) - 3(-1)}{3 + (-1)} = \frac{9}{2}$$

$$23. \lim_{x \rightarrow a} \sqrt[3]{g(x)[f(x) + 3]} = \lim_{x \rightarrow a} \sqrt[3]{g(x)} \cdot \lim_{x \rightarrow a} [f(x) + 3] \\ = \sqrt[3]{\lim_{x \rightarrow a} g(x)} \cdot \left[\lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} 3 \right] = \sqrt[3]{-1} \cdot (3+3) \\ = -6$$

$$24. \lim_{x \rightarrow a} [f(x) - 3]^4 = \left[\lim_{x \rightarrow a} (f(x) - 3) \right]^4 \\ = \left[\lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} 3 \right]^4 = (3-3)^4 = 0$$

$$25. \lim_{t \rightarrow a} [|f(t)| + |3g(t)|] = \lim_{t \rightarrow a} |f(t)| + 3 \lim_{t \rightarrow a} |g(t)| \\ = \left| \lim_{t \rightarrow a} f(t) \right| + 3 \left| \lim_{t \rightarrow a} g(t) \right| \\ = |3| + 3|-1| = 6$$

$$26. \lim_{u \rightarrow a} [f(u) + 3g(u)]^3 = \left(\lim_{u \rightarrow a} [f(u) + 3g(u)] \right)^3 \\ = \left[\lim_{u \rightarrow a} f(u) + 3 \lim_{u \rightarrow a} g(u) \right]^3 = [3 + 3(-1)]^3 = 0$$

$$27. \lim_{x \rightarrow 2} \frac{3x^2 - 12}{x - 2} = \lim_{x \rightarrow 2} \frac{3(x-2)(x+2)}{x-2} \\ = 3 \lim_{x \rightarrow 2} (x+2) = 3(2+2) = 12$$

$$28. \lim_{x \rightarrow 2} \frac{(3x^2 + 2x + 1) - 17}{x - 2} = \lim_{x \rightarrow 2} \frac{3x^2 + 2x - 16}{x - 2} \\ = \lim_{x \rightarrow 2} \frac{(3x+8)(x-2)}{x-2} = \lim_{x \rightarrow 2} (3x+8) \\ = 3 \lim_{x \rightarrow 2} x + 8 = 3(2) + 8 = 14$$

$$29. \lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = \lim_{x \rightarrow 2} \frac{\frac{2-x}{2x}}{x-2} = \lim_{x \rightarrow 2} \frac{-\frac{x-2}{2x}}{x-2} \\ = \lim_{x \rightarrow 2} -\frac{1}{2x} = \frac{-1}{2 \lim_{x \rightarrow 2} x} = \frac{-1}{2(2)} = -\frac{1}{4}$$

$$\begin{aligned}
 30. \quad \lim_{x \rightarrow 2} \frac{\frac{3}{x^2} - \frac{3}{4}}{x-2} &= \lim_{x \rightarrow 2} \frac{\frac{3(4-x^2)}{4x^2}}{x-2} = \lim_{x \rightarrow 2} \frac{-3(x+2)(x-2)}{4x^2(x-2)} \\
 &= \lim_{x \rightarrow 2} \frac{-3(x+2)}{4x^2} = \frac{-3\left(\lim_{x \rightarrow 2} x + 2\right)}{4\left(\lim_{x \rightarrow 2} x\right)^2} = \frac{-3(2+2)}{4(2)^2} \\
 &= -\frac{3}{4}
 \end{aligned}$$

31. Suppose $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$.
 $|f(x)g(x) - LM| \leq |g(x)||f(x) - L| + |L||g(x) - M|$
 as shown in the text. Choose $\varepsilon_1 = 1$. Since
 $\lim_{x \rightarrow c} g(x) = M$, there is some $\delta_1 > 0$ such that if
 $0 < |x - c| < \delta_1$, $|g(x) - M| < \varepsilon_1 = 1$ or
 $M - 1 < g(x) < M + 1$, $|M - 1| \leq |M| + 1$ and
 $|M + 1| \leq |M| + 1$ so for
 $0 < |x - c| < \delta_1$, $|g(x)| < |M| + 1$. Choose $\varepsilon > 0$.
 Since $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$, there
 exist δ_2 and δ_3 such that $0 < |x - c| < \delta_2 \Rightarrow$

$$|f(x) - L| < \frac{\varepsilon}{|L| + |M| + 1} \text{ and } 0 < |x - c| < \delta_3 \Rightarrow$$

$$|g(x) - M| < \frac{\varepsilon}{|L| + |M| + 1}. \text{ Let}$$

$$\delta = \min\{\delta_1, \delta_2, \delta_3\}, \text{ then } 0 < |x - c| < \delta \Rightarrow$$

$$\begin{aligned}
 |f(x)g(x) - LM| &\leq |g(x)||f(x) - L| + |L||g(x) - M| \\
 &< (|M| + 1) \frac{\varepsilon}{|L| + |M| + 1} + |L| \frac{\varepsilon}{|L| + |M| + 1} = \varepsilon
 \end{aligned}$$

Hence,

$$\lim_{x \rightarrow c} f(x)g(x) = LM = \left(\lim_{x \rightarrow c} f(x)\right)\left(\lim_{x \rightarrow c} g(x)\right)$$

32. Say $\lim_{x \rightarrow c} g(x) = M$ and choose $\varepsilon_1 = 1$. There is

some $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow |g(x) - M| < \varepsilon_1 = 1 \text{ or}$$

$$M - 1 < g(x) < M + 1$$

$$|M - 1| \geq |M| - 1 \text{ and } |M + 1| \geq |M| + 1$$

$$\text{so } |g(x)| > |M| - 1 \text{ and } \frac{1}{|g(x)|} < \frac{1}{|M| - 1}$$

Choose $\varepsilon > 0$.

Since $\lim_{x \rightarrow c} g(x) = M$ there is $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon |M| (|M| - 1)$$

Let $\delta = \min\{\delta_1, \delta_2\}$, then

$$\begin{aligned}
 0 < |x - c| < \delta &\Rightarrow \left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{M - g(x)}{g(x)M} \right| \\
 &= \frac{1}{|M||g(x)|} |g(x) - M| < \frac{1}{|M|(|M| - 1)} \varepsilon |M| (|M| - 1) \\
 &= \varepsilon
 \end{aligned}$$

$$\text{Thus, } \lim_{x \rightarrow c} \frac{1}{g(x)} = \frac{1}{M} = \frac{1}{\lim_{x \rightarrow c} g(x)}.$$

Using statement 6 and the above result,

$$\begin{aligned}
 \lim_{x \rightarrow c} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} \frac{1}{g(x)} \\
 &= \lim_{x \rightarrow c} f(x) \cdot \frac{1}{\lim_{x \rightarrow c} g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}.
 \end{aligned}$$

33. $\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} L$
 $\Leftrightarrow \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} L = 0$
 $\Leftrightarrow \lim_{x \rightarrow c} [f(x) - L] = 0$

34. $\lim_{x \rightarrow c} f(x) = 0 \Leftrightarrow \left[\lim_{x \rightarrow c} f(x) \right]^2 = 0$
 $\Leftrightarrow \lim_{x \rightarrow c} f^2(x) = 0$
 $\Leftrightarrow \sqrt{\lim_{x \rightarrow c} f^2(x)} = 0$
 $\Leftrightarrow \lim_{x \rightarrow c} \sqrt{f^2(x)} = 0$
 $\Leftrightarrow \lim_{x \rightarrow c} |f(x)| = 0$

35. $\lim_{x \rightarrow c} |x| = \sqrt{\left(\lim_{x \rightarrow c} |x|\right)^2} = \sqrt{\lim_{x \rightarrow c} |x|^2} = \sqrt{\lim_{x \rightarrow c} x^2}$
 $= \sqrt{\left(\lim_{x \rightarrow c} x\right)^2} = \sqrt{c^2} = |c|$

36. a. If $f(x) = \frac{x+1}{x-2}$, $g(x) = \frac{x-5}{x-2}$ and $c = 2$, then
 $\lim_{x \rightarrow c} [f(x) + g(x)]$ exists, but neither
 $\lim_{x \rightarrow c} f(x)$ nor $\lim_{x \rightarrow c} g(x)$ exists.

- b. If $f(x) = \frac{2}{x}$, $g(x) = x$, and $c = 0$, then
 $\lim_{x \rightarrow c} [f(x) \cdot g(x)]$ exists, but $\lim_{x \rightarrow c} f(x)$ does
 not exist.

$$37. \lim_{x \rightarrow -3^+} \frac{\sqrt{3+x}}{x} = \frac{\sqrt{3-3}}{-3} = 0$$

$$38. \lim_{x \rightarrow -\pi^+} \frac{\sqrt{\pi^3 + x^3}}{x} = \frac{\sqrt{\pi^3 + (-\pi)^3}}{-\pi} = 0$$

$$\begin{aligned} 39. \lim_{x \rightarrow 3^+} \frac{x-3}{\sqrt{x^2-9}} &= \lim_{x \rightarrow 3^+} \frac{(x-3)\sqrt{x^2-9}}{x^2-9} \\ &= \lim_{x \rightarrow 3^+} \frac{(x-3)\sqrt{x^2-9}}{(x-3)(x+3)} = \lim_{x \rightarrow 3^+} \frac{\sqrt{x^2-9}}{x+3} \\ &= \frac{\sqrt{3^2-9}}{3+3} = 0 \end{aligned}$$

$$40. \lim_{x \rightarrow 1^-} \frac{\sqrt{1+x}}{4+4x} = \frac{\sqrt{1+1}}{4+4(1)} = \frac{\sqrt{2}}{8}$$

$$41. \lim_{x \rightarrow 2^+} \frac{(x^2+1)\llbracket x \rrbracket}{(3x-1)^2} = \frac{(2^2+1)\llbracket 2 \rrbracket}{(3 \cdot 2-1)^2} = \frac{5 \cdot 2}{5^2} = \frac{2}{5}$$

$$42. \lim_{x \rightarrow 3^-} (x - \llbracket x \rrbracket) = \lim_{x \rightarrow 3^-} x - \lim_{x \rightarrow 3^-} \llbracket x \rrbracket = 3 - 2 = 1$$

$$43. \lim_{x \rightarrow 0^-} \frac{x}{|x|} = -1$$

$$44. \lim_{x \rightarrow 3^+} \llbracket x^2 + 2x \rrbracket = \llbracket 3^2 + 2 \cdot 3 \rrbracket = 15$$

$$45. f(x)g(x) = 1; g(x) = \frac{1}{f(x)}$$

$$\lim_{x \rightarrow a} g(x) = 0 \Leftrightarrow \lim_{x \rightarrow a} \frac{1}{f(x)} = 0$$

$$\Leftrightarrow \frac{1}{\lim_{x \rightarrow a} f(x)} = 0$$

No value satisfies this equation, so $\lim_{x \rightarrow a} f(x)$ must not exist.

$$46. R \text{ has the vertices } \left(\pm \frac{x}{2}, \pm \frac{1}{2} \right)$$

Each side of Q has length $\sqrt{x^2+1}$ so the perimeter of Q is $4\sqrt{x^2+1}$. R has two sides of length 1 and two sides of length $\sqrt{x^2}$ so the perimeter of R is $2+2\sqrt{x^2}$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\text{perimeter of } R}{\text{perimeter of } Q} &= \lim_{x \rightarrow 0^+} \frac{2\sqrt{x^2}+2}{4\sqrt{x^2+1}} \\ &= \frac{2\sqrt{0^2}+2}{4\sqrt{0^2+1}} = \frac{2}{4} = \frac{1}{2} \end{aligned}$$

$$47. \text{ a. } NO = \sqrt{(0-0)^2 + (1-0)^2} = 1$$

$$\begin{aligned} OP &= \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2} \\ &= \sqrt{x^2+x} \end{aligned}$$

$$\begin{aligned} NP &= \sqrt{(x-0)^2 + (y-1)^2} = \sqrt{x^2+y^2-2y+1} \\ &= \sqrt{x^2+x-2\sqrt{x}+1} \end{aligned}$$

$$MO = \sqrt{(1-0)^2 + (0-0)^2} = 1$$

$$\begin{aligned} MP &= \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{y^2+x^2-2x+1} \\ &= \sqrt{x^2-x+1} \end{aligned}$$

$$\lim_{x \rightarrow 0^+} \frac{\text{perimeter of } \triangle NOP}{\text{perimeter of } \triangle MOP}$$

$$\begin{aligned} &= \lim_{x \rightarrow 0^+} \frac{1 + \sqrt{x^2+x} + \sqrt{x^2+x-2\sqrt{x}+1}}{1 + \sqrt{x^2+x} + \sqrt{x^2-x+1}} \\ &= \frac{1 + \sqrt{1}}{1 + \sqrt{1}} = 1 \end{aligned}$$

$$\text{ b. Area of } \triangle NOP = \frac{1}{2}(1)(x) = \frac{x}{2}$$

$$\text{Area of } \triangle MOP = \frac{1}{2}(1)(y) = \frac{\sqrt{x}}{2}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\text{area of } \triangle NOP}{\text{area of } \triangle MOP} &= \lim_{x \rightarrow 0^+} \frac{\frac{x}{2}}{\frac{\sqrt{x}}{2}} = \lim_{x \rightarrow 0^+} \frac{x}{\sqrt{x}} \\ &= \lim_{x \rightarrow 0^+} \sqrt{x} = 0 \end{aligned}$$

2.7 Concepts Review

- 0
- 1
- the denominator is 0 when $t = 0$.
- 1

Problem Set 2.7

$$1. \lim_{x \rightarrow 0} \frac{\cos x}{x+1} = \frac{1}{1} = 1$$

$$2. \lim_{\theta \rightarrow \pi/2} \theta \cos \theta = \frac{\pi}{2} \cdot 0 = 0$$

$$\begin{aligned} 3. \lim_{t \rightarrow 0} \frac{\cos^2 t}{1 + \sin t} &= \lim_{t \rightarrow 0} \frac{1 - \sin^2 t}{1 + \sin t} \\ &= \lim_{t \rightarrow 0} \frac{(1 - \sin t)(1 + \sin t)}{1 + \sin t} \\ &= \lim_{t \rightarrow 0} (1 - \sin t) = 1 \end{aligned}$$

$$\begin{aligned} 4. \lim_{x \rightarrow 0} \frac{3x \tan x}{\sin x} &= \lim_{x \rightarrow 0} \frac{3x(\sin x / \cos x)}{\sin x} = \lim_{x \rightarrow 0} \frac{3x}{\cos x} \\ &= \frac{0}{1} = 0 \end{aligned}$$

$$5. \lim_{x \rightarrow 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$\begin{aligned} 6. \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{2\theta} &= \lim_{\theta \rightarrow 0} \frac{3}{2} \cdot \frac{\sin 3\theta}{3\theta} = \frac{3}{2} \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta} \\ &= \frac{3}{2} \cdot 1 = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} 7. \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\tan \theta} &= \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\frac{\sin \theta}{\cos \theta}} = \lim_{\theta \rightarrow 0} \frac{\cos \theta \sin 3\theta}{\sin \theta} \\ &= \lim_{\theta \rightarrow 0} \left[\cos \theta \cdot 3 \cdot \frac{\sin 3\theta}{3\theta} \cdot \frac{1}{\frac{\sin \theta}{\theta}} \right] \\ &= 3 \lim_{\theta \rightarrow 0} \left[\cos \theta \cdot \frac{\sin 3\theta}{3\theta} \cdot \frac{1}{\frac{\sin \theta}{\theta}} \right] \\ &= 3 \cdot 1 \cdot 1 \cdot 1 = 3 \end{aligned}$$

$$\begin{aligned} 8. \lim_{\theta \rightarrow 0} \frac{\tan 5\theta}{\sin 2\theta} &= \lim_{\theta \rightarrow 0} \frac{\frac{\sin 5\theta}{\cos 5\theta}}{\sin 2\theta} = \lim_{\theta \rightarrow 0} \frac{\sin 5\theta}{\cos 5\theta \sin 2\theta} \\ &= \lim_{\theta \rightarrow 0} \left[\frac{1}{\cos 5\theta} \cdot 5 \cdot \frac{\sin 5\theta}{5\theta} \cdot \frac{1}{2} \cdot \frac{2\theta}{\sin 2\theta} \right] \\ &= \frac{5}{2} \lim_{\theta \rightarrow 0} \left[\frac{1}{\cos 5\theta} \cdot \frac{\sin 5\theta}{5\theta} \cdot \frac{2\theta}{\sin 2\theta} \right] \\ &= \frac{5}{2} \cdot 1 \cdot 1 \cdot 1 = \frac{5}{2} \end{aligned}$$

$$\begin{aligned} 9. \lim_{\theta \rightarrow 0} \frac{\cot \pi \theta \sin \theta}{2 \sec \theta} &= \lim_{\theta \rightarrow 0} \frac{\frac{\cos \pi \theta}{\sin \pi \theta} \sin \theta}{\frac{2}{\cos \theta}} \\ &= \lim_{\theta \rightarrow 0} \frac{\cos \pi \theta \sin \theta \cos \theta}{2 \sin \pi \theta} \\ &= \lim_{\theta \rightarrow 0} \left[\frac{\cos \pi \theta \cos \theta}{2} \cdot \frac{\sin \theta}{\theta} \cdot \frac{1}{\pi} \cdot \frac{\pi \theta}{\sin \pi \theta} \right] \\ &= \frac{1}{2\pi} \lim_{\theta \rightarrow 0} \left[\cos \pi \theta \cos \theta \cdot \frac{\sin \theta}{\theta} \cdot \frac{\pi \theta}{\sin \pi \theta} \right] \\ &= \frac{1}{2\pi} \cdot 1 \cdot 1 \cdot 1 \cdot 1 = \frac{1}{2\pi} \end{aligned}$$

$$\begin{aligned} 10. \lim_{t \rightarrow 0} \frac{\sin^2 3t}{2t} &= \lim_{t \rightarrow 0} \frac{9t}{2} \cdot \frac{\sin 3t}{3t} \cdot \frac{\sin 3t}{3t} \\ &= 0 \cdot 1 \cdot 1 = 0 \end{aligned}$$

$$\begin{aligned} 11. \lim_{t \rightarrow 0} \frac{\tan^2 3t}{2t} &= \lim_{t \rightarrow 0} \frac{\sin^2 3t}{(2t)(\cos^2 3t)} \\ &= \lim_{t \rightarrow 0} \frac{3(\sin 3t)}{2 \cos^2 3t} \cdot \frac{\sin 3t}{3t} = 0 \cdot 1 = 0 \end{aligned}$$

$$12. \lim_{t \rightarrow 0} \frac{\tan 2t}{\sin 2t - 1} = \frac{0}{-1} = 0$$

$$\begin{aligned} 13. \lim_{t \rightarrow 0} \frac{\sin(3t) + 4t}{t \sec t} &= \lim_{t \rightarrow 0} \left(\frac{\sin 3t}{t \sec t} + \frac{4t}{t \sec t} \right) \\ &= \lim_{t \rightarrow 0} \frac{\sin 3t}{t \sec t} + \lim_{t \rightarrow 0} \frac{4t}{t \sec t} \\ &= \lim_{t \rightarrow 0} 3 \cos t \cdot \frac{\sin 3t}{3t} + \lim_{t \rightarrow 0} 4 \cos t \\ &= 3 \cdot 1 + 4 = 7 \end{aligned}$$

14. The result that $\lim_{t \rightarrow 0} \cos t = 1$ was established in the proof of the theorem. Then

$$\begin{aligned}
 \lim_{t \rightarrow c} \cos t &= \lim_{h \rightarrow 0} \cos(c+h) \\
 &= \lim_{h \rightarrow 0} (\cos c \cosh - \sin c \sinh) \\
 &= \lim_{h \rightarrow 0} \cos c \lim_{h \rightarrow 0} \cosh - \sin c \lim_{h \rightarrow 0} \sinh \\
 &= \cos c
 \end{aligned}$$

$$15. \lim_{t \rightarrow c} \tan t = \lim_{t \rightarrow c} \frac{\sin t}{\cos t} = \frac{\lim_{t \rightarrow c} \sin t}{\lim_{t \rightarrow c} \cos t} = \frac{\sin c}{\cos c} = \tan c$$

$$\lim_{t \rightarrow c} \cot t = \lim_{t \rightarrow c} \frac{\cos t}{\sin t} = \frac{\lim_{t \rightarrow c} \cos t}{\lim_{t \rightarrow c} \sin t} = \frac{\cos c}{\sin c} = \cot c$$

$$16. \lim_{t \rightarrow c} \sec t = \lim_{t \rightarrow c} \frac{1}{\cos t} = \frac{1}{\cos c} = \sec c$$

$$\lim_{t \rightarrow c} \csc t = \lim_{t \rightarrow c} \frac{1}{\sin t} = \frac{1}{\sin c} = \csc c$$

$$\begin{aligned}
 17. \quad \overline{BP} &= \sin t, \overline{OB} = \cos t \\
 \text{area}(\triangle OBP) &\leq \text{area}(\text{sector } OAP) \\
 &\leq \text{area}(\triangle OBP) + \text{area}(\triangle BPQ) \\
 \frac{1}{2} \overline{OB} \cdot \overline{BP} &\leq \frac{1}{2} t(1)^2 \leq \frac{1}{2} \overline{OB} \cdot \overline{BP} + (1 - \overline{OB}) \overline{BP} \\
 \frac{1}{2} \sin t \cos t &\leq \frac{1}{2} t \leq \frac{1}{2} \sin t \cos t + (1 - \cos t) \sin t
 \end{aligned}$$

$$\cos t \leq \frac{t}{\sin t} \leq 2 - \cos t$$

$$\frac{1}{2 - \cos t} \leq \frac{\sin t}{t} \leq \frac{1}{\cos t}$$

$$\lim_{t \rightarrow 0} \frac{1}{2 - \cos t} \leq \lim_{t \rightarrow 0} \frac{\sin t}{t} \leq \lim_{t \rightarrow 0} \frac{1}{\cos t}$$

$$1 \leq \lim_{t \rightarrow 0} \frac{\sin t}{t} \leq 1$$

$$\text{Thus, } \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1.$$

18. a. Written response

$$\begin{aligned}
 b. \quad D &= \frac{1}{2} \overline{AB} \cdot \overline{BP} = \frac{1}{2} (1 - \cos t) \sin t \\
 &= \frac{\sin t (1 - \cos t)}{2}
 \end{aligned}$$

$$E = \frac{1}{2} t(1)^2 - \frac{1}{2} \overline{OB} \cdot \overline{BP} = \frac{t}{2} - \frac{\sin t \cos t}{2}$$

$$\frac{D}{E} = \frac{\sin t (1 - \cos t)}{t - \sin t \cos t}$$

$$c. \lim_{t \rightarrow 0^+} \left(\frac{D}{E} \right) \approx 0.75$$

2.8 Concepts Review

1. x increases without bound; $f(x)$ gets close to L as x increases without bound
2. $f(x)$ increases without bound as x approaches c from the right; $f(x)$ decreases without bound as x approaches c from the left
3. $y = 6$; horizontal
4. $x = 6$; vertical

Problem Set 2.8

$$1. \lim_{x \rightarrow \infty} \frac{x}{x-5} = \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{5}{x}} = 1$$

$$2. \lim_{x \rightarrow \infty} \frac{x^2}{5 - x^3} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{5}{x^3} - 1} = 0$$

$$3. \lim_{t \rightarrow -\infty} \frac{t^2}{7 - t^2} = \lim_{t \rightarrow -\infty} \frac{1}{\frac{7}{t^2} - 1} = -1$$

$$4. \lim_{t \rightarrow -\infty} \frac{t}{t-5} = \lim_{t \rightarrow -\infty} \frac{1}{1 - \frac{5}{t}} = 1$$

$$\begin{aligned}
 5. \quad \lim_{x \rightarrow \infty} \frac{x^2}{(x-5)(3-x)} &= \lim_{x \rightarrow \infty} \frac{x^2}{-x^2 + 8x - 15} \\
 &= \lim_{x \rightarrow \infty} \frac{1}{-1 + \frac{8}{x} - \frac{15}{x^2}} = -1
 \end{aligned}$$

$$6. \lim_{x \rightarrow \infty} \frac{x^2}{x^2 - 8x + 15} = \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{8}{x} + \frac{15}{x^2}} = 1$$

$$7. \lim_{x \rightarrow \infty} \frac{x^3}{2x^3 - 100x^2} = \lim_{x \rightarrow \infty} \frac{1}{2 - \frac{100}{x}} = \frac{1}{2}$$

$$8. \lim_{\theta \rightarrow -\infty} \frac{\pi \theta^5}{\theta^5 - 5\theta^4} = \lim_{\theta \rightarrow -\infty} \frac{\pi}{1 - \frac{5}{\theta}} = \pi$$

$$9. \lim_{x \rightarrow \infty} \frac{3x^3 - x^2}{\pi x^3 - 5x^2} = \lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x}}{\pi - \frac{5}{x}} = \frac{3}{\pi}$$

$$10. \lim_{\theta \rightarrow \infty} \frac{\sin^2 \theta}{\theta^2 - 5}; 0 \leq \sin^2 \theta \leq 1 \text{ for all } \theta \text{ and}$$

$$\lim_{\theta \rightarrow \infty} \frac{1}{\theta^2 - 5} = \lim_{\theta \rightarrow \infty} \frac{\frac{1}{\theta^2}}{1 - \frac{5}{\theta^2}} = 0 \text{ so } \lim_{\theta \rightarrow \infty} \frac{\sin^2 \theta}{\theta^2 - 5} = 0$$

$$11. \lim_{x \rightarrow \infty} \frac{3\sqrt{x^3} + 3x}{\sqrt{2x^3}} = \lim_{x \rightarrow \infty} \frac{3x^{3/2} + 3x}{\sqrt{2}x^{3/2}}$$

$$= \lim_{x \rightarrow \infty} \frac{3 + \frac{3}{\sqrt{x}}}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$12. \lim_{x \rightarrow \infty} \sqrt[3]{\frac{\pi x^3 + 3x}{\sqrt{2}x^3 + 7x}} = \sqrt[3]{\lim_{x \rightarrow \infty} \frac{\pi x^3 + 3x}{\sqrt{2}x^3 + 7x}}$$

$$= \sqrt[3]{\lim_{x \rightarrow \infty} \frac{\pi + \frac{3}{x^2}}{\sqrt{2} + \frac{7}{x^2}}} = \sqrt[3]{\frac{\pi}{\sqrt{2}}}$$

$$13. \lim_{x \rightarrow \infty} \sqrt[3]{\frac{1 + 8x^2}{x^2 + 4}} = \sqrt[3]{\lim_{x \rightarrow \infty} \frac{1 + 8x^2}{x^2 + 4}}$$

$$= \sqrt[3]{\lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} + 8}{1 + \frac{4}{x^2}}} = \sqrt[3]{8} = 2$$

$$14. \lim_{x \rightarrow \infty} \sqrt{\frac{x^2 + x + 3}{(x-1)(x+1)}} = \sqrt{\lim_{x \rightarrow \infty} \frac{x^2 + x + 3}{x^2 - 1}}$$

$$= \sqrt{\lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x} + \frac{3}{x^2}}{1 - \frac{1}{x^2}}} = \sqrt{1} = 1$$

$$15. \text{ For } x > 0, x = \sqrt{x^2}.$$

$$\lim_{x \rightarrow \infty} \frac{2x+1}{\sqrt{x^2+3}} = \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x}}{\frac{\sqrt{x^2+3}}{\sqrt{x^2}}} = \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x}}{\sqrt{1 + \frac{3}{x^2}}}$$

$$= \frac{2}{\sqrt{1}} = 2$$

$$16. \lim_{x \rightarrow \infty} \frac{\sqrt{2x+1}}{x+4} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{2x+1}}{\sqrt{x^2}}}{1 + \frac{4}{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{\frac{2}{x} + \frac{1}{x^2}}}{1 + \frac{4}{x}} = 0$$

$$17. \lim_{x \rightarrow \infty} \left(\sqrt{2x^2+3} - \sqrt{2x^2-5} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{(\sqrt{2x^2+3} - \sqrt{2x^2-5})(\sqrt{2x^2+3} + \sqrt{2x^2-5})}{\sqrt{2x^2+3} + \sqrt{2x^2-5}}$$

$$= \lim_{x \rightarrow \infty} \frac{2x^2+3 - (2x^2-5)}{\sqrt{2x^2+3} + \sqrt{2x^2-5}}$$

$$= \lim_{x \rightarrow \infty} \frac{8}{\sqrt{2x^2+3} + \sqrt{2x^2-5}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{8}{x}}{\frac{\sqrt{2x^2+3} + \sqrt{2x^2-5}}{\sqrt{x^2}}} = \lim_{x \rightarrow \infty} \frac{\frac{8}{x}}{\sqrt{2 + \frac{3}{x^2}} + \sqrt{2 - \frac{5}{x^2}}} = 0$$

$$18. \lim_{x \rightarrow \infty} \left(\sqrt{x^2+2x} - x \right)$$

$$= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2+2x} - x)(\sqrt{x^2+2x} + x)}{\sqrt{x^2+2x} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{x^2+2x - x^2}{\sqrt{x^2+2x} + x} = \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2+2x} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1 + \frac{2}{x}} + 1} = \frac{2}{2} = 1$$

$$19. \lim_{y \rightarrow -\infty} \frac{9y^3+1}{y^2-2y+2} = \lim_{y \rightarrow -\infty} \frac{9y + \frac{1}{y^2}}{1 - \frac{2}{y} + \frac{2}{y^2}} = -\infty$$

$$20. \lim_{x \rightarrow \infty} \frac{a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n}{b_0x^n + b_1x^{n-1} + \dots + b_{n-1}x + b_n}$$

$$= \lim_{x \rightarrow \infty} \frac{a_0 + \frac{a_1}{x} + \dots + \frac{a_{n-1}}{x^{n-1}} + \frac{a_n}{x^n}}{b_0 + \frac{b_1}{x} + \dots + \frac{b_{n-1}}{x^{n-1}} + \frac{b_n}{x^n}} = \frac{a_0}{b_0}$$

21. As $x \rightarrow 4^+$, $x \rightarrow 4$ while $x-4 \rightarrow 0^+$.

$$\lim_{x \rightarrow 4^+} \frac{x}{x-4} = \infty$$

$$22. \lim_{t \rightarrow -3^+} \frac{t^2-9}{t+3} = \lim_{t \rightarrow -3^+} \frac{(t+3)(t-3)}{t+3}$$

$$= \lim_{t \rightarrow -3^+} (t-3) = -6$$

23. As $t \rightarrow 3^-$, $t^2 \rightarrow 9$ while $9 - t^2 \rightarrow 0^+$.

$$\lim_{t \rightarrow 3^-} \frac{t^2}{9 - t^2} = \infty$$

24. As $x \rightarrow \sqrt[3]{5}^+$, $x^2 \rightarrow 5^{2/3}$ while $5 - x^3 \rightarrow 0^-$.

$$\lim_{x \rightarrow \sqrt[3]{5}^+} \frac{x^2}{5 - x^3} = -\infty$$

25. As $x \rightarrow 5^-$, $x^2 \rightarrow 25$, $x - 5 \rightarrow 0^-$, and $3 - x \rightarrow -2$.

$$\lim_{x \rightarrow 5^-} \frac{x^2}{(x - 5)(3 - x)} = \infty$$

26. As $\theta \rightarrow \pi^+$, $\theta^2 \rightarrow \pi^2$ while $\sin \theta \rightarrow 0^-$.

$$\lim_{\theta \rightarrow \pi^+} \frac{\theta^2}{\sin \theta} = -\infty$$

27. As $x \rightarrow 3^-$, $x^3 \rightarrow 27$, while $x - 3 \rightarrow 0^-$.

$$\lim_{x \rightarrow 3^-} \frac{x^3}{x - 3} = -\infty$$

28. As $\theta \rightarrow \frac{\pi^+}{2}$, $\pi\theta \rightarrow \frac{\pi^2}{2}$ while $\cos \theta \rightarrow 0^-$.

$$\lim_{\theta \rightarrow \frac{\pi^+}{2}} \frac{\pi\theta}{\cos \theta} = -\infty$$

29. $\lim_{x \rightarrow 3^-} \frac{x^2 - x - 6}{x - 3} = \lim_{x \rightarrow 3^-} \frac{(x + 2)(x - 3)}{x - 3}$
 $= \lim_{x \rightarrow 3^-} (x + 2) = 5$

30. $\lim_{x \rightarrow 2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \lim_{x \rightarrow 2^+} \frac{(x + 4)(x - 2)}{(x + 2)(x - 2)}$
 $= \lim_{x \rightarrow 2^+} \frac{x + 4}{x + 2} = \frac{6}{4} = \frac{3}{2}$

31. For $0 \leq x < 1$, $\lfloor x \rfloor = 0$, so for $0 < x < 1$, $\frac{\lfloor x \rfloor}{x} = 0$

$$\text{thus } \lim_{x \rightarrow 0^+} \frac{\lfloor x \rfloor}{x} = 0$$

32. For $-1 \leq x < 0$, $\lfloor x \rfloor = -1$, so for $-1 < x < 0$,

$$\frac{\lfloor x \rfloor}{x} = -\frac{1}{x} \text{ thus } \lim_{x \rightarrow 0^-} \frac{\lfloor x \rfloor}{x} = \infty.$$

(Since $x < 0$, $-\frac{1}{x} > 0$.)

33. For $x < 0$, $|x| = -x$, thus

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$$

34. For $x > 0$, $|x| = x$, thus $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1$

35. As $x \rightarrow 0^-$, $1 + \cos x \rightarrow 2$ while $\sin x \rightarrow 0^-$.

$$\lim_{x \rightarrow 0^-} \frac{1 + \cos x}{\sin x} = -\infty$$

36. $-1 \leq \sin x \leq 1$ for all x , and

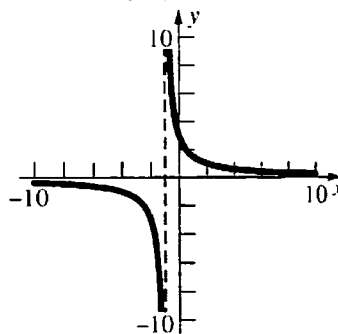
$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0, \text{ so } \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0.$$

37. $\lim_{x \rightarrow \infty} \frac{3}{x + 1} = 0$, $\lim_{x \rightarrow -\infty} \frac{3}{x + 1} = 0$;

Horizontal asymptote $y = 0$.

$$\lim_{x \rightarrow -1^+} \frac{3}{x + 1} = \infty, \lim_{x \rightarrow -1^-} \frac{3}{x + 1} = -\infty;$$

Vertical asymptote $x = -1$

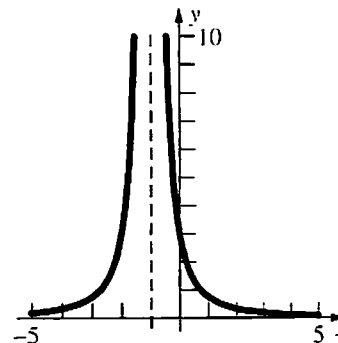


38. $\lim_{x \rightarrow \infty} \frac{3}{(x + 1)^2} = 0$, $\lim_{x \rightarrow -\infty} \frac{3}{(x + 1)^2} = 0$;

Horizontal asymptote $y = 0$.

$$\lim_{x \rightarrow -1^+} \frac{3}{(x + 1)^2} = \infty, \lim_{x \rightarrow -1^-} \frac{3}{(x + 1)^2} = \infty;$$

Vertical asymptote $x = -1$



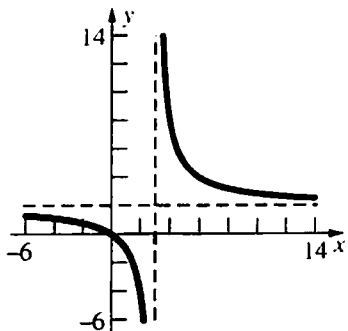
$$39. \lim_{x \rightarrow \infty} \frac{2x}{x-3} = \lim_{x \rightarrow \infty} \frac{2}{1-\frac{3}{x}} = 2,$$

$$\lim_{x \rightarrow -\infty} \frac{2x}{x-3} = \lim_{x \rightarrow -\infty} \frac{2}{1-\frac{3}{x}} = 2,$$

Horizontal asymptote $y = 2$

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3} = \infty, \lim_{x \rightarrow 3^-} \frac{2x}{x-3} = -\infty;$$

Vertical asymptote $x = 3$



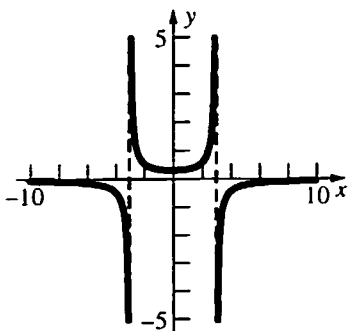
$$40. \lim_{x \rightarrow \infty} \frac{3}{9-x^2} = 0, \lim_{x \rightarrow -\infty} \frac{3}{9-x^2} = 0;$$

Horizontal asymptote $y = 0$

$$\lim_{x \rightarrow 3^+} \frac{3}{9-x^2} = -\infty, \lim_{x \rightarrow 3^-} \frac{3}{9-x^2} = \infty,$$

$$\lim_{x \rightarrow -3^+} \frac{3}{9-x^2} = \infty, \lim_{x \rightarrow -3^-} \frac{3}{9-x^2} = -\infty;$$

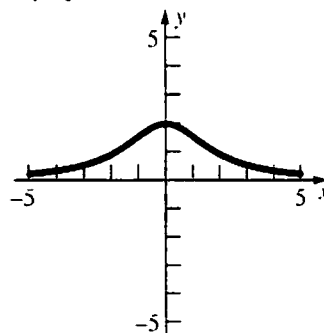
Vertical asymptotes $x = -3, x = 3$



$$41. \lim_{x \rightarrow \infty} \frac{14}{2x^2+7} = 0, \lim_{x \rightarrow -\infty} \frac{14}{2x^2+7} = 0;$$

Horizontal asymptote $y = 0$

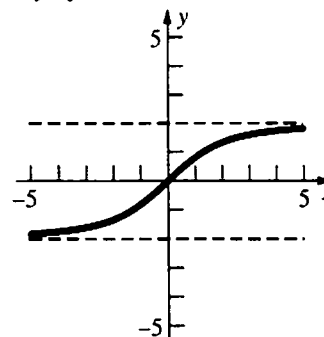
Since $2x^2 + 7 > 0$ for all x , $g(x)$ has no vertical asymptotes.



$$42. \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2+5}} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1+\frac{5}{x^2}}} = \frac{2}{\sqrt{1}} = 2,$$

$$\lim_{x \rightarrow -\infty} \frac{2x}{\sqrt{x^2+5}} = \lim_{x \rightarrow -\infty} \frac{2}{-\sqrt{1+\frac{5}{x^2}}} = \frac{2}{-\sqrt{1}} = -2$$

Since $\sqrt{x^2+5} > 0$ for all x , $g(x)$ has no vertical asymptotes.



$$43. f(x) = 2x + 3 - \frac{1}{x^3-1}, \text{ thus}$$

$$\lim_{x \rightarrow \infty} [f(x) - (2x+3)] = \lim_{x \rightarrow \infty} \left[-\frac{1}{x^3-1} \right] = 0$$

The oblique asymptote is $y = 2x + 3$.

$$44. f(x) = 3x + 4 - \frac{4x+3}{x^2+1}, \text{ thus}$$

$$\lim_{x \rightarrow \infty} [f(x) - (3x+4)] = \lim_{x \rightarrow \infty} \left[-\frac{4x+3}{x^2+1} \right]$$

$$= \lim_{x \rightarrow \infty} \left[-\frac{\frac{4}{x} + \frac{3}{x^2}}{1 + \frac{1}{x^2}} \right] = 0.$$

The oblique asymptote is $y = 3x + 4$.

45. a. We say that $\lim_{x \rightarrow c^+} f(x) = -\infty$ if for each negative number M there corresponds a $\delta > 0$ such that $0 < x - c < \delta \Rightarrow f(x) < M$.

b. We say that $\lim_{x \rightarrow c^-} f(x) = \infty$ if for each positive number M there corresponds a $\delta > 0$ such that $0 < c - x < \delta \Rightarrow f(x) > M$.

46. a. We say that $\lim_{x \rightarrow \infty} f(x) = \infty$ if for each positive number M there corresponds an $N > 0$ such that $N < x \Rightarrow f(x) > M$.

b. We say that $\lim_{x \rightarrow -\infty} f(x) = \infty$ if for each positive number M there corresponds an $N < 0$ such that $x < N \Rightarrow f(x) > M$.

47. Let $\varepsilon > 0$ be given. Since $\lim_{x \rightarrow \infty} f(x) = A$, there is a corresponding number M_1 such that

$x > M_1 \Rightarrow |f(x) - A| < \frac{\varepsilon}{2}$. Similarly, there is a number M_2 such that $x > M_2 \Rightarrow |g(x) - B| < \frac{\varepsilon}{2}$.

Let $M = \max\{M_1, M_2\}$, then

$$\begin{aligned} x > M &\Rightarrow |f(x) + g(x) - (A + B)| \\ &= |f(x) - A + g(x) - B| \leq |f(x) - A| + |g(x) - B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, $\lim_{x \rightarrow \infty} [f(x) + g(x)] = A + B$

48. Written response

49. a. $\lim_{x \rightarrow \infty} \sin x$ does not exist as $\sin x$ oscillates between -1 and 1 as x increases.

b. Let $u = \frac{1}{x}$, then as $x \rightarrow \infty$, $u \rightarrow 0^+$.

$$\lim_{x \rightarrow \infty} \sin \frac{1}{x} = \lim_{u \rightarrow 0^+} \sin u = 0$$

c. Let $u = \frac{1}{x}$, then as $x \rightarrow \infty$, $u \rightarrow 0^+$.

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{u \rightarrow 0^+} \frac{1}{u} \sin u = \lim_{u \rightarrow 0^+} \frac{\sin u}{u} = 1$$

d. Let $u = \frac{1}{x}$, then

$$\begin{aligned} \lim_{x \rightarrow \infty} x^{3/2} \sin \frac{1}{x} &= \lim_{u \rightarrow 0^+} \left(\frac{1}{u} \right)^{3/2} \sin u \\ &= \lim_{u \rightarrow 0^+} \left[\left(\frac{1}{\sqrt{u}} \right) \left(\frac{\sin u}{u} \right) \right] = \infty \end{aligned}$$

e. As $x \rightarrow \infty$, $\sin x$ oscillates between -1 and 1 , while $x^{-1/2} = \frac{1}{\sqrt{x}} \rightarrow 0$.

$$\lim_{x \rightarrow \infty} x^{-1/2} \sin x = 0$$

f. Let $u = \frac{1}{x}$, then

$$\begin{aligned} \lim_{x \rightarrow \infty} \sin \left(\frac{\pi}{6} + \frac{1}{x} \right) &= \lim_{u \rightarrow 0^+} \sin \left(\frac{\pi}{6} + u \right) \\ &= \sin \frac{\pi}{6} = \frac{1}{2} \end{aligned}$$

g. As $x \rightarrow \infty$, $x + \frac{1}{x} \rightarrow \infty$, so $\lim_{x \rightarrow \infty} \sin \left(x + \frac{1}{x} \right)$ does not exist. (See part a.)

$$\begin{aligned} \text{h. } \sin \left(x + \frac{1}{x} \right) &= \sin x \cos \frac{1}{x} + \cos x \sin \frac{1}{x} \\ \lim_{x \rightarrow \infty} \left[\sin \left(x + \frac{1}{x} \right) - \sin x \right] \\ &= \lim_{x \rightarrow \infty} \left[\sin x \left(\cos \frac{1}{x} - 1 \right) + \cos x \sin \frac{1}{x} \right] \end{aligned}$$

As $x \rightarrow \infty$, $\cos \frac{1}{x} \rightarrow 1$ so $\cos \frac{1}{x} - 1 \rightarrow 0$.

From part b., $\lim_{x \rightarrow \infty} \sin \frac{1}{x} = 0$.

As $x \rightarrow \infty$ both $\sin x$ and $\cos x$ oscillate between -1 and 1 .

$$\lim_{x \rightarrow \infty} \left[\sin \left(x + \frac{1}{x} \right) - \sin x \right] = 0.$$

$$50. \lim_{v \rightarrow c^-} m(v) = \lim_{v \rightarrow c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \infty$$

$$51. \lim_{x \rightarrow \infty} \frac{3x^2 + x + 1}{2x^2 - 1} = \frac{3}{2}$$

$$52. \lim_{x \rightarrow -\infty} \sqrt{\frac{2x^2 - 3x}{5x^2 + 1}} = \sqrt{\frac{2}{5}}$$

$$53. \lim_{x \rightarrow -\infty} \left(\sqrt{2x^2 + 3x} - \sqrt{2x^2 - 5} \right) = -\frac{3}{2\sqrt{2}}$$

$$54. \lim_{x \rightarrow \infty} \frac{2x+1}{\sqrt{3x^2+1}} = \frac{2}{\sqrt{3}}$$

$$55. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{10} = 1$$

$$56. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e \approx 2.718$$

$$57. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{x^2} = \infty$$

$$58. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{\sin x} = 1$$

$$59. \lim_{x \rightarrow 3^-} \frac{\sin|x-3|}{x-3} = -1$$

$$60. \lim_{x \rightarrow 3^-} \frac{\sin|x-3|}{\tan(x-3)} = -1$$

$$61. \lim_{x \rightarrow 3^-} \frac{\cos(x-3)}{x-3} = -\infty$$

$$62. \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\cos x}{x - \frac{\pi}{2}} = -1$$

$$63. \lim_{x \rightarrow 0^+} (1 + \sqrt{x})^{\frac{1}{\sqrt{x}}} = e \approx 2.718$$

$$64. \lim_{x \rightarrow 0^+} (1 + \sqrt{x})^{1/x} = \infty$$

$$65. \lim_{x \rightarrow 0^+} (1 + \sqrt{x})^x = 1$$

2.9 Concepts Review

$$1. \lim_{x \rightarrow c} f(x)$$

2. every integer

$$3. \lim_{x \rightarrow a^+} f(x) = f(a); \lim_{x \rightarrow b^-} f(x) = f(b)$$

$$4. a; b; f(c) = W$$

$$7. \lim_{t \rightarrow 3} |t| = 3 = f(3); \text{ continuous}$$

$$8. \lim_{t \rightarrow 3} |t - 2| = 1 = g(3); \text{ continuous}$$

9. $h(3)$ does not exist, so $h(t)$ is not continuous at 3.

10. $f(3)$ does not exist, so $f(x)$ is not continuous at 3.

Problem Set 2.9

$$1. \lim_{x \rightarrow 3} [(x-3)(x-4)] = 0 = f(3); \text{ continuous}$$

$$2. \lim_{x \rightarrow 3} (x^2 - 9) = 0 = g(3); \text{ continuous}$$

$$3. \lim_{x \rightarrow 3} \frac{3}{x-3} \text{ and } h(3) \text{ do not exist, so } h(x) \text{ is not continuous at 3.}$$

$$4. \lim_{t \rightarrow 3} \sqrt{t-4} \text{ and } g(3) \text{ do not exist, so } g(t) \text{ is not continuous at 3.}$$

$$5. \lim_{t \rightarrow 3} \frac{|t-3|}{t-3} \text{ and } h(3) \text{ do not exist, so } h(t) \text{ is not continuous at 3.}$$

6. $h(3)$ does not exist, so $h(t)$ is not continuous at 3.

$$\begin{aligned} 11. \lim_{t \rightarrow 3} \frac{t^3 - 27}{t - 3} &= \lim_{t \rightarrow 3} \frac{(t-3)(t^2 + 3t + 9)}{t - 3} \\ &= \lim_{t \rightarrow 3} (t^2 + 3t + 9) = (3)^2 + 3(3) + 9 = 27 = r(3) \\ &\text{continuous} \end{aligned}$$

12. From Problem 11, $\lim_{t \rightarrow 3} r(t) = 27$, so $r(t)$ is not continuous at 3 because $\lim_{t \rightarrow 3} r(t) \neq r(3)$.

$$\begin{aligned} 13. \lim_{t \rightarrow 3^+} f(t) &= \lim_{t \rightarrow 3^+} (3 - t) = 0 \\ \lim_{t \rightarrow 3^-} f(t) &= \lim_{t \rightarrow 3^-} (t - 3) = 0 \\ \lim_{t \rightarrow 3} f(t) &= f(3); \text{ continuous} \end{aligned}$$

$$14. \lim_{t \rightarrow 3^+} f(t) = \lim_{t \rightarrow 3^+} (3-t)^2 = 0$$

$$\lim_{t \rightarrow 3^-} f(t) = \lim_{t \rightarrow 3^-} (t^2 - 9) = 0$$

$$\lim_{t \rightarrow 3} f(t) = f(3); \text{ continuous}$$

$$15. \lim_{t \rightarrow 3} f(x) = -2 = f(3); \text{ continuous}$$

$$16. g \text{ is discontinuous at } x = -3, 4, 6, 8; g \text{ is left continuous at } x = 4, 8; g \text{ is right continuous at } x = -3, 6$$

$$17. h \text{ is continuous on the intervals}$$

$$(-\infty, -5) \cup \left[-\frac{5}{2}, 4\right] \cup (4, 6) \cup [6, 8] \cup (8, \infty)$$

$$18. \lim_{x \rightarrow 7} \frac{x^2 - 49}{x - 7} = \lim_{x \rightarrow 7} \frac{(x-7)(x+7)}{x-7} = \lim_{x \rightarrow 7} (x+7) = 7+7 = 14$$

Define $f(7) = 14$.

$$19. \lim_{x \rightarrow 3} \frac{2x^2 - 18}{3 - x} = \lim_{x \rightarrow 3} \frac{2(x+3)(x-3)}{3-x}$$

$$= \lim_{x \rightarrow 3} [-2(x+3)] = -2(3+3) = -12$$

Define $f(3) = -12$.

$$20. \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1$$

Define $g(0) = 1$

$$21. \lim_{t \rightarrow 1} \frac{\sqrt{t} - 1}{t - 1} = \lim_{t \rightarrow 1} \frac{(\sqrt{t} - 1)(\sqrt{t} + 1)}{(t-1)(\sqrt{t} + 1)}$$

$$= \lim_{t \rightarrow 1} \frac{t - 1}{(t-1)(\sqrt{t} + 1)} = \lim_{t \rightarrow 1} \frac{1}{\sqrt{t} + 1} = \frac{1}{2}$$

Define $H(1) = \frac{1}{2}$.

$$22. \lim_{x \rightarrow -1} \frac{x^4 + 2x^2 - 3}{x + 1} = \lim_{x \rightarrow -1} \frac{(x^2 - 1)(x^2 + 3)}{x + 1}$$

$$= \lim_{x \rightarrow -1} \frac{(x+1)(x-1)(x^2 + 3)}{x + 1}$$

$$= \lim_{x \rightarrow -1} [(x-1)(x^2 + 3)]$$

$$= (-1-1)[(-1)^2 + 3] = -8$$

Define $\phi(-1) = -8$.

$$23. \lim_{x \rightarrow -1} \sin\left(\frac{x^2 - 1}{x + 1}\right) = \lim_{x \rightarrow -1} \sin\left(\frac{(x-1)(x+1)}{x+1}\right)$$

$$= \lim_{x \rightarrow -1} \sin(x-1) = \sin(-1-1) = \sin(-2) = -\sin 2$$

Define $F(-1) = -\sin 2$.

$$24. \text{Discontinuous at } x = \pi, 30$$

$$25. f(x) = \frac{33 - x^2}{(\pi - x)(x - 3)}$$

Discontinuous at $x = 3, \pi$

$$26. \text{Continuous at all points}$$

$$27. \text{Discontinuous at all } \theta = n\pi + \frac{\pi}{2} \text{ where } n \text{ is any integer.}$$

$$28. \text{Discontinuous at all } u \leq -5$$

$$29. \text{Discontinuous at } u = -1$$

$$30. \text{Continuous at all points}$$

$$31. G(x) = \frac{1}{\sqrt{(2-x)(2+x)}}$$

Discontinuous on $(-\infty, -2] \cup [2, \infty)$

$$32. \text{Continuous at all points since}$$

$$\lim_{x \rightarrow 0} f(x) = 0 = f(0) \text{ and } \lim_{x \rightarrow 1} f(x) = 1 = f(1).$$

$$33. \lim_{x \rightarrow 0} g(x) = 0 = g(0)$$

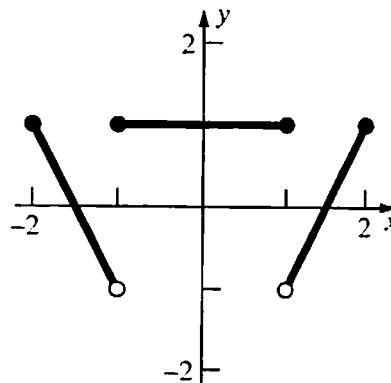
$$\lim_{x \rightarrow 1^+} g(x) = 1, \lim_{x \rightarrow 1^-} g(x) = -1$$

$\lim_{x \rightarrow 1} g(x)$ does not exist, so $g(x)$ is discontinuous at $x = 1$.

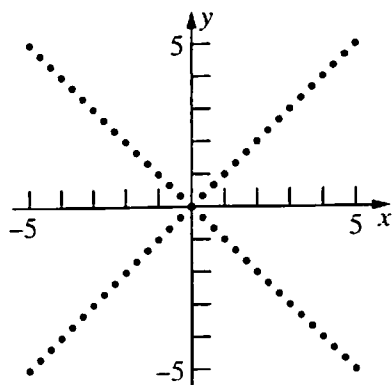
$$34. \text{Discontinuous at every integer}$$

$$35. \text{Discontinuous at } t = n + \frac{1}{2} \text{ where } n \text{ is a } y \text{ integer}$$

$$36.$$



37.



Discontinuous at all points except $x = 0$, because $\lim_{x \rightarrow c} f(x) \neq f(c)$ for $c \neq 0$. $\lim_{x \rightarrow c} f(x)$ exists only at $c = 0$ and $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$.

38. Let $f(x) = x^3 + 3x - 2$. f is continuous on $[0, 1]$.

$f(0) = -2 < 0$ and $f(1) = 2 > 0$. Thus, there is at least one number c between 0 and 1 such that $x^3 + 3x - 2 = 0$.

39. Because the function is continuous on $[0, 2\pi]$ and

$$(\cos 0)^3 + 6\sin^5 0 - 3 = -3 < 0,$$

$(\cos 2\pi)^3 + 6\sin^5(2\pi) - 3 = 8\pi^3 - 3 > 0$, there is at least one number c between 0 and 2π such that $(\cos t)^3 + 6\sin^5 t - 3 = 0$.

40. Let $f(x) = x^5 + 4x^3 - 7x + 14$

$f(x)$ is continuous at all values of x .

$$f(-2) = -36, f(0) = 14$$

Because 0 is between -36 and 14, there is at least one number c between -2 and 0 such that

$$f(x) = x^5 + 4x^3 - 7x + 14 = 0.$$

41. Suppose that f is continuous at c , so

$$\lim_{x \rightarrow c} f(x) = f(c). \text{ Let } x = t + c, \text{ so } t = x - c, \text{ then}$$

as $x \rightarrow c$, $t \rightarrow 0$ and the statement

$$\lim_{x \rightarrow c} f(x) = f(c) \text{ becomes } \lim_{t \rightarrow 0} f(t + c) = f(c).$$

Suppose that $\lim_{t \rightarrow 0} f(t + c) = f(c)$ and let $x = t +$

c , so $t = x - c$. Since c is fixed, $t \rightarrow 0$ means that $x \rightarrow c$ and the statement $\lim_{t \rightarrow 0} f(t + c) = f(c)$

becomes $\lim_{x \rightarrow c} f(x) = f(c)$, so f is continuous at

c .

42. Since $f(x)$ is continuous at c ,

$$\lim_{x \rightarrow c} f(x) = f(c) > 0. \text{ Thus, } f(x) > 0 \text{ for all } x \text{ in}$$

some deleted interval $(c - \delta, c + \delta)$ about c . Since also $f(c) > 0$, $f(x) > 0$ for all x in $(c - \delta, c + \delta)$.

43. Let $g(x) = x - f(x)$. Then,

$$g(0) = 0 - f(0) = -f(0) \leq 0 \text{ and } g(1) = 1 - f(1) \geq 0$$

since $0 \leq f(x) \leq 1$ on $[0, 1]$. If $g(0) = 0$, then

$$f(0) = 0 \text{ and } c = 0 \text{ is a fixed point of } f. \text{ If } g(1) = 0,$$

then $f(1) = 1$ and $c = 1$ is a fixed point of f . If

neither $g(0) = 0$ nor $g(1) = 0$, then $g(0) < 0$ and

$g(1) > 0$ so there is some c in $[0, 1]$ such that

$$g(c) = 0. \text{ If } g(c) = 0 \text{ then } c - f(c) = 0 \text{ or}$$

$$f(c) = c \text{ and } c \text{ is a fixed point of } f.$$

44. For $f(x)$ to be continuous everywhere,

$$f(1) = a(1) + b = 2 \text{ and } f(2) = 6 = a(2) + b$$

$$a + b = 2$$

$$2a + b = 6$$

$$-a = -4$$

$$a = 4, b = -2$$

45. For x in $[0, 1]$, let $f(x)$ indicate where the string originally at x ends up. Thus $f(0) = a$, $f(1) = b$.

$f(x)$ is continuous since the string is unbroken.

Since $0 \leq a$, $b \leq 1$, $f(x)$ satisfies the conditions of

Problem 43, so there is some c in $[0, 1]$ with

$f(c) = c$, i.e., the point of string originally at c ends up at c .

46. The Intermediate Value Theorem does not imply the existence of a number c between -2 and 2

such that $f(c) = 0$. The reason is that the

function $f(x)$ is not continuous on $[-2, 2]$.

47. Let $f(x)$ be the difference in times on the hiker's watch where x is a point on the path, and suppose $x = 0$ at the bottom and $x = 1$ at the top of the mountain.

So $f(x) = (\text{time on watch on the way up}) - (\text{time on watch on the way down})$.

$$f(0) = 4 - 11 = -7, f(1) = 12 - 5 = 7. \text{ Since time is}$$

continuous, $f(x)$ is continuous, hence there is

some c between 0 and 1 where $f(c) = 0$. This c is

the point where the hiker's watch showed the same time on both days.

48. Let f be the function on $\left[0, \frac{\pi}{2}\right]$ such that $f(\theta)$ is the length of the side of the rectangle which makes angle θ with the x -axis minus the length of the sides perpendicular to it. f is continuous on $\left[0, \frac{\pi}{2}\right]$. If $f(0) = 0$ then the region is circumscribed by a square. If $f(0) \neq 0$, then observe that $f(0) = -f\left(\frac{\pi}{2}\right)$. Thus, by the Intermediate Value Theorem, there is an angle θ_0 between 0 and $\frac{\pi}{2}$ such that $f(\theta_0) = 0$. Hence, D can be circumscribed by a square.

49. a. $f(x) = f(x+0) = f(x) + f(0)$, so $f(0) = 0$. We want to prove that $\lim_{x \rightarrow c} f(x) = f(c)$, or, equivalently, $\lim_{x \rightarrow c} [f(x) - f(c)] = 0$. But $f(x) - f(c) = f(x-c)$, so $\lim_{x \rightarrow c} [f(x) - f(c)] = \lim_{x \rightarrow c} f(x-c)$. Let $h = x - c$ then as $x \rightarrow c$, $h \rightarrow 0$ and $\lim_{x \rightarrow c} f(x-c) = \lim_{h \rightarrow 0} f(h) = f(0) = 0$. Hence $\lim_{x \rightarrow c} f(x) = f(c)$ and f is continuous at c . Thus, f is continuous everywhere, since c was arbitrary.

- b. By Problem 41 of Section 2.1, $f(t) = mt$ for all t in $\mathbb{Q}$. Since $g(t) = mt$ is a polynomial function, it is continuous for all real numbers. $f(t) = g(t)$ for all t in $\mathbb{Q}$, thus $f(t) = g(t)$ for all t in $\mathbb{R}$, i.e. $f(t) = mt$.

50. $y_1(x) = m_1x + b_1$, $y_2(x) = m_2x + b_2$
 $y_1(x) + y_2(x) = m_1x + b_1 + m_2x + b_2$
 $= (m_1 + m_2)x + (b_1 + b_2)$
This is a linear function.

51. $y_1(x) = m_1x + b_1$, $y_2(x) = m_2x + b_2$
 $y_1(y_2(x)) = y_1(m_2x + b_2) = m_1(m_2x + b_2) + b_1$
 $= m_1m_2x + m_1b_2 + b_1 = (m_1m_2)x + (m_1b_2 + b_1)$
This is a linear function.

52. $y_1(x) = m_1x + b_1$, $y_2(x) = m_2x + b_2$
 $y_1(x)y_2(x) = (m_1x + b_1)(m_2x + b_2)$
 $= m_1m_2x^2 + m_1b_2x + b_1m_2x + b_1b_2$
This is not a linear function unless $m_1m_2 = 0$. Thus, the product of two linear functions is not linear unless at least one of the functions is a constant function.

53. $y_1(x) = m_1x + b_1$, $y_2(x) = m_2x + b_2$

$$\frac{y_1(x)}{y_2(x)} = \frac{m_1x + b_1}{m_2x + b_2}$$

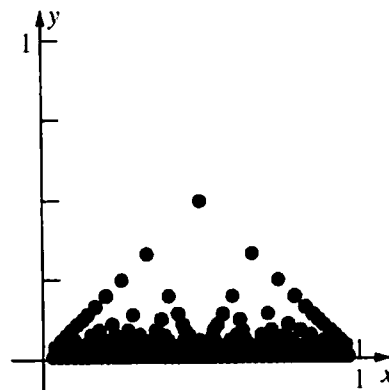
This is a linear function only when $m_2 = 0$ and $b_2 \neq 0$.

54. If $f(x)$ is continuous on an interval then $\lim_{x \rightarrow c} f(x) = f(c)$ for all points in the interval:
 $\lim_{x \rightarrow c} f(x) = f(c) \Rightarrow \lim_{x \rightarrow c} |f(x)|$
 $= \lim_{x \rightarrow c} \sqrt{f^2(x)} = \sqrt{\left(\lim_{x \rightarrow c} f(x)\right)^2}$
 $= \sqrt{(f(c))^2} = |f(c)|$

55. Suppose $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$. $f(x)$ is discontinuous at $x = 0$, but $g(x) = |f(x)| = 1$ is continuous everywhere.

56. If f is continuous at c , $\lim_{x \rightarrow c} f(x) = f(c)$
 $= f\left(\lim_{x \rightarrow c} x\right)$ since $\lim_{x \rightarrow c} x = c$.

57. a.



- b. If r is any rational number, then any deleted interval about r contains an irrational number. Thus, if $f(r) = \frac{1}{q}$, any deleted interval about r contains at least one point c such that $|f(r) - f(c)| = \left|\frac{1}{q} - 0\right| = \frac{1}{q}$. Hence, $\lim_{x \rightarrow r} f(x)$ does not exist. If c is any irrational number in $(0, 1)$, then as $x = \frac{p}{q} \rightarrow c$ (where $\frac{p}{q}$ is the reduced form of the rational number) $q \rightarrow \infty$, so

$f(x) \rightarrow 0$ as $x \rightarrow c$. Thus, $\lim_{x \rightarrow c} f(x) = 0 = f(c)$ for any irrational number c .

58. a. Suppose the block rotates to the left. Using geometry, $f(x) = -\frac{3}{4}$. Suppose the block rotates to the right. Using geometry, $f(x) = \frac{3}{4}$. If $x = 0$, the block does not rotate, so $f(x) = 0$.

$$\text{Domain: } \left[-\frac{3}{4}, \frac{3}{4}\right];$$

$$\text{Range: } \left\{-\frac{3}{4}, 0, \frac{3}{4}\right\}$$

b. At $x = 0$

- c. If $x = -\frac{3}{4}$, $f(x) = -\frac{3}{4}$ and if $x = \frac{3}{4}$, $f(x) = \frac{3}{4}$, so $x = -\frac{3}{4}, \frac{3}{4}$ are fixed points of f .

2.10 Chapter Review

Concepts Test

- True: $xy + x^2 = 3y$
 $xy - 3y = -x^2$
 $y = -\frac{x^2}{x-3}$
- False: $xy^2 + x^2 = 3x$
 $xy^2 = 3x - x^2$
 $y^2 = \frac{3x - x^2}{x}$
 $y = \pm \sqrt{\frac{3x - x^2}{x}}$
- True: $\theta \sin \theta + 1 - \cos \theta = 0$
 $1 = \cos \theta - \theta \sin \theta$
- False: $\Phi + \Psi = |\Phi + \Psi|$ cannot be written in the form $\Phi = f(\Psi)$.
- False: The equation determines T as a function of θ .
- True: $f(x) = \sqrt{\frac{x}{4-x}}; \frac{x}{4-x}$ cannot be negative and $4-x \neq 0$, so the domain is $[0, 4)$.
- True: $f(x) = \sqrt{-(x^2 + 4x + 3)}$
 $= \sqrt{-(x+3)(x+1)}$
 $-(x^2 + 4x + 3) \geq 0$ on $-3 \leq x \leq -1$.
- False: The domain does not include $n\pi + \frac{\pi}{2}$ where n is an integer.
- True: The domain is $(-\infty, \infty)$ and the range is $[-6, \infty)$.
- False: The range is $(-\infty, \infty)$.
- False: The range $(-\infty, \infty)$.
- True: If $f(x)$ and $g(x)$ are even functions, $f(x) + g(x)$ is even.
 $f(-x) + g(-x) = f(x) + g(x)$
- True: If $f(x)$ and $g(x)$ are odd functions, $f(-x) + g(-x) = -f(x) - g(x) = -(f(x) + g(x))$, so $f(x) + g(x)$ is odd.
- False: If $f(x)$ and $g(x)$ are odd functions, $f(-x)g(-x) = -f(x)[-g(x)] = f(x)g(x)$, so $f(x)g(x)$ is even.
- True: If $f(x)$ is even and $g(x)$ is odd, $f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x)$, so $f(x)g(x)$ is odd.
- False: If $f(x)$ is even and $g(x)$ is odd, $f(g(-x)) = f(-g(x)) = f(g(x))$; while if $f(x)$ is odd and $g(x)$ is even, $f(g(-x)) = f(g(x))$; so $f(g(x))$ is even.
- False: If $f(x)$ and $g(x)$ are odd functions, $f(g(-x)) = f(-g(x)) = -f(g(x))$, so $f(g(x))$ is odd.

18. True:
$$f(-x) = \frac{2(-x)^3 + (-x)}{(-x)^2 + 1} = \frac{-2x^3 - x}{x^2 + 1}$$
$$= -\frac{2x^3 + x}{x^2 + 1}$$
19. True:
$$f(-t) = \frac{(\sin(-t))^2 + \cos(-t)}{\tan(-t) \csc(-t)}$$
$$= \frac{(-\sin t)^2 + \cos t}{-\tan t (-\csc t)} = \frac{(\sin t)^2 + \cos t}{\tan t \csc t}$$
20. False: $f(x) = c$ has domain $(-\infty, \infty)$ and the only value of the range is c .
21. False: $f(x) = c$ has domain $(-\infty, \infty)$, yet the range has only one value, c .
22. True: $g(-1.8) = \left\lceil \frac{-1.8}{2} \right\rceil = \lceil -0.9 \rceil = -1$
23. True: $(f \circ g)(x) = (x^3)^2 = x^6$
 $(g \circ f)(x) = (x^2)^3 = x^6$
24. False: $(f \circ g)(x) = (x^3)^2 = x^6$
 $f(x) \cdot g(x) = x^2 x^3 = x^5$
25. False: The domain of $\frac{f}{g}$ excludes any values where $g = 0$.
26. True: $f(a) = 0$
Let $F(x) = f(x + h)$, then
 $F(a - h) = f(a - h + h) = f(a) = 0$
27. True: $\cot x = \frac{\cos x}{\sin x}$
 $\cot(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos x}{-\sin x} = -\cot x$
28. False: The domain of the tangent function excludes all $n\pi + \frac{\pi}{2}$ where n is an integer.
29. False: The cosine function is periodic, so $\cos s = \cos t$ does not necessarily imply $s = t$; e.g., $\cos 0 = \cos 2\pi = 1$, but $0 \neq 2\pi$.
30. False: c may not be in the domain of $f(x)$, or it may be defined separately.

31. False: If $f(c)$ is not defined, $\lim_{x \rightarrow c} f(x)$ might exist; e.g., $f(x) = \frac{x^2 - 4}{x + 2}$.
 $f(-2)$ does not exist,
but $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2} = -4$.
32. True:
$$\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5} = \lim_{x \rightarrow 5} \frac{(x - 5)(x + 5)}{x - 5}$$
$$= \lim_{x \rightarrow 5} (x + 5) = 5 + 5 = 10$$
33. True: Substitution Theorem
34. False: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
35. False: The tangent function is not defined for all values of c .
36. True: Since both $\sin x$ and $\cos x$ are continuous for all real numbers, by Theorem C we can conclude that $f(x) = 2 \sin^2 x - \cos x$ is also continuous for all real numbers.
37. False: Since $-1 \leq \sin x \leq 1$ for all x and $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$, we get $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$.
38. False: Consider $f(x) = \sin x$.
39. True: As $x \rightarrow 1^+$ both the numerator and denominator are positive. Since the numerator approaches a constant and the denominator approaches zero, the limit goes to $+\infty$.
40. False: $\lim_{x \rightarrow c} f(x)$ must equal $f(c)$ for f to be continuous at $x = c$.
41. True: $\lim_{x \rightarrow c} f(x) = f\left(\lim_{x \rightarrow c} x\right) = f(c)$, so f is continuous at $x = c$.
42. True: $\lim_{x \rightarrow 2.3} \left\lceil \frac{x}{2} \right\rceil = 1 = f(2.3)$
43. True: Choose $\varepsilon = 0.001f(2)$ then since $\lim_{x \rightarrow 2} f(x) = f(2)$, there is some δ such that $0 < |x - 2| < \delta \Rightarrow |f(x) - f(2)| < 0.001f(2)$, or

$$-0.001f(2) < f'(x) - f(2) < 0.001f(2)$$

Thus, $0.999f(2) < f(x) < 1.001f(2)$ and $f(x) < 1.001f(2)$ for $0 < |x - 2| < \delta$.

Since $f(2) < 1.001f(2)$, as $f(2) > 0$, $f(x) < 1.001f(2)$ on $(2 - \delta, 2 + \delta)$.

44. False: That $\lim_{x \rightarrow c} [f(x) + g(x)]$ exists does not imply that $\lim_{x \rightarrow c} f(x)$ and

$\lim_{x \rightarrow c} g(x)$ exist; e.g., $f(x) = \frac{x-3}{x+2}$ and

$$g(x) = \frac{x+7}{x+2}.$$

45. True: Squeeze Theorem

46. True: A function has only one limit at a point, so if $\lim_{x \rightarrow a} f(x) = L$ and

$$\lim_{x \rightarrow a} f(x) = M, \quad L = M$$

47. False: That $f(x) \neq g(x)$ for all x does not imply that $\lim_{x \rightarrow c} f(x) \neq \lim_{x \rightarrow c} g(x)$. For

example, if $f(x) = \frac{x^2 + x - 6}{x - 2}$ and

$$g(x) = \frac{5}{2}x, \text{ then } f(x) \neq g(x) \text{ for all } x.$$

$$\text{but } \lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} g(x) = 5.$$

48. False: If $f(x) < 10$, $\lim_{x \rightarrow 2} f(x)$ could equal 10 if there is a discontinuity point $(2, 10)$. For example,

$$f(x) = \frac{-x^3 + 6x^2 - 2x - 12}{x - 2} < 10 \text{ for}$$

all x , but $\lim_{x \rightarrow 2} f(x) = 10$.

49. True: $\lim_{x \rightarrow a} |f(x)| = \lim_{x \rightarrow a} \sqrt{f^2(x)}$

$$= \sqrt{\left[\lim_{x \rightarrow a} f(x) \right]^2} = \sqrt{(b)^2} = |b|$$

50. True: If f is continuous and positive on $[a, b]$, the reciprocal is also continuous, so it will assume all values between $\frac{1}{f(a)}$ and $\frac{1}{f(b)}$.

Sample Test Problems

1. a. $f(1) = \frac{1}{1+1} - \frac{1}{1} = -\frac{1}{2}$

b. $f\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}+1} - \frac{1}{-\frac{1}{2}} = 4$

c. $f(-1)$ does not exist.

d. $f(t-1) = \frac{1}{t-1+1} - \frac{1}{t-1} = \frac{1}{t} - \frac{1}{t-1}$

e. $f\left(\frac{1}{t}\right) = \frac{1}{\frac{1}{t}+1} - \frac{1}{\frac{1}{t}} = \frac{t}{1+t} - t$

2. a. $g(2) = \frac{2+1}{2} = \frac{3}{2}$

b. $g\left(\frac{1}{2}\right) = \frac{\frac{1}{2}+1}{\frac{1}{2}} = 3$

c. $g\left(\frac{1}{10}\right) = \frac{\frac{1}{10}+1}{\frac{1}{10}} = 11$

d. $\frac{g(2+h) - g(2)}{h} = \frac{\frac{2+h+1}{2+h} - \frac{2+1}{2}}{h}$
 $= \frac{\frac{2h+6-3h-6}{2(h+2)}}{h} = \frac{-\frac{h}{2(h+2)}}{h} = \frac{-1}{2(h+2)}$

3. a. $\{x \in \mathbb{R} : x \neq -1, 1\}$

b. $\{x \in \mathbb{R} : |x| \leq 2\}$

c. $\left\{x \in \mathbb{R} : x \neq -\frac{3}{2}\right\}$

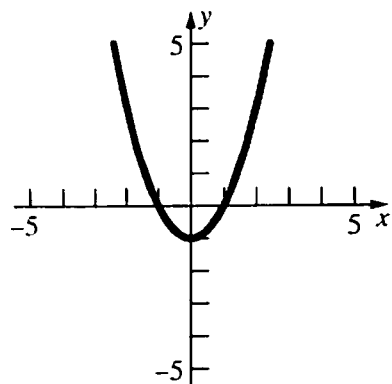
4. a. $f(-x) = \frac{3(-x)}{(-x)^2 + 1} = -\frac{3x}{x^2 + 1}$; odd

b. $g(-x) = |\sin(-x)| + \cos(-x)$
 $= |-\sin x| + \cos x = |\sin x| + \cos x$; even

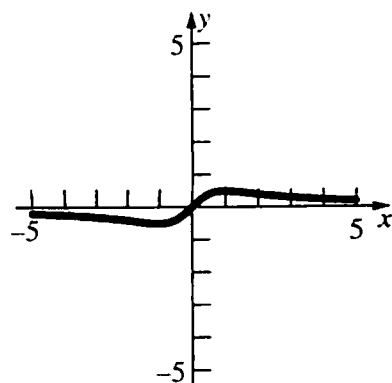
c. $h(-x) = (-x)^3 + \sin(-x) = -x^3 - \sin x$; odd

d. $k(-x) = \frac{(-x)^2 + 1}{|-x| + (-x)^4} = \frac{x^2 + 1}{|x| + x^4}$; even

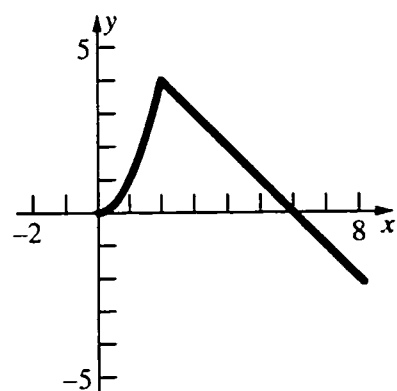
5. a. $f(x) = x^2 - 1$



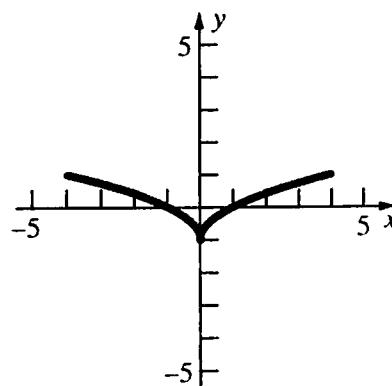
b. $g(x) = \frac{x}{x^2 + 1}$



c. $h(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 2 \\ 6 - x & \text{if } x > 2 \end{cases}$



6.



7. $V(x) = x(32 - 2x)(24 - 2x)$
Domain $[0, 12]$

8. a. $(f + g)(2) = \left(2 - \frac{1}{2}\right) + (2^2 + 1) = \frac{13}{2}$

b. $(f \cdot g)(2) = \left(\frac{3}{2}\right)(5) = \frac{15}{2}$

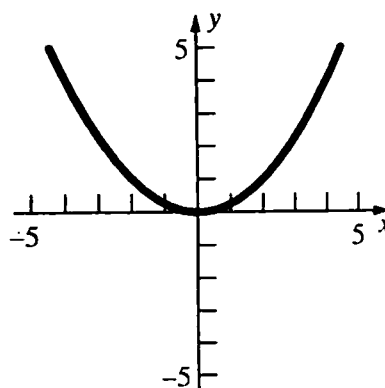
c. $(f \circ g)(2) = f(5) = 5 - \frac{1}{5} = \frac{24}{5}$

d. $(g \circ f)(2) = g\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 + 1 = \frac{13}{4}$

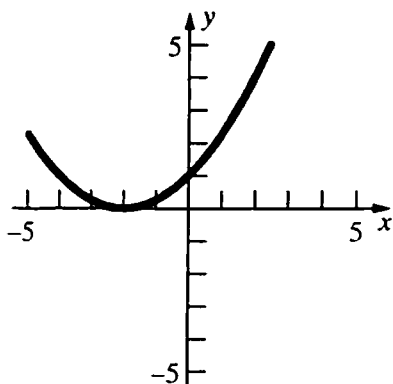
e. $f^3(-1) = \left(-1 + \frac{1}{1}\right)^3 = 0$

f. $f^2(2) + g^2(2) = \left(\frac{3}{2}\right)^2 + (5)^2 = \frac{9}{4} + 25 = \frac{109}{4}$

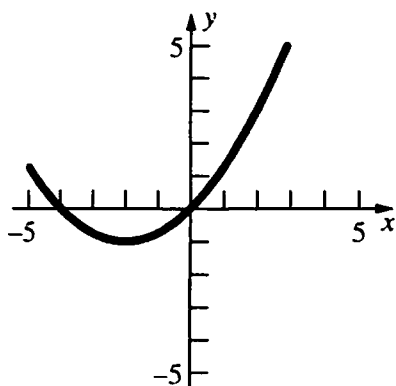
9. a. $y = \frac{1}{4}x^2$



b. $y = \frac{1}{4}(x+2)^2$



c. $y = -1 + \frac{1}{4}(x+2)^2$



10. a. $(-\infty, 16]$

b. $f \circ g = \sqrt{16 - x^4}$; domain $[-2, 2]$

c. $g \circ f = (\sqrt{16 - x})^4 = (16 - x)^2$;
domain $(-\infty, 16]$

11. $f(x) = \sqrt{x}$, $g(x) = 1 + x$, $h(x) = x^2$,
 $k(x) = \sin x$, $F(x) = \sqrt{1 + \sin^2 x} = f \circ g \circ h \circ k$

12. a. $\sin(570^\circ) = \sin(210^\circ) = -\frac{1}{2}$

b. $\cos\left(\frac{9\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

c. $\sin^2(5) + \cos^2(5) = 1$

d. $\cos\left(-\frac{13\pi}{6}\right) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

13. a. $\sin(-t) = -\sin t = -0.8$

b. $\sin^2 t + \cos^2 t = 1$
 $\cos^2 t = 1 - (0.8)^2 = 0.36$
 $\cos t = -0.6$

c. $\sin 2t = 2 \sin t \cos t = 2(0.8)(-0.6) = -0.96$

d. $\tan t = \frac{\sin t}{\cos t} = \frac{0.8}{-0.6} = -\frac{4}{3} \approx -1.333$

e. $\cos\left(\frac{\pi}{2} - t\right) = \sin t = 0.8$

f. $\sin(\pi + t) = -\sin t = -0.8$

14. $\sin 3t = \sin(2t + t) = \sin 2t \cos t + \cos 2t \sin t$
 $= 2 \sin t \cos^2 t + (1 - 2 \sin^2 t) \sin t$
 $= 2 \sin t (1 - \sin^2 t) + \sin t - 2 \sin^3 t$
 $= 2 \sin t - 2 \sin^3 t + \sin t - 2 \sin^3 t$
 $= 3 \sin t - 4 \sin^3 t$

15. $s = rt$
 $= 9 \left(20 \frac{\text{rev}}{\text{min}} \right) \left(2\pi \frac{\text{rad}}{\text{rev}} \right) \left(\frac{1 \text{ min}}{60 \text{ sec}} \right) (1 \text{ sec}) = 6\pi$
 $\approx 18.85 \text{ in.}$

16. $\lim_{u \rightarrow 1} \frac{u^2 - 1}{u + 1} = \frac{1^2 - 1}{1 + 1} = 0$

17. $\lim_{u \rightarrow 1} \frac{u^2 - 1}{u - 1} = \lim_{u \rightarrow 1} \frac{(u - 1)(u + 1)}{u - 1} = \lim_{u \rightarrow 1} (u + 1)$
 $= 1 + 1 = 2$

18. $\lim_{u \rightarrow 1} \frac{u + 1}{u^2 - 1} = \lim_{u \rightarrow 1} \frac{u + 1}{(u + 1)(u - 1)} = \lim_{u \rightarrow 1} \frac{1}{u - 1}$;
does not exist

19. $\lim_{x \rightarrow 2} \frac{1 - \frac{2}{x}}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{\frac{x - 2}{x}}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2} \frac{1}{x(x + 2)}$
 $= \frac{1}{2(2 + 2)} = \frac{1}{8}$

20. $\lim_{z \rightarrow 2} \frac{z^2 - 4}{z^2 + z - 6} = \lim_{z \rightarrow 2} \frac{(z + 2)(z - 2)}{(z + 3)(z - 2)}$
 $= \lim_{z \rightarrow 2} \frac{z + 2}{z + 3} = \frac{2 + 2}{2 + 3} = \frac{4}{5}$

21. $\lim_{x \rightarrow 0} \frac{\tan x}{\sin 2x} = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x}}{2 \sin x \cos x} = \lim_{x \rightarrow 0} \frac{1}{2 \cos^2 x}$
 $= \frac{1}{2 \cos^2 0} = \frac{1}{2}$

$$22. \lim_{y \rightarrow 1} \frac{y^3 - 1}{y^2 - 1} = \lim_{y \rightarrow 1} \frac{(y-1)(y^2 + y + 1)}{(y-1)(y+1)} \\ = \lim_{y \rightarrow 1} \frac{y^2 + y + 1}{y + 1} = \frac{1^2 + 1 + 1}{1 + 1} = \frac{3}{2}$$

$$23. \lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x \rightarrow 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2} \\ = \lim_{x \rightarrow 4} (\sqrt{x}+2) = \sqrt{4}+2 = 4$$

$$24. \lim_{x \rightarrow 0} \frac{\cos x}{x} \text{ does not exist.}$$

$$25. \lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} (-1) = -1$$

$$26. \lim_{x \rightarrow (1/2)^+} \lfloor 4x \rfloor = 2$$

$$27. \lim_{t \rightarrow 2^-} (\lfloor t \rfloor - t) = \lim_{t \rightarrow 2^-} \lfloor t \rfloor - \lim_{t \rightarrow 2^-} t = 1 - 2 = -1$$

$$28. \text{ a. } f(1) = 0$$

$$\text{ b. } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (1-x) = 0$$

$$\text{ c. } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x = 1$$

$$\text{ d. } \lim_{x \rightarrow -1} f(x) = -1 \text{ because} \\ \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} x^3 = -1 \text{ and} \\ \lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} x = -1$$

$$29. \text{ a. } f \text{ is discontinuous at } x = 1 \text{ because } f(1) = 0, \\ \text{ but } \lim_{x \rightarrow 1} f(x) \text{ does not exist. } f \text{ is} \\ \text{ discontinuous at } x = -1 \text{ because } f(-1) \text{ does} \\ \text{ not exist.}$$

$$\text{ b. Define } f(-1) = -1$$

$$30. \text{ a. } 0 < |u-a| < \delta \Rightarrow |g(u) - M| < \varepsilon$$

$$\text{ b. } 0 < a-x < \delta \Rightarrow |f(x) - L| < \varepsilon$$

$$31. \text{ a. } \lim_{x \rightarrow 3} [2f(x) - 4g(x)] \\ = 2 \lim_{x \rightarrow 3} f(x) - 4 \lim_{x \rightarrow 3} g(x) \\ = 2(3) - 4(-2) = 14$$

$$\text{ b. } \lim_{x \rightarrow 3} g(x) \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} g(x)(x+3) \\ = \lim_{x \rightarrow 3} g(x) \cdot \lim_{x \rightarrow 3} (x+3) = -2 \cdot (3+3) = -12$$

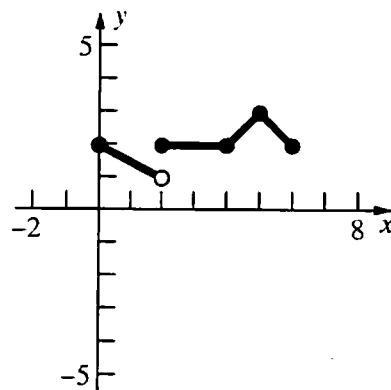
$$\text{ c. } g(3) = -2$$

$$\text{ d. } \lim_{x \rightarrow 3} g(f(x)) = g\left(\lim_{x \rightarrow 3} f(x)\right) = g(3) = -2$$

$$\text{ e. } \lim_{x \rightarrow 3} \sqrt{f^2(x) - 8g(x)} \\ = \sqrt{\left[\lim_{x \rightarrow 3} f(x)\right]^2 - 8 \lim_{x \rightarrow 3} g(x)} \\ = \sqrt{(3)^2 - 8(-2)} = 5$$

$$\text{ f. } \lim_{x \rightarrow 3} \frac{|g(x) - g(3)|}{f(x)} = \frac{|-2 - g(3)|}{3} = \frac{|-2 - (-2)|}{3} \\ = 0$$

32.



$$33. \begin{aligned} a(0) + b &= -1 \text{ and } a(1) + b = 1 \\ b &= -1; \quad a + b = 1 \\ a - 1 &= 1 \\ a &= 2 \end{aligned}$$

$$34. \text{ Let } f(x) = x^5 - 4x^3 - 3x + 1 \\ f(2) = -5, f(3) = 127 \\ \text{Because } f(x) \text{ is continuous on } [2, 3] \text{ and} \\ f(2) < 0 < f(3), \text{ there exists some number } c \\ \text{between 2 and 3 such that } f(c) = 0.$$

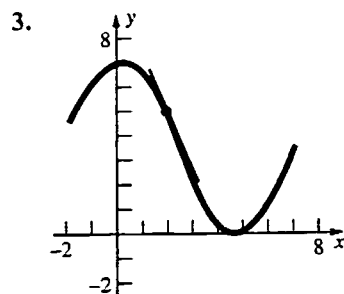
3.1 Concepts Review

1. tangent line
2. secant line
3. $\frac{f(c+h)-f(c)}{h}$
4. average velocity

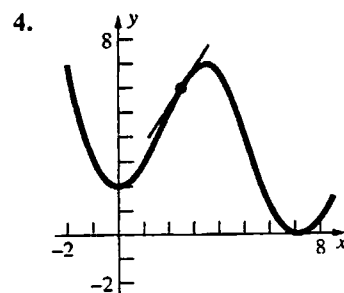
Problem Set 3.1

1. Slope $= \frac{5-3}{2-\frac{3}{2}} = 4$

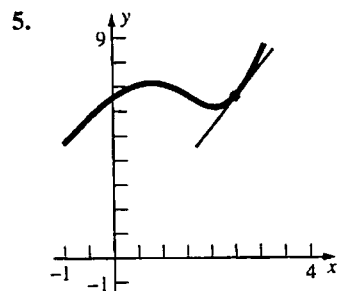
2. Slope $= \frac{6-4}{4-6} = -2$



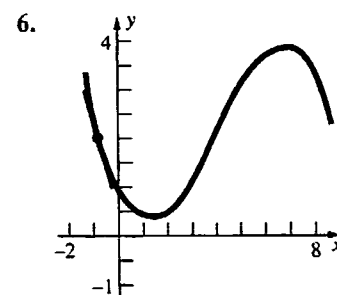
Slope ≈ -2



Slope ≈ 1.5



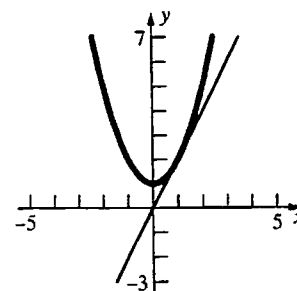
Slope $\approx \frac{5}{2}$



Slope $\approx -\frac{3}{2}$

7. $y = x^2 + 1$

a., b.



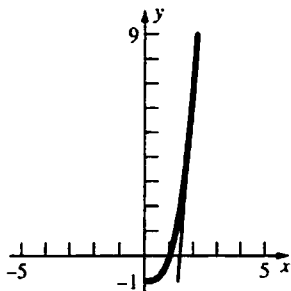
c. $m_{\tan} = 2$

d. $m_{\sec} = \frac{(1.01)^2 + 1.0 - 2}{1.01 - 1} = \frac{0.0201}{.01} = 2.01$

$$\begin{aligned}
 \text{e. } m_{\tan} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(1+h)^2 + 1] - (1^2 + 1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2 + 2h + h^2 - 2}{h} = \lim_{h \rightarrow 0} \frac{h(2+h)}{h} \\
 &= \lim_{h \rightarrow 0} (2+h) = 2
 \end{aligned}$$

8. $y = x^3 - 1$

a., b.



c. $m_{\tan} = 12$

$$\begin{aligned}
 \text{d. } m_{\sec} &= \frac{[(2.01)^3 - 1.0] - 7}{2.01 - 2} = \frac{0.120601}{0.01} \\
 &= 12.0601
 \end{aligned}$$

$$\begin{aligned}
 \text{e. } m_{\tan} &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(2+h)^3 - 1] - (2^3 - 1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h \rightarrow 0} \frac{h(12 + 6h + h^2)}{h} \\
 &= 12
 \end{aligned}$$

9. $f(x) = x^2 - 1$

$$\begin{aligned}
 m_{\tan} &= \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(c+h)^2 - 1] - (c^2 - 1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{c^2 + 2ch + h^2 - 1 - c^2 + 1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(2c + h)}{h} = 2c
 \end{aligned}$$

At $x = -2$, $m_{\tan} = -4$

$x = -1$, $m_{\tan} = -2$

$x = 1$, $m_{\tan} = 2$

$x = 2$, $m_{\tan} = 4$

10. $f(x) = x^3 - 3x$

$$\begin{aligned}
 m_{\tan} &= \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(c+h)^3 - 3(c+h)] - (c^3 - 3c)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{c^3 + 3c^2h + 3ch^2 + h^3 - 3c - 3h - c^3 + 3c}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(3c^2 + 3ch + h^2 - 3)}{h} = 3c^2 - 3
 \end{aligned}$$

At $x = -2$, $m_{\tan} = 9$

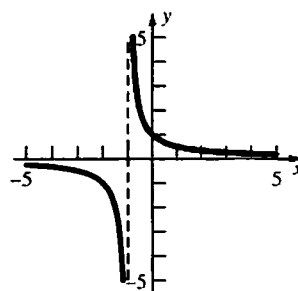
$x = -1$, $m_{\tan} = 0$

$x = 0$, $m_{\tan} = -3$

$x = 1$, $m_{\tan} = 0$

$x = 2$, $m_{\tan} = 9$

11.



$$f(x) = \frac{1}{x+1}$$

$$\begin{aligned}
 m_{\tan} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h \rightarrow 0} \frac{-\frac{h}{2(2+h)}}{h} = \lim_{h \rightarrow 0} -\frac{1}{2(2+h)} \\
 &= -\frac{1}{4}
 \end{aligned}$$

$$y - \frac{1}{2} = -\frac{1}{4}(x - 1)$$

12. $f(x) = \frac{1}{x-1}$

$$m_{\tan} = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{h-1} + 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{h}{h-1}}{h} = \lim_{h \rightarrow 0} \frac{1}{h-1} = -1$$

$$y + 1 = -1(x - 0); y = -x - 1$$

13. a. $16(1^2) - 16(0^2) = 16 \text{ ft}$

b. $16(2^2) - 16(1^2) = 48 \text{ ft}$

c. $V_{\text{ave}} = \frac{144 - 64}{3 - 2} = 80 \text{ ft/sec}$

$$\begin{aligned} \text{d. } V_{\text{ave}} &= \frac{16(3.01)^2 - 16(3)^2}{3.01 - 3} = \frac{0.9616}{0.01} \\ &= 96.16 \text{ ft/s} \end{aligned}$$

$$\begin{aligned} \text{e. } f(t) &= 16t^2; v = 32c \\ v &= 32(3) = 96 \text{ ft/s} \end{aligned}$$

$$14. \text{ a. } V_{\text{ave}} = \frac{(3^2 + 1) - (2^2 + 1)}{3 - 2} = 5 \text{ m/sec}$$

$$\begin{aligned} \text{b. } V_{\text{ave}} &= \frac{[(2.003)^2 + 1] - (2^2 + 1)}{2.003 - 2} = \frac{0.012009}{0.003} \\ &= 4.003 \text{ m/sec} \end{aligned}$$

$$\begin{aligned} 15. \text{ a. } v &= \lim_{h \rightarrow 0} \frac{f(\alpha + h) - f(\alpha)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{2(\alpha + h) + 1} - \sqrt{2\alpha + 1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{2\alpha + 2h + 1} - \sqrt{2\alpha + 1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{2\alpha + 2h + 1} - \sqrt{2\alpha + 1})(\sqrt{2\alpha + 2h + 1} + \sqrt{2\alpha + 1})}{h(\sqrt{2\alpha + 2h + 1} + \sqrt{2\alpha + 1})} \\ &= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{2\alpha + 2h + 1} + \sqrt{2\alpha + 1})} \\ &= \frac{2}{\sqrt{2\alpha + 1} + \sqrt{2\alpha + 1}} = \frac{1}{\sqrt{2\alpha + 1}} \text{ ft/s} \end{aligned}$$

$$\begin{aligned} \text{b. } \frac{1}{\sqrt{2\alpha + 1}} &= \frac{1}{2} \\ \sqrt{2\alpha + 1} &= 2 \\ 2\alpha + 1 &= 4; \alpha = \frac{3}{2} \end{aligned}$$

The object reaches a velocity of $\frac{1}{2}$ ft/s when $t = \frac{3}{2}$.

$$\begin{aligned} 16. f(t) &= -t^2 + 4t \\ v &= \lim_{h \rightarrow 0} \frac{[-(c+h)^2 + 4(c+h)] - (-c^2 + 4c)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-c^2 - 2ch - h^2 + 4c + 4h + c^2 - 4c}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(-2c - h + 4)}{h} = -2c + 4 \\ -2c + 4 &= 0 \text{ when } c = 2 \\ \text{The particle comes to a momentary stop at } t &= 2. \end{aligned}$$

$$17. \text{ a. } \left[\frac{1}{2}(2.01)^2 + 1 \right] - \left[\frac{1}{2}(2)^2 + 1 \right] = 0.02005 \text{ g}$$

$$\begin{aligned} \text{c. } V_{\text{ave}} &= \frac{[(2+h)^2 + 1] - (2^2 + 1)}{2 + h - 2} = \frac{4h + h^2}{h} \\ &= 4 + h \end{aligned}$$

$$\begin{aligned} \text{d. } f(t) &= t^2 + 1 \\ v &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(2+h)^2 + 1] - (2^2 + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4h + h^2}{h} = \lim_{h \rightarrow 0} (4 + h) = 4 \end{aligned}$$

$$\text{b. } r_{\text{ave}} = \frac{0.02005}{2.01 - 2} = 2.005 \text{ g/hr}$$

$$\begin{aligned} \text{c. } f(t) &= \frac{1}{2}t^2 + 1 \\ r &= \lim_{h \rightarrow 0} \frac{\left[\frac{1}{2}(2+h)^2 + 1 \right] - \left[\frac{1}{2}2^2 + 1 \right]}{h} \\ &= \lim_{h \rightarrow 0} \frac{2 + 2h + \frac{1}{2}h^2 + 1 - 2 - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{h\left(2 + \frac{1}{2}h\right)}{h} = 2 \\ \text{At } t = 2, r &= 2 \end{aligned}$$

18. a. $1000(3)^2 - 1000(2)^2 = 5000$

b. $\frac{1000(2.5)^2 - 1000(2)^2}{2.5 - 2} = \frac{2250}{0.5} = 4500$

c. $f(t) = 1000t^2$

$$r = \lim_{h \rightarrow 0} \frac{1000(2+h)^2 - 1000(2)^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4000 + 4000h + 1000h^2 - 4000}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(4000 + 1000h)}{h} = 4000$$

20. $MR = \lim_{h \rightarrow 0} \frac{R(c+h) - R(c)}{h}$

$$= \lim_{h \rightarrow 0} \frac{[0.4(c+h) - 0.001(c+h)^2] - (0.4c - 0.001c^2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{0.4c + 0.4h - 0.001c^2 - 0.002ch - 0.001h^2 - 0.4c + 0.001c^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(0.4 - 0.002c - 0.001h)}{h} = 0.4 - 0.002c$$

When $n = 10$, $MR = 0.38$; when $n = 100$, $MR = 0.2$

21. $a = \lim_{h \rightarrow 0} \frac{2(1+h)^2 - 2(1)^2}{h}$

$$= \lim_{h \rightarrow 0} \frac{2 + 4h + 2h^2 - 2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(4 + 2h)}{h} = 4$$

22. $r = \lim_{h \rightarrow 0} \frac{p(c+h) - p(c)}{h}$

$$= \lim_{h \rightarrow 0} \frac{[120(c+h)^2 - 2(c+h)^3] - (120c^2 - 2c^3)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(240c - 6c^2 + 120h - 6ch - 2h^2)}{h}$$

$$= 240c - 6c^2$$

When $t = 10$, $r = 240(10) - 6(10)^2 = 1800$

$t = 20$, $r = 240(20) - 6(20)^2 = 2400$

$t = 40$, $r = 240(40) - 6(40)^2 = 0$

23. $r_{\text{ave}} = \frac{100 - 800}{24 - 0} = -\frac{175}{6} \approx -29.167$
 29,167 gal/hr

19. a. $d_{\text{ave}} = \frac{5^3 - 3^3}{5 - 3} = \frac{98}{2} = 49 \text{ g/cm}$

b. $f(x) = x^3$

$$d = \lim_{h \rightarrow 0} \frac{(3+h)^3 - 3^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{27 + 27h + 9h^2 + h^3 - 27}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(27 + 9h + h^2)}{h} = 27 \text{ g/cm}$$

At 8 o'clock, $r \approx \frac{700 - 400}{6 - 10} \approx -75$

75,000 gal/hr

24. a. The elevator reached the seventh floor at time $t = 80$. The average velocity is
 $v_{\text{avg}} = (84 - 0) / 80 = 1.05$ feet per second

b. The slope of the line is approximately
 $\frac{60 - 12}{55 - 15} = 1.2$. The velocity is
 approximately 1.2 feet per second.

c. The building averages $84/7 = 12$ feet from floor to floor. Since the velocity is zero for two intervals between time 0 and time 85, the elevator stopped twice. The heights are approximately 12 and 60. Thus, the elevator stopped at floors 1 and 5.

25. a. A tangent line at $t = 91$ has slope
 approximately $(63 - 48) / (91 - 61) = 0.5$. The
 normal high temperature increases at the rate
 of 0.5 degree F per day.

- b. A tangent line at $t = 191$ has approximate slope $(90 - 88)/30 \approx 0.067$. The normal high temperature increases at the rate of 0.067 degree per day.
- c. There is a time in January, about January 15, when the rate of change is zero. There is also a time in July, about July 15, when the rate of change is zero.
- d. The greatest rate of increase occurs around day 61, that is, some time in March. The greatest rate of decrease occurs between day 301 and 331, that is, sometime in November.
26. The slope of the tangent line at $t = 1930$ is approximately $(10 - 3.5)/(1980 - 1919) \approx 0.107$. The rate of growth in 1930 is approximately 0.107 million, or 107,000, persons per year. In 1990, the tangent line has approximate slope $(24 - 16)/(20000 - 1980) \approx 0.4$. Thus, the rate of growth in 1990 is 0.4 million, or 400,000, persons per year. The approximate percentage growth in 1930 is $0.107/6 \approx 0.018$ and in 1990 it is approximately $0.4/20 \approx 0.02$.
27. In both (a) and (b), the tangent line is always positive. In (a) the tangent line becomes steeper and steeper as t increases; thus, the velocity is increasing. In (b) the tangent line becomes flatter and flatter as t increases; thus, the velocity is decreasing.

28. $f(t) = \frac{1}{3}t^3 + t$

$$\begin{aligned}\text{current} &= \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left[\frac{1}{3}(c+h)^3 + (c+h)\right] - \left(\frac{1}{3}c^3 + c\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h\left(c^2 + ch + \frac{1}{3}h^2 + 1\right)}{h} = c^2 + 1\end{aligned}$$

When $t = 3$, the current = 10

$$c^2 + 1 = 20$$

$$c^2 = 19$$

$$c = \sqrt{19} \approx 4.4$$

A 20-amp fuse will blow at $t = 4.4$ s.

29. $A = \pi r^2$, $r = 2t$

$$A = 4\pi t^2$$

$$\text{rate} = \lim_{h \rightarrow 0} \frac{4\pi(3+h)^2 - 4\pi(3)^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(24\pi + 4\pi h)}{h} = 24\pi \text{ km}^2/\text{day}$$

30. $V = \frac{4}{3}\pi r^3$, $r = \frac{1}{4}t$

$$V = \frac{1}{48}\pi t^3$$

$$\text{rate} = \frac{1}{48}\pi \lim_{h \rightarrow 0} \frac{(3+h)^3 - 3^3}{h} = \frac{27}{48}\pi$$

$$= \frac{9}{16}\pi \text{ inch}^3/\text{sec}$$

31. $y = f(x) = x^3 - 2x^2 + 1$

a. $m_{\tan} = 7$

b. $m_{\tan} = 0$

c. $m_{\tan} = -1$

d. $m_{\tan} = 17.92$

32. $y = f(x) = \sin x \sin^2 2x$

a. $m_{\tan} = -1.125$

b. $m_{\tan} \approx -1.0315$

c. $m_{\tan} = 0$

d. $m_{\tan} \approx 1.1891$

33. $s = f(t) = t + t \cos^2 t$

At $t = 3$, $v \approx 2.818$

34. $s = f(t) = \frac{(t+1)^3}{t+2}$

At $t = 1.6$, $v \approx 4.277$

3.2 Concepts Review

1. $\frac{f(c+h) - f(c)}{h}$; $\frac{f(t) - f(c)}{t - c}$

2. $\frac{f(x+h) - f(x)}{h}$

3. continuous; $f(x) = |x|$

4. $2x^2$; c

Problem Set 3.2

$$\begin{aligned} 1. \quad f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \rightarrow 0} \frac{2h + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2 + h) = 2 \end{aligned}$$

$$\begin{aligned} 2. \quad f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[2(2+h)]^2 - [2(2)]^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{16h + 4h^2}{h} = \lim_{h \rightarrow 0} (16 + 4h) = 16 \end{aligned}$$

$$\begin{aligned} 3. \quad f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(3+h)^2 - (3+h)] - (3^2 - 3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{5h + h^2}{h} = \lim_{h \rightarrow 0} (5 + h) = 5 \end{aligned}$$

$$\begin{aligned} 4. \quad f'(4) &= \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{3+h} - \frac{1}{4}}{h} = \lim_{h \rightarrow 0} \frac{\frac{3-(3+h)}{3(3+h)}}{h} = \lim_{h \rightarrow 0} \frac{-1}{3(3+h)} \\ &= -\frac{1}{9} \end{aligned}$$

$$\begin{aligned} 5. \quad s'(x) &= \lim_{h \rightarrow 0} \frac{s(x+h) - s(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[2(x+h) + 1] - (2x + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h}{h} = 2 \end{aligned}$$

$$\begin{aligned} 6. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[\alpha(x+h) + \beta] - (\alpha x + \beta)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\alpha h}{h} = \alpha \end{aligned}$$

$$\begin{aligned} 7. \quad r'(x) &= \lim_{h \rightarrow 0} \frac{r(x+h) - r(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3(x+h)^2 + 4] - (3x^2 + 4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2}{h} = \lim_{h \rightarrow 0} (6x + 3h) = 6x \end{aligned}$$

$$\begin{aligned} 8. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + (x+h) + 1] - (x^2 + x + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + h}{h} = \lim_{h \rightarrow 0} (2x + h + 1) = 2x + 1 \end{aligned}$$

$$\begin{aligned} 9. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[a(x+h)^2 + b(x+h) + c] - (ax^2 + bx + c)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2axh + ah^2 + bh}{h} = \lim_{h \rightarrow 0} (2ax + ah + b) \\ &= 2ax + b \end{aligned}$$

$$\begin{aligned} 10. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{4hx^3 + 6h^2x^2 + 4h^3x + h^4}{h} \\ &= \lim_{h \rightarrow 0} (4x^3 + 6hx^2 + 4h^2x + h^3) = 4x^3 \end{aligned}$$

$$\begin{aligned} 11. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 + 2(x+h)^2 + 1] - (x^3 + 2x^2 + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3hx^2 + 3h^2x + h^3 + 4hx + 2h^2}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3hx + h^2 + 4x + 2h) = 3x^2 + 4x \end{aligned}$$

$$\begin{aligned} 12. \quad g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^4 + (x+h)^2] - (x^4 + x^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4hx^3 + 6h^2x^2 + 4h^3x + h^4 + 2hx + h^2}{h} \\ &= \lim_{h \rightarrow 0} (4x^3 + 6hx^2 + 4h^2x + h^3 + 2x + h) \\ &= 4x^3 + 2x \end{aligned}$$

$$\begin{aligned}
 13. \quad h'(x) &= \lim_{h \rightarrow 0} \frac{h(x+h) - h(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\left(\frac{2}{x+h} - \frac{2}{x} \right) \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-2h}{x(x+h)} \cdot \frac{1}{h} \right] = \lim_{h \rightarrow 0} \frac{-2}{x(x+h)} \\
 &= -\frac{2}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad S'(x) &= \lim_{h \rightarrow 0} \frac{S(x+h) - S(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\left(\frac{1}{x+h+1} - \frac{1}{x+1} \right) \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-h}{(x+1)(x+h+1)} \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{-1}{(x+1)(x+h+1)} = -\frac{1}{(x+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 17. \quad G'(x) &= \lim_{h \rightarrow 0} \frac{G(x+h) - G(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\left(\frac{2(x+h)-1}{x+h-4} - \frac{2x-1}{x-4} \right) \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{2x^2 + 2hx - 9x - 8h + 4 - (2x^2 + 2hx - 9x - h + 4)}{(x+h-4)(x-4)} \cdot \frac{1}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{-7h}{(x+h-4)(x-4)} \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{-7}{(x+h-4)(x-4)} = -\frac{7}{(x-4)^2}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad G'(x) &= \lim_{h \rightarrow 0} \frac{G(x+h) - G(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\left(\frac{2(x+h)}{(x+h)^2 - (x+h)} - \frac{2x}{x^2 - x} \right) \cdot \frac{1}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{(2x+2h)(x^2-x) - 2x(x^2+2xh+h^2-x-h)}{(x^2+2hx+h^2-x-h)(x^2-x)} \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-2h^2x - 2hx^2}{(x^2+2hx+h^2-x-h)(x^2-x)} \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{-2hx - 2x^2}{(x^2+2hx+h^2-x-h)(x^2-x)} \\
 &= \frac{-2x^2}{(x^2-x)^2} = -\frac{2}{(x-1)^2}
 \end{aligned}$$

$$\begin{aligned}
 15. \quad F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\left(\frac{6}{(x+h)^2 + 1} - \frac{6}{x^2 + 1} \right) \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{6(x^2+1) - 6(x^2+2hx+h^2+1)}{(x^2+1)(x^2+2hx+h^2+1)} \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-12hx - 6h^2}{(x^2+1)(x^2+2hx+h^2+1)} \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{-12x - 6h}{(x^2+1)(x^2+2hx+h^2+1)} = -\frac{12x}{(x^2+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\left(\frac{x+h-1}{x+h+1} - \frac{x-1}{x+1} \right) \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{x^2 + hx + h - 1 - (x^2 + hx - h - 1)}{(x+h+1)(x+1)} \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{2h}{(x+h+1)(x+1)} \cdot \frac{1}{h} \right] = \frac{2}{(x+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 19. \quad g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt{3x+3h} - \sqrt{3x})(\sqrt{3x+3h} + \sqrt{3x})}{h(\sqrt{3x+3h} + \sqrt{3x})} \\
 &= \lim_{h \rightarrow 0} \frac{3h}{h(\sqrt{3x+3h} + \sqrt{3x})} = \lim_{h \rightarrow 0} \frac{3}{\sqrt{3x+3h} + \sqrt{3x}} = \frac{3}{2\sqrt{3x}}
 \end{aligned}$$

$$\begin{aligned}
 20. \quad g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\left(\frac{1}{\sqrt{3(x+h)}} - \frac{1}{\sqrt{3x}} \right) \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{\sqrt{3x} - \sqrt{3x+3h}}{\sqrt{9x(x+h)}} \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{(\sqrt{3x} - \sqrt{3x+3h})(\sqrt{3x} + \sqrt{3x+3h})}{\sqrt{9x(x+h)}(\sqrt{3x} + \sqrt{3x+3h})} \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{-3h}{h\sqrt{9x(x+h)}(\sqrt{3x} + \sqrt{3x+3h})} = \frac{-3}{3x \cdot 2\sqrt{3x}} = -\frac{1}{2x\sqrt{3x}}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad H'(x) &= \lim_{h \rightarrow 0} \frac{H(x+h) - H(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\left(\frac{3}{\sqrt{x+h-2}} - \frac{3}{\sqrt{x-2}} \right) \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{3\sqrt{x-2} - 3\sqrt{x+h-2}}{\sqrt{(x+h-2)(x-2)}} \cdot \frac{1}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{3(\sqrt{x-2} - \sqrt{x+h-2})(\sqrt{x-2} + \sqrt{x+h-2})}{h\sqrt{(x+h-2)(x-2)}(\sqrt{x-2} + \sqrt{x+h-2})} \\
 &= \lim_{h \rightarrow 0} \frac{-3h}{h[(x-2)\sqrt{x+h-2} + (x+h-2)\sqrt{x-2}]} \\
 &= \lim_{h \rightarrow 0} \frac{-3}{(x-2)\sqrt{x+h-2} + (x+h-2)\sqrt{x-2}} \\
 &= -\frac{3}{2(x-2)\sqrt{x-2}} = -\frac{3}{2(x-2)^{3/2}}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad H'(x) &= \lim_{h \rightarrow 0} \frac{H(x+h) - H(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2 + 4} - \sqrt{x^2 + 4}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt{x^2 + 2hx + h^2 + 4} - \sqrt{x^2 + 4})(\sqrt{x^2 + 2hx + h^2 + 4} + \sqrt{x^2 + 4})}{h(\sqrt{x^2 + 2hx + h^2 + 4} + \sqrt{x^2 + 4})}
 \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{2hx + h^2}{h(\sqrt{x^2 + 2hx + h^2 + 4} + \sqrt{x^2 + 4})} \\
&= \lim_{h \rightarrow 0} \frac{2x + h}{\sqrt{x^2 + 2hx + h^2 + 4} + \sqrt{x^2 + 4}} \\
&= \frac{2x}{2\sqrt{x^2 + 4}} = \frac{x}{\sqrt{x^2 + 4}}
\end{aligned}$$

$$\begin{aligned}
23. \quad f'(x) &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \\
&= \lim_{t \rightarrow x} \frac{(t^2 - 3t) - (x^2 - 3x)}{t - x} \\
&= \lim_{t \rightarrow x} \frac{t^2 - x^2 - (3t - 3x)}{t - x} \\
&= \lim_{t \rightarrow x} \frac{(t - x)(t + x) - 3(t - x)}{t - x} \\
&= \lim_{t \rightarrow x} \frac{(t - x)(t + x - 3)}{t - x} = \lim_{t \rightarrow x} (t + x - 3) \\
&= 2x - 3
\end{aligned}$$

$$\begin{aligned}
24. \quad f'(x) &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \\
&= \lim_{t \rightarrow x} \frac{(t^3 + 5t) - (x^3 + 5x)}{t - x} \\
&= \lim_{t \rightarrow x} \frac{t^3 - x^3 + 5t - 5x}{t - x} \\
&= \lim_{t \rightarrow x} \frac{(t - x)(t^2 + tx + x^2) + 5(t - x)}{t - x} \\
&= \lim_{t \rightarrow x} \frac{(t - x)(t^2 + tx + x^2 + 5)}{t - x} \\
&= \lim_{t \rightarrow x} (t^2 + tx + x^2 + 5) = 3x^2 + 5
\end{aligned}$$

$$\begin{aligned}
25. \quad f'(x) &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \\
&= \lim_{t \rightarrow x} \left[\left(\frac{t}{t-5} - \frac{x}{x-5} \right) \left(\frac{1}{t-x} \right) \right] \\
&= \lim_{t \rightarrow x} \frac{tx - 5t - tx + 5x}{(t-5)(x-5)(t-x)} \\
&= \lim_{t \rightarrow x} \frac{-5(t-x)}{(t-5)(x-5)(t-x)} = \lim_{t \rightarrow x} \frac{-5}{(t-5)(x-5)} \\
&= -\frac{5}{(x-5)^2}
\end{aligned}$$

$$\begin{aligned}
26. \quad f'(x) &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \\
&= \lim_{t \rightarrow x} \left[\left(\frac{t+3}{t} - \frac{x+3}{x} \right) \left(\frac{1}{t-x} \right) \right] \\
&= \lim_{t \rightarrow x} \frac{3x - 3t}{xt(t-x)} = \lim_{t \rightarrow x} \frac{-3}{xt} = -\frac{3}{x^2}
\end{aligned}$$

$$27. \quad f(x) = 2x^3 \text{ at } x = 5$$

$$28. \quad f(x) = x^2 + 2x \text{ at } x = 3$$

$$29. \quad f(x) = x^2 \text{ at } x = 2$$

$$30. \quad f(x) = x^3 + x \text{ at } x = 3$$

$$31. \quad f(x) = x^2 \text{ at } x$$

$$32. \quad f(x) = x^3 \text{ at } x$$

$$33. \quad f(t) = \frac{2}{t} \text{ at } t$$

$$34. \quad f(y) = \sin y \text{ at } y$$

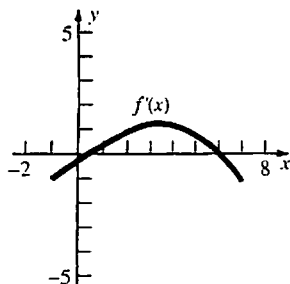
$$35. \quad f(x) = \cos x \text{ at } x$$

$$36. \quad f(t) = \tan t \text{ at } t$$

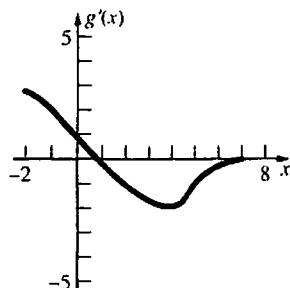
$$\begin{aligned}
37. \quad f'(0) &\approx -\frac{1}{2}; f'(2) \approx 1 \\
f'(5) &\approx \frac{2}{3}; f'(7) \approx -3
\end{aligned}$$

$$\begin{aligned}
38. \quad g'(-1) &\approx 2; g'(1) \approx 0 \\
g'(4) &\approx -2; g'(6) \approx -\frac{1}{3}
\end{aligned}$$

39.



40.



41. a. $f(2) \approx \frac{5}{2}$; $f'(2) \approx \frac{3}{2}$
 $f(0.5) \approx 1.8$; $f'(0.5) \approx -0.6$

b. $\frac{2.9 - 1.9}{2.5 - 0.5} = 0.5$

c. $x = 5$

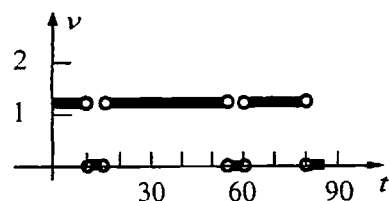
d. $x = 3, 5$

e. $x = 1, 3, 5$

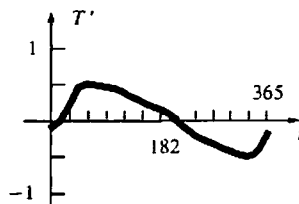
f. $x = 0$

g. $x \approx -0.7, \frac{3}{2}$ and $5 < x < 7$

42. The derivative fails to exist at the corners of the graph; that is, at $t = 10, 15, 55, 60, 80$. The derivative exists at all other points on the interval $(0, 85)$.



43. The derivative is 0 at approximately $t = 15$ and $t = 201$. The greatest rate of increase occurs at about $t = 61$ and it is about 0.5 degree F per day. The greatest rate of decrease occurs at about $t = 320$ and it is about 0.5 degree F per day. The derivative is positive on $(15, 201)$ and negative on $(0, 15)$ and $(201, 365)$.



44. The slope of a tangent line for the dashed function is zero when x is approximately 0.3 or 1.9. The solid function is zero at both of these points. The graph indicates that the solid function is negative when the dashed function has a tangent line with a negative slope and positive when the dashed function has a tangent line with a positive slope. Thus, the solid function is the derivative of the dashed function.

45. The short-dash function has a tangent line with zero slope at about $x = 2.1$, where the solid function is zero. The solid function has a tangent line with zero slope at about $x = 0.4, 1.2$ and 3.5 . The long-dash function is zero at these points. The graph shows that the solid function is positive (negative) when the slope of the tangent line of the short-dash function is positive (negative). Also, the long-dash function is positive (negative) when the slope of the tangent line of the solid function is positive (negative). Thus, the short-dash function is f , the solid function is $f' = g$, and the dash function is g' .

46. Note that since $x = 0 + x$, $f(x) = f(0 + x) = f(0)f(x)$, hence $f(0) = 1$.

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(a)f(h) - f(a)}{h} \\ &= f(a) \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} = f(a) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= f(a)f'(0) \\ f'(a) \text{ exists since } f'(0) \text{ exists.} \end{aligned}$$

47. If f is differentiable everywhere, then it is continuous everywhere, so

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (mx + b) = 2m + b = f(2) = 4$$

and $b = 4 - 2m$.

For f to be differentiable everywhere,

$$f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \text{ must exist.}$$

$$\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2^+} (x + 2) = 4$$

$$\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{mx + b - 4}{x - 2}$$

$$= \lim_{x \rightarrow 2^-} \frac{mx + 4 - 2m - 4}{x - 2} = \lim_{x \rightarrow 2^-} \frac{m(x - 2)}{x - 2} = m$$

Thus $m = 4$ and $b = 4 - 2(4) = -4$

$$\begin{aligned} 48. \quad f'_s(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + f(x) - f(x-h)}{2h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{2h} + \frac{f(x) - f(x-h)}{-2h} \right] \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \frac{1}{2} \lim_{h \rightarrow 0} \frac{f(x) - f(x-h)}{-h} \\ &= \frac{1}{2} f'(x) + \frac{1}{2} f'(x) = f'(x). \end{aligned}$$

For the converse, let $f(x) = |x|$. Then

$$f'_s(0) = \lim_{h \rightarrow 0} \frac{|h| - |-h|}{2h} = \lim_{h \rightarrow 0} \frac{|h| - |h|}{2h} = 0$$

but $f'(0)$ does not exist.

$$49. \quad f'(x_0) = \lim_{t \rightarrow x_0} \frac{f(t) - f(x_0)}{t - x_0}, \text{ so}$$

$$f'(-x_0) = \lim_{t \rightarrow -x_0} \frac{f(t) - f(-x_0)}{t - (-x_0)}$$

$$= \lim_{t \rightarrow -x_0} \frac{f(t) - f(-x_0)}{t + x_0}$$

a. If f is an odd function,

$$f'(-x_0) = \lim_{t \rightarrow -x_0} \frac{f(t) - [-f(-x_0)]}{t + x_0}$$

$$= \lim_{t \rightarrow -x_0} \frac{f(t) + f(-x_0)}{t + x_0}.$$

Let $u = -t$. As $t \rightarrow -x_0$, $u \rightarrow x_0$ and so

$$f'(-x_0) = \lim_{u \rightarrow x_0} \frac{f(-u) + f(x_0)}{-u + x_0}$$

$$= \lim_{u \rightarrow x_0} \frac{-f(u) + f(x_0)}{-(u - x_0)} = \lim_{u \rightarrow x_0} \frac{[f(u) - f(x_0)]}{-(u - x_0)}$$

$$= \lim_{u \rightarrow x_0} \frac{f(u) - f(x_0)}{u - x_0} = f'(x_0) = m.$$

b. If f is an even function,

$$f'(-x_0) = \lim_{t \rightarrow -x_0} \frac{f(t) - f(x_0)}{t + x_0}. \text{ Let } u = -t, \text{ as}$$

$$\text{above, then } f'(-x_0) = \lim_{u \rightarrow x_0} \frac{f(-u) - f(x_0)}{-u + x_0}$$

$$= \lim_{u \rightarrow x_0} \frac{f(u) - f(x_0)}{-(u - x_0)} = - \lim_{u \rightarrow x_0} \frac{f(u) - f(x_0)}{u - x_0}$$

$$= -f'(x_0) = -m.$$

50. Say $f(-x) = -f(x)$. Then

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-f(x-h) + f(x)}{h} = - \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h}$$

$$= \lim_{-h \rightarrow 0} \frac{f[x+(-h)] - f(x)}{-h} = f'(x) \text{ so } f'(x) \text{ is}$$

an even function if $f(x)$ is an odd function.

Say $f(-x) = f(x)$. Then

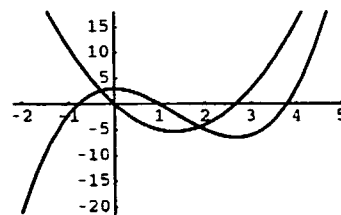
$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h}$$

$$= - \lim_{-h \rightarrow 0} \frac{f[x+(-h)] - f(x)}{-h} = -f'(x) \text{ so } f'(x)$$

is an odd function if $f(x)$ is an even function.

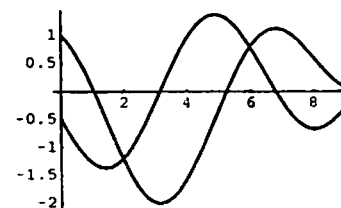
51.



a. $0 < x < \frac{8}{3}$ b. $0 < x < \frac{8}{3}$

c. A function $f(x)$ decreases as x increases when $f'(x) < 0$.

52.



a. $\pi < x < 6.8$ b. $\pi < x < 6.8$

c. A function $f(x)$ increases as x increases when $f'(x) > 0$.

3.3 Concepts Review

1. the derivative of the second; second;
 $f(x)g'(x) + g(x)f'(x)$
2. denominator; denominator; square of the
denominator; $\frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$
3. $nx^{n-1}h$; nx^{n-1}
4. $kL(f)$; $L(f) + L(g)$; D_x

Problem Set 3.3

1. $D_x(2x^2) = 2D_x(x^2) = 2 \cdot 2x = 4x$
2. $D_x(3x^3) = 3D_x(x^3) = 3 \cdot 3x^2 = 9x^2$
3. $D_x(\pi x) = \pi D_x(x) = \pi \cdot 1 = \pi$
4. $D_x(\pi x^3) = \pi D_x(x^3) = \pi \cdot 3x^2 = 3\pi x^2$
5. $D_x(2x^{-2}) = 2D_x(x^{-2}) = 2(-2x^{-3}) = -4x^{-3}$
6. $D_x(-3x^{-4}) = -3D_x(x^{-4}) = -3(-4x^{-5}) = 12x^{-5}$
7. $D_x\left(\frac{\pi}{x}\right) = \pi D_x(x^{-1}) = \pi(-1x^{-2}) = -\pi x^{-2}$
 $= -\frac{\pi}{x^2}$
8. $D_x\left(\frac{\alpha}{x^3}\right) = \alpha D_x(x^{-3}) = \alpha(-3x^{-4}) = -3\alpha x^{-4}$
 $= -\frac{3\alpha}{x^4}$
9. $D_x\left(\frac{100}{x^5}\right) = 100D_x(x^{-5}) = 100(-5x^{-6})$
 $= -500x^{-6} = -\frac{500}{x^6}$
10. $D_x\left(\frac{3\alpha}{4x^5}\right) = \frac{3\alpha}{4} D_x(x^{-5}) = \frac{3\alpha}{4}(-5x^{-6})$
 $= -\frac{15\alpha}{4}x^{-6} = -\frac{15\alpha}{4x^6}$
11. $D_x(x^2 + 2x) = D_x(x^2) + 2D_x(x) = 2x + 2$
12. $D_x(3x^4 + x^3) = 3D_x(x^4) + D_x(x^3)$
 $= 3(4x^3) + 3x^2 = 12x^3 + 3x^2$
13. $D_x(x^4 + x^3 + x^2 + x + 1)$
 $= D_x(x^4) + D_x(x^3) + D_x(x^2) + D_x(x) + D_x(1)$
 $= 4x^3 + 3x^2 + 2x + 1$
14. $D_x(3x^4 - 2x^3 - 5x^2 + \pi x + \pi^2)$
 $= 3D_x(x^4) - 2D_x(x^3) - 5D_x(x^2)$
 $+ \pi D_x(x) + D_x(\pi^2)$
 $= 3(4x^3) - 2(3x^2) - 5(2x) + \pi(1) + 0$
 $= 12x^3 - 6x^2 - 10x + \pi$
15. $D_x(\pi x^7 - 2x^5 - 5x^{-2})$
 $= \pi D_x(x^7) - 2D_x(x^5) - 5D_x(x^{-2})$
 $= \pi(7x^6) - 2(5x^4) - 5(-2x^{-3})$
 $= 7\pi x^6 - 10x^4 + 10x^{-3}$
16. $D_x(x^{12} + 5x^{-2} - \pi x^{-10})$
 $= D_x(x^{12}) + 5D_x(x^{-2}) - \pi D_x(x^{-10})$
 $= 12x^{11} + 5(-2x^{-3}) - \pi(-10x^{-11})$
 $= 12x^{11} - 10x^{-3} + 10\pi x^{-11}$
17. $D_x\left(\frac{3}{x^3} + x^{-4}\right) = 3D_x(x^{-3}) + D_x(x^{-4})$
 $= 3(-3x^{-4}) + (-4x^{-5}) = -\frac{9}{x^4} - 4x^{-5}$
18. $D_x(2x^{-6} + x^{-1}) = 2D_x(x^{-6}) + D_x(x^{-1})$
 $= 2(-6x^{-7}) + (-1x^{-2}) = -12x^{-7} - x^{-2}$
19. $D_x\left(\frac{2}{x} - \frac{1}{x^2}\right) = 2D_x(x^{-1}) - D_x(x^{-2})$
 $= 2(-1x^{-2}) - (-2x^{-3}) = -\frac{2}{x^2} + \frac{2}{x^3}$
20. $D_x\left(\frac{3}{x^3} - \frac{1}{x^4}\right) = 3D_x(x^{-3}) - D_x(x^{-4})$
 $= 3(-3x^{-4}) - (-4x^{-5}) = -\frac{9}{x^4} + \frac{4}{x^5}$
21. $D_x\left(\frac{1}{2x} + 2x\right) = \frac{1}{2}D_x(x^{-1}) + 2D_x(x)$
 $= \frac{1}{2}(-1x^{-2}) + 2(1) = -\frac{1}{2x^2} + 2$

$$22. D_x \left(\frac{2}{3x} - \frac{2}{3} \right) = \frac{2}{3} D_x(x^{-1}) - D_x \left(\frac{2}{3} \right) \\ = \frac{2}{3} (-1x^{-2}) - 0 = -\frac{2}{3x^2}$$

$$23. D_x[x(x^2+1)] = x D_x(x^2+1) + (x^2+1) D_x(x) \\ = x(2x) + (x^2+1)(1) = 3x^2 + 1$$

$$24. D_x[3x(x^3-1)] = 3x D_x(x^3-1) + (x^3-1) D_x(3x) \\ = 3x(3x^2) + (x^3-1)(3) = 12x^3 - 3$$

$$25. D_x[(2x+1)^2] \\ = (2x+1) D_x(2x+1) + (2x+1) D_x(2x+1) \\ = (2x+1)(2) + (2x+1)(2) = 8x + 4$$

$$26. D_x[(-3x+2)^2] \\ = (-3x+2) D_x(-3x+2) + (-3x+2) D_x(-3x+2) \\ = (-3x+2)(-3) + (-3x+2)(-3) = 18x - 12$$

$$29. D_x[(x^2+17)(x^3-3x+1)] \\ = (x^2+17) D_x(x^3-3x+1) + (x^3-3x+1) D_x(x^2+17) \\ = (x^2+17)(3x^2-3) + (x^3-3x+1)(2x) \\ = 3x^4 + 48x^2 - 51 + 2x^4 - 6x^2 + 2x \\ = 5x^4 + 42x^2 + 2x - 51$$

$$30. D_x[(x^4+2x)(x^3+2x^2+1)] = (x^4+2x) D_x(x^3+2x^2+1) + (x^3+2x^2+1) D_x(x^4+2x) \\ = (x^4+2x)(3x^2+4x) + (x^3+2x^2+1)(4x^3+2) \\ = 7x^6 + 12x^5 + 12x^3 + 12x^2 + 2$$

$$31. D_x[(5x^2-7)(3x^2-2x+1)] = (5x^2-7) D_x(3x^2-2x+1) + (3x^2-2x+1) D_x(5x^2-7) \\ = (5x^2-7)(6x-2) + (3x^2-2x+1)(10x) \\ = 60x^3 - 30x^2 - 32x + 14$$

$$32. D_x[(3x^2+2x)(x^4-3x+1)] = (3x^2+2x) D_x(x^4-3x+1) + (x^4-3x+1) D_x(3x^2+2x) \\ = (3x^2+2x)(4x^3-3) + (x^4-3x+1)(6x+2) \\ = 18x^5 + 10x^4 - 27x^2 - 6x + 2$$

$$33. D_x \left(\frac{1}{3x^2+1} \right) = \frac{(3x^2+1) D_x(1) - (1) D_x(3x^2+1)}{(3x^2+1)^2} \\ = \frac{(3x^2+1)(0) - (6x)}{(3x^2+1)^2} = -\frac{6x}{(3x^2+1)^2}$$

$$27. D_x[(x^2+2)(x^3+1)] \\ = (x^2+2) D_x(x^3+1) + (x^3+1) D_x(x^2+2) \\ = (x^2+2)(3x^2) + (x^3+1)(2x) \\ = 3x^4 + 6x^2 + 2x^4 + 2x \\ = 5x^4 + 6x^2 + 2x$$

$$28. D_x[(x^4-1)(x^2+1)] \\ = (x^4-1) D_x(x^2+1) + (x^2+1) D_x(x^4-1) \\ = (x^4-1)(2x) + (x^2+1)(4x^3) \\ = 2x^5 - 2x + 4x^5 + 4x^3 = 6x^5 + 4x^3 - 2x$$

34. $D_x \left(\frac{2}{5x^2 - 1} \right) = \frac{(5x^2 - 1)D_x(2) - (2)D_x(5x^2 - 1)}{(5x^2 - 1)^2}$
 $= \frac{(5x^2 - 1)(0) - 2(10x)}{(5x^2 - 1)^2} = -\frac{20x}{(5x^2 - 1)^2}$
35. $D_x \left(\frac{1}{4x^2 - 3x + 9} \right) = \frac{(4x^2 - 3x + 9)D_x(1) - (1)D_x(4x^2 - 3x + 9)}{(4x^2 - 3x + 9)^2}$
 $= \frac{(4x^2 - 3x + 9)(0) - (8x - 3)}{(4x^2 - 3x + 9)^2} = -\frac{8x - 3}{(4x^2 - 3x + 9)^2}$
 $= \frac{-8x + 3}{(4x^2 - 3x + 9)^2}$
36. $D_x \left(\frac{4}{2x^3 - 3x} \right) = \frac{(2x^3 - 3x)D_x(4) - (4)D_x(2x^3 - 3x)}{(2x^3 - 3x)^2}$
 $= \frac{(2x^3 - 3x)(0) - 4(6x^2 - 3)}{(2x^3 - 3x)^2} = \frac{-24x^2 + 12}{(2x^3 - 3x)^2}$
37. $D_x \left(\frac{x-1}{x+1} \right) = \frac{(x+1)D_x(x-1) - (x-1)D_x(x+1)}{(x+1)^2}$
 $= \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$
38. $D_x \left(\frac{2x-1}{x-1} \right) = \frac{(x-1)D_x(2x-1) - (2x-1)D_x(x-1)}{(x-1)^2}$
 $= \frac{(x-1)(2) - (2x-1)(1)}{(x-1)^2} = -\frac{1}{(x-1)^2}$
39. $D_x \left(\frac{2x^2 - 1}{3x + 5} \right) = \frac{(3x + 5)D_x(2x^2 - 1) - (2x^2 - 1)D_x(3x + 5)}{(3x + 5)^2}$
 $= \frac{(3x + 5)(4x) - (2x^2 - 1)(3)}{(3x + 5)^2}$
 $= \frac{6x^2 + 20x + 3}{(3x + 5)^2}$
40. $D_x \left(\frac{5x - 4}{3x^2 + 1} \right) = \frac{(3x^2 + 1)D_x(5x - 4) - (5x - 4)D_x(3x^2 + 1)}{(3x^2 + 1)^2}$
 $= \frac{(3x^2 + 1)(5) - (5x - 4)(6x)}{(3x^2 + 1)^2}$
 $= \frac{-15x^2 + 24x + 5}{(3x^2 + 1)^2}$

$$\begin{aligned}
 41. \quad D_x \left(\frac{2x^2 - 3x + 1}{2x + 1} \right) &= \frac{(2x + 1)D_x(2x^2 - 3x + 1) - (2x^2 - 3x + 1)D_x(2x + 1)}{(2x + 1)^2} \\
 &= \frac{(2x + 1)(4x - 3) - (2x^2 - 3x + 1)(2)}{(2x + 1)^2} \\
 &= \frac{4x^2 + 4x - 5}{(2x + 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 42. \quad D_x \left(\frac{5x^2 + 2x - 6}{3x - 1} \right) &= \frac{(3x - 1)D_x(5x^2 + 2x - 6) - (5x^2 + 2x - 6)D_x(3x - 1)}{(3x - 1)^2} \\
 &= \frac{(3x - 1)(10x + 2) - (5x^2 + 2x - 6)(3)}{(3x - 1)^2} \\
 &= \frac{15x^2 - 10x + 16}{(3x - 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 43. \quad D_x \left(\frac{x^2 - x + 1}{x^2 + 1} \right) &= \frac{(x^2 + 1)D_x(x^2 - x + 1) - (x^2 - x + 1)D_x(x^2 + 1)}{(x^2 + 1)^2} \\
 &= \frac{(x^2 + 1)(2x - 1) - (x^2 - x + 1)(2x)}{(x^2 + 1)^2} \\
 &= \frac{x^2 - 1}{(x^2 + 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 44. \quad D_x \left(\frac{x^2 - 2x + 5}{x^2 + 2x - 3} \right) &= \frac{(x^2 + 2x - 3)D_x(x^2 - 2x + 5) - (x^2 - 2x + 5)D_x(x^2 + 2x - 3)}{(x^2 + 2x - 3)^2} \\
 &= \frac{(x^2 + 2x - 3)(2x - 2) - (x^2 - 2x + 5)(2x + 2)}{(x^2 + 2x - 3)^2} \\
 &= \frac{4x^2 - 16x - 4}{(x^2 + 2x - 3)^2}
 \end{aligned}$$

$$\begin{aligned}
 45. \quad \text{a.} \quad (f \cdot g)'(0) &= f(0)g'(0) + g(0)f'(0) \\
 &= 4(5) + (-3)(-1) = 23
 \end{aligned}$$

$$\text{b.} \quad (f + g)'(0) = f'(0) + g'(0) = -1 + 5 = 4$$

$$\begin{aligned}
 \text{c.} \quad (f/g)'(0) &= \frac{g(0)f'(0) - f(0)g'(0)}{g^2(0)} \\
 &= \frac{-3(-1) - 4(5)}{(-3)^2} = -\frac{17}{9}
 \end{aligned}$$

$$46. \quad \text{a.} \quad (f - g)'(3) = f'(3) - g'(3) = 2 - (-10) = 12$$

$$\begin{aligned}
 \text{b.} \quad (f \cdot g)'(3) &= f(3)g'(3) + g(3)f'(3) \\
 &= 7(-10) + 6(2) = -58
 \end{aligned}$$

$$\begin{aligned}
 \text{c.} \quad (g/f)'(3) &= \frac{f(3)g'(3) - g(3)f'(3)}{f^2(3)} \\
 &= \frac{7(-10) - 6(2)}{(7)^2} = -\frac{82}{49}
 \end{aligned}$$

$$\begin{aligned}
 47. \quad D_x[f(x)]^2 &= D_x[f(x)f(x)] \\
 &= f(x)D_x[f(x)] + f(x)D_x[f(x)] \\
 &= 2 \cdot f(x) \cdot D_x f(x)
 \end{aligned}$$

$$\begin{aligned}
 48. \quad D_x[f(x)g(x)h(x)] &= f(x)D_x[g(x)h(x)] + g(x)h(x)D_x f(x) \\
 &= f(x)[g(x)D_x h(x) + h(x)D_x g(x)] + g(x)h(x)D_x f(x) \\
 &= f(x)g(x)D_x h(x) + f(x)h(x)D_x g(x) + g(x)h(x)D_x f(x)
 \end{aligned}$$

$$\begin{aligned}
 49. \quad D_x(x^2 - 2x + 2) &= 2x - 2 \\
 \text{At } x = 1: m_{\tan} &= 2(1) - 2 = 0 \\
 \text{Tangent line: } y &= 1
 \end{aligned}$$

$$\begin{aligned}
 50. \quad D_x\left(\frac{1}{x^2 + 4}\right) &= \frac{(x^2 + 4)D_x(1) - (1)D_x(x^2 + 4)}{(x^2 + 4)^2} \\
 &= \frac{(x^2 + 4)(0) - (2x)}{(x^2 + 4)^2} = -\frac{2x}{(x^2 + 4)^2} \\
 \text{At } x = 1: m_{\tan} &= -\frac{2(1)}{(1^2 + 4)^2} = -\frac{2}{25} \\
 \text{Tangent line: } y - \frac{1}{5} &= -\frac{2}{25}(x - 1) \\
 y &= -\frac{2}{25}x + \frac{7}{25}
 \end{aligned}$$

$$\begin{aligned}
 51. \quad D_x(x^3 - x^2) &= 3x^2 - 2x \\
 \text{The tangent line is horizontal when } m_{\tan} &= 0: \\
 m_{\tan} = 3x^2 - 2x &= 0 \\
 x(3x - 2) &= 0 \\
 x = 0 \text{ and } x &= \frac{2}{3} \\
 (0, 0) \text{ and } \left(\frac{2}{3}, -\frac{4}{27}\right)
 \end{aligned}$$

$$\begin{aligned}
 52. \quad D_x\left(\frac{1}{3}x^3 + x^2 - x\right) &= x^2 + 2x - 1 \\
 m_{\tan} = x^2 + 2x - 1 &= 1 \\
 x^2 + 2x - 2 &= 0 \\
 x = \frac{-2 \pm \sqrt{4 - 4(1)(-2)}}{2} &= \frac{-2 \pm \sqrt{12}}{2} \\
 &= -1 - \sqrt{3}, -1 + \sqrt{3} \\
 x &= -1 \pm \sqrt{3} \\
 \left(-1 + \sqrt{3}, \frac{5}{3} - \sqrt{3}\right), \left(-1 - \sqrt{3}, \frac{5}{3} + \sqrt{3}\right)
 \end{aligned}$$

$$\begin{aligned}
 53. \quad y &= 100/x^5 = 100x^{-5} \\
 y' &= -500x^{-6}
 \end{aligned}$$

Set y' equal to -1 , the negative reciprocal of the slope of the line $y = x$. Solving for x gives

$$\begin{aligned}
 x &= \pm 500^{1/6} \approx \pm 2.817 \\
 y &= \pm 100(500)^{-5/6} \approx \pm 0.563
 \end{aligned}$$

The points are $(2.817, 0.563)$ and $(-2.817, -0.563)$.

54. Proof #1:

$$\begin{aligned}
 D_x[f(x) - g(x)] &= D_x[f(x) + (-1)g(x)] \\
 &= D_x[f(x)] + D_x[(-1)g(x)] \\
 &= D_x f(x) - D_x g(x)
 \end{aligned}$$

Proof #2:

Let $F(x) = f(x) - g(x)$. Then

$$\begin{aligned}
 F'(x) &= \lim_{h \rightarrow 0} \frac{[f(x+h) - g(x+h)] - [f(x) - g(x)]}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h} \right] \\
 &= f'(x) - g'(x)
 \end{aligned}$$

$$\begin{aligned}
 55. \quad \text{a. } D_t(-16t^2 + 40t + 100) &= -32t + 40 \\
 v &= -32(2) + 40 = -24 \text{ ft/s}
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } v &= -32t + 40 = 0 \\
 t &= \frac{5}{4} \text{ s}
 \end{aligned}$$

$$\begin{aligned}
 56. \quad D_t(4.5t^2 + 2t) &= 9t + 2 \\
 9t + 2 &= 30 \\
 t &= \frac{28}{9} \text{ s}
 \end{aligned}$$

57. $m_{\tan} = D_x(4x - x^2) = 4 - 2x$
The line through $(2, 5)$ and (x_0, y_0) has slope

$$\frac{y_0 - 5}{x_0 - 2}$$

$$4 - 2x_0 = \frac{4x_0 - x_0^2 - 5}{x_0 - 2}$$

$$-2x_0^2 + 8x_0 - 8 = -x_0^2 + 4x_0 - 5$$

$$x_0^2 - 4x_0 + 3 = 0$$

$$(x_0 - 3)(x_0 - 1) = 0$$

$$x_0 = 1, x_0 = 3$$

$$\text{At } x_0 = 1: y_0 = 4(1) - (1)^2 = 3$$

$$m_{\tan} = 4 - 2(1) = 2$$

$$\text{Tangent line: } y - 3 = 2(x - 1); y = 2x + 1$$

$$\text{At } x_0 = 3: y_0 = 4(3) - (3)^2 = 3$$

$$m_{\tan} = 4 - 2(3) = -2$$

$$\text{Tangent line: } y - 3 = -2(x - 3); y = -2x + 9$$

58. $D_x(x^2) = 2x$

The line through $(4, 15)$ and (x_0, y_0) has slope

$$\frac{y_0 - 15}{x_0 - 4}. \text{ If } (x_0, y_0) \text{ is on the curve } y = x^2, \text{ then}$$

$$m_{\tan} = 2x_0 = \frac{x_0^2 - 15}{x_0 - 4}$$

$$2x_0^2 - 8x_0 = x_0^2 - 15$$

$$x_0^2 - 8x_0 + 15 = 0$$

$$(x_0 - 3)(x_0 - 5) = 0$$

$$\text{At } x_0 = 3: y_0 = (3)^2 = 9$$

She should shut off the engines at $(3, 9)$. (At $x_0 = 5$ she would not go to $(4, 15)$ since she is moving left to right.)

59. $D_x(7 - x^2) = -2x$

The line through $(4, 0)$ and (x_0, y_0) has

$$\text{slope } \frac{y_0 - 0}{x_0 - 4}. \text{ If the fly is at } (x_0, y_0) \text{ when the}$$

$$\text{spider sees it, then } m_{\tan} = -2x_0 = \frac{7 - x_0^2 - 0}{x_0 - 4}.$$

$$-2x_0^2 + 8x_0 = 7 - x_0^2$$

$$x_0^2 - 8x_0 + 7 = 0$$

$$(x_0 - 7)(x_0 - 1) = 0$$

$$\text{At } x_0 = 1: y_0 = 6$$

$$d = \sqrt{(4-1)^2 + (0-6)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5} \approx 6.7$$

They are 6.7 units apart when they see each other.

60. $P(a, b)$ is $\left(a, \frac{1}{a}\right)$. $D_x y = -\frac{1}{x^2}$ so the slope of

the tangent line at P is $-\frac{1}{a^2}$. The tangent line is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \text{ or } y = -\frac{1}{a^2}(x - 2a) \text{ which}$$

has x -intercept $(2a, 0)$.

$$d(O, P) = \sqrt{a^2 + \frac{1}{a^2}}, d(P, A) = \sqrt{(a - 2a)^2 + \frac{1}{a^2}}$$

$$= \sqrt{a^2 + \frac{1}{a^2}} = d(O, P) \text{ so } AOP \text{ is an isosceles}$$

triangle. The height of AOP is a while the base,

$$\overline{OA} \text{ has length } 2a, \text{ so the area is } \frac{1}{2}(2a)(a) = a^2.$$

61. The watermelon has volume $\frac{4}{3}\pi r^3$; the volume of the rind is

$$V = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi \left(r - \frac{r}{10}\right)^3 = \frac{271}{750}\pi r^3.$$

At the end of the fifth week $r = 10$, so

$$D_r V = \frac{271}{250}\pi r^2 = \frac{271}{250}\pi(10)^2 = \frac{542\pi}{5} \approx 340 \text{ cm}^3$$

per cm of radius growth. Since the radius is growing 2 cm per week, the volume of the rind is

$$\text{growing at the rate of } \frac{542\pi}{5}(2) \approx 681 \text{ cm}^3 \text{ per}$$

week.

3.4 Concepts Review

1. $\frac{\sin(x+h) - \sin(x)}{h}$

2. 0; 1

3. $\cos x; -\sin x$

4. $\cos \frac{\pi}{3} = \frac{1}{2}; y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$

Problem Set 3.4

1. $D_x(2 \sin x + 3 \cos x) = 2 D_x(\sin x) + 3 D_x(\cos x)$
 $= 2 \cos x - 3 \sin x$
2. $D_x(\sin^2 x) = \sin x D_x(\sin x) + \sin x D_x(\sin x)$
 $= \sin x \cos x + \sin x \cos x = 2 \sin x \cos x = \sin 2x$
3. $D_x(\sin^2 x + \cos^2 x) = D_x(1) = 0$
4. $D_x(1 - \cos^2 x) = D_x(\sin^2 x)$
 $= \sin x D_x(\sin x) + \sin x D_x(\sin x)$
 $= \sin x \cos x + \sin x \cos x$
 $= 2 \sin x \cos x = \sin 2x$
5. $D_x(\sec x) = D_x\left(\frac{1}{\cos x}\right)$
 $= \frac{\cos x D_x(1) - (1) D_x(\cos x)}{\cos^2 x}$
 $= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$
6. $D_x(\csc x) = D_x\left(\frac{1}{\sin x}\right)$
 $= \frac{\sin x D_x(1) - (1) D_x(\sin x)}{\sin^2 x}$
 $= \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$
7. $D_x(\tan x) = D_x\left(\frac{\sin x}{\cos x}\right)$
 $= \frac{\cos x D_x(\sin x) - \sin x D_x(\cos x)}{\cos^2 x}$
 $= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
8. $D_x(\cot x) = D_x\left(\frac{\cos x}{\sin x}\right)$
 $= \frac{\sin x D_x(\cos x) - \cos x D_x(\sin x)}{\sin^2 x}$
 $= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$
 $= -\frac{1}{\sin^2 x} = -\csc^2 x$
9. $D_x\left(\frac{\sin x + \cos x}{\cos x}\right)$
 $= \frac{\cos x D_x(\sin x + \cos x) - (\sin x + \cos x) D_x(\cos x)}{\cos^2 x}$
 $= \frac{\cos x(\cos x - \sin x) - (-\sin^2 x - \sin x \cos x)}{\cos^2 x}$
 $= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
10. $D_x\left(\frac{\sin x + \cos x}{\tan x}\right)$
 $= \frac{\tan x D_x(\sin x + \cos x) - (\sin x + \cos x) D_x(\tan x)}{\tan^2 x}$
 $= \frac{\tan x(\cos x - \sin x) - \sec^2 x(\sin x + \cos x)}{\tan^2 x}$
 $= \left(\sin x - \frac{\sin^2 x}{\cos x} - \frac{\sin x}{\cos^2 x} - \frac{1}{\cos x}\right) \div \left(\frac{\sin^2 x}{\cos^2 x}\right)$
 $= \left(\sin x - \frac{\sin^2 x}{\cos x} - \frac{\sin x}{\cos^2 x} - \frac{1}{\cos x}\right) \left(\frac{\cos^2 x}{\sin^2 x}\right)$
 $= \frac{\cos^2 x}{\sin x} - \cos x - \frac{1}{\sin x} - \frac{\cos x}{\sin^2 x}$
11. $D_x(x^2 \cos x) = x^2 D_x(\cos x) + \cos x D_x(x^2)$
 $= -x^2 \sin x + 2x \cos x$
12. $D_x\left(\frac{x \cos x + \sin x}{x^2 + 1}\right)$
 $= \frac{(x^2 + 1) D_x(x \cos x + \sin x) - (x \cos x + \sin x) D_x(x^2 + 1)}{(x^2 + 1)^2}$
 $= \frac{(x^2 + 1)(-x \sin x + \cos x + \cos x) - 2x(x \cos x + \sin x)}{(x^2 + 1)^2} = \frac{-x^3 \sin x - 3x \sin x + 2 \cos x}{(x^2 + 1)^2}$
13. $y = \tan^2 x = (\tan x)(\tan x)$
 $y' = (\tan x)(\sec^2 x) + (\tan x)(\sec^2 x)$
 $= 2 \tan x \sec^2 x$

$$14. y = \sec^3 x = (\sec^2 x)(\sec x)$$

$$\begin{aligned} y' &= (\sec^2 x) \sec x \tan x + (\sec x)(\sec^2 x)' \\ &= \sec^3 x \tan x + \sec x (\sec x \sec x \tan x \\ &\quad + \sec x \sec x \tan x) \\ &= \sec^3 x \tan x + 2 \sec^3 x \tan x \\ &= 3 \sec^3 x \tan x \end{aligned}$$

$$15. D_x(\cos x) = -\sin x$$

$$\text{At } x = 1: m_{\tan} = -\sin 1 \approx -0.8415$$

$$y = \cos 1 \approx 0.5403$$

$$\text{Tangent line: } y - 0.5403 = -0.8415(x - 1)$$

$$16. D_x(\cot x) = -\csc^2 x$$

$$\text{At } x = \frac{\pi}{4}: m_{\tan} = -2;$$

$$y = 1$$

$$\text{Tangent line: } y - 1 = -2\left(x - \frac{\pi}{4}\right)$$

$$17. D_t(30 \cos 2t) = 30 D_t(\cos^2 t - \sin^2 t)$$

$$= 30[\cos t D_t(\cos t) + \cos t D_t(\cos t) - D_t(\sin^2 t)]$$

$$= 30[-2 \sin t \cos t - \sin t D_t(\sin t) - \sin t D_t(\sin t)]$$

$$= 30[-2 \sin t \cos t - 2 \sin t \cos t]$$

$$= -60 \sin 2t$$

$$\text{At } t = \frac{\pi}{4}: -60 \sin\left(2 \cdot \frac{\pi}{4}\right) = -60 \text{ ft/s}$$

The seat is moving to the left at the rate of 60 ft/s.

$$18. \text{ The coordinates of the seat at time } t \text{ are } (20 \cos t, 20 \sin t).$$

$$\begin{aligned} \text{a. } \left(20 \cos \frac{\pi}{6}, 20 \sin \frac{\pi}{6}\right) &= (10\sqrt{3}, 10) \\ &\approx (17.32, 10) \end{aligned}$$

$$\text{b. } D_t(20 \sin t) = 20 \cos t$$

$$\text{At } t = \frac{\pi}{6}: \text{rate} = 20 \cos \frac{\pi}{6} = 10\sqrt{3}$$

$$\approx 17.32 \text{ ft/s}$$

$$\text{c. The fastest rate } 20 \cos t \text{ can obtain is } 20 \text{ ft/s.}$$

$$19. y = \tan x$$

$$y' = \sec^2 x$$

$$\text{When } y = 0, y = \tan 0 = 0 \text{ and } y' = \sec^2 0 = 1.$$

$$\text{The tangent line at } x = 0 \text{ is } y = x.$$

$$20. y = \tan^2 x = (\tan x)(\tan x)$$

$$y' = (\tan x)(\sec^2 x) + (\tan x)(\sec^2 x)$$

$$= 2 \tan x \sec^2 x$$

Now, $\sec^2 x$ is never 0, but $\tan x = 0$ at $x = k\pi$ where k is an integer.

$$21. y = 9 \sin x \cos x$$

$$y' = 9[\sin x(-\sin x) + \cos x(\cos x)]$$

$$= 9[\sin^2 x - \cos^2 x]$$

$$= 9[-\cos 2x]$$

The tangent line is horizontal when $y' = 0$ or, in this case, where $\cos 2x = 0$. This occurs when

$$x = \frac{\pi}{4} + k\frac{\pi}{2} \text{ where } k \text{ is an integer.}$$

$$22. f(x) = x - \sin x$$

$$f'(x) = 1 - \cos x$$

$$f'(x) = 0 \text{ when } \cos x = 1; \text{ i.e. when } x = 2k\pi \text{ where } k \text{ is an integer.}$$

$$f'(x) = 2 \text{ when } x = (2k+1)\pi \text{ where } k \text{ is an integer.}$$

$$23. \text{ The curves intersect when } \sqrt{2} \sin x = \sqrt{2} \cos x, \sin x = \cos x \text{ at } x = \frac{\pi}{4} \text{ for } 0 < x < \frac{\pi}{2}.$$

$$D_x(\sqrt{2} \sin x) = \sqrt{2} \cos x; \sqrt{2} \cos \frac{\pi}{4} = 1$$

$$D_x(\sqrt{2} \cos x) = -\sqrt{2} \sin x; -\sqrt{2} \sin \frac{\pi}{4} = -1$$

$$1(-1) = -1 \text{ so the curves intersect at right angles.}$$

$$24. v = D_t(2 \sin t) = 2 \cos t$$

$$\text{At } t = 0: v = 2 \text{ cm/s}$$

$$t = \frac{\pi}{2}: v = 0 \text{ cm/s}$$

$$t = \pi: v = -2 \text{ cm/s}$$

$$\begin{aligned}
25. D_x(\sin x^2) &= \lim_{h \rightarrow 0} \frac{\sin(x+h)^2 - \sin x^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(x^2 + 2xh + h^2) - \sin x^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin x^2 \cos(2xh + h^2) + \cos x^2 \sin(2xh + h^2) - \sin x^2}{h} = \lim_{h \rightarrow 0} \frac{\sin x^2 [\cos(2xh + h^2) - 1] + \cos x^2 \sin(2xh + h^2)}{h} \\
&= \lim_{h \rightarrow 0} (2x + h) \left[\sin x^2 \frac{\cos(2xh + h^2) - 1}{2xh + h^2} + \cos x^2 \frac{\sin(2xh + h^2)}{2xh + h^2} \right] = 2x(\sin x^2 \cdot 0 + \cos x^2 \cdot 1) = 2x \cos x^2
\end{aligned}$$

$$\begin{aligned}
26. D_x(\sin 5x) &= \lim_{h \rightarrow 0} \frac{\sin(5(x+h)) - \sin 5x}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(5x + 5h) - \sin 5x}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin 5x \cos 5h + \cos 5x \sin 5h - \sin 5x}{h} \\
&= \lim_{h \rightarrow 0} \left[\sin 5x \frac{\cos 5h - 1}{h} + \cos 5x \frac{\sin 5h}{h} \right] \\
&= \lim_{h \rightarrow 0} \left[5 \sin 5x \frac{\cos 5h - 1}{5h} + 5 \cos 5x \frac{\sin 5h}{5h} \right] \\
&= 0 + 5 \cos 5x \cdot 1 = 5 \cos 5x
\end{aligned}$$

$$27. \sin x_0 = \sin 2x_0$$

$$\sin x_0 = 2 \sin x_0 \cos x_0$$

$$\cos x_0 = \frac{1}{2} \text{ [if } \sin x_0 \neq 0 \text{]}$$

$$x_0 = \frac{\pi}{3}$$

$D_x(\sin x) = \cos x$, $D_x(\sin 2x) = 2 \cos 2x$, so at x_0 , the tangent lines to $y = \sin x$ and $y = \sin 2x$ have

$$\text{slopes of } m_1 = \frac{1}{2} \text{ and } m_2 = 2 \left(-\frac{1}{2} \right) = -1,$$

respectively. From Problem 40 of Section 2.3,

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2} \text{ where } \theta \text{ is the angle between}$$

$$\text{the tangent lines. } \tan \theta = \frac{-1 - \frac{1}{2}}{1 + \left(\frac{1}{2}\right)(-1)} = \frac{-\frac{3}{2}}{\frac{1}{2}} = -3,$$

so $\theta \approx -1.25$. The curves intersect at an angle of 1.25 radians.

$$28. \frac{1}{2} \overline{AB} = \overline{OA} \sin \frac{t}{2}$$

$$D = \frac{1}{2} \overline{OA} \cos \frac{t}{2} \cdot \overline{AB} = \overline{OA}^2 \cos \frac{t}{2} \sin \frac{t}{2}$$

$$E = D + \text{area (semi-circle)}$$

$$= \overline{OA}^2 \cos \frac{t}{2} \sin \frac{t}{2} + \frac{1}{2} \pi \left(\frac{1}{2} \overline{AB} \right)^2$$

$$= \overline{OA}^2 \cos \frac{t}{2} \sin \frac{t}{2} + \frac{1}{2} \pi \overline{OA}^2 \sin^2 \frac{t}{2}$$

$$= \overline{OA}^2 \sin \frac{t}{2} \left(\cos \frac{t}{2} + \frac{1}{2} \pi \sin \frac{t}{2} \right)$$

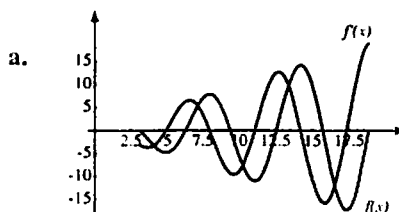
$$\frac{D}{E} = \frac{\cos \frac{t}{2}}{\cos \frac{t}{2} + \frac{1}{2} \pi \sin \frac{t}{2}}$$

$$\lim_{t \rightarrow 0^+} \frac{D}{E} = \frac{1}{1+0} = 1$$

$$\lim_{t \rightarrow \pi^-} \frac{D}{E} = \lim_{t \rightarrow \pi^-} \frac{\cos(t/2)}{\cos(t/2) + \frac{\pi}{2} \sin(t/2)}$$

$$= \frac{0}{0 + \frac{\pi}{2}} = 0$$

$$29. f(x) = x \sin x$$



b. $f(x) = 0$ has 6 solutions on $[\pi, 6\pi]$
 $f'(x) = 0$ has 5 solutions on $[\pi, 6\pi]$

c. $f(x) = x \sin x$ is a counterexample.

d. The maximum value of $|f(x) - f'(x)|$ on $[\pi, 6\pi]$ is about 24.93.

$$30. x_0 \approx 1.95$$

$$f'(x_0) \approx -1.24$$

3.5 Concepts Review

1. $D_t u; f'(g(t))g'(t)$
2. $D_v w; G'(H(s))H'(s)$
3. $(f(x))^2; (f(x))^2$
4. $2x \cos(x^2); 6(2x+1)^2$

Problem Set 3.5

1. $y = u^{15}$ and $u = 1 + x$
 $D_x y = D_u y \cdot D_x u$
 $= (15u^{14})(1)$
 $= 15(1+x)^{14}$
2. $y = u^5$ and $u = 7 + x$
 $D_x y = D_u y \cdot D_x u$
 $= (5u^4)(1)$
 $= 5(7+x)^4$
3. $y = u^5$ and $u = 3 - 2x$
 $D_x y = D_u y \cdot D_x u$
 $= (5u^4)(-2) = -10(3-2x)^4$
4. $y = u^7$ and $u = 4 + 2x^2$
 $D_x y = D_u y \cdot D_x u$
 $= (7u^6)(4x) = 28x(4+2x^2)^6$
5. $y = u^{11}$ and $u = x^3 - 2x^2 + 3x + 1$
 $D_x y = D_u y \cdot D_x u$
 $= (11u^{10})(3x^2 - 4x + 3)$
 $= 11(3x^2 - 4x + 3)(x^3 - 2x^2 + 3x + 1)^{10}$
6. $y = u^{101}$ and $u = x^5 - 5x^3 + \pi x + 1$
 $D_x y = D_u y \cdot D_x u$
 $= (101u^{100})(5x^4 - 15x^2 + \pi)$
 $= 101(5x^4 - 15x^2 + \pi)(x^5 - 5x^3 + \pi x + 1)^{100}$
7. $y = u^{111}$ and $u = x^3 - 2x^2 + 3x + 1$
 $D_x y = D_u y \cdot D_x u$
 $= (111u^{110})(3x^2 - 4x + 3)$
 $= 111(3x^2 - 4x + 3)(x^3 - 2x^2 + 3x + 1)^{110}$

8. $y = u^{-7}$ and $u = x^2 - x + 1$
 $D_x y = D_u y \cdot D_x u$
 $= (-7u^{-8})(2x-1)$
 $= -7(2x-1)(x^2-x+1)^{-8}$
9. $y = u^{-5}$ and $u = x+3$
 $D_x y = D_u y \cdot D_x u$
 $= (-5u^{-6})(1) = -5(x+3)^{-6} = -\frac{5}{(x+3)^6}$
10. $y = u^{-9}$ and $u = 3x^2 + x - 3$
 $D_x y = D_u y \cdot D_x u$
 $= (-9u^{-10})(6x+1)$
 $= -9(6x+1)(3x^2+x-3)^{-10}$
 $= -\frac{9(6x+1)}{(3x^2+x-3)^{10}}$
11. $y = \sin u$ and $u = x^2 + x$
 $D_x y = D_u y \cdot D_x u$
 $= (\cos u)(2x+1)$
 $= (2x+1)\cos(x^2+x)$
12. $y = \cos u$ and $u = 3x^2 - 2x$
 $D_x y = D_u y \cdot D_x u$
 $= (-\sin u)(6x-2)$
 $= -(6x-2)\sin(3x^2-2x)$
13. $y = u^3$ and $u = \cos x$
 $D_x y = D_u y \cdot D_x u$
 $= (3u^2)(-\sin x)$
 $= -3\sin x \cos^2 x$
14. $y = u^4$, $u = \sin v$, and $v = 3x^2$
 $D_x y = D_u y \cdot D_v u \cdot D_x v$
 $= (4u^3)(\cos v)(6x)$
 $= 24x \sin^3(3x^2) \cos(3x^2)$
15. $y = u^3$ and $u = \frac{x+1}{x-1}$
 $D_x y = D_u y \cdot D_x u$
 $= (3u^2) \frac{(x-1)D_x(x+1) - (x+1)D_x(x-1)}{(x-1)^2}$
 $= 3\left(\frac{x+1}{x-1}\right)^2 \left(\frac{-2}{(x-1)^2}\right) = -\frac{6(x+1)^2}{(x-1)^4}$

$$16. \quad y = u^{-3} \text{ and } u = \frac{x-2}{x-\pi}$$

$$D_x y = D_u y \cdot D_x u$$

$$= (-3u^{-4}) \cdot \frac{(x-\pi)D_x(x-2) - (x-2)D_x(x-\pi)}{(x-\pi)^2}$$

$$= -3 \left(\frac{x-2}{x-\pi} \right)^{-4} \frac{(2-\pi)}{(x-\pi)^2} = -3 \frac{(x-\pi)^2}{(x-2)^4} (2-\pi)$$

$$17. \quad y = \cos u \text{ and } u = \frac{3x^2}{x+2}$$

$$D_x y = D_u y \cdot D_x u = (-\sin u) \frac{(x+2)D_x(3x^2) - (3x^2)D_x(x+2)}{(x+2)^2}$$

$$= -\sin \left(\frac{3x^2}{x+2} \right) \frac{(x+2)(6x) - (3x^2)(1)}{(x+2)^2} = -\frac{3x^2 + 12x}{(x+2)^2} \sin \left(\frac{3x^2}{x+2} \right)$$

$$18. \quad y = u^3, \quad u = \cos v, \text{ and } v = \frac{x^2}{1-x}$$

$$D_x y = D_u y \cdot D_v u \cdot D_x v = (3u^2)(-\sin v) \frac{(1-x)D_x(x^2) - (x^2)D_x(1-x)}{(1-x)^2}$$

$$= -3 \cos^2 \left(\frac{x^2}{1-x} \right) \sin \left(\frac{x^2}{1-x} \right) \frac{(1-x)(2x) - (x^2)(-1)}{(1-x)^2} = \frac{-3(2x-x^2)}{(1-x)^2} \cos^2 \left(\frac{x^2}{1-x} \right) \sin \left(\frac{x^2}{1-x} \right)$$

$$19. \quad D_x[(3x-2)^2(3-x^2)^2] = (3x-2)^2 D_x(3-x^2)^2 + (3-x^2)^2 D_x(3x-2)^2$$

$$= (3x-2)^2(2)(3-x^2)(-2x) + (3-x^2)^2(2)(3x-2)(3)$$

$$= 2(3x-2)(3-x^2)[(3x-2)(-2x) + (3-x^2)(3)] = 2(3x-2)(3-x^2)(9+4x-9x^2)$$

$$20. \quad D_x[(2-3x^2)^4(x^7+3)^3] = (2-3x^2)^4 D_x(x^7+3)^3 + (x^7+3)^3 D_x(2-3x^2)^4$$

$$= (2-3x^2)^4(3)(x^7+3)^2(7x^6) + (x^7+3)^3(4)(2-3x^2)^3(-6x) = 3x(3x^2-2)^3(x^7+3)^2(29x^7-14x^5+24)$$

$$21. \quad D_x \left[\frac{(x+1)^2}{3x-4} \right] = \frac{(3x-4)D_x(x+1)^2 - (x+1)^2 D_x(3x-4)}{(3x-4)^2} = \frac{(3x-4)(2)(x+1)(1) - (x+1)^2(3)}{(3x-4)^2} = \frac{3x^2-8x-11}{(3x-4)^2}$$

$$= \frac{(x+1)(3x-11)}{(3x-4)^2}$$

$$22. \quad D_x \left[\frac{2x-3}{(x^2+4)^2} \right] = \frac{(x^2+4)^2 D_x(2x-3) - (2x-3)D_x(x^2+4)^2}{(x^2+4)^4}$$

$$= \frac{(x^2+4)^2(2) - (2x-3)(2)(x^2+4)(2x)}{(x^2+4)^4} = \frac{-6x^2+12x+8}{(x^2+4)^3}$$

$$23. \quad D_t \left(\frac{3t-2}{t+5} \right)^3 = 3 \left(\frac{3t-2}{t+5} \right)^2 \frac{(t+5)D_t(3t-2) - (3t-2)D_t(t+5)}{(t+5)^2}$$

$$= 3 \left(\frac{3t-2}{t+5} \right)^2 \frac{(t+5)(3) - (3t-2)(1)}{(t+5)^2} = \frac{51(3t-2)^2}{(t+5)^4}$$

$$24. \quad D_s \left(\frac{s^2-9}{s+4} \right) = \frac{(s+4)D_s(s^2-9) - (s^2-9)D_s(s+4)}{(s+4)^2} = \frac{(s+4)(2s) - (s^2-9)(1)}{(s+4)^2} = \frac{s^2+8s+9}{(s+4)^2}$$

$$25. D_t \left(\frac{(3t-2)^3}{t+5} \right) = \frac{(t+5)D_t(3t-2)^3 - (3t-2)^3 D_t(t+5)}{(t+5)^2} = \frac{(t+5)(3)(3t-2)^2(3) - (3t-2)^3(1)}{(t+5)^2}$$

$$= \frac{(6t+47)(3t-2)^2}{(t+5)^2}$$

$$26. D_\theta(\sin^3 \theta) = 3 \sin^2 \theta \cos \theta$$

$$27. D_x \left(\frac{\sin x}{\cos 2x} \right)^3 = 3 \left(\frac{\sin x}{\cos 2x} \right)^2 \frac{(\cos 2x)D_x(\sin x) - (\sin x)D_x(\cos 2x)}{\cos^2 2x}$$

$$= 3 \left(\frac{\sin x}{\cos 2x} \right)^2 \frac{\cos x \cos 2x + 2 \sin x \sin 2x}{\cos^2 2x} = \frac{3 \sin^2 x \cos x \cos 2x + 6 \sin^3 x \sin 2x}{\cos^4 2x}$$

$$= \frac{3(\sin^2 x)(\cos x \cos 2x + 2 \sin x \sin 2x)}{\cos^4 2x}$$

$$28. D_t[\sin t \tan(t^2 + 1)] = \sin t D_t[\tan(t^2 + 1)] + \tan(t^2 + 1)D_t(\sin t)$$

$$= (\sin t)[\sec^2(t^2 + 1)](2t) + \tan(t^2 + 1)\cos t = 2t \sin t \sec^2(t^2 + 1) + \cos t \tan(t^2 + 1)$$

$$29. f'(x) = 3 \left(\frac{x^2 + 1}{x+2} \right)^2 \frac{(x+2)D_x(x^2 + 1) - (x^2 + 1)D_x(x+2)}{(x+2)^2}$$

$$= 3 \left(\frac{x^2 + 1}{x+2} \right)^2 \frac{2x^2 + 4x - x^2 - 1}{(x+2)^2} = \frac{3(x^2 + 1)^2(x^2 + 4x - 1)}{(x+2)^4}$$

$$f'(3) = 9.6$$

$$30. G'(t) = (t^2 + 9)^3 D_t(t^2 - 2)^4 + (t^2 - 2)^4 D_t(t^2 + 9)^3 = (t^2 + 9)^3(4)(t^2 - 2)^3(2t) + (t^2 - 2)^4(3)(t^2 + 9)^2(2t)$$

$$= 2t(7t^2 + 30)(t^2 + 9)^2(t^2 - 2)^3$$

$$G'(1) = -7400$$

$$31. F'(t) = [\cos(t^2 + 3t + 1)](2t + 3) = (2t + 3)\cos(t^2 + 3t + 1)$$

$$F'(1) = 5 \cos 5 \approx 1.4183$$

$$32. g'(s) = (\cos \pi s)D_s(\sin^2 \pi s) + (\sin^2 \pi s)D_s(\cos \pi s) = (\cos \pi s)(2 \sin \pi s)(\cos \pi s)(\pi) + (\sin^2 \pi s)(-\sin \pi s)(\pi)$$

$$= \pi \sin \pi s [2 \cos^2 \pi s - \sin^2 \pi s]$$

$$g'\left(\frac{1}{2}\right) = -\pi$$

$$33. D_x[\sin^4(x^2 + 3x)] = 4 \sin^3(x^2 + 3x)D_x \sin(x^2 + 3x) = 4 \sin^3(x^2 + 3x)\cos(x^2 + 3x)D_x(x^2 + 3x)$$

$$= 4 \sin^3(x^2 + 3x)\cos(x^2 + 3x)(2x + 3) = 4(2x + 3)\sin^3(x^2 + 3x)\cos(x^2 + 3x)$$

$$34. D_t[\cos^5(4t - 19)] = 5 \cos^4(4t - 19)D_t \cos(4t - 19) = 5 \cos^4(4t - 19)[- \sin(4t - 19)]D_t(4t - 19)$$

$$= -5 \cos^4(4t - 19)\sin(4t - 19)(4) = -20 \cos^4(4t - 19)\sin(4t - 19)$$

$$35. D_t[\sin^3(\cos t)] = 3 \sin^2(\cos t)D_t \sin(\cos t) = 3 \sin^2(\cos t)\cos(\cos t)D_t(\cos t)$$

$$= 3 \sin^2(\cos t)\cos(\cos t)(-\sin t) = -3 \sin t \sin^2(\cos t)\cos(\cos t)$$

$$36. D_u \left[\cos^4 \left(\frac{u+1}{u-1} \right) \right] = 4 \cos^3 \left(\frac{u+1}{u-1} \right) D_u \cos \left(\frac{u+1}{u-1} \right) = 4 \cos^3 \left(\frac{u+1}{u-1} \right) \left[-\sin \left(\frac{u+1}{u-1} \right) \right] D_u \left(\frac{u+1}{u-1} \right) \\ = -4 \cos^3 \left(\frac{u+1}{u-1} \right) \sin \left(\frac{u+1}{u-1} \right) \frac{(u-1)D_u(u+1) - (u+1)D_u(u-1)}{(u-1)^2} = \frac{8}{(u-1)^2} \cos^3 \left(\frac{u+1}{u-1} \right) \sin \left(\frac{u+1}{u-1} \right)$$

$$37. D_\theta [\cos^4(\sin \theta^2)] = 4 \cos^3(\sin \theta^2) D_\theta \cos(\sin \theta^2) = 4 \cos^3(\sin \theta^2) [-\sin(\sin \theta^2)] D_\theta (\sin \theta^2) \\ = -4 \cos^3(\sin \theta^2) \sin(\sin \theta^2) (\cos \theta^2) D_\theta (\theta^2) = -8 \theta \cos^3(\sin \theta^2) \sin(\sin \theta^2) (\cos \theta^2)$$

$$38. D_x [x \sin^2(2x)] = x D_x \sin^2(2x) + \sin^2(2x) D_x x = x [2 \sin(2x) D_x \sin(2x)] + \sin^2(2x) (1) \\ = x [2 \sin(2x) \cos(2x) D_x (2x)] + \sin^2(2x) = x [4 \sin(2x) \cos(2x)] + \sin^2(2x) = 2x \sin(4x) + \sin^2(2x)$$

$$39. D_x \{\sin[\cos(\sin 2x)]\} = \cos[\cos(\sin 2x)] D_x \cos(\sin 2x) = \cos[\cos(\sin 2x)] [-\sin(\sin 2x)] D_x (\sin 2x) \\ = -\cos[\cos(\sin 2x)] \sin(\sin 2x) (\cos 2x) D_x (2x) = -2 \cos[\cos(\sin 2x)] \sin(\sin 2x) (\cos 2x)$$

$$40. D_t \{\cos^2[\cos(\cos t)]\} = 2 \cos[\cos(\cos t)] D_t \cos[\cos(\cos t)] = 2 \cos[\cos(\cos t)] \{-\sin[\cos(\cos t)]\} D_t \cos(\cos t) \\ = -2 \cos[\cos(\cos t)] \sin[\cos(\cos t)] [-\sin(\cos t)] D_t (\cos t) = 2 \cos[\cos(\cos t)] \sin[\cos(\cos t)] \sin(\cos t) (-\sin t) \\ = -2 \sin t \cos[\cos(\cos t)] \sin[\cos(\cos t)] \sin(\cos t)$$

$$41. D_x y = (x^2 + 1)^3 D_x (x^4 + 1)^2 + (x^4 + 1)^2 D_x (x^2 + 1)^3 = (x^2 + 1)^3 (2)(x^4 + 1)(4x^3) + (x^4 + 1)^2 (3)(x^2 + 1)(2x) \\ = 8x^3(x^2 + 1)^3(x^4 + 1) + 6x(x^2 + 1)(x^4 + 1)^2$$

$$\text{At } x = 1: m_{\tan} = 224$$

$$\text{Tangent line: } y - 32 = 224(x - 1)$$

$$42. \text{ a. } \left(\frac{x}{4} \right)^2 + \left(\frac{y}{7} \right)^2 = \left(\frac{4 \cos 2t}{4} \right)^2 + \left(\frac{7 \sin 2t}{7} \right)^2 = \cos^2 2t + \sin^2 2t = 1$$

$$\text{ b. } L = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2} = \sqrt{(4 \cos 2t)^2 + (7 \sin 2t)^2} = \sqrt{16 \cos^2 2t + 49 \sin^2 2t}$$

$$\text{ c. } D_t L = \frac{1}{2\sqrt{16 \cos^2 2t + 49 \sin^2 2t}} D_t (16 \cos^2 2t + 49 \sin^2 2t) = \frac{32 \cos 2t D_t (\cos 2t) + 98 \sin 2t D_t (\sin 2t)}{2\sqrt{16 \cos^2 2t + 49 \sin^2 2t}} \\ = \frac{-64 \cos 2t \sin 2t + 196 \sin 2t \cos 2t}{2\sqrt{16 \cos^2 2t + 49 \sin^2 2t}} = \frac{-16 \sin 4t + 49 \sin 4t}{\sqrt{16 \cos^2 2t + 49 \sin^2 2t}} = \frac{33 \sin 4t}{\sqrt{16 \cos^2 2t + 49 \sin^2 2t}}$$

$$\text{At } t = \frac{\pi}{8}: \text{ rate} = \frac{33}{\sqrt{16 \cdot \frac{1}{2} + 49 \cdot \frac{1}{2}}} \approx 5.8 \text{ ft/sec}$$

$$43. \text{ a. } (10 \cos 8\pi t, 10 \sin 8\pi t)$$

$$\text{ b. } D_t (10 \sin 8\pi t) = 10 \cos(8\pi t) D_t (8\pi t) \\ = 80\pi \cos(8\pi t)$$

$$\text{At } t = 1: \text{ rate} = 80\pi \approx 251 \text{ cm/s}$$

P is rising at the rate of 251 cm/s.

$$44. \text{ a. } (\cos 2t, \sin 2t)$$

$$\text{ b. } (0 - \cos 2t)^2 + (y - \sin 2t)^2 = 5^2, \text{ so}$$

$$y = \sin 2t + \sqrt{25 - \cos^2 2t}$$

$$\text{ c. } D_t \left(\sin 2t + \sqrt{25 - \cos^2 2t} \right)$$

$$= 2 \cos 2t + \frac{1}{2\sqrt{25 - \cos^2 2t}} \cdot 4 \cos 2t \sin 2t$$

$$= 2 \cos 2t \left(1 + \frac{\sin 2t}{\sqrt{25 - \cos^2 2t}} \right)$$

45. 60 revolutions per minute is 120π radians per minute or 2π radians per second.

a. $(\cos 2\pi t, \sin 2\pi t)$

b. $(0 - \cos 2\pi t)^2 + (y - \sin 2\pi t)^2 = 5^2$, so
 $y = \sin 2\pi t + \sqrt{25 - \cos^2 2\pi t}$

c. $D_t \left(\sin 2\pi t + \sqrt{25 - \cos^2 2\pi t} \right)$
 $= 2\pi \cos 2\pi t$
 $+ \frac{1}{2\sqrt{25 - \cos^2 2\pi t}} \cdot 4\pi \cos 2\pi t \sin 2\pi t$
 $= 2\pi \cos 2\pi t \left(1 + \frac{\sin 2\pi t}{\sqrt{25 - \cos^2 2\pi t}} \right)$

46. $V(t) = \frac{4}{3}\pi [r(t)]^3$
 $V'(t) = 4\pi [r(t)]^2 r'(t)$
 Let t_0 be the time when $r = 6$.
 $V'(t_0) = 4\pi \cdot 6^2 \cdot 0.75$
 $= 108\pi \text{ cm}^3/\text{sec}$

47. $V(t) = \frac{4}{3}\pi [r(t)]^3$
 $[r(t)]^3 = \frac{3V(t)}{4\pi}$
 $r(t) = \left(\frac{3V(t)}{4\pi} \right)^{1/3}$
 $r'(t) = \frac{1}{3} \left(\frac{3V(t)}{4\pi} \right)^{-2/3} \cdot \frac{3}{4\pi} V'(t)$
 Let t_0 be the time when $V = 60$.
 $r'(t_0) = \left(\frac{3 \cdot 60}{4\pi} \right)^{-2/3} \cdot \frac{1}{4\pi} \cdot 3$
 $\approx 0.0405 \text{ cm/sec}$

48. $y = \sqrt{u}$ and $u = x^2$
 $D_x y = D_u y \cdot D_x u$
 $= \frac{1}{2\sqrt{u}} \cdot 2x = \frac{2x}{2\sqrt{x^2}} = \frac{x}{|x|} = \frac{|x|}{x}$

49. a. $D_x |x^2 - 1| = \frac{|x^2 - 1|}{x^2 - 1} D_x (x^2 - 1)$
 $= \frac{|x^2 - 1|}{x^2 - 1} (2x) = \frac{2x|x^2 - 1|}{x^2 - 1}$

b. $D_x |\sin x| = \frac{|\sin x|}{\sin x} D_x (\sin x)$
 $= \frac{|\sin x|}{\sin x} \cos x = \cot x |\sin x|$

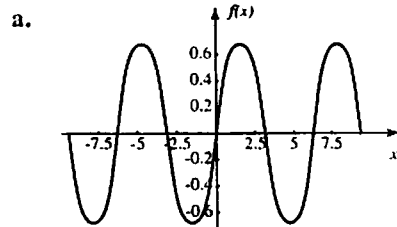
50. a. $D_x L\left(\frac{x}{x+1}\right) = \left(\frac{x+1}{x}\right) D_x \left(\frac{x}{x+1}\right)$
 $= \left(\frac{x+1}{x}\right) \frac{(x+1)D_x x - xD_x(x+1)}{(x+1)^2}$
 $= \frac{x+1}{x(x+1)^2} = \frac{1}{x(x+1)} = \frac{1}{x^2 + x}$

b. $D_x L(\cos^4 x) = \sec^4 x D_x (\cos^4 x)$
 $= \sec^4 x (4 \cos^3 x) D_x (\cos x)$
 $= 4 \sec^4 x \cos^3 x (-\sin x)$
 $= -4 \sec x \sin x = -4 \tan x$

51. $[f(f(f(f(0))))]' = f'(f(f(f(0)))) \cdot f'(f(f(0))) \cdot f'(f(0)) \cdot f'(0)$
 $= 2 \cdot 2 \cdot 2 \cdot 2 = 16$

52. Let $g(x) = -x$, so $g'(x) = -1$
 Suppose $f(x)$ is odd. Then $-f(x) = f(g(x))$. Take the derivative of both sides.
 $-f'(x) = f'(g(x))g'(x)$, so
 $-f'(x) = -f'(-x)$ or $f'(x) = f'(-x)$. Thus, the derivative is an even function.
 Suppose $f(x)$ is even. Then $f(x) = f(g(x))$. Take the derivative of both sides. $f'(x) = f'(g(x))g'(x)$, so $f'(x) = -f'(-x)$. Thus, the derivative is an odd function.
 $f'(-x) = f'(-x) \cdot (-1) = -f'(-x)$
 If $f(x)$ is odd, then $-f'(-x) = -f'(x)$ is even.
 If $f(x)$ is even, then $-f'(-x) = f'(x)$ is odd.

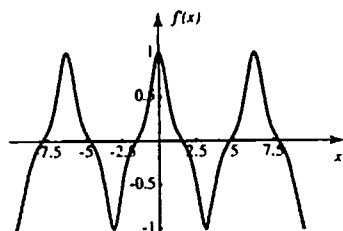
53. $f(x) = \sin(\sin(\sin(\sin x)))$



Odd function

$f(-x) = \sin(\sin(\sin(\sin(-x))))$
 $= \sin(\sin(\sin(-\sin x))) = \sin(\sin(-\sin(\sin x)))$
 $= \sin(-\sin(\sin(\sin x))) = -\sin(\sin(\sin(\sin x)))$

b.

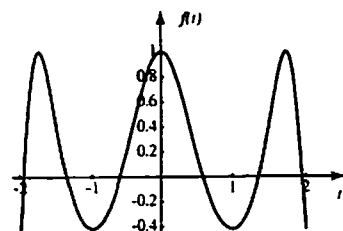


Even; the derivative of an odd function is even.

c. Largest value of $f(x) \approx 0.678$ d. Largest value of $|f'(x)| = 1$

54. $f(t) = \cos(t^3 - 3t)$

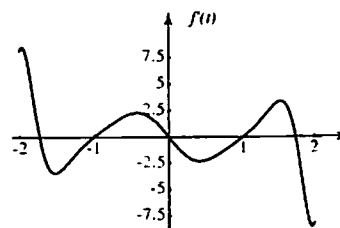
a.



Even function

$$\begin{aligned} f(-t) &= \cos[(-t)^3 - 3(-t)] \\ &= \cos[-t^3 + 3t] \\ &= \cos[-(t^3 - 3t)] \\ &= \cos(t^3 - 3t) \end{aligned}$$

b.



Odd

The derivative of an even function is odd.

c. Largest value of $f(t) = 1$ d. Largest value of $|f'(t)| \approx 8.53$

3.6 Concepts Review

1. Increment; $\frac{\Delta y}{\Delta x}; \frac{dy}{dx}$

2. $f'(x); D_x y; \frac{dy}{dx}$

3. $\frac{dy}{du} \frac{du}{dx}$

4. $\frac{dw}{dt} \frac{dt}{ds} \frac{ds}{dr}$

Problem Set 3.6

1. $\Delta y = [3(1.5) + 2] - [3(1) + 2] = 1.5$

2. $\Delta y = [3(0.1)^2 + 2(0.1) + 1] - [3(0.0)^2 + 2(0.0) + 1] = 0.23$

3. $\Delta y = \frac{3}{2.31+1} - \frac{3}{2.34+1} \approx 0.0081$

4. $\Delta y = \cos[2(0.573)] - \cos[2(0.571)] \approx -0.0036$

5. $\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x$$

6.
$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{[(x + \Delta x)^3 - 3(x + \Delta x)^2] - (x^3 - 3x^2)}{\Delta x} \\ &= \frac{3x^2\Delta x + 3x(\Delta x)^2 - 6x\Delta x - 3(\Delta x)^2 + \Delta x^3}{\Delta x} \end{aligned}$$

$$= 3x^2 + 3x\Delta x - 6x - 3\Delta x + (\Delta x)^2$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x - 6x - 3\Delta x + (\Delta x)^2) \\ &= 3x^2 - 6x \end{aligned}$$

7.
$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{\frac{1}{x+\Delta x+1} - \frac{1}{x+1}}{\Delta x} \\ &= \left(\frac{x+1 - (x+\Delta x+1)}{(x+\Delta x+1)(x+1)} \right) \left(\frac{1}{\Delta x} \right) \\ &= \frac{-\Delta x}{(x+\Delta x+1)(x+1)\Delta x} \\ &= -\frac{1}{(x+\Delta x+1)(x+1)} \end{aligned}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left[-\frac{1}{(x+\Delta x+1)(x+1)} \right] = -\frac{1}{(x+1)^2}$$

$$\begin{aligned}
 8. \quad \frac{\Delta y}{\Delta x} &= \left[\frac{(x+\Delta x)^3 + (x+\Delta x)^2}{(x+\Delta x)^3} - \frac{x^3 + x^2}{x^3} \right] \frac{1}{\Delta x} \\
 &= \left(1 + \frac{1}{x+\Delta x} - 1 - \frac{1}{x} \right) \left(\frac{1}{\Delta x} \right) \\
 &= \frac{-\Delta x}{x(x+\Delta x)\Delta x} = -\frac{1}{x(x+\Delta x)} \\
 \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \left(-\frac{1}{x(x+\Delta x)} \right) = -\frac{1}{x^2}
 \end{aligned}$$

$$9. \quad \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (2u)(\cos x) = 2 \sin x \cos x = \sin 2x$$

$$\begin{aligned}
 10. \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = (-\sin u) \frac{(x+1)(0) - (1)(1)}{(x+1)^2} \\
 &= -\sin \left(\frac{1}{x+1} \right) \left(-\frac{1}{(x+1)^2} \right) = \frac{1}{(x+1)^2} \sin \left(\frac{1}{x+1} \right)
 \end{aligned}$$

$$\begin{aligned}
 11. \quad y &= \tan u \text{ and } u = x^2 \\
 \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = (\sec^2 u)(2x) = 2x \sec^2(x^2)
 \end{aligned}$$

$$\begin{aligned}
 12. \quad y &= u^2 \text{ and } u = \tan x \\
 \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = (2u)(\sec^2 x) = 2 \tan x \sec^2 x
 \end{aligned}$$

$$\begin{aligned}
 13. \quad y &= u^4 \text{ and } u = \frac{x^2+1}{\cos x} \\
 \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = (4u^3) \frac{(\cos x)(2x) - (x^2+1)(-\sin x)}{\cos^2 x} \\
 &= 4 \left(\frac{x^2+1}{\cos x} \right)^3 \left(\frac{2x \cos x + \sin x + x^2 \sin x}{\cos^2 x} \right) \\
 &= \frac{4(x^2+1)^3 (2x \cos x + \sin x + x^2 \sin x)}{\cos^5 x}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad y &= u^3 \text{ and } u = (x^2+1)\sin x \\
 \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = (3u^2)[(x^2+1)(\cos x) + (\sin x)(2x)] \\
 &= [(x^2+1)\sin x]^2 (3x^2 \cos x + 3 \cos x + 6x \sin x)
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \frac{dy}{dx} &= \cos(x^2) \frac{d}{dx}(\sin^2 x) + \sin^2 x \frac{d}{dx}[\cos(x^2)] \\
 &= \cos(x^2) 2 \sin x \cos x + \sin^2 x [-\sin(x^2)](2x) \\
 &= \sin 2x \cos(x^2) - 2x \sin^2 x \sin(x^2)
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \frac{dy}{dx} &= \frac{(x^4+1) \frac{d}{dx}(x^3+2x)^4 - (x^3+2x)^4 \frac{d}{dx}(x^4+1)}{(x^4+1)^2} \\
 &= \frac{(x^4+1)4(x^3+2x)^3(3x^2+2) - (x^3+2x)^4(4x^3)}{(x^4+1)^2} \\
 &= \frac{4x^3(x^2+2)^3(2x^6+3x^2+2)}{(x^4+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 17. \quad y &= u^4, \quad u = \sin v, \text{ and } v = x^2 + 3 \\
 \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx} = (4u^3)(\cos v)(2x) \\
 &= 4 \sin^3(x^2+3) \cos(x^2+3)(2x) \\
 &= 8x \sin^3(x^2+3) \cos(x^2+3)
 \end{aligned}$$

$$\begin{aligned}
 18. \quad y &= \sin u, \quad u = v^4, \text{ and } v = x^2 + 3 \\
 \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx} = (\cos u)(4v^3)(2x) \\
 &= 8x \cos[(x^2+3)^4](x^2+3)^3
 \end{aligned}$$

$$\begin{aligned}
 19. \quad y &= u^2, \quad u = \cos v, \text{ and } v = \frac{x^2+2}{x^2-2} \\
 \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx} \\
 &= (2u)(-\sin v) \frac{(x^2-2)(2x) - (x^2+2)(2x)}{(x^2-2)^2} \\
 &= -2 \cos \left(\frac{x^2+2}{x^2-2} \right) \sin \left(\frac{x^2+2}{x^2-2} \right) \left[\frac{-8x}{(x^2-2)^2} \right] \\
 &= \frac{16x}{(x^2-2)^2} \cos \left(\frac{x^2+2}{x^2-2} \right) \sin \left(\frac{x^2+2}{x^2-2} \right)
 \end{aligned}$$

$$\begin{aligned}
 20. \quad y &= t^2, \quad t = \sin u, \quad u = v^2, \quad v = \cos w, \text{ and } w = x^2 \\
 \frac{dy}{dx} &= \frac{dy}{dt} \frac{dt}{du} \frac{du}{dv} \frac{dv}{dw} \frac{dw}{dx} \\
 &= (2t)(\cos u)(2v)(-\sin w)(2x) \\
 &= -8x \sin(\cos^2(x^2)) \cos(\cos^2(x^2)) (\cos(x^2)) (\sin(x^2)) \\
 &= -8x \cos(x^2) \sin(x^2) \sin(\cos^2(x^2)) \cos(\cos^2(x^2))
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \frac{d}{dt}(\sin^3 t + \cos^3 t) &= 3 \sin^2 t \cos t + 3 \cos^2 t (-\sin t) \\
 &= 3(\sin^2 t \cos t - \sin t \cos^2 t)
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \frac{d}{ds}[(s^2+3)^3 - (s^2+3)^{-3}] \\
 &= 3(s^2+3)^2(2s) - (-3)(s^2+3)^{-4}(2s) \\
 &= 6s[(s^2+3)^2 + (s^2+3)^{-4}]
 \end{aligned}$$

$$\begin{aligned}
 23. \quad D_r[\pi(r+3)^2 - 3\pi r(r+2)^2] \\
 &= 2\pi(r+3)(1) - [3\pi r(2)(r+2) + (r+2)^2(3\pi)] \\
 &= 2\pi r + 6\pi - 6\pi r^2 - 12\pi r - 3\pi r^2 - 12\pi r - 12\pi \\
 &= -9\pi r^2 - 22\pi r - 6\pi
 \end{aligned}$$

$$\begin{aligned}
 24. \quad D_t[u^3 + 3u] &= 3u^2 D_t u + 3D_t u \\
 &= 3(t^2)^2(2t) + 3(2t) = 6t^5 + 6t
 \end{aligned}$$

$$\begin{aligned}
 25. \quad f'(x) &= 4\left(x + \frac{1}{x}\right)^3 [1 + (-1)x^{-2}] \\
 &= 4\left(x + \frac{1}{x}\right)^3 \left(1 - \frac{1}{x^2}\right) \\
 f'(2) &= 4\left(\frac{5}{2}\right)^3 \left(\frac{3}{4}\right) = \frac{1500}{32} = 46.875
 \end{aligned}$$

$$\begin{aligned}
 26. \quad F'(t) &= \cos(t^2)(\cos 3t)(3) + (\sin 3t)[- \sin(t^2)(2t)] \\
 &= 3\cos(t^2)(\cos 3t) - 2t\sin(t^2)(\sin 3t) \\
 F'(0) &= 3
 \end{aligned}$$

$$27. \text{ a. } (f+g)'(3) = f'(3) + g'(3) = -1 + (-4) = -5$$

$$\begin{aligned}
 \text{b. } (f \cdot g)'(3) &= f(3)g'(3) + g(3)f'(3) \\
 &= (2)(-4) + (3)(-1) = -11
 \end{aligned}$$

$$\begin{aligned}
 \text{c. } (f/g)'(3) &= \frac{g(3)f'(3) - f(3)g'(3)}{g^2(3)} \\
 &= \frac{(3)(-1) - (2)(-4)}{(3)^2} = \frac{5}{9}
 \end{aligned}$$

$$\begin{aligned}
 \text{d. } (f \circ g)'(3) &= f'(g(3)) \cdot g'(3) \\
 &= f'(3) \cdot g'(3) = (-1)(-4) = 4
 \end{aligned}$$

$$\begin{aligned}
 28. \text{ a. } \frac{d}{dx}[f(x)]^3 &= 3[f(x)]^2 f'(x) \\
 \text{At } x = 2, \quad 3[f(2)]^2 f'(2) &= 3(4)^2(-2) = -96
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } \frac{d}{dx}\left[\frac{3}{f(x)}\right] &= \frac{f(x)(0) - 3f'(x)}{f^2(x)} = -\frac{3f'(x)}{f^2(x)} \\
 \text{At } x = 2, \quad -\frac{3f'(2)}{f^2(2)} &= -\frac{3(-2)}{(4)^2} = \frac{3}{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{c. } (f \circ f)'(2) &= f'(f(2))f'(2) = f'(4)f'(2) \\
 &= (6)(-2) = -12
 \end{aligned}$$

$$\begin{aligned}
 29. \text{ a. } (f+g)'(4) &= f'(4) + g'(4) \\
 &\approx \frac{1}{2} + \frac{3}{2} \approx 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } (f \circ g)'(6) &= f'(g(6))g'(6) \\
 &= f'(2)g'(6) \approx (1)(-1) = -1
 \end{aligned}$$

$$\begin{aligned}
 30. \text{ a. } (f/g)'(2) &= \frac{g(2)f'(2) - f(2)g'(2)}{g^2(2)} \\
 &\approx \frac{(1)(1) - (3)(0)}{(1)^2} = 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } (g \circ f)'(3) &= g'(f(3))f'(3) \\
 &= g'(4)f'(3) \approx \left(\frac{3}{2}\right)(1) = \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 31. \text{ a. } V &= s^3 \text{ where } s \text{ is the length of an edge of} \\
 &\text{the cube. } \frac{ds}{dt} = 16 \text{ cm/min.} \\
 \frac{dV}{dt} &= \frac{dV}{ds} \frac{ds}{dt} = (3s^2)(16) = 48s^2 \text{ cm}^3/\text{min.} \\
 \text{When } s = 20, \\
 \frac{dV}{dt} &= 48(20)^2 = 19,200 \text{ cm}^3/\text{min.}
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } A &= 6s^2 \text{ cm}^2. \\
 \frac{dA}{dt} &= \frac{dA}{ds} \frac{ds}{dt} = (12s)(16) = 192s \text{ cm}^2/\text{min.} \\
 \text{When } s = 15, \\
 \frac{dA}{dt} &= 192(15) = 2880 \text{ cm}^2/\text{min.}
 \end{aligned}$$

$$\begin{aligned}
 32. \quad D &= \sqrt{(20t)^2 + (12t)^2} = \sqrt{400t^2 + 144t^2} \\
 &= \sqrt{544t^2} = t\sqrt{544} \\
 D_t(t\sqrt{544}) &= \sqrt{544} \approx 23.3 \text{ mph at } t = 3 \text{ and} \\
 &t = 6 \text{ hours}
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \frac{dy}{dx} &= (x^2)(2)\sin(x^2)\cos(x^2)(2x) + 2x\sin^2(x^2) \\
 &= 4x^3\sin(x^2)\cos(x^2) + 2x\sin^2(x^2) \\
 \text{At } x &= \sqrt{\frac{\pi}{2}}: m_{\tan} = \sqrt{2\pi} \approx 2.507 \\
 y &= \frac{\pi}{2}
 \end{aligned}$$

$$\text{Tangent line: } y - \frac{\pi}{2} = \sqrt{2\pi}\left(x - \sqrt{\frac{\pi}{2}}\right)$$

The line intersects the x -axis when

$$-\frac{\pi}{2} = \sqrt{2\pi}x - \pi$$

$$\frac{\pi}{2} = \sqrt{2\pi}x$$

$$x = \frac{\sqrt{2\pi}}{4}$$

34. The minute hand makes 1 revolution every hour, so at t minutes after the hour, it makes an angle of $\frac{\pi t}{30}$ radians with the vertical. By the Law of Cosines, the length of the elastic string is

$$s = \sqrt{10^2 + 10^2 - 2(10)(10)\cos\frac{\pi t}{30}}$$

$$= 10\sqrt{2 - 2\cos\frac{\pi t}{30}}$$

35. $g'(x) = f'(f(f(f(x)))) \cdot f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$
 $g'(x_1) = f'(f(f(f(x_1)))) \cdot f'(f(f(x_1))) \cdot f'(f(x_1)) \cdot f'(x_1)$
 $= f'(f(f(x_2))) \cdot f'(f(x_2)) \cdot f'(x_2) \cdot f'(x_1)$
 $= f'(f(x_1)) \cdot f'(x_1) \cdot f'(x_2) \cdot f'(x_1)$
 $= f'(x_2)f'(x_1)f'(x_2)f'(x_1) = [f'(x_1)f'(x_2)]^2$
 $g'(x_2) = f'(f(f(f(x_2)))) \cdot f'(f(f(x_2))) \cdot f'(f(x_2)) \cdot f'(x_2)$
 $= f'(f(f(x_1))) \cdot f'(f(x_1)) \cdot f'(x_1) \cdot f'(x_2)$
 $= f'(f(x_2)) \cdot f'(x_2) \cdot f'(x_1) \cdot f'(x_2)$
 $= f'(x_1)f'(x_2)f'(x_1)f'(x_2) = [f'(x_1)f'(x_2)]^2$

36. a. $f'(x) = x^2 \left(\cos \frac{1}{x} \right) \left(-\frac{1}{x^2} \right) + \left(\sin \frac{1}{x} \right) (2x)$

$$= -\cos \frac{1}{x} + 2x \sin \frac{1}{x}$$

b. $f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0$$

- c. $f'(x)$ is discontinuous at $x = 0$ because $\lim_{x \rightarrow 0} f'(x)$ does not exist. $\lim_{x \rightarrow 0} 2x \sin \frac{1}{x} = 0$ but $\cos \frac{1}{x}$ oscillates between -1 and 1 increasingly rapidly as $x \rightarrow 0$.

37. The minute hand makes 1 revolution every hour, so at t minutes after noon it makes an angle of $\frac{\pi t}{30}$ radians with the vertical. Similarly, at t minutes after noon the hour hand makes an angle of $\frac{\pi t}{360}$ with the vertical. Thus, by the Law of Cosines, the distance between the tips of the hands is

$$\frac{ds}{dt} = 10 \cdot \frac{1}{2\sqrt{2 - 2\cos\frac{\pi t}{30}}} \cdot \frac{\pi}{15} \sin \frac{\pi t}{30}$$

$$= \frac{\pi \sin \frac{\pi t}{30}}{3\sqrt{2 - 2\cos\frac{\pi t}{30}}}$$

At 12:15, the string is stretching at the rate of

$$\frac{\pi \sin \frac{\pi}{2}}{3\sqrt{2 - 2\cos\frac{\pi}{2}}} = \frac{\pi}{3\sqrt{2}} \approx 0.74 \text{ cm/min}$$

$$s = \sqrt{6^2 + 8^2 - 2 \cdot 6 \cdot 8 \cos \left(\frac{\pi t}{30} - \frac{\pi t}{360} \right)}$$

$$= \sqrt{100 - 96 \cos \frac{11\pi t}{360}}$$

$$\frac{ds}{dt} = \frac{1}{2\sqrt{100 - 96 \cos \frac{11\pi t}{360}}} \cdot \frac{44\pi}{15} \sin \frac{11\pi t}{360}$$

$$= \frac{22\pi \sin \frac{11\pi t}{360}}{15\sqrt{100 - 96 \cos \frac{11\pi t}{360}}}$$

At 12:20,

$$\frac{ds}{dt} = \frac{22\pi \sin \frac{11\pi}{18}}{15\sqrt{100 - 96 \cos \frac{11\pi}{18}}} \approx 0.38 \text{ in./min}$$

38. From Problem 37, $\frac{ds}{dt} = \frac{22\pi \sin \frac{11\pi t}{360}}{15\sqrt{100 - 96 \cos \frac{11\pi t}{360}}}$.

Using a computer algebra system or graphing

utility to view $\frac{ds}{dt}$ for $0 \leq t \leq 60$, $\frac{ds}{dt}$ is largest

when $t \approx 7.5$. Thus, the distance between the tips of the hands is increasing most rapidly at about 12:08.

$$\begin{aligned}
 39. \quad D_x \left(\frac{f(x)}{g(x)} \right) &= D_x \left(f(x) \cdot \frac{1}{g(x)} \right) \\
 &= D_x \left(f(x) \cdot (g(x))^{-1} \right) \\
 &= f(x) D_x \left((g(x))^{-1} \right) \\
 &\quad + (g(x))^{-1} D_x f(x) \\
 &= f(x) \cdot (-1)(g(x))^{-2} D_x g(x) \\
 &\quad + (g(x))^{-1} D_x f(x) \\
 &= -f(x)(g(x))^{-2} D_x g(x) \\
 &\quad + (g(x))^{-1} D_x f(x) \\
 &= \frac{-f(x) D_x g(x)}{g^2(x)} + \frac{D_x f(x)}{g(x)} \\
 &= \frac{-f(x) D_x g(x)}{g^2(x)} + \frac{g(x)}{g(x)} \cdot \frac{D_x f(x)}{g(x)} \\
 &= \frac{-f(x) D_x g(x)}{g^2(x)} + \frac{g(x) D_x f(x)}{g^2(x)} \\
 &= \frac{g(x) D_x f(x) - f(x) D_x g(x)}{g^2(x)}
 \end{aligned}$$

$$\begin{aligned}
 40. \quad a. \quad \frac{d}{dx} f^{[2]} &= f'(f(x)) \cdot f'(x) \\
 &= f'(f^{[1]}) \cdot \frac{d}{dx} f^{[1]}(x)
 \end{aligned}$$

$$\begin{aligned}
 b. \quad \frac{d}{dx} f^{[3]} &= f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x) \\
 &= f'(f^{[2]}(x)) \cdot f'(f^{[1]}(x)) \cdot \frac{d}{dx} f^{[1]}(x) \\
 &= f'(f^{[2]}(x)) \cdot \frac{d}{dx} f^{[2]}(x)
 \end{aligned}$$

$$\begin{aligned}
 c. \quad \frac{d}{dx} f^{[4]} &= f'(f(f(f(x)))) \cdot f'(f(f(x))) \\
 &\quad \cdot f'(f(x)) \cdot f'(x) \\
 &= f'(f^{[3]}(x)) \cdot f'(f^{[2]}(x)) \\
 &\quad \cdot f'(f^{[1]}(x)) \cdot \frac{d}{dx} f^{[1]}(x) \\
 &= f'(f^{[3]}(x)) \cdot \frac{d}{dx} f^{[3]}(x)
 \end{aligned}$$

$$\begin{aligned}
 d. \quad \text{Conjecture:} \\
 \frac{d}{dx} f^{[n]}(x) &= f'(f^{[n-1]}(x)) \cdot \frac{d}{dx} f^{[n-1]}(x)
 \end{aligned}$$

Proof Applying the chain rule gives

$$\begin{aligned}
 \frac{d}{dx} f^{[k+1]}(x) &= \frac{d}{dx} f(f^{[k]}(x)) \\
 &= f'(f^{[k]}(x)) \cdot \frac{d}{dx} f^{[k]}(x)
 \end{aligned}$$

3.7 Concepts Review

$$1. \quad f''(x), D_x^3 y, \frac{d^3 y}{dx^3}$$

$$2. \quad \frac{ds}{dt}; \left| \frac{ds}{dt} \right|; \frac{d^2 s}{dt^2}$$

$$3. \quad 0; < 0$$

$$4. \quad \text{positive; negative}$$

Problem Set 3.7

$$\begin{aligned}
 1. \quad \frac{dy}{dx} &= 3x^2 + 6x + 6 \\
 \frac{d^2 y}{dx^2} &= 6x + 6 \\
 \frac{d^3 y}{dx^3} &= 6
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \frac{dy}{dx} &= 5x^4 + 4x^3 \\
 \frac{d^2 y}{dx^2} &= 20x^3 + 12x^2 \\
 \frac{d^3 y}{dx^3} &= 60x^2 + 24x
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \frac{dy}{dx} &= 3(3x+5)^2(3) = 9(3x+5)^2 \\
 \frac{d^2 y}{dx^2} &= 18(3x+5)(3) = 162x + 270 \\
 \frac{d^3 y}{dx^3} &= 162
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \frac{dy}{dx} &= 5(3-5x)^4(-5) = -25(3-5x)^4 \\
 \frac{d^2 y}{dx^2} &= -100(3-5x)^3(-5) = 500(3-5x)^3 \\
 \frac{d^3 y}{dx^3} &= 1500(3-5x)^2(-5) = -7500(3-5x)^2
 \end{aligned}$$

$$5. \frac{dy}{dx} = 7 \cos(7x)$$

$$\frac{d^2y}{dx^2} = -7^2 \sin(7x)$$

$$\frac{d^3y}{dx^3} = -7^3 \cos(7x) = -343 \cos(7x)$$

$$6. \frac{dy}{dx} = 3x^2 \cos(x^3)$$

$$\frac{d^2y}{dx^2} = 3x^2[-3x^2 \sin(x^3)] + 6x \cos(x^3) = -9x^4 \sin(x^3) + 6x \cos(x^3)$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= -9x^4 \cos(x^3)(3x^2) + \sin(x^3)(-36x^3) + 6x[-\sin(x^3)(3x^2)] + 6 \cos(x^3) \\ &= -27x^6 \cos(x^3) - 36x^3 \sin(x^3) - 18x^3 \sin(x^3) + 6 \cos(x^3) = (6 - 27x^6) \cos(x^3) - 54x^3 \sin(x^3) \end{aligned}$$

$$7. \frac{dy}{dx} = \frac{(x-1)(0) - (1)(1)}{(x-1)^2} = -\frac{1}{(x-1)^2}$$

$$\frac{d^2y}{dx^2} = -\frac{(x-1)^2(0) - 2(x-1)}{(x-1)^4} = \frac{2}{(x-1)^3}$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= \frac{(x-1)^3(0) - 2[3(x-1)^2]}{(x-1)^6} \\ &= -\frac{6}{(x-1)^4} \end{aligned}$$

$$8. \frac{dy}{dx} = \frac{(1-x)(3) - (3x)(-1)}{(1-x)^2} = \frac{3}{(x-1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(x-1)^2(0) - 3[2(x-1)]}{(x-1)^4} = -\frac{6}{(x-1)^3}$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= -\frac{(x-1)^3(0) - 6(3)(x-1)^2}{(x-1)^6} \\ &= \frac{18}{(x-1)^4} \end{aligned}$$

$$9. f'(x) = 2x; f''(x) = 2; f''(2) = 2$$

$$10. f'(x) = 15x^2 + 4x + 1$$

$$f''(x) = 30x + 4$$

$$f''(2) = 64$$

$$11. f'(t) = -\frac{2}{t^2}$$

$$f''(t) = \frac{4}{t^3}$$

$$f''(2) = \frac{4}{8} = \frac{1}{2}$$

$$12. f'(u) = \frac{(5-u)(4u) - (2u^2)(-1)}{(5-u)^2} = \frac{20u - 2u^2}{(5-u)^2}$$

$$f''(u) = \frac{(5-u)^2(20-4u) - (20u-2u^2)2(5-u)(-1)}{(5-u)^4}$$

$$= \frac{100}{(5-u)^3}$$

$$f''(2) = \frac{100}{3^3} = \frac{100}{27}$$

$$13. f'(\theta) = -2(\cos \theta \pi)^{-3}(-\sin \theta \pi) \pi = 2\pi(\cos \theta \pi)^{-3}(\sin \theta \pi)$$

$$f''(\theta) = 2\pi[(\cos \theta \pi)^{-3}(\pi)(\cos \theta \pi) + (\sin \theta \pi)(-3)(\cos \theta \pi)^{-4}(-\sin \theta \pi)(\pi)] = 2\pi^2[(\cos \theta \pi)^{-2} + 3\sin^2 \theta \pi(\cos \theta \pi)^{-4}]$$

$$f''(2) = 2\pi^2[1 + 3(0)(1)] = 2\pi^2$$

$$14. f'(t) = t \cos\left(\frac{\pi}{t}\right) \left(-\frac{\pi}{t^2}\right) + \sin\left(\frac{\pi}{t}\right) = \left(-\frac{\pi}{t}\right) \cos\left(\frac{\pi}{t}\right) + \sin\left(\frac{\pi}{t}\right)$$

$$f''(t) = \left(-\frac{\pi}{t}\right) \left[-\sin\left(\frac{\pi}{t}\right) \left(-\frac{\pi}{t^2}\right)\right] + \left(\frac{\pi}{t^2}\right) \cos\left(\frac{\pi}{t}\right) + \left(-\frac{\pi}{t^2}\right) \cos\left(\frac{\pi}{t}\right) = -\frac{\pi^2}{t^3} \sin\left(\frac{\pi}{t}\right)$$

$$f''(2) = -\frac{\pi^2}{8} \sin\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{8} \approx -1.23$$

$$15. f'(s) = s(3)(1-s^2)^2(-2s) + (1-s^2)^3 = -6s^2(1-s^2)^2 + (1-s^2)^3 = -7s^6 + 15s^4 - 9s^2 + 1$$

$$f''(s) = -42s^5 + 60s^3 - 18s$$

$$f''(2) = -900$$

$$16. f'(x) = \frac{(x-1)2(x+1) - (x+1)^2}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2}$$

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2 - 2x - 3)2(x-1)}{(x-1)^4} = \frac{(x-1)(2x-2) - (x^2 - 2x - 3)(2)}{(x-1)^3} = \frac{8}{(x-1)^3}$$

$$f''(2) = \frac{8}{1^3} = 8$$

$$17. D_x(x^n) = nx^{n-1}$$

$$D_x^2(x^n) = n(n-1)x^{n-2}$$

$$D_x^3(x^n) = n(n-1)(n-2)x^{n-3}$$

$$D_x^4(x^n) = n(n-1)(n-2)(n-3)x^{n-4}$$

⋮

$$D_x^{n-1}(x^n) = n(n-1)(n-2)(n-3)\dots(2)x$$

$$D_x^n(x^n) = n(n-1)(n-2)(n-3)\dots 2(1)x^0$$

$$= n!$$

$$18. \text{ Let } k < n.$$

$$D_x^n(x^k) = D_x^{n-k}[D_x^k(x^k)] = D_x(k!) = 0$$

$$\text{so } D_x^n[a_n x^{n-1} + \dots + a_1 x + a_0] = 0$$

$$19. \text{ a. } D_x^4(3x^3 + 2x - 19) = 0$$

$$\text{b. } D_x^{12}(100x^{11} - 79x^{10}) = 0$$

$$\text{c. } D_x^{11}(x^2 - 3)^5 = 0$$

$$20. D_x\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$D_x^2\left(\frac{1}{x}\right) = D_x(-x^{-2}) = 2x^{-3} = \frac{2}{x^3}$$

$$D_x^3\left(\frac{1}{x}\right) = D_x(2x^{-3}) = -\frac{3(2)}{x^4}$$

$$D_x^4\left(\frac{1}{x}\right) = \frac{4(3)(2)}{x^5}$$

$$D_x^n\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$$

$$21. f'(x) = 3x^2 + 6x - 45 = 3(x+5)(x-3)$$

$$3(x+5)(x-3) = 0$$

$$x = -5, x = 3$$

$$f''(x) = 6x + 6$$

$$f''(-5) = -24$$

$$f''(3) = 24$$

$$22. g(1) = a + b + c = 5$$

$$g'(t) = 2at + b$$

$$g'(1) = 2a + b = 3$$

$$g''(t) = 2a$$

$$g''(1) = 2a = -4$$

$$a = -2$$

$$2(-2) + b = 3$$

$$b = 7$$

$$(-2) + (7) + c = 5$$

$$c = 0$$

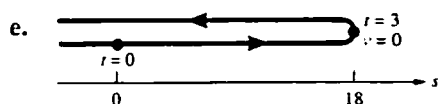
$$23. \text{ a. } v(t) = \frac{ds}{dt} = 12 - 4t$$

$$a(t) = \frac{d^2s}{dt^2} = -4$$

b. $12 - 4t > 0$
 $4t < 12$
 $t < 3; (-\infty, 3)$

c. $12 - 4t < 0$
 $t > 3; (3, \infty)$

d. $a(t) = -4 < 0$ for all t

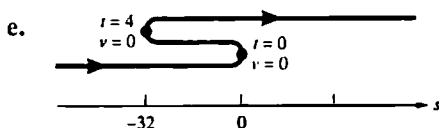


24. a. $v(t) = \frac{ds}{dt} = 3t^2 - 12t$
 $a(t) = \frac{d^2s}{dt^2} = 6t - 12$

b. $3t^2 - 12t > 0$
 $3t(t - 4) > 0; (-\infty, 0) \cup (4, \infty)$

c. $3t^2 - 12t < 0$
 $(0, 4)$

d. $6t - 12 < 0$
 $6t < 12$
 $t < 2; (-\infty, 2)$

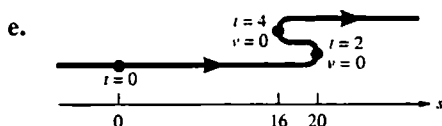


25. a. $v(t) = \frac{ds}{dt} = 3t^2 - 18t + 24$
 $a(t) = \frac{d^2s}{dt^2} = 6t - 18$

b. $3t^2 - 18t + 24 > 0$
 $3(t - 2)(t - 4) > 0$
 $(-\infty, 2) \cup (4, \infty)$

c. $3t^2 - 18t + 24 < 0$
 $(2, 4)$

d. $6t - 18 < 0$
 $6t < 18$
 $t < 3; (-\infty, 3)$



26. a. $v(t) = \frac{ds}{dt} = 6t^2 - 6$
 $a(t) = \frac{d^2s}{dt^2} = 12t$

b. $6t^2 - 6 > 0$
 $6(t + 1)(t - 1) > 0$
 $(-\infty, -1) \cup (1, \infty)$

c. $6t^2 - 6 < 0$
 $(-1, 1)$

d. $12t < 0$
 $t < 0$
The acceleration is negative for negative t .

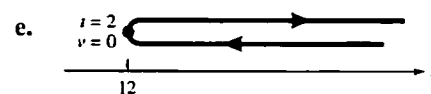


27. a. $v(t) = \frac{ds}{dt} = 2t - \frac{16}{t^2}$
 $a(t) = \frac{d^2s}{dt^2} = 2 + \frac{32}{t^3}$

b. $2t - \frac{16}{t^2} > 0$
 $\frac{2t^3 - 16}{t^2} > 0; (2, \infty)$

c. $2t - \frac{16}{t^2} < 0; (0, 2)$

d. $2 + \frac{32}{t^3} < 0$
 $\frac{2t^3 + 32}{t^3} < 0$; The acceleration is not negative for any positive t .

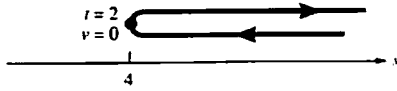


28. a. $v(t) = \frac{ds}{dt} = 1 - \frac{4}{t^2}$
 $a(t) = \frac{d^2s}{dt^2} = \frac{8}{t^3}$

b. $1 - \frac{4}{t^2} > 0$
 $\frac{t^2 - 4}{t^2} > 0; (2, \infty)$

c. $1 - \frac{4}{t^2} < 0$; $(0, 2)$

d. $\frac{8}{t^3} < 0$; The acceleration is not negative for any positive t .

e. 

29. $v(t) = \frac{ds}{dt} = 2t^3 - 15t^2 + 24t$

$$a(t) = \frac{d^2s}{dt^2} = 6t^2 - 30t + 24$$

$$6t^2 - 30t + 24 = 0$$

$$6(t-4)(t-1) = 0$$

$$t = 4, 1$$

$$v(4) = -16, v(1) = 11$$

30. $v(t) = \frac{ds}{dt} = \frac{1}{10}(4t^3 - 42t^2 + 120t)$

$$a(t) = \frac{d^2s}{dt^2} = \frac{1}{10}(12t^2 - 84t + 120)$$

$$\frac{1}{10}(12t^2 - 84t + 120) = 0$$

$$\frac{12}{10}(t-2)(t-5) = 0$$

$$t = 2, t = 5$$

$$v(2) = 10.4, v(5) = 5$$

31. $v_1(t) = \frac{ds_1}{dt} = 4 - 6t$

$$v_2(t) = \frac{ds_2}{dt} = 2t - 2$$

a. $4 - 6t = 2t - 2$

$$8t = 6$$

$$t = \frac{3}{4} \text{ sec}$$

b. $|4 - 6t| = |2t - 2|$; $4 - 6t = -2t + 2$

$$t = \frac{1}{2} \text{ sec and } t = \frac{3}{4} \text{ sec}$$

c. $4t - 3t^2 = t^2 - 2t$

$$4t^2 - 6t = 0$$

$$2t(2t - 3) = 0$$

$$t = 0 \text{ sec and } t = \frac{3}{2} \text{ sec}$$

32. $v_1(t) = \frac{ds_1}{dt} = 9t^2 - 24t + 18$

$$v_2(t) = \frac{ds_2}{dt} = -3t^2 + 18t - 12$$

$$9t^2 - 24t + 18 = -3t^2 + 18t - 12$$

$$12t^2 - 42t + 30 = 0$$

$$2t^2 - 7t + 5 = 0$$

$$(2t - 5)(t - 1) = 0$$

$$t = 1, \frac{5}{2}$$

33. a. $v(t) = -32t + 48$

$$\text{initial velocity} = v_0 = 48 \text{ ft/sec}$$

b. $-32t + 48 = 0$

$$t = \frac{3}{2} \text{ sec}$$

c. $s = -16(1.5)^2 + 48(1.5) + 256 = 292 \text{ ft}$

d. $-16t^2 + 48t + 256 = 0$

$$t = \frac{-48 \pm \sqrt{48^2 - 4(-16)(256)}}{-32} \approx -2.77, 5.77$$

The object hits the ground at $t = 5.77 \text{ sec}$.

e. $v(5.77) \approx -137 \text{ ft/sec}$;

$$\text{speed} = |-137| = 137 \text{ ft/sec}.$$

34. $v(t) = 48 - 32t$

a. $48 - 32t = 0$

$$t = 1.5$$

$$s = 48(1.5) - 16(1.5)^2 = 36 \text{ ft}$$

b. $v(1) = 16 \text{ ft/sec upward}$

c. $48t - 16t^2 = 0$

$$-16t(-3 + t) = 0$$

$$t = 3 \text{ sec}$$

35. $v(t) = v_0 - 32t$

$$v_0 - 32t = 0$$

$$t = \frac{v_0}{32}$$

$$v_0 \left(\frac{v_0}{32} \right) - 16 \left(\frac{v_0}{32} \right)^2 = 5280$$

$$\frac{v_0^2}{32} - \frac{v_0^2}{64} = 5280$$

$$\frac{v_0^2}{64} = 5280$$

$$v_0 = \sqrt{337,920} \approx 581 \text{ ft/sec}$$

36. $v(t) = v_0 + 32t$

$v_0 + 32t = 140$

$v_0 + 32(3) = 140$

$v_0 = 44$

$s = 44(3) + 16(3)^2 = 276 \text{ ft}$

37. $v(t) = 3t^2 - 6t - 24$

$$\frac{d}{dt} |3t^2 - 6t - 24| = \frac{|3t^2 - 6t - 24|}{3t^2 - 6t - 24} (6t - 6)$$

$$= \frac{|(t-4)(t+2)|}{(t-4)(t+2)} (6t-6)$$

$$\frac{|(t-4)(t+2)|(6t-6)}{(t-4)(t+2)} < 0$$

$t < -2, 1 < t < 4; (-\infty, -2) \cup (1, 4)$

38. Point slowing down when

$$\frac{d}{dt} |v(t)| < 0$$

$$\frac{d}{dt} |v(t)| = \frac{|v(t)|a(t)}{v(t)}$$

$$\frac{|v(t)|a(t)}{v(t)} < 0 \text{ when } a(t) \text{ and } v(t) \text{ have opposite signs.}$$

39. Let s be the distance traveled. Then $\frac{ds}{dt}$ is the speed of the car.

a. $\frac{ds}{dt} = ks, k$ a constant

b. $\frac{d^2s}{dt^2} > 0$

c. $\frac{d^3s}{dt^3} < 0, \frac{d^2s}{dt^2} > 0$

d. $\frac{d^2s}{dt^2} = 10 \text{ mph/min}$

e. $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ are approaching zero.

f. $\frac{ds}{dt}$ is constant.

40. a. $\frac{dV}{dt} = k < 0$, V is the volume of water in the tank, k is a constant.

b. $\frac{dV}{dt} = 3 - \frac{1}{2} = 2\frac{1}{2} \text{ gal/min}$

c. $\frac{dV}{dt} = k, \frac{dh}{dt} > 0, \frac{d^2h}{dt^2} < 0$

d. $I(t) = k$ now, but $\frac{dI}{dt}, \frac{d^2I}{dt^2} > 0$ in the future where I is inflation.

e. $\frac{dp}{dt} < 0$, but $\frac{d^2p}{dt^2} > 0$ and at $t = 2$: $\frac{dp}{dt} > 0$. where p is the price of oil.

f. $\frac{dT}{dt} > 0, \frac{d^2T}{dt^2} < 0$, where T is David's temperature.

41. a. $\frac{dC}{dt} > 0, \frac{d^2C}{dt^2} > 0$, where C is the car's cost.

b. $f(t)$ is oil consumption at time t .

$$\frac{df}{dt} < 0, \frac{d^2f}{dt^2} > 0$$

c. $\frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$, where P is world population.

d. $\frac{d\theta}{dt} > 0, \frac{d^2\theta}{dt^2} > 0$, where θ is the angle that the tower makes with the vertical.

e. $P = f(t)$ is profit at time t .

$$\frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$$

f. R is revenue at time t .

$$R < 0, \frac{dR}{dt} > 0$$

42. a. $R(t) \approx 0.28, t < 1981$

b. On $[1981, 1983]$, $\frac{dR}{dt} > 0, \frac{d^2R}{dt^2} > 0$, $R(1983) \approx 0.36$

43. $D_x(uv) = uv' + u'v$

$$D_x^2(uv) = uv'' + u'v' + u'v' + u''v \\ = uv'' + 2u'v' + u''v$$

$$D_x^3(uv) = uv''' + u'v'' + 2(u'v'' + u''v') + u''v' + u'''v \\ = uv''' + 3u'v'' + 3u''v' + u'''v$$

$$D_x^n(uv) = \sum_{k=0}^n \binom{n}{k} D_x^{n-k}(u) D_x^k(v)$$

where $\binom{n}{k}$ is the binomial coefficient

$$\frac{n!}{(n-k)!k!}$$

44. $D_x^4(x^4 \sin x) = \binom{4}{0} D_x^4(x^4) D_x^0(\sin x) \\ + \binom{4}{1} D_x^3(x^4) D_x^1(\sin x) + \binom{4}{2} D_x^2(x^4) D_x^2(\sin x) \\ + \binom{4}{3} D_x^1(x^4) D_x^3(\sin x) + \binom{4}{4} D_x^0(x^4) D_x^4(\sin x) \\ = 24 \sin x + 96x \cos x - 72x^2 \sin x - 16x^3 \cos x + x^4 \sin x$

3.8 Concepts Review

1. $\frac{9}{x^3 - 3}$

2. $3y^2 \frac{dy}{dx}$

3. $x(2y) \frac{dy}{dx} + y^2 + 3y^2 \frac{dy}{dx} - \frac{dy}{dx} = 3x^2$

4. $\frac{p}{q} x^{p/q-1}; \frac{5}{3} (x^2 - 5x)^{2/3} (2x - 5)$

Problem Set 3.8

1. $2y D_x y - 2x = 0$

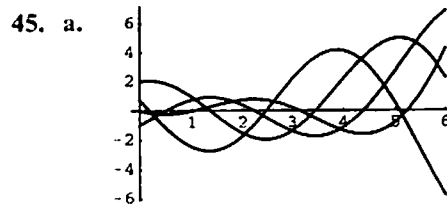
$$D_x y = \frac{2x}{2y} = \frac{x}{y}$$

2. $18x + 8y D_x y = 0$

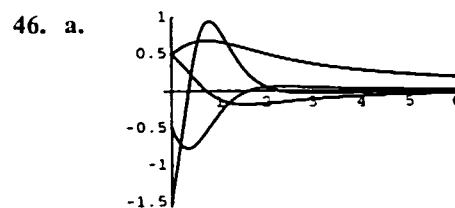
$$D_x y = \frac{-18x}{8y} = -\frac{9x}{4y}$$

3. $x D_x y + y = 0$

$$D_x y = -\frac{y}{x}$$



b. $f'''(2.13) \approx -1.2826$



b. $f'''(2.13) \approx 0.0271$

4. $2x + 2\alpha^2 y D_x y = 0$

$$D_x y = -\frac{2x}{2\alpha^2 y} = -\frac{x}{\alpha^2 y}$$

5. $x(2y) D_x y + y^2 = 1$

$$D_x y = \frac{1 - y^2}{2xy}$$

6. $2x + 2x^2 D_x y + 4xy + 3x D_x y + 3y = 0$

$$D_x y(2x^2 + 3x) = -2x - 4xy - 3y$$

$$D_x y = \frac{-2x - 4xy - 3y}{2x^2 + 3x}$$

7. $12x^2 + 7x(2y) D_x y + 7y^2 = 6y^2 D_x y$

$$12x^2 + 7y^2 = 6y^2 D_x y - 14xy D_x y$$

$$D_x y = \frac{12x^2 + 7y^2}{6y^2 - 14xy}$$

8. $x^2 D_x y + 2xy = y^2 + x(2y) D_x y$

$$x^2 D_x y - 2xy D_x y = y^2 - 2xy$$

$$D_x y = \frac{y^2 - 2xy}{x^2 - 2xy}$$

$$\begin{aligned}
 9. \quad & \frac{1}{2\sqrt{5xy}} \cdot (5x D_x y + 5y) + 2 D_x y \\
 &= 2y D_x y + x(3y^2) D_x y + y^3 \\
 & \frac{5x}{2\sqrt{5xy}} D_x y + 2 D_x y - 2y D_x y - 3xy^2 D_x y \\
 &= y^3 - \frac{5y}{2\sqrt{5xy}} \\
 & D_x y = \frac{y^3 - \frac{5y}{2\sqrt{5xy}}}{\frac{5x}{2\sqrt{5xy}} + 2 - 2y - 3xy^2}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad & x \frac{1}{2\sqrt{y+1}} D_x y + \sqrt{y+1} = x D_x y + y \\
 & \frac{x}{2\sqrt{y+1}} D_x y - x D_x y = y - \sqrt{y+1} \\
 & D_x y = \frac{y - \sqrt{y+1}}{\frac{x}{2\sqrt{y+1}} - x}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad & x D_x y + y + \cos(xy)(x D_x y + y) = 0 \\
 & x D_x y + x \cos(xy) D_x y = -y - y \cos(xy) \\
 & D_x y = \frac{-y - y \cos(xy)}{x + x \cos(xy)} = -\frac{y}{x}
 \end{aligned}$$

$$\begin{aligned}
 12. \quad & -\sin(xy^2)(2xy D_x y + y^2) = 2y D_x y + 1 \\
 & -2xy \sin(xy^2) D_x y - 2y D_x y = 1 + y^2 \sin(xy^2) \\
 & D_x y = \frac{1 + y^2 \sin(xy^2)}{-2xy \sin(xy^2) - 2y}
 \end{aligned}$$

$$\begin{aligned}
 13. \quad & x^3 y' + 3x^2 y + y^3 + 3xy^2 y' = 0 \\
 & y'(x^3 + 3xy^2) = -3x^2 y - y^3 \\
 & y' = \frac{-3x^2 y - y^3}{x^3 + 3xy^2} \\
 & \text{At } (1, 3), y' = -\frac{36}{28} = -\frac{9}{7} \\
 & \text{Tangent line: } y - 3 = -\frac{9}{7}(x - 1)
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & x^2(2y)y' + 2xy^2 + 4xy' + 4y = 12y' \\
 & y'(2x^2 y + 4x - 12) = -2xy^2 - 4y \\
 & y' = \frac{-2xy^2 - 4y}{2x^2 y + 4x - 12} = \frac{-xy^2 - 2y}{x^2 y + 2x - 6} \\
 & \text{At } (2, 1), y' = -2 \\
 & \text{Tangent line: } y - 1 = -2(x - 2)
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & \cos(xy)(xy' + y) = y' \\
 & y'[x \cos(xy) - 1] = -y \cos(xy) \\
 & y' = \frac{-y \cos(xy)}{x \cos(xy) - 1} = \frac{y \cos(xy)}{1 - x \cos(xy)} \\
 & \text{At } \left(\frac{\pi}{2}, 1\right), y' = 0 \\
 & \text{Tangent line: } y - 1 = 0 \left(x - \frac{\pi}{2}\right) \\
 & y = 1
 \end{aligned}$$

$$\begin{aligned}
 16. \quad & y' + [-\sin(xy^2)][2xyy' + y^2] + 6x = 0 \\
 & y'[1 - 2xy \sin(xy^2)] = y^2 \sin(xy^2) - 6x \\
 & y' = \frac{y^2 \sin(xy^2) - 6x}{1 - 2xy \sin(xy^2)} \\
 & \text{At } (1, 0), y' = -\frac{6}{1} = -6 \\
 & \text{Tangent line: } y - 0 = -6(x - 1)
 \end{aligned}$$

$$\begin{aligned}
 17. \quad & \frac{2}{3} x^{-1/3} - \frac{2}{3} y^{-1/3} y' - 2y' = 0 \\
 & \frac{2}{3} x^{-1/3} = y' \left(\frac{2}{3} y^{-1/3} + 2 \right) \\
 & y' = \frac{\frac{2}{3} x^{-1/3}}{\frac{2}{3} y^{-1/3} + 2} \\
 & \text{At } (1, -1), y' = \frac{\frac{2}{3}}{\frac{4}{3} + 2} = \frac{1}{2} \\
 & \text{Tangent line: } y + 1 = \frac{1}{2}(x - 1)
 \end{aligned}$$

$$\begin{aligned}
 18. \quad & \frac{1}{2\sqrt{y}} y' + 2xyy' + y^2 = 0 \\
 & y' \left(\frac{1}{2\sqrt{y}} + 2xy \right) = -y^2 \\
 & y' = \frac{-y^2}{\frac{1}{2\sqrt{y}} + 2xy} \\
 & \text{At } (4, 1), y' = \frac{-1}{\frac{17}{2}} = -\frac{2}{17} \\
 & \text{Tangent line: } y - 1 = -\frac{2}{17}(x - 4)
 \end{aligned}$$

$$19. \quad \frac{dy}{dx} = 5x^{2/3} + \frac{1}{2\sqrt{x}}$$

$$20. \quad \frac{dy}{dx} = \frac{1}{3} x^{-2/3} - 7x^{5/2} = \frac{1}{3\sqrt[3]{x^2}} - 7x^{5/2}$$

$$21. \frac{dy}{dx} = \frac{1}{3}x^{-2/3} - \frac{1}{3}x^{-4/3} = \frac{1}{3\sqrt[3]{x^2}} - \frac{1}{3\sqrt[3]{x^4}}$$

$$22. \frac{dy}{dx} = \frac{1}{4}(2x+1)^{-3/4}(2) = \frac{1}{2\sqrt[4]{(2x+1)^3}}$$

$$23. \frac{dy}{dx} = \frac{1}{4}(3x^2-4x)^{-3/4}(6x-4) \\ = \frac{6x-4}{4\sqrt[4]{(3x^2-4x)^3}} = \frac{3x-2}{2\sqrt[4]{(3x^2-4x)^3}}$$

$$24. \frac{dy}{dx} = \frac{1}{3}(x^3-2x)^{-2/3}(3x^2-2)$$

$$25. \frac{dy}{dx} = \frac{d}{dx}[(x^3+2x)^{-2/3}] \\ = -\frac{2}{3}(x^3+2x)^{-5/3}(3x^2+2) = -\frac{6x^2+4}{3\sqrt[3]{(x^3+2x)^5}}$$

$$26. \frac{dy}{dx} = -\frac{5}{3}(3x-9)^{-8/3}(3) = -5(3x-9)^{-8/3}$$

$$27. \frac{dy}{dx} = \frac{1}{2\sqrt{x^2+\sin x}}(2x+\cos x) \\ = \frac{2x+\cos x}{2\sqrt{x^2+\sin x}}$$

$$28. \frac{dy}{dx} = \frac{1}{2\sqrt{x^2\cos x}}[x^2(-\sin x)+2x\cos x] \\ = \frac{2x\cos x - x^2\sin x}{2\sqrt{x^2\cos x}}$$

$$29. \frac{dy}{dx} = \frac{d}{dx}[(x^2\sin x)^{-1/3}] \\ = -\frac{1}{3}(x^2\sin x)^{-4/3}(x^2\cos x + 2x\sin x) \\ = -\frac{x^2\cos x + 2x\sin x}{3\sqrt[3]{(x^2\sin x)^4}}$$

$$30. \frac{dy}{dx} = \frac{1}{4}(1+\sin 5x)^{-3/4}(\cos 5x)(5) \\ = \frac{5\cos 5x}{4\sqrt[4]{(1+\sin 5x)^3}}$$

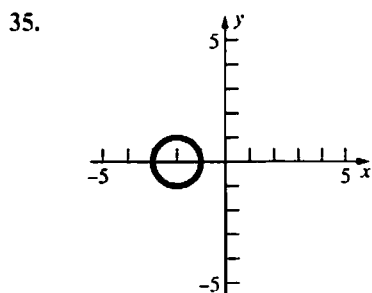
$$31. \frac{dy}{dx} = \frac{[1+\cos(x^2+2x)]^{-3/4}[-\sin(x^2+2x)(2x+2)]}{4}$$

$$= -\frac{(x+1)\sin(x^2+2x)}{2\sqrt[4]{[1+\cos(x^2+2x)]^3}}$$

$$32. \frac{dy}{dx} = \frac{(\tan^2 x + \sin^2 x)^{-1/2}(2\tan x \sec^2 x + 2\sin x \cos x)}{2} \\ = \frac{\tan x \sec^2 x + \sin x \cos x}{\sqrt{\tan^2 x + \sin^2 x}}$$

$$33. s^2 + 2st \frac{ds}{dt} + 3t^2 = 0 \\ \frac{ds}{dt} = \frac{-s^2 - 3t^2}{2st} = -\frac{s^2 + 3t^2}{2st} \\ s^2 \frac{dt}{ds} + 2st + 3t^2 \frac{dt}{ds} = 0 \\ \frac{dt}{ds}(s^2 + 3t^2) = -2st \\ \frac{dt}{ds} = -\frac{2st}{s^2 + 3t^2}$$

$$34. 1 = \cos(x^2)(2x) \frac{dx}{dy} + 6x^2 \frac{dx}{dy} \\ \frac{dx}{dy} = \frac{1}{2x\cos(x^2) + 6x^2}$$



$$2x + 4 + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x+4}{2y} = -\frac{x+2}{y}$$

The tangent line at (x_0, y_0) has equation

$$y - y_0 = -\frac{x_0 + 2}{y_0}(x - x_0) \text{ which simplifies to}$$

$$2x_0 - yy_0 - 2x - xx_0 + y_0^2 + x_0^2 = 0. \text{ Since}$$

$$(x_0, y_0) \text{ is on the circle, } x_0^2 + y_0^2 = -3 - 4x_0,$$

so the equation of the tangent line is

$$-yy_0 - 2x_0 - 2x - xx_0 = 3.$$

If $(0, 0)$ is on the tangent line, then $x_0 = -\frac{3}{2}$.

Solve for y_0 in the equation of the circle to get

$y_0 = \pm \frac{\sqrt{3}}{2}$. Put these values into the equation of the tangent line to get that the tangent lines are $\sqrt{3}y + x = 0$ and $\sqrt{3}y - x = 0$.

$$36. \quad 16(x^2 + y^2)(2x + 2yy') = 100(2x - 2yy')$$

$$32x^3 + 32x^2yy' + 32xy^2 + 32y^3y' = 200x - 200yy'$$

$$y'(4x^2y + 4y^3 + 25y) = 25x - 4x^3 - 4xy^2$$

$$y' = \frac{25x - 4x^3 - 4xy^2}{4x^2y + 4y^3 + 25y}$$

The slope of the normal line $= -\frac{1}{y'}$

$$= \frac{4x^2y + 4y^3 + 25y}{4x^3 + 4xy^2 - 25x}$$

$$\text{At } (3, 1), \text{ slope} = \frac{65}{45} = \frac{13}{9}$$

$$\text{Normal line: } y - 1 = \frac{13}{9}(x - 3)$$

$$37. \quad a. \quad xy' + y + 3y^2y' = 0$$

$$y'(x + 3y^2) = -y$$

$$y' = -\frac{y}{x + 3y^2}$$

$$b. \quad xy'' + \left(\frac{-y}{x + 3y^2}\right) + \left(\frac{-y}{x + 3y^2}\right) + 3y^2y''$$

$$+ 6y\left(\frac{-y}{x + 3y^2}\right)^2 = 0$$

$$xy'' + 3y^2y'' - \frac{2y}{x + 3y^2} + \frac{6y^3}{(x + 3y^2)^2} = 0$$

$$y''(x + 3y^2) = \frac{2y}{x + 3y^2} - \frac{6y^3}{(x + 3y^2)^2}$$

$$y''(x + 3y^2) = \frac{2xy}{(x + 3y^2)^2}$$

$$y'' = \frac{2xy}{(x + 3y^2)^3}$$

$$38. \quad 3x^2 - 8yy' = 0$$

$$y' = \frac{3x^2}{8y}$$

$$6x - 8(yy'' + (y')^2) = 0$$

$$6x - 8yy'' - 8\left(\frac{3x^2}{8y}\right)^2 = 0$$

$$6x - 8yy'' - \frac{9x^4}{8y^2} = 0$$

$$\frac{48xy^2 - 9x^4}{8y^2} = 8yy''$$

$$y'' = \frac{48xy^2 - 9x^4}{64y^3}$$

$$39. \quad 2(x^2y' + 2xy) - 12y^2y' = 0$$

$$2x^2y' - 12y^2y' = -4xy$$

$$y' = \frac{2xy}{6y^2 - x^2}$$

$$2(x^2y'' + 2xy' + 2xy' + 2y) - 12[y^2y'' + 2y(y')^2] = 0$$

$$2x^2y'' - 12y^2y'' = -8xy' - 4y + 24y(y')^2$$

$$y''(2x^2 - 12y^2) = -\frac{16x^2y}{6y^2 - x^2} - 4y + \frac{96x^2y^3}{(6y^2 - x^2)^2}$$

$$y''(2x^2 - 12y^2) = \frac{12x^4y + 48x^2y^3 - 144y^5}{(6y^2 - x^2)^2}$$

$$y''(6y^2 - x^2) = \frac{72y^5 - 6x^4y - 24x^2y^3}{(6y^2 - x^2)^2}$$

$$y'' = \frac{72y^5 - 6x^4y - 24x^2y^3}{(6y^2 - x^2)^3}$$

$$\text{At } (2, 1), y'' = \frac{-120}{8} = -15$$

$$40. \quad 2x + 2yy' = 0$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

$$2 + 2[yy'' + (y')^2] = 0$$

$$2 + 2yy'' + 2\left(-\frac{x}{y}\right)^2 = 0$$

$$2yy'' = -2 - \frac{2x^2}{y^2}$$

$$y'' = -\frac{1}{y} - \frac{x^2}{y^3} = -\frac{y^2 + x^2}{y^3}$$

$$\text{At } (3, 4), y'' = -\frac{25}{64}$$

41. $3x^2 + 3y^2 y' = 3(xy' + y)$

$$y'(3y^2 - 3x) = 3y - 3x^2$$

$$y' = \frac{y - x^2}{y^2 - x}$$

At $\left(\frac{3}{2}, \frac{3}{2}\right)$, $y' = -1$

Slope of the normal line is 1.

Normal line: $y - \frac{3}{2} = 1\left(x - \frac{3}{2}\right)$; $y = x$

This line includes the point (0, 0).

42. $xy' + y = 0$

$$y' = -\frac{y}{x}$$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

The slopes of the tangents are negative reciprocals, so the hyperbolas intersect at right angles.

43. Implicitly differentiate the first equation.

$$4x + 2yy' = 0$$

$$y' = -\frac{2x}{y}$$

Implicitly differentiate the second equation.

$$2yy' = 4$$

$$y' = \frac{2}{y}$$

Solve for the points of intersection.

$$2x^2 + 4x = 6$$

$$2(x^2 + 2x - 3) = 0$$

$$(x + 3)(x - 1) = 0$$

$$x = -3, x = 1$$

$x = -3$ is extraneous, and $y = -2, 2$ when $x = 1$.

The graphs intersect at (1, -2) and (1, 2).

At (1, -2): $m_1 = 1, m_2 = -1$

At (1, 2): $m_1 = -1, m_2 = 1$

44. $(x - 1)^2 + (1 - x^2) = 1$

$$x^2 - 2x + 1 + 1 - x^2 = 1$$

$$x = \frac{1}{2}$$

Points of intersection: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Implicitly differentiate the first equation.

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

Implicitly differentiate the second equation.

$$2(x - 1) + 2yy' = 0$$

$$y' = \frac{1 - x}{y}$$

At $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$: $m_1 = -\frac{1}{\sqrt{3}}, m_2 = \frac{1}{\sqrt{3}}$

$$\tan \theta = \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \sqrt{3}$$

$$\theta = \frac{\pi}{3}$$

At $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$: $m_1 = \frac{1}{\sqrt{3}}, m_2 = -\frac{1}{\sqrt{3}}$

$$\tan \theta = \frac{-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} = \frac{-\frac{2}{\sqrt{3}}}{\frac{2}{3}} = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3}$$

45. $x^2 - x(2x) + 2(2x)^2 = 28$

$$7x^2 = 28$$

$$x^2 = 4$$

$$x = -2, 2$$

Intersection point in first quadrant: (2, 4)

$$y'_1 = 2$$

$$2x - xy'_2 - y + 4yy'_2 = 0$$

$$y'_2(4y - x) = y - 2x$$

$$y'_2 = \frac{y - 2x}{4y - x}$$

At (2, 4): $m_1 = 2, m_2 = 0$

$$\tan \theta = \frac{0 - 2}{1 + (0)(2)} = -2; \theta = \pi + \tan^{-1}(-2) \approx 2.034$$

46. The equation is $mv^2 - mv_0^2 = kx_0^2 - kx^2$.

Differentiate implicitly with respect to t to get

$$2mv \frac{dv}{dt} = -2kx \frac{dx}{dt}. \text{ Since } v = \frac{dx}{dt} \text{ this simplifies}$$

$$\text{to } 2mv \frac{dv}{dt} = -2kxv \text{ or } m \frac{dv}{dt} = -kx.$$

47. $x^2 - xy + y^2 = 16$, when $y = 0$,

$$x^2 = 16$$

$$x = -4, 4$$

The ellipse intersects the x -axis at $(-4, 0)$ and $(4, 0)$.

$$2x - xy' - y + 2yy' = 0$$

$$y'(2y - x) = y - 2x$$

$$y' = \frac{y - 2x}{2y - x}$$

At $(-4, 0)$, $y' = 2$

At $(4, 0)$, $y' = 2$

Tangent lines: $y = 2(x + 4)$ and $y = 2(x - 4)$

48. $x^2 + 2xy \frac{dx}{dy} - 2xy - y^2 \frac{dx}{dy} = 0$

$$\frac{dx}{dy}(2xy - y^2) = 2xy - x^2;$$

$$\frac{dx}{dy} = \frac{2xy - x^2}{2xy - y^2}$$

$$\frac{2xy - x^2}{2xy - y^2} = 0 \text{ if } x(2y - x) = 0, \text{ which occurs}$$

when $x = 0$ or $y = \frac{x}{2}$. There are no points on

$$x^2y - xy^2 = 2 \text{ where } x = 0. \text{ If } y = \frac{x}{2}, \text{ then}$$

$$2 = x^2 \left(\frac{x}{2} \right) - x \left(\frac{x}{2} \right)^2 = \frac{x^3}{2} - \frac{x^3}{4} = \frac{x^3}{4} \text{ so } x = 2,$$

$$y = \frac{2}{2} = 1.$$

The tangent line is vertical at $(2, 1)$.

49. $2x + 2y \frac{dy}{dx} = 0; \frac{dy}{dx} = -\frac{x}{y}$

The tangent line at (x_0, y_0) has slope $-\frac{x_0}{y_0}$,

hence the equation of the tangent line is

$$y - y_0 = -\frac{x_0}{y_0}(x - x_0) \text{ which simplifies to}$$

$$yy_0 + xx_0 - (x_0^2 + y_0^2) = 0 \text{ or } yy_0 + xx_0 = 1$$

since (x_0, y_0) is on $x^2 + y^2 = 1$. If $(1.25, 0)$ is on the tangent line through (x_0, y_0) , $x_0 = 0.8$.

Put this into $x^2 + y^2 = 1$ to get $y_0 = 0.6$, since $y_0 > 0$. The line is $6y + 8x = 10$. When $x = -2$,

$$y = \frac{13}{3}, \text{ so the light bulb must be } \frac{13}{3} \text{ units high.}$$

3.9 Concepts Review

1. $\frac{du}{dt}; t = 2$

2. 400 mi/hr

3. negative

4. negative; positive

Problem Set 3.9

1. $V = x^3; \frac{dx}{dt} = 3$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

When $x = 12$, $\frac{dV}{dt} = 3(12)^2(3) = 1296 \text{ in.}^3/\text{s}$.

2. $V = \frac{4}{3}\pi r^3; \frac{dV}{dt} = 3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

When $r = 3$, $3 = 4\pi(3)^2 \frac{dr}{dt}$.

$$\frac{dr}{dt} = \frac{1}{12\pi} \approx 0.027 \text{ in./s}$$

3. $y^2 = x^2 + 1^2; \frac{dx}{dt} = 400$

$$2y \frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{x}{y} \frac{dx}{dt} \text{ mi/hr}$$

When $x = 5$, $y = \sqrt{26}$, $\frac{dy}{dt} = \frac{5}{\sqrt{26}}(400)$

$$\approx 392 \text{ mi/h.}$$

$$4. V = \frac{1}{3}\pi r^2 h; \frac{r}{h} = \frac{3}{10}; r = \frac{3h}{10}$$

$$V = \frac{1}{3}\pi \left(\frac{3h}{10}\right)^2 h = \frac{3\pi h^3}{100}; \frac{dV}{dt} = 3, h = 5$$

$$\frac{dV}{dt} = \frac{9\pi h^2}{100} \frac{dh}{dt}$$

$$\text{When } h = 5, 3 = \frac{9\pi(5)^2}{100} \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{3\pi} \approx 0.42 \text{ cm/s}$$

$$5. s^2 = (x+300)^2 + y^2; \frac{dx}{dt} = 300, \frac{dy}{dt} = 400,$$

$$2s \frac{ds}{dt} = 2(x+300) \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$s \frac{ds}{dt} = (x+300) \frac{dx}{dt} + y \frac{dy}{dt}$$

$$\text{When } x = 300, y = 400, s = 200\sqrt{13}, \text{ so}$$

$$200\sqrt{13} \frac{ds}{dt} = (300+300)(300) + 400(400)$$

$$\frac{ds}{dt} \approx 471 \text{ mi/h}$$

$$6. y^2 = x^2 + (10)^2; \frac{dy}{dt} = 2$$

$$2y \frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$\text{When } y = 25, x \approx 22.9, \text{ so}$$

$$\frac{dx}{dt} = \frac{y}{x} \frac{dy}{dt} \approx \frac{25}{22.9}(2) \approx 2.18 \text{ ft/s}$$

$$7. 20^2 = x^2 + y^2; \frac{dx}{dt} = 1$$

$$0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\text{When } x = 5, y = \sqrt{375} = 5\sqrt{15}, \text{ so}$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = -\frac{5}{5\sqrt{15}}(1) \approx -0.258 \text{ ft/s}$$

The top of the ladder is moving down at 0.258 ft/s.

$$8. \frac{dV}{dt} = -4 \text{ ft}^3/\text{h}; V = \pi h r^2; \frac{dh}{dt} = -0.0005 \text{ ft/h}$$

$$A = \pi r^2 = \frac{V}{h} = Vh^{-1}, \text{ so } \frac{dA}{dt} = h^{-1} \frac{dV}{dt} - \frac{V}{h^2} \frac{dh}{dt}$$

$$\text{When } h = 0.001 \text{ ft}, V = \pi(0.001)(250)^2 = 62.5\pi$$

$$\text{and } \frac{dA}{dt} = 1000(-4) - 1,000,000(62.5\pi)(-0.0005)$$

$$= -4000 + 31,250\pi \approx 94,175 \text{ ft}^2/\text{h}.$$

(The height is decreasing due to the spreading of the oil rather than the bacteria.)

$$9. V = \frac{1}{3}\pi r^2 h; h = \frac{d}{4} = \frac{r}{2}, r = 2h$$

$$V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3; \frac{dV}{dt} = 16$$

$$\frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$$

$$\text{When } h = 4, 16 = 4\pi(4)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{4\pi} \approx 0.0796 \text{ ft/s}$$

$$10. y^2 = x^2 + (90)^2; \frac{dx}{dt} = 5$$

$$2y \frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$\text{When } y = 150, x = 120, \text{ so}$$

$$\frac{dy}{dt} = \frac{x}{y} \frac{dx}{dt} = \frac{120}{150}(5) = 4 \text{ ft/s}$$

$$11. V = \frac{hx}{2}(20); \frac{40}{5} = \frac{x}{h}, x = 8h$$

$$V = 10h(8h) = 80h^2; \frac{dV}{dt} = 40$$

$$\frac{dV}{dt} = 160h \frac{dh}{dt}$$

$$\text{When } h = 3, 40 = 160(3) \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{12} \text{ ft/s}$$

$$12. y = \sqrt{x^2 - 4}; \frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x^2 - 4}}(2x) \frac{dx}{dt} = \frac{x}{\sqrt{x^2 - 4}} \frac{dx}{dt}$$

$$\text{When } x = 3, \frac{dy}{dt} = \frac{3}{\sqrt{3^2 - 4}}(5) = \frac{15}{\sqrt{5}} \approx 6.7 \text{ units/s}$$

$$13. A = \pi r^2; \frac{dr}{dt} = 0.02$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\text{When } r = 8.1, \frac{dA}{dt} = 2\pi(0.02)(8.1) = 0.324\pi$$

$$\approx 1.018 \text{ in.}^2/\text{s}$$

$$14. \quad s^2 = x^2 + (y + 48)^2; \frac{dx}{dt} = 30, \frac{dy}{dt} = 24$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2(y + 48) \frac{dy}{dt}$$

$$s \frac{ds}{dt} = x \frac{dx}{dt} + (y + 48) \frac{dy}{dt}$$

At 2:00 p.m., $x = 3(30) = 90$, $y = 3(24) = 72$,
so $s = 150$.

$$(150) \frac{ds}{dt} = 90(30) + (72 + 48)(24)$$

$$\frac{ds}{dt} = \frac{5580}{150} = 37.2 \text{ knots/h}$$

15. Let x be the distance from the beam to the point opposite the lighthouse and θ be the angle between the beam and the line from the lighthouse to the point opposite.

$$\tan \theta = \frac{x}{1}; \frac{d\theta}{dt} = 2(2\pi) = 4\pi \text{ rad/min.}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\text{At } x = \frac{1}{2}, \theta = \tan^{-1} \frac{1}{2} \text{ and } \sec^2 \theta = \frac{5}{4}.$$

$$\frac{dx}{dt} = \frac{5}{4}(4\pi) \approx 15.71 \text{ km/min}$$

$$16. \quad \tan \theta = \frac{4000}{x}$$

$$\sec^2 \theta \frac{d\theta}{dt} = -\frac{4000}{x^2} \frac{dx}{dt}$$

$$\text{When } \theta = \frac{1}{2}, \frac{d\theta}{dt} = \frac{1}{10} \text{ and } x = \frac{4000}{\tan \frac{1}{2}} \approx 7322.$$

$$\frac{dx}{dt} \approx \sec^2 \frac{1}{2} \left(\frac{1}{10} \right) \left[-\frac{(7322)^2}{4000} \right]$$

$$\approx -1740 \text{ ft/s or } -1186 \text{ mi/h}$$

The plane's ground speed is 1186 mi/h.

17. a. Let x be the distance along the ground from the light pole to Chris, and let s be the distance from Chris to the tip of his shadow.

$$\text{By similar triangles, } \frac{6}{s} = \frac{30}{x+s}, \text{ so } s = \frac{x}{4}$$

$$\text{and } \frac{ds}{dt} = \frac{1}{4} \frac{dx}{dt}. \quad \frac{dx}{dt} = 2 \text{ ft/s, hence}$$

$$\frac{ds}{dt} = \frac{1}{2} \text{ ft/s no matter how far from the light pole Chris is.}$$

- b. Let $l = x + s$, then

$$\frac{dl}{dt} = \frac{dx}{dt} + \frac{ds}{dt} = 2 + \frac{1}{2} = \frac{5}{2} \text{ ft/s.}$$

- c. The angular rate at which Chris must lift his head to follow his shadow is the same as the rate at which the angle that the light makes with the ground is decreasing. Let θ be the angle that the light makes with the ground at the tip of Chris' shadow.

$$\tan \theta = \frac{6}{s} \text{ so } \sec^2 \theta \frac{d\theta}{dt} = -\frac{6}{s^2} \frac{ds}{dt} \text{ and}$$

$$\frac{d\theta}{dt} = -\frac{6 \cos^2 \theta}{s^2} \frac{ds}{dt} = -\frac{1}{2} \text{ ft/s}$$

$$\text{When } s = 6, \theta = \frac{\pi}{4}, \text{ so}$$

$$\frac{d\theta}{dt} = -\frac{6 \left(\frac{1}{\sqrt{2}} \right)^2}{6^2} \left(\frac{1}{2} \right) = -\frac{1}{24}.$$

Chris must lift his head at the rate of

$$\frac{1}{24} \text{ rad/s.}$$

18. Let θ be the measure of the vertex angle, a be the measure of the equal sides, and b be the measure of the base. Observe that $b = 2a \sin \frac{\theta}{2}$ and the

$$\text{height of the triangle is } a \cos \frac{\theta}{2}.$$

$$A = \frac{1}{2} \left(2a \sin \frac{\theta}{2} \right) \left(a \cos \frac{\theta}{2} \right) = \frac{1}{2} a^2 \sin \theta$$

$$A = \frac{1}{2} (100)^2 \sin \theta = 5000 \sin \theta; \frac{dA}{dt} = \frac{1}{10}$$

$$\frac{dA}{dt} = 5000 \cos \theta \frac{d\theta}{dt}$$

$$\text{When } \theta = \frac{\pi}{6}, \frac{dA}{dt} = 5000 \left(\cos \frac{\pi}{6} \right) \left(\frac{1}{10} \right) = 250\sqrt{3}$$

$$\approx 433 \text{ cm}^2/\text{min}.$$

19. Let p be the point on the bridge directly above the railroad tracks. If a is the distance between p and the automobile, then $\frac{da}{dt} = 66 \text{ ft/s}$. If l is the distance between the train and the point directly below p , then $\frac{dl}{dt} = 88 \text{ ft/s}$. The distance from the

train to p is $\sqrt{100^2 + l^2}$, while the distance from p to the automobile is a . The distance between the train and automobile is

$$D = \sqrt{a^2 + \left(\sqrt{100^2 + l^2} \right)^2} = \sqrt{a^2 + l^2 + 100^2}.$$

$$\frac{dD}{dt} = \frac{1}{2\sqrt{a^2 + l^2 + 100^2}} \cdot \left(2a \frac{da}{dt} + 2l \frac{dl}{dt} \right)$$

$$= \frac{a \frac{da}{dt} + l \frac{dl}{dt}}{\sqrt{a^2 + l^2 + 100^2}}. \text{ After 10 seconds, } a = 660$$

and $l = 880$, so

$$\frac{dD}{dt} = \frac{660(66) + 880(88)}{\sqrt{660^2 + 880^2 + 100^2}} \approx 110 \text{ ft/s.}$$

20. $V = \frac{1}{3}\pi h \cdot (a^2 + ab + b^2); a = 20, b = \frac{h}{4} + 20,$

$$V = \frac{1}{3}\pi h \left(400 + 5h + 400 + \frac{h^2}{16} + 10h + 400 \right)$$

$$= \frac{1}{3}\pi \left(1200h + 15h^2 + \frac{h^3}{16} \right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi \left(1200 + 30h + \frac{3h^2}{16} \right) \frac{dh}{dt}$$

When $h = 30$ and $\frac{dV}{dt} = 2000$,

$$2000 = \frac{1}{3}\pi \left(1200 + 900 + \frac{675}{4} \right) \frac{dh}{dt} = \frac{3025\pi}{4} \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{320}{121\pi} \approx 0.84 \text{ cm/min.}$$

21. $V = \pi h^2 \left[r - \frac{h}{3} \right]; \frac{dV}{dt} = -2, r = 8$

$$V = \pi r h^2 - \frac{\pi h^3}{3} = 8\pi h^2 - \frac{\pi h^3}{3}$$

$$\frac{dV}{dt} = 16\pi h \frac{dh}{dt} - \pi h^2 \frac{dh}{dt}$$

When $h = 3$, $-2 = \frac{dh}{dt} [16\pi(3) - \pi(3)^2]$

$$\frac{dh}{dt} = \frac{-2}{39\pi} \approx -0.016 \text{ ft/hr}$$

22. $s^2 = a^2 + b^2 - 2ab \cos \theta;$

$$a = 5, b = 4, \frac{d\theta}{dt} = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \text{ rad/h}$$

$$s^2 = 41 - 40 \cos \theta$$

$$2s \frac{ds}{dt} = 40 \sin \theta \frac{d\theta}{dt}$$

At 3:00, $\theta = \frac{\pi}{2}$ and $s = \sqrt{41}$, so

$$2\sqrt{41} \frac{ds}{dt} = 40 \sin \left(\frac{\pi}{2} \right) \left(\frac{11\pi}{6} \right) = \frac{220\pi}{3}$$

$$\frac{ds}{dt} \approx 18 \text{ in./hr}$$

23. $PV = k$

$$P \frac{dV}{dt} + V \frac{dP}{dt} = 0$$

At $t = 6.5$, $P \approx 67$, $\frac{dP}{dt} \approx -30$, $V = 300$

$$\frac{dV}{dt} = -\frac{V}{P} \frac{dP}{dt} = -\frac{300}{67}(-30) \approx 134 \text{ in.}^3/\text{min}$$

24. $V = \pi h^2 \left(r - \frac{h}{3} \right); r = 20$

$$V = \pi h^2 \left(20 - \frac{h}{3} \right) = 20\pi h^2 - \frac{\pi}{3} h^3$$

$$\frac{dV}{dt} = (40\pi h - \pi h^2) \frac{dh}{dt}$$

At 7:00 a.m., $h = 15$, $\frac{dh}{dt} \approx -3$, so

$$\frac{dV}{dt} = (40\pi(15) - \pi(15)^2)(-3) \approx -1125\pi \approx -3534.$$

Webster City residents used water at the rate of $2400 + 3534 = 5934 \text{ ft}^3/\text{h}$.

25. Assuming that the tank is now in the shape of an upper hemisphere with radius r , we again let t be the number of hours past midnight and h be the height of the water at time t . The volume, V , of water in the tank at that time is given by

$$V = \frac{2}{3}\pi r^3 - \frac{\pi}{3}(r-h)^2(2r+h)$$

and so $V = \frac{16000}{3}\pi - \frac{\pi}{3}(20-h)^2(40+h)$

from which

$$\frac{dV}{dt} = -\frac{\pi}{3}(20-h)^2 \frac{dh}{dt} + \frac{2\pi}{3}(20-h)(40+h) \frac{dh}{dt}$$

At $t = 7$, $\frac{dV}{dt} \approx -525\pi \approx -1649$

Thus Webster City residents were using water at the rate of $2400 + 1649 = 4049$ cubic feet per hour at 7:00 A.M.

26. The amount of water used by Webster City can be found by

$$\text{usage} = \text{beginning amount} + \text{added amount} \\ - \text{remaining amount}$$

Thus the usage is

$$\approx \pi(20)^2(9) + 2400(12) - \pi(20)^2(10.5) \approx 26,915 \text{ ft}^3$$

over the 12 hour period.

27. a. Let x be the distance from the bottom of the wall to the end of the ladder on the ground, so $\frac{dx}{dt} = 2$ ft/s. Let y

be the height of the opposite end of the ladder. By similar triangles, $\frac{y}{12} = \frac{18}{\sqrt{144+x^2}}$, so $y = \frac{216}{\sqrt{144+x^2}}$.

$$\frac{dy}{dt} = -\frac{216}{2(144+x^2)^{3/2}} 2x \frac{dx}{dt} = -\frac{216x}{(144+x^2)^{3/2}} \frac{dx}{dt}$$

When the ladder makes an angle of 60° with the ground, $x = 4\sqrt{3}$ and $\frac{dy}{dt} = -\frac{216(4\sqrt{3})}{(144+48)^{3/2}} \cdot 2 = -1.125$ ft/s.

$$\text{b. } \frac{d^2y}{dt^2} = \frac{d}{dt} \left(-\frac{216x}{(144+x^2)^{3/2}} \frac{dx}{dt} \right) = \frac{d}{dt} \left(-\frac{216x}{(144+x^2)^{3/2}} \right) \frac{dx}{dt} - \frac{216x}{(144+x^2)^{3/2}} \cdot \frac{d^2x}{dt^2}$$

Since $\frac{dx}{dt} = 2$, $\frac{d^2x}{dt^2} = 0$, thus

$$\begin{aligned} \frac{d^2y}{dt^2} &= \left[\frac{-216(144+x^2)^{3/2} \frac{dx}{dt} + 216x \left(\frac{3}{2} \right) \sqrt{144+x^2} (2x) \frac{dx}{dt}}{(144+x^2)^3} \right] \frac{dx}{dt} \\ &= \frac{-216(144+x^2) + 648x^2}{(144+x^2)^{5/2}} \left(\frac{dx}{dt} \right)^2 = \frac{432x^2 - 31,104}{(144+x^2)^{5/2}} \left(\frac{dx}{dt} \right)^2 \end{aligned}$$

When the ladder makes an angle of 60° with the ground,

$$\frac{d^2y}{dt^2} = \frac{432 \cdot 48 - 31,104}{(144+48)^{5/2}} (2)^2 \approx -0.08 \text{ ft/s}^2$$

$$\frac{dV}{dt} = 12h\pi \frac{dh}{dt}$$

28. a. If the ball has radius 6 in., the volume of the water in the tank is

$$V = 8\pi h^2 - \frac{\pi h^3}{3} - \frac{4}{3}\pi \left(\frac{1}{2} \right)^3$$

$$= 8\pi h^2 - \frac{\pi h^3}{3} - \frac{\pi}{6}$$

$$\frac{dV}{dt} = 16\pi h \frac{dh}{dt} - \pi h^2 \frac{dh}{dt}$$

This is the same as in Problem 21, so $\frac{dh}{dt}$ is again -0.016 ft/hr.

- b. If the ball has radius 2 ft, and the height of the water in the tank is h feet with $2 \leq h \leq 3$, the part of the ball in the water has volume

$$\frac{4}{3}\pi(2)^3 - \pi(4-h)^2 \left[2 - \frac{4-h}{3} \right] = \frac{(6-h)h^2\pi}{3}.$$

The volume of water in the tank is

$$V = 8\pi h^2 - \frac{\pi h^3}{3} - \frac{(6-h)h^2\pi}{3} = 6h^2\pi$$

$$\frac{dh}{dt} = \frac{1}{12h\pi} \frac{dV}{dt}$$

When $h = 3$, $\frac{dh}{dt} = \frac{1}{36\pi}(-2) \approx -0.018$ ft/hr.

$$29. \frac{dV}{dt} = k(4\pi r^2)$$

$$\text{a. } V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$k(4\pi r^2) = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = k$$

- b. If the original volume was V_0 , the volume after 1 hour is $\frac{8}{27}V_0$. The original radius was $r_0 = \sqrt[3]{\frac{3}{4\pi}V_0}$ while the radius after 1 hour is $r_1 = \sqrt[3]{\frac{8}{27}V_0 \cdot \frac{3}{4\pi}} = \frac{2}{3}r_0$. Since $\frac{dr}{dt}$ is constant, $\frac{dr}{dt} = -\frac{1}{3}r_0$ unit/hr. The snowball will take 3 hours to melt completely.

30. Let P be the point on the ground where the ball hits. Then the distance from P to the bottom of the light pole is 10 ft. Let s be the distance between P and the shadow of the ball. The height of the ball t seconds after it is dropped is $64 - 16t^2$.

$$\text{By similar triangles, } \frac{48}{64 - 16t^2} = \frac{10 + s}{s}$$

$$(\text{for } t > 1), \text{ so } s = \frac{10t^2 - 40}{1 - t^2}.$$

$$\frac{ds}{dt} = \frac{20t(1 - t^2) - (10t^2 - 40)(-2t)}{(1 - t^2)^2} = -\frac{60t}{(1 - t^2)^2}$$

$$\text{The ball hits the ground when } t = 2, \frac{ds}{dt} = -\frac{120}{9}.$$

$$\text{The shadow is moving } \frac{120}{9} \approx 13.33 \text{ ft/s.}$$

31. Let l be the distance along the ground from the brother to the tip of the shadow. The shadow is

controlled by both siblings when $\frac{3}{l} = \frac{5}{l+4}$ or $l = 6$. Again using similar triangles, this occurs when $\frac{y}{20} = \frac{6}{3}$, so $y = 40$. Thus, the girl controls the tip of the shadow when $y \geq 40$ and the boy controls it when $y < 40$.

Let x be the distance along the ground from the light pole to the girl. $\frac{dx}{dt} = -4$

$$\text{When } y \geq 40, \frac{20}{y} = \frac{5}{y-x} \text{ or } y = \frac{4}{3}x.$$

$$\text{When } y < 40, \frac{20}{y} = \frac{3}{y-(x+4)} \text{ or } y = \frac{20}{17}(x+4).$$

$x = 30$ when $y = 40$. Thus,

$$y = \begin{cases} \frac{4}{3}x & \text{if } x \geq 30 \\ \frac{20}{17}(x+4) & \text{if } x < 30 \end{cases}$$

and

$$\frac{dy}{dt} = \begin{cases} \frac{4}{3} \frac{dx}{dt} & \text{if } x \geq 30 \\ \frac{20}{17} \frac{dx}{dt} & \text{if } x < 30 \end{cases}$$

Hence, the tip of the shadow is moving at the rate of $\frac{4}{3}(4) = \frac{16}{3}$ ft/s when the girl is at least 30 feet

from the light pole, and it is moving

$$\frac{20}{17}(4) = \frac{80}{17} \text{ ft/s when the girl is less than 30 ft}$$

from the light pole.

3.10 Concepts Review

1. $f'(x)dx$
2. $\Delta y; dy$
3. Δx is small.
4. larger ; smaller

Problem Set 3.10

1. $dy = (2x + 1)dx$
2. $dy = (21x^2 + 6x)dx$

$$3. \quad dy = -4(2x+3)^{-5}(2)dx = -8(2x+3)^{-5}dx$$

$$4. \quad dy = -2(3x^2 + x + 1)^{-3}(6x+1)dx \\ = -2(6x+1)(3x^2 + x + 1)^{-3}dx$$

$$5. \quad dy = 3(\sin x + \cos x)^2(\cos x - \sin x)dx$$

$$6. \quad dy = 3(\tan x + 1)^2(\sec^2 x)dx \\ = 3\sec^2 x(\tan x + 1)^2dx$$

$$7. \quad dy = -\frac{3}{2}(7x^2 + 3x - 1)^{-5/2}(14x+3)dx \\ = -\frac{3}{2}(14x+3)(7x^2 + 3x - 1)^{-5/2}dx$$

$$8. \quad dy = 2(x^{10} + \sqrt{\sin 2x})(10x^9 + \frac{1}{2\sqrt{\sin 2x}} \cdot (\cos 2x)(2))dx$$

$$= 2\left(10x^9 + \frac{\cos 2x}{\sqrt{\sin 2x}}\right)(x^{10} + \sqrt{\sin 2x})dx$$

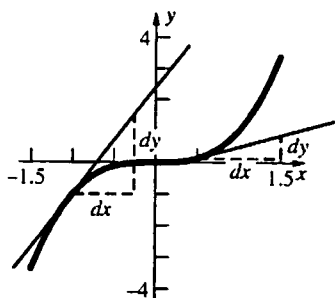
$$9. \quad ds = \frac{3}{2}(t^2 - \cot t + 2)^{1/2}(2t + \csc^2 t)dt$$

$$= \frac{3}{2}(2t + \csc^2 t)\sqrt{t^2 - \cot t + 2}dt$$

$$10. \quad a. \quad dy = 3x^2 dx = 3(0.5)^2(1) = 0.75$$

$$b. \quad dy = 3x^2 dx = 3(-1)^2(0.75) = 2.25$$

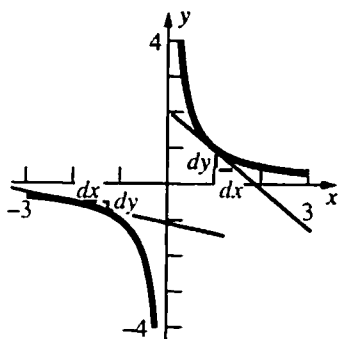
11.



$$12. \quad a. \quad dy = -\frac{dx}{x^2} = -\frac{0.5}{(1)^2} = -0.5$$

$$b. \quad dy = -\frac{dx}{x^2} = -\frac{0.75}{(-2)^2} = -0.1875$$

13.



$$14. \quad a. \quad \Delta y = (1.5)^3 - (0.5)^3 = 3.25$$

$$b. \quad \Delta y = (-0.25)^3 - (-1)^3 = 0.984375$$

$$15. \quad a. \quad \Delta y = \frac{1}{1.5} - \frac{1}{1} = -\frac{1}{3}$$

$$b. \quad \Delta y = \frac{1}{-1.25} + \frac{1}{2} = -0.3$$

$$16. \quad a. \quad \Delta y = [(2.5)^2 - 3] - [(2)^2 - 3] = 2.25$$

$$dy = 2x dx = 2(2)(0.5) = 2$$

$$b. \quad \Delta y = [(2.88)^2 - 3] - [(3)^2 - 3] = -0.7056$$

$$dy = 2x dx = 2(3)(-0.12) = -0.72$$

$$17. \quad a. \quad \Delta y = [(3)^4 + 2(3)] - [(2)^4 + 2(2)] = 67$$

$$dy = (4x^3 + 2)dx = [4(2)^3 + 2](1) = 34$$

$$b. \quad \Delta y = [(2.005)^4 + 2(2.005)] - [(2)^4 + 2(2)]$$

$$\approx 0.1706$$

$$dy = (4x^3 + 2)dx = [4(2)^3 + 2](0.005) = 0.17$$

$$18. \quad y = \sqrt{x}; dy = \frac{1}{2\sqrt{x}} dx; x = 400, dx = 2$$

$$dy = \frac{1}{2\sqrt{400}}(2) = 0.05$$

$$\sqrt{402} \approx \sqrt{400} + dy = 20 + 0.05 = 20.05$$

$$19. \quad y = \sqrt{x}; dy = \frac{1}{2\sqrt{x}} dx; x = 36, dx = -0.1$$

$$dy = \frac{1}{2\sqrt{36}}(-0.1) \approx -0.0083$$

$$\sqrt{35.9} \approx \sqrt{36} + dy = 6 - 0.0083 = 5.9917$$

$$20. \quad y = \sqrt[3]{x}; dy = \frac{1}{3}x^{-2/3} dx = \frac{1}{3\sqrt[3]{x^2}} dx;$$

$$x = 27, dx = -0.09$$

$$dy = \frac{1}{3\sqrt[3]{(27)^2}}(-0.09) \approx -0.0033$$

$$\sqrt[3]{26.91} \approx \sqrt[3]{27} + dy = 3 - 0.0033 = 2.9967$$

$$21. \quad V = \frac{4}{3}\pi r^3; r = 5, dr = 0.125$$

$$dV = 4\pi r^2 dr = 4\pi(5)^2(0.125) \approx 39.27 \text{ cm}^3$$

$$22. \quad V = x^3; x = \sqrt[3]{40}, dx = 0.5$$

$$dV = 3x^2 dx = 3(\sqrt[3]{40})^2(0.5) \approx 17.54 \text{ in}^3$$

$$23. \quad V = \frac{4}{3}\pi r^3; r = 6 \text{ ft} = 72 \text{ in}, dr = -0.3$$

$$dV = 4\pi r^2 dr = 4\pi(72)^2(-0.3) \approx -19,543$$

$$V \approx \frac{4}{3}\pi(72)^3 - 19,543$$

$$\approx 1,543,915 \text{ in}^3 \approx 893 \text{ ft}^3$$

24. $V = \pi r^2 h$; $r = 6 \text{ ft} = 72 \text{ in.}$, $dr = -0.05$,
 $h = 8 \text{ ft} = 96 \text{ in.}$
 $dV = 2\pi r h dr = 2\pi(72)(96)(-0.05) \approx -2171 \text{ in.}^3$
 About 9.4 gal of paint are needed.

25. $C = 2\pi r$; $r = 4000 \text{ mi} = 21,120,000 \text{ ft.}$, $dr = 2$
 $dC = 2\pi dr = 2\pi(2) = 4\pi \approx 12.6 \text{ ft}$

26. $T = 2\pi\sqrt{\frac{L}{32}}$; $L = 4$, $dL = -0.03$
 $dT = \frac{2\pi}{2\sqrt{\frac{L}{32}}} \cdot \frac{1}{32} \cdot dL = \frac{\pi}{\sqrt{32L}} dL$

$$dT = \frac{\pi}{\sqrt{32(4)}}(-0.03) \approx -0.0083$$

The time change in 24 hours is
 $(0.0083)(60)(60)(24) \approx 717 \text{ sec}$

27. $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(10)^3 \approx 4189$

$$dV = 4\pi r^2 dr = 4\pi(10)^2(0.05) \approx 62.8$$

The volume is $4189 \pm 62.8 \text{ cm}^3$.

The absolute error is ≈ 62.8 while the relative error is $62.8/4189 \approx 0.015$ or 1.5% .

28. $V = \pi r^2 h = \pi(3)^2(12) \approx 339$

$$dV = 24\pi r dr = 24\pi(3)(0.0025) \approx 0.565$$

The volume is $339 \pm 0.565 \text{ in.}^3$

The absolute error is ≈ 0.565 while the relative error is $0.565/339 \approx 0.0017$ or 0.17% .

29. $s = \sqrt{a^2 + b^2 - 2ab \cos \theta}$
 $= \sqrt{151^2 + 151^2 - 2(151)(151) \cos 0.53} \approx 79.097$

$$s = \sqrt{45,602 - 45,602 \cos \theta}$$

$$ds = \frac{1}{2\sqrt{45,602 - 45,602 \cos \theta}} \cdot 45,602 \sin \theta d\theta$$

$$= \frac{22,801 \sin \theta}{\sqrt{45,602 - 45,602 \cos \theta}} d\theta$$

$$= \frac{22,801 \sin 0.53}{\sqrt{45,602 - 45,602 \cos 0.53}} (0.005) \approx 0.729$$

$$s \approx 79.097 \pm 0.729 \text{ cm}$$

The absolute error is ≈ 0.729 while the relative error is $0.729/79.097 \approx 0.0092$ or 0.92% .

30. $A = \frac{1}{2}ab \sin \theta = \frac{1}{2}(151)(151) \sin 0.53 \approx 5763.33$

$$A = \frac{22,801}{2} \sin \theta; \theta = 0.53, d\theta = 0.005$$

$$dA = \frac{22,801}{2} (\cos \theta) d\theta$$

$$= \frac{22,801}{2} (\cos 0.53)(0.005) \approx 49.18$$

$$A \approx 5763.33 \pm 49.18 \text{ cm}^2$$

The absolute error is ≈ 49.18 while the relative error is $49.18/5763.33 \approx 0.0085$ or 0.85% .

31. $y = 3x^2 - 2x + 1$; $x = 2$, $dx = 0.001$

$$dy = (6x - 2)dx = [6(2) - 2](0.001) = 0.01$$

$$\frac{d^2y}{dx^2} = 6, \text{ so with } \Delta x = 0.001,$$

$$|\Delta y - dy| \leq \frac{1}{2}(6)(0.001)^2 = 0.000003$$

32. Using the approximation

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

we let $x = 1.02$ and $\Delta x = -0.02$. We can rewrite the above form as

$$f(x) \approx f(x + \Delta x) - f'(x)\Delta x$$

which gives

$$f(1.02) \approx f(1) - f'(1.02)(-0.02)$$

$$= 10 + 12(0.02) = 10.24$$

33. $V = \pi r^2 h + \frac{4}{3}\pi r^3$

$$V = 100\pi r^2 + \frac{4}{3}\pi r^3; r = 10, dr = 0.1$$

$$dV = (200\pi r + 4\pi r^2)dr$$

$$= (2000\pi + 400\pi)(0.1) = 240\pi \approx 754 \text{ cm}^3$$

34. From similar triangles, the radius at height h is

$$\frac{2}{5}h. \text{ Thus, } V = \frac{1}{3}\pi r^2 h = \frac{4}{75}\pi h^3, \text{ so}$$

$$dV = \frac{4}{25}\pi h^2 dh, h = 10, dh = -1:$$

$$dV = \frac{4}{25}\pi(100)(-1) \approx -50 \text{ cm}^3$$

The ice cube has volume $3^3 = 27 \text{ cm}^3$, so there is room for the ice cube without the cup overflowing.

35. The percent increase in mass is $\frac{dm}{m}$.

$$dm = -\frac{m_0}{2} \left(1 - \frac{v^2}{c^2} \right)^{-3/2} \left(-\frac{2v}{c^2} \right) dv$$

$$= \frac{m_0 v}{c^2} \left(1 - \frac{v^2}{c^2} \right)^{-3/2} dv$$

$$\frac{dm}{m} = \frac{v}{c^2} \left(1 - \frac{v^2}{c^2}\right)^{-1} dv = \frac{v}{c^2} \left(\frac{c^2}{c^2 - v^2}\right) dv$$

$$= \frac{v}{c^2 - v^2} dv$$

$$v = 0.9c, dv = 0.02c$$

$$\frac{dm}{m} = \frac{0.9c}{c^2 - 0.81c^2} (0.02c) = \frac{0.018}{0.19} \approx 0.095$$

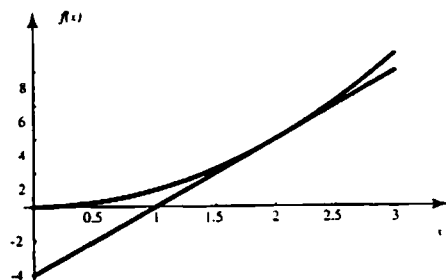
The percent increase in mass is about 9.5.

36. a. $f(x) = x^2$; $f'(x) = 2x$; $a = 2$

The linear approximation is then

$$L(x) = f(2) + f'(2)(x-2)$$

$$= 4 + 4(x-2) = 4x - 4$$



b. $g(x) = x^2 \cos x$; $g'(x) = -x^2 \sin x + 2x \cos x$

$$a = \pi/2$$

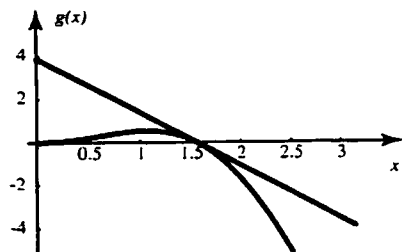
The linear approximation is then

$$L(x) = 0 + -\left(\frac{\pi}{2}\right)^2 \left(x - \frac{\pi}{2}\right)$$

$$= -\frac{\pi^2}{4}x + \frac{\pi^3}{8}$$

$$L(x) = 0 + -\frac{\pi^2}{4} \left(x - \frac{\pi}{2}\right)$$

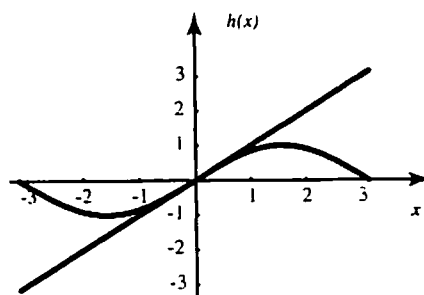
$$= -\frac{\pi^2}{4}x + \frac{\pi^3}{8}$$



37. a. $h(x) = \sin x$; $h'(x) = \cos x$; $a = 0$

The linear approximation is then

$$L(x) = 0 + 1(x-0) = x$$

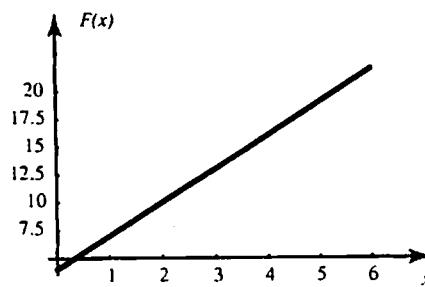


b. $F(x) = 3x + 4$; $F'(x) = 3$; $a = 3$

The linear approximation is then

$$L(x) = 13 + 3(x-3) = 13 + 3x - 9$$

$$= 3x + 4$$



38. The linear approximation to $f(x)$ at a is

$$L(x) = f(a) + f'(a)(x-a)$$

$$= a^2 + 2a(x-a)$$

$$= 2ax - a^2$$

Thus,

$$f(x) - L(x) = x^2 - (2ax - a^2)$$

$$= x^2 - 2ax + a^2$$

$$= (x-a)^2$$

$$\geq 0$$

3.11 Chapter Review

Concepts Test

1. False: If $f(x) = x^3$, $f'(x) = 3x^2$ and the tangent line $y = 0$ at $x = 0$ crosses the curve at the point of tangency.
2. False: The tangent line can touch the curve at infinitely many points.
3. True: $m_{\text{tan}} = 4x^3$, which is unique for each value of x .
4. False: $m_{\text{tan}} = -\sin x$, which is periodic.
5. True: If the velocity is negative and increasing, the speed is decreasing.
6. True: If the velocity is negative and decreasing, the speed is increasing.
7. True: If the tangent line is horizontal, the slope must be 0.
8. False: $f(x) = ax^2 + b$, $g(x) = ax^2 + c$, $b \neq c$. Then $f'(x) = 2ax = g'(x)$, but $f(x) \neq g(x)$.
9. True: $D_x f(g(x)) = f'(g(x))g'(x)$; since $g(x) = x$, $g'(x) = 1$, so $D_x f(g(x)) = f'(g(x))$.
10. False: $D_x y = 0$ because π is a constant, not a variable.
11. True: Theorem 3.2.A
12. True: The derivative does not exist when the tangent line is vertical.
13. False: $(f \cdot g)'(x) = f'(x)g'(x) + g(x)f'(x)$
14. True: Negative acceleration indicates decreasing velocity.
15. True: If $f(x) = x^3 g(x)$, then $D_x f(x) = x^3 g'(x) + 3x^2 g(x) = x^2 [xg'(x) + 3g(x)]$.
16. False: $D_x y = 3x^2$; At $(1, 1)$: $m_{\text{tan}} = 3(1)^2 = 3$
Tangent line: $y - 1 = 3(x - 1)$
17. False: $D_x y = f(x)g'(x) + g(x)f'(x)$
 $D_x^2 y = f(x)g''(x) + g'(x)f'(x) + g(x)f''(x) + f'(x)g'(x)$
 $= f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x)$
18. True: The degree of $y = (x^3 + x)^8$ is 24, so $D_x^{25} y = 0$.
19. True: $f(x) = ax^n$; $f'(x) = anx^{n-1}$
20. True: $D_x \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$
21. True: $h'(x) = f(x)g'(x) + g(x)f'(x)$
 $h'(c) = f(c)g'(c) + g(c)f'(c)$
 $= f(c)(0) + g(c)(0) = 0$
22. True: $f'\left(\frac{\pi}{2}\right) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - \sin\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{2}}$
 $= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{x - \frac{\pi}{2}}$
23. True: $D^2(kf) = kD^2 f$ and $D^2(f + g) = D^2 f + D^2 g$
24. True: $h'(x) = f'(g(x)) \cdot g'(x)$
 $h'(c) = f'(g(c)) \cdot g'(c) = 0$
25. True: $(f \circ g)'(2) = f'(g(2)) \cdot g'(2)$
 $= f'(2) \cdot g'(2) = 2 \cdot 2 = 4$
26. False: Consider $f(x) = \sqrt{x}$. The curve always lies below the tangent.
27. False: The rate of volume change depends on the radius of the sphere.
28. True: $c = 2\pi r$; $\frac{dr}{dt} = 4$
 $\frac{dc}{dt} = 2\pi \frac{dr}{dt} = 2\pi(4) = 8\pi$
29. True: $D_x(\sin x) = \cos x$;
 $D_x^2(\sin x) = -\sin x$;
 $D_x^3(\sin x) = -\cos x$;
 $D_x^4(\sin x) = \sin x$;
 $D_x^5(\sin x) = \cos x$

30. False: $D_x(\cos x) = -\sin x$; $\frac{dr}{dt} > 0$.
 $D_x^2(\cos x) = -\cos x$;
 $D_x^3(\cos x) = \sin x$;
 $D_x^4(\cos x) = D_x[D_x^3(\cos x)] = D_x(\sin x)$
 Since $D_x^{1+3}(\cos x) = D_x^1(\sin x)$,
 $D_x^{n+3}(\cos x) = D_x^n(\sin x)$.
31. True: $\lim_{x \rightarrow 0} \frac{\tan x}{3x} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x \cos x}$
 $= \frac{1}{3} \cdot 1 = \frac{1}{3}$
32. True: $v = \frac{ds}{dt} = 15t^2 + 6$ which is greater than 0 for all t .
33. True: $V = \frac{4}{3}\pi r^3$
 $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$
 If $\frac{dV}{dt} = 3$, then $\frac{dr}{dt} = \frac{3}{4\pi r^2}$ so
34. True: When $h > r$, then $\frac{d^2h}{dt^2} > 0$
35. True: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$
 $dV = 4\pi r^2 dr = S \cdot dr$
 If $\Delta r = dr$, then $dV = S \cdot \Delta r$
36. False: $dy = 5x^4 dx$, so $dy > 0$ when $dx > 0$, but $dy < 0$ when $dx < 0$.
37. False: The slope of the linear approximation is equal to
 $f'(a) = f'(0) = -\sin(0) = 0$.

Sample Test Problems

1. a. $f'(x) = \lim_{h \rightarrow 0} \frac{3(x+h)^3 - 3x^3}{h} = \lim_{h \rightarrow 0} \frac{9x^2h + 9xh^2 + 3h^3}{h} = \lim_{h \rightarrow 0} (9x^2 + 9xh + 3h^2) = 9x^2$
- b. $f'(x) = \lim_{h \rightarrow 0} \frac{[2(x+h)^5 + 3(x+h)] - (2x^5 + 3x)}{h} = \lim_{h \rightarrow 0} \frac{10x^4h + 20x^3h^2 + 20x^2h^3 + 10xh^4 + 2h^5 + 3h}{h}$
 $= \lim_{h \rightarrow 0} (10x^4 + 20x^3h + 20x^2h^2 + 10xh^3 + 2h^4 + 3) = 10x^4 + 3$
- c. $f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{3(x+h)} - \frac{1}{3x}}{h} = \lim_{h \rightarrow 0} \left[-\frac{h}{3(x+h)x} \right] \frac{1}{h} = \lim_{h \rightarrow 0} -\left(\frac{1}{3x(x+h)} \right) = -\frac{1}{3x^2}$
- d. $f'(x) = \lim_{h \rightarrow 0} \left[\left(\frac{1}{3(x+h)^2 + 2} - \frac{1}{3x^2 + 2} \right) \frac{1}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{3x^2 + 2 - 3(x+h)^2 - 2}{(3(x+h)^2 + 2)(3x^2 + 2)} \cdot \frac{1}{h} \right]$
 $= \lim_{h \rightarrow 0} \left[\frac{-6xh - 3h^2}{(3(x+h)^2 + 2)(3x^2 + 2)} \cdot \frac{1}{h} \right] = \lim_{h \rightarrow 0} \frac{-6x - 3h}{(3(x+h)^2 + 2)(3x^2 + 2)} = -\frac{6x}{(3x^2 + 2)^2}$
- e. $f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{3x+3h} - \sqrt{3x})(\sqrt{3x+3h} + \sqrt{3x})}{h(\sqrt{3x+3h} + \sqrt{3x})}$
 $= \lim_{h \rightarrow 0} \frac{3h}{h(\sqrt{3x+3h} + \sqrt{3x})} = \lim_{h \rightarrow 0} \frac{3}{\sqrt{3x+3h} + \sqrt{3x}} = \frac{3}{2\sqrt{3x}}$

$$\begin{aligned}
 \text{f. } f'(x) &= \lim_{h \rightarrow 0} \frac{\sin[3(x+h)] - \sin 3x}{h} = \lim_{h \rightarrow 0} \frac{\sin(3x+3h) - \sin 3x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin 3x \cos 3h + \sin 3h \cos 3x - \sin 3x}{h} = \lim_{h \rightarrow 0} \frac{\sin 3x(\cos 3h - 1)}{h} + \lim_{h \rightarrow 0} \frac{\sin 3h \cos 3x}{h} \\
 &= 3 \sin 3x \lim_{h \rightarrow 0} \frac{\cos 3h - 1}{3h} + \cos 3x \lim_{h \rightarrow 0} \frac{\sin 3h}{h} = (3 \sin 3x)(0) + (\cos 3x)3 \lim_{h \rightarrow 0} \frac{\sin 3h}{3h} = (\cos 3x)(3)(1) = 3 \cos 3x
 \end{aligned}$$

$$\begin{aligned}
 \text{g. } f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2 + 5} - \sqrt{x^2 + 5}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{(x+h)^2 + 5} - \sqrt{x^2 + 5})(\sqrt{(x+h)^2 + 5} + \sqrt{x^2 + 5})}{h(\sqrt{(x+h)^2 + 5} + \sqrt{x^2 + 5})} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h(\sqrt{(x+h)^2 + 5} + \sqrt{x^2 + 5})} = \lim_{h \rightarrow 0} \frac{2x + h}{\sqrt{(x+h)^2 + 5} + \sqrt{x^2 + 5}} = \frac{2x}{2\sqrt{x^2 + 5}} = \frac{x}{\sqrt{x^2 + 5}}
 \end{aligned}$$

$$\begin{aligned}
 \text{h. } f'(x) &= \lim_{h \rightarrow 0} \frac{\cos[\pi(x+h)] - \cos \pi x}{h} = \lim_{h \rightarrow 0} \frac{\cos(\pi x + \pi h) - \cos \pi x}{h} = \lim_{h \rightarrow 0} \frac{\cos \pi x \cos \pi h - \sin \pi x \sin \pi h - \cos \pi x}{h} \\
 &= \lim_{h \rightarrow 0} \left(-\pi \cos \pi x \frac{1 - \cos \pi h}{\pi h} \right) - \lim_{h \rightarrow 0} \left(\pi \sin \pi x \frac{\sin \pi h}{\pi h} \right) = (-\pi \cos \pi x)(0) - (\pi \sin \pi x) = -\pi \sin \pi x
 \end{aligned}$$

$$\begin{aligned}
 \text{2. a. } g'(x) &= \lim_{t \rightarrow x} \frac{2t^2 - 2x^2}{t - x} = \lim_{t \rightarrow x} \frac{2(t-x)(t+x)}{t-x} \\
 &= 2 \lim_{t \rightarrow x} (t+x) = 2(2x) = 4x
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } g'(x) &= \lim_{t \rightarrow x} \frac{(t^3 + t) - (x^3 + x)}{t - x} \\
 &= \lim_{t \rightarrow x} \frac{(t-x)(t^2 + tx + x^2) + (t-x)}{t-x} \\
 &= \lim_{t \rightarrow x} (t^2 + tx + x^2 + 1) = 3x^2 + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{c. } g'(x) &= \lim_{t \rightarrow x} \frac{\frac{1}{t} - \frac{1}{x}}{t - x} = \lim_{t \rightarrow x} \frac{\frac{x-t}{tx}}{t-x} \\
 &= \lim_{t \rightarrow x} \frac{-1}{tx} = -\frac{1}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{d. } g'(x) &= \lim_{t \rightarrow x} \left[\left(\frac{1}{t^2 + 1} - \frac{1}{x^2 + 1} \right) \left(\frac{1}{t-x} \right) \right] \\
 &= \lim_{t \rightarrow x} \frac{x^2 - t^2}{(t^2 + 1)(x^2 + 1)(t-x)} \\
 &= \lim_{t \rightarrow x} \frac{-(x+t)(t-x)}{(t^2 + 1)(x^2 + 1)(t-x)} \\
 &= \lim_{t \rightarrow x} \frac{-(x+t)}{(t^2 + 1)(x^2 + 1)} = -\frac{2x}{(x^2 + 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{e. } g'(x) &= \lim_{t \rightarrow x} \frac{\sqrt{t} - \sqrt{x}}{t - x} \\
 &= \lim_{t \rightarrow x} \frac{(\sqrt{t} - \sqrt{x})(\sqrt{t} + \sqrt{x})}{(t-x)(\sqrt{t} + \sqrt{x})} \\
 &= \lim_{t \rightarrow x} \frac{t - x}{(t-x)(\sqrt{t} + \sqrt{x})} = \lim_{t \rightarrow x} \frac{1}{\sqrt{t} + \sqrt{x}} \\
 &= \frac{1}{2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{f. } g'(x) &= \lim_{t \rightarrow x} \frac{\sin \pi t - \sin \pi x}{t - x} \\
 \text{Let } v &= t - x, \text{ then } t = v + x \text{ and as } t \rightarrow x, v \rightarrow 0. \\
 \lim_{t \rightarrow x} \frac{\sin \pi t - \sin \pi x}{t - x} &= \lim_{v \rightarrow 0} \frac{\sin \pi(v+x) - \sin \pi x}{v} \\
 &= \lim_{v \rightarrow 0} \frac{\sin \pi v \cos \pi x + \sin \pi x \cos \pi v - \sin \pi x}{v} \\
 &= \lim_{v \rightarrow 0} \left[\pi \cos \pi x \frac{\sin \pi v}{\pi v} + \pi \sin \pi x \frac{\cos \pi v - 1}{\pi v} \right] \\
 &= \pi \cos \pi x \cdot 1 + \pi \sin \pi x \cdot 0 = \pi \cos \pi x
 \end{aligned}$$

Other method:

Use the subtraction formula

$$\sin \pi t - \sin \pi x = 2 \cos \frac{\pi(t+x)}{2} \sin \frac{\pi(t-x)}{2}$$

$$\begin{aligned}
 \text{g. } g'(x) &= \lim_{t \rightarrow x} \frac{\sqrt{t^3 + C} - \sqrt{x^3 + C}}{t - x} \\
 &= \lim_{t \rightarrow x} \frac{(\sqrt{t^3 + C} - \sqrt{x^3 + C})(\sqrt{t^3 + C} + \sqrt{x^3 + C})}{(t - x)(\sqrt{t^3 + C} + \sqrt{x^3 + C})} \\
 &= \lim_{t \rightarrow x} \frac{t^3 - x^3}{(t - x)(\sqrt{t^3 + C} + \sqrt{x^3 + C})} \\
 &= \lim_{t \rightarrow x} \frac{t^2 + tx + x^2}{\sqrt{t^3 + C} + \sqrt{x^3 + C}} = \frac{3x^2}{2\sqrt{x^3 + C}}
 \end{aligned}$$

$$\begin{aligned}
 \text{h. } g'(x) &= \lim_{t \rightarrow x} \frac{\cos 2t - \cos 2x}{t - x} \\
 \text{Let } v &= t - x, \text{ then } t = v + x \text{ and as } \\
 t \rightarrow x, v &\rightarrow 0. \\
 \lim_{t \rightarrow x} \frac{\cos 2t - \cos 2x}{t - x} &= \lim_{v \rightarrow 0} \frac{\cos 2(v + x) - \cos 2x}{v} \\
 &= \lim_{v \rightarrow 0} \frac{\cos 2v \cos 2x - \sin 2v \sin 2x - \cos 2x}{v} \\
 &= \lim_{v \rightarrow 0} \left[2 \cos 2x \frac{\cos 2v - 1}{2v} - 2 \sin 2x \frac{\sin 2v}{2v} \right] \\
 &= 2 \cos 2x \cdot 0 - 2 \sin 2x \cdot 1 = -2 \sin 2x \\
 \text{Other method:} \\
 \text{Use the subtraction formula} \\
 \cos 2t - \cos 2x &= -2 \sin(t + x) \sin(t - x).
 \end{aligned}$$

3. a. $f(x) = 3x$ at $x = 1$
- b. $f(x) = 4x^3$ at $x = 2$
- c. $f(x) = \sqrt{x^3}$ at $x = 1$
- d. $f(x) = \sin x$ at $x = \pi$
- e. $f(x) = \frac{4}{x}$ at x
- f. $f(x) = -\sin 3x$ at x
- g. $f(x) = \tan x$ at $x = \frac{\pi}{4}$
- h. $f(x) = \frac{1}{\sqrt{x}}$ at $x = 5$
4. a. $f'(2) \approx -\frac{3}{4}$
- b. $f'(6) \approx \frac{3}{2}$

$$\text{c. } V_{\text{avg}} = \frac{6 - \frac{3}{2}}{7 - 3} = \frac{9}{8}$$

$$\begin{aligned}
 \text{d. } \frac{d}{dt} f(t^2) &= f'(t^2)(2t) \\
 \text{At } t = 2, 4f'(4) &\approx 4\left(\frac{2}{3}\right) = \frac{8}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{e. } \frac{d}{dt} [f^2(t)] &= 2f(t)f'(t) \\
 \text{At } t = 2, \\
 2f(2)f'(2) &\approx 2(2)\left(-\frac{3}{4}\right) = -3
 \end{aligned}$$

$$\begin{aligned}
 \text{f. } \frac{d}{dt} (f(f(t))) &= f'(f(t))f'(t) \\
 \text{At } t = 2, f'(f(2))f'(2) &= f'(2)f'(2) \\
 &\approx \left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right) = \frac{9}{16}
 \end{aligned}$$

$$5. D_x(3x^5) = 15x^4$$

$$\begin{aligned}
 6. D_x(x^3 - 3x^2 + x^{-2}) &= 3x^2 - 6x + (-2)x^{-3} \\
 &= 3x^2 - 6x - 2x^{-3}
 \end{aligned}$$

$$7. D_z(z^3 + 4z^2 + 2z) = 3z^2 + 8z + 2$$

$$\begin{aligned}
 8. D_x \left(\frac{3x-5}{x^2+1} \right) &= \frac{(x^2+1)(3) - (3x-5)(2x)}{(x^2+1)^2} \\
 &= \frac{-3x^2 + 10x + 3}{(x^2+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 9. D_t \left(\frac{4t-5}{6t^2+2t} \right) &= \frac{(6t^2+2t)(4) - (4t-5)(12t+2)}{(6t^2+2t)^2} \\
 &= \frac{-24t^2 + 60t + 10}{(6t^2+2t)^2}
 \end{aligned}$$

$$\begin{aligned}
 10. D_x(3x+2)^{2/3} &= \frac{2}{3}(3x+2)^{-1/3}(3) \\
 &= 2(3x+2)^{-1/3} \\
 D_x^2(3x+2)^{2/3} &= -\frac{2}{3}(3x+2)^{-4/3}(3) \\
 &= -2(3x+2)^{-4/3}
 \end{aligned}$$

$$11. \frac{d}{dx} \left(\frac{4x^2 - 2}{x^3 + x} \right) = \frac{(x^3 + x)(8x) - (4x^2 - 2)(3x^2 + 1)}{(x^3 + x)^2}$$

$$= \frac{-4x^4 + 10x^2 + 2}{(x^3 + x)^2}$$

$$12. D_t(t\sqrt{2t+6}) = t \frac{1}{2\sqrt{2t+6}}(2) + \sqrt{2t+6}$$

$$= \frac{t}{\sqrt{2t+6}} + \sqrt{2t+6}$$

$$13. \frac{d}{dx} \left(\frac{1}{\sqrt{x^2 + 4}} \right) = \frac{d}{dx} (x^2 + 4)^{-1/2}$$

$$= -\frac{1}{2} (x^2 + 4)^{-3/2} (2x)$$

$$= -\frac{x}{\sqrt{(x^2 + 4)^3}}$$

$$14. \frac{d}{dx} \sqrt{\frac{x^2 - 1}{x^3 - x}} = \frac{d}{dx} \frac{1}{\sqrt{x}} = \frac{d}{dx} x^{-1/2} = -\frac{1}{2x^{3/2}}$$

$$19. \frac{d}{d\theta} [\sin^2(\sin(\pi\theta))] = 2 \sin(\sin(\pi\theta)) \cos(\sin(\pi\theta)) (\cos(\pi\theta)) (\pi) = 2\pi \sin(\sin(\pi\theta)) \cos(\sin(\pi\theta)) \cos(\pi\theta)$$

$$20. \frac{d}{dt} [\sin^2(\cos 4t)] = 2 \sin(\cos 4t) (\cos(\cos 4t)) (-\sin 4t) (4) = -8 \sin(\cos 4t) \cos(\cos 4t) \sin 4t$$

$$21. D_\theta \tan 3\theta = (\sec^2 3\theta)(3) = 3 \sec^2 3\theta$$

$$22. \frac{d}{dx} \left(\frac{\sin 3x}{\cos 5x^2} \right) = \frac{(\cos 5x^2)(\cos 3x)(3) - (\sin 3x)(-\sin 5x^2)(10x)}{\cos^2 5x^2} = \frac{3 \cos 5x^2 \cos 3x + 10x \sin 3x \sin 5x^2}{\cos^2 5x^2}$$

$$23. f'(x) = (x^2 - 1)^2 (9x^2 - 4) + (3x^3 - 4x)(2)(x^2 - 1)(2x) = (x^2 - 1)^2 (9x^2 - 4) + 4x(x^2 - 1)(3x^3 - 4x)$$

$$f'(2) = 672$$

$$24. g'(x) = 3 \cos 3x + 2(\sin 3x)(\cos 3x)(3) = 3 \cos 3x + 3 \sin 6x$$

$$g''(x) = -9 \sin 3x + 18 \cos 6x$$

$$g''(0) = 18$$

$$25. \frac{d}{dx} \left(\frac{\cot x}{\sec x^2} \right) = \frac{(\sec x^2)(-\csc^2 x) - (\cot x)(\sec x^2)(\tan x^2)(2x)}{\sec^2 x^2} = \frac{-\csc^2 x - 2x \cot x \tan x^2}{\sec x^2}$$

$$26. D_t \left(\frac{4t \sin t}{\cos t - \sin t} \right) = \frac{(\cos t - \sin t)(4t \cos t + 4 \sin t) - (4t \sin t)(-\sin t - \cos t)}{(\cos t - \sin t)^2}$$

$$= \frac{4t \cos^2 t + 2 \sin 2t - 4 \sin^2 t + 4t \sin^2 t}{(\cos t - \sin t)^2} = \frac{4t + 2 \sin 2t - 4 \sin^2 t}{(\cos t - \sin t)^2}$$

$$15. D_\theta (\sin \theta + \cos^3 \theta) = \cos \theta + 3 \cos^2 \theta (-\sin \theta)$$

$$= \cos \theta - 3 \sin \theta \cos^2 \theta$$

$$D_\theta^2 (\sin \theta + \cos^3 \theta)$$

$$= -\sin \theta - 3[\sin \theta (2)(\cos \theta)(-\sin \theta) + \cos^3 \theta]$$

$$= -\sin \theta + 6 \sin^2 \theta \cos \theta - 3 \cos^3 \theta$$

$$16. \frac{d}{dt} [\sin(t^2) - \sin^2(t)] = \cos(t^2)(2t) - (2 \sin t)(\cos t)$$

$$= 2t \cos(t^2) - \sin(2t)$$

$$17. D_\theta [\sin(\theta^2)] = \cos(\theta^2)(2\theta) = 2\theta \cos(\theta^2)$$

$$18. \frac{d}{dx} (\cos^3 5x) = (3 \cos^2 5x)(-\sin 5x)(5)$$

$$= -15 \cos^2 5x \sin 5x$$

$$\begin{aligned}
 27. \quad f'(x) &= (x-1)^3 2(\sin \pi x - x)(\pi \cos \pi x - 1) + (\sin \pi x - x)^2 3(x-1)^2 \\
 &= 2(x-1)^3 (\sin \pi x - x)(\pi \cos \pi x - 1) + 3(\sin \pi x - x)^2 (x-1)^2 \\
 f'(2) &= 16 - 4\pi \approx 3.43
 \end{aligned}$$

$$\begin{aligned}
 28. \quad h'(t) &= 5(\sin(2t) + \cos(3t))^4 (2 \cos(2t) - 3 \sin(3t)) \\
 h''(t) &= 5(\sin(2t) + \cos(3t))^4 (-4 \sin(2t) - 9 \cos(3t)) + 20(\sin(2t) + \cos(3t))^3 (2 \cos(2t) - 3 \sin(3t))^2 \\
 h''(0) &= 5 \cdot 1^4 \cdot (-9) + 20 \cdot 1^3 \cdot 2^2 = 35
 \end{aligned}$$

$$\begin{aligned}
 29. \quad g'(r) &= 3(\cos^2 5r)(-\sin 5r)(5) = -15 \cos^2 5r \sin 5r \\
 g''(r) &= -15[(\cos^2 5r)(\cos 5r)(5) + (\sin 5r)2(\cos 5r)(-\sin 5r)(5)] = -15[5 \cos^3 5r - 10(\sin^2 5r)(\cos 5r)] \\
 g'''(r) &= -15[5(3)(\cos^2 5r)(-\sin 5r)(5) - (10 \sin^2 5r)(-\sin 5r)(5) - (\cos 5r)(20 \sin 5r)(\cos 5r)(5)] \\
 &= -15[-175(\cos^2 5r)(\sin 5r) + 50 \sin^3 5r] \\
 g'''(1) &\approx 458.8
 \end{aligned}$$

$$30. \quad f'(t) = h'(g(t))g'(t) + 2g(t)g'(t)$$

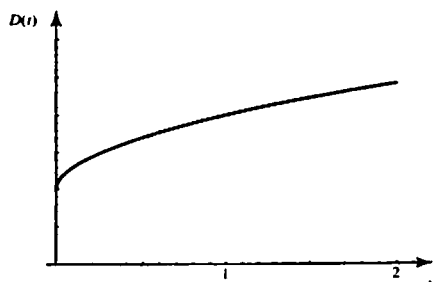
$$\begin{aligned}
 31. \quad G'(x) &= F'(r(x) + s(x))(r'(x) + s'(x)) + s'(x) \\
 G''(x) &= F''(r(x) + s(x))(r''(x) + s''(x)) + (r'(x) + s'(x))F''(r(x) + s(x))(r'(x) + s'(x)) + s''(x) \\
 &= F''(r(x) + s(x))(r''(x) + s''(x)) + (r'(x) + s'(x))^2 F''(r(x) + s(x)) + s''(x)
 \end{aligned}$$

$$\begin{aligned}
 32. \quad F'(x) &= Q'(R(x))R'(x) = 3[R(x)]^2(-\sin x) \\
 &= -3 \cos^2 x \sin x
 \end{aligned}$$

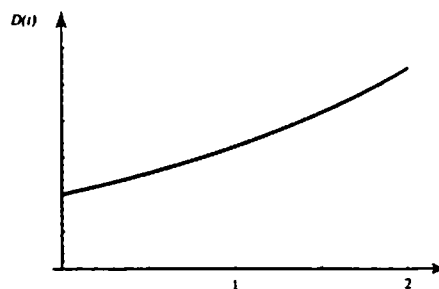
$$\begin{aligned}
 33. \quad F'(z) &= r'(s(z))s'(z) = [3 \cos(3s(z))](9z^2) \\
 &= 27z^2 \cos(9z^3)
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \frac{d}{d\beta} F(\beta) &= \sum'(\Phi(\beta))\Phi'(\beta) \\
 &= \frac{3}{2\sqrt{3\Phi(\beta)}}(3\beta^2 - 1) = \frac{3}{2\sqrt{3(\beta^3 - \beta)}}(3\beta^2 - 1)
 \end{aligned}$$

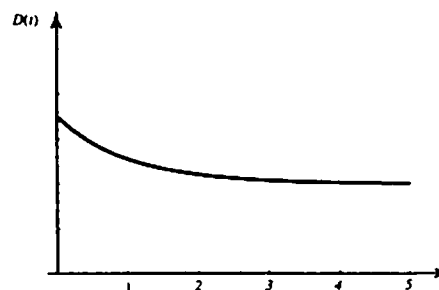
35. One possibility:



36. One possibility:



37. One possibility:



$$\begin{aligned}
 38. \quad \frac{dy}{dx} &= 2(x-2) \\
 2x - y + 2 &= 0; y = 2x + 2; m = 2 \\
 2(x-2) &= -\frac{1}{2}
 \end{aligned}$$

$$x = \frac{7}{4}$$

$$y = \left(\frac{7}{4} - 2\right)^2 = \frac{1}{16}; \left(\frac{7}{4}, \frac{1}{16}\right)$$

$$39. V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

When $r = 5$, $\frac{dV}{dr} = 4\pi(5)^2 = 100\pi \approx 314$ m³ per meter of increase in the radius.

$$40. V = \frac{4}{3}\pi r^3; \frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\text{When } r = 5, 10 = 4\pi(5)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{10\pi} \approx 0.0318 \text{ m/h}$$

$$41. V = \frac{1}{2}bh(12); \frac{6}{4} = \frac{b}{h}; b = \frac{3h}{2}$$

$$V = 6\left(\frac{3h}{2}\right)h = 9h^2; \frac{dV}{dt} = 9$$

$$\frac{dV}{dt} = 18h \frac{dh}{dt}$$

$$\text{When } h = 3, 9 = 18(3) \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{6} \approx 0.167 \text{ ft/min}$$

$$42. \text{ a. } v = 128 - 32t$$

$$v = 0, \text{ when } t = 4\text{ s}$$

$$s = 128(4) - 16(4)^2 = 256 \text{ ft}$$

$$\text{ b. } 128t - 16t^2 = 0$$

$$-16t(t - 8) = 0$$

The object hits the ground when $t = 8\text{ s}$

$$v = 128 - 32(8) = -128 \text{ ft/s}$$

$$43. s = t^3 - 6t^2 + 9t$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

$$a(t) = \frac{d^2s}{dt^2} = 6t - 12$$

$$\text{ a. } 3t^2 - 12t + 9 < 0$$

$$3(t - 3)(t - 1) < 0$$

$$1 < t < 3; (1, 3)$$

$$\text{ b. } 3t^2 - 12t + 9 = 0$$

$$3(t - 3)(t - 1) = 0$$

$$t = 1, 3$$

$$a(1) = -6, a(3) = 6$$

$$\text{ c. } 6t - 12 > 0$$

$$t > 2; (2, \infty)$$

$$44. \text{ a. } D_x^{20}(x^{19} + x^{12} + x^5 + 100) = 0$$

$$\text{ b. } D_x^{20}(x^{20} + x^{19} + x^{18}) = 20!$$

$$\text{ c. } D_x^{20}(7x^{21} + 3x^{20}) = (7 \cdot 21!)x + (3 \cdot 20!)$$

$$\text{ d. } D_x^{20}(\sin x + \cos x) = D_x^4(\sin x + \cos x) = \sin x + \cos x$$

$$\text{ e. } D_x^{20}(\sin 2x) = 2^{20} \sin 2x = 1,048,576 \sin 2x$$

$$\text{ f. } D_x^{20}\left(\frac{1}{x}\right) = \frac{(-1)^{20}(20!)}{x^{21}} = \frac{20!}{x^{21}}$$

$$45. \text{ a. } 2(x - 1) + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(x - 1)}{y} = \frac{1 - x}{y}$$

$$\text{ b. } x(2y) \frac{dy}{dx} + y^2 + y(2x) + x^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2xy + x^2) = -(y^2 + 2xy)$$

$$\frac{dy}{dx} = -\frac{y^2 + 2xy}{x^2 + 2xy}$$

$$\text{ c. } 3x^2 + 3y^2 \frac{dy}{dx} = x^3(3y^2) \frac{dy}{dx} + 3x^2 y^3$$

$$\frac{dy}{dx}(3y^2 - 3x^3 y^2) = 3x^2 y^3 - 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2 y^3 - 3x^2}{3y^2 - 3x^3 y^2} = \frac{x^2 y^3 - x^2}{y^2 - x^3 y^2}$$

$$\text{ d. } x \cos(xy) \left[x \frac{dy}{dx} + y \right] + \sin(xy) = 2x$$

$$x^2 \cos(xy) \frac{dy}{dx} = 2x - \sin(xy) - xy \cos(xy)$$

$$\frac{dy}{dx} = \frac{2x - \sin(xy) - xy \cos(xy)}{x^2 \cos(xy)}$$

$$\begin{aligned} \text{e. } x \sec^2(xy) \left(x \frac{dy}{dx} + y \right) + \tan(xy) &= 0 \\ x^2 \sec^2(xy) \frac{dy}{dx} &= -[\tan(xy) + xy \sec^2(xy)] \\ \frac{dy}{dx} &= -\frac{\tan(xy) + xy \sec^2(xy)}{x^2 \sec^2(xy)} \end{aligned}$$

$$46. \quad 2yy'_1 = 12x^2$$

$$y'_1 = \frac{6x^2}{y}$$

$$\text{At } (1, 2): y'_1 = 3$$

$$4x + 6yy'_2 = 0$$

$$y'_2 = -\frac{2x}{3y}$$

$$\text{At } (1, 2): y'_2 = -\frac{1}{3}$$

Since $(y'_1)(y'_2) = -1$ at $(1, 2)$, the tangents are perpendicular.

$$47. \quad dy = [\pi \cos(\pi x) + 2x]dx; x = 2, dx = 0.01$$

$$\begin{aligned} dy &= [\pi \cos(2\pi) + 2(2)](0.01) = (4 + \pi)(0.01) \\ &\approx 0.0714 \end{aligned}$$

$$48. \quad x(2y) \frac{dy}{dx} + y^2 + 2y[2(x+2)] + (x+2)^2(2) \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} [2xy + 2(x+2)^2] = -[y^2 + 2y(2x+4)]$$

$$\frac{dy}{dx} = \frac{-(y^2 + 4xy + 8y)}{2xy + 2(x+2)^2}$$

$$dy = -\frac{y^2 + 4xy + 8y}{2xy + 2(x+2)^2} dx$$

$$\text{When } x = -2, y = \pm 1$$

$$\begin{aligned} \text{a. } dy &= -\frac{(1)^2 + 4(-2)(1) + 8(1)}{2(-2)(1) + 2(-2+2)^2} (-0.01) \\ &= -0.0025 \end{aligned}$$

$$\begin{aligned} \text{b. } dy &= -\frac{(-1)^2 + 4(-2)(-1) + 8(-1)}{2(-2)(-1) + 2(-2+2)^2} (-0.01) \\ &= 0.0025 \end{aligned}$$

$$\begin{aligned} 49. \quad \text{a. } \frac{d}{dx}[f^2(x) + g^3(x)] \\ &= 2f(x)f'(x) + 3g^2(x)g'(x) \\ &= 2f(2)f'(2) + 3g^2(2)g'(2) \\ &= 2(3)(4) + 3(2)^2(5) = 84 \end{aligned}$$

c.

$$\begin{aligned} \text{b. } \frac{d}{dx}[f(x)g(x)] &= f(x)g'(x) + g(x)f'(x) \\ f(2)g'(2) + g(2)f'(2) &= (3)(5) + (2)(4) = 23 \end{aligned}$$

$$\begin{aligned} \text{c. } \frac{d}{dx}[f(g(x))] &= f'(g(x))g'(x) \\ f'(g(2))g'(2) &= f'(2)g'(2) = (4)(5) = 20 \end{aligned}$$

$$\begin{aligned} \text{d. } D_x[f^2(x)] &= 2f(x)f'(x) \\ D_x^2[f^2(x)] &= 2[f(x)f''(x) + f'(x)f'(x)] \\ 2f(2)f''(2) + 2[f'(2)]^2 \\ &= 2(3)(-1) + 2(4)^2 = 26 \end{aligned}$$

$$50. \quad (13)^2 = x^2 + y^2; \frac{dx}{dt} = 2$$

$$0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

$$\text{When } y = 5, x = 12, \text{ so}$$

$$\frac{dy}{dt} = -\frac{12}{5}(2) = -\frac{24}{5} = -4.8 \text{ ft/s}$$

$$51. \quad \sin 15^\circ = \frac{y}{x}, \frac{dx}{dt} = 400$$

$$y = x \sin 15^\circ$$

$$\frac{dy}{dt} = \sin 15^\circ \frac{dx}{dt}$$

$$\frac{dy}{dt} = 400 \sin 15^\circ \approx 104 \text{ mi/hr}$$

$$52. \quad \text{a. } D_x(|x|^2) = 2|x| \cdot \frac{|x|}{x} = \frac{2(|x|^2)}{x} = \frac{2x^2}{x} = 2x$$

$$\text{b. } D_x^2|x| = D_x\left(\frac{|x|}{x}\right) = \frac{x\left(\frac{|x|}{x}\right)' - |x|}{x^2} = \frac{|x| - |x|}{x^2} = 0$$

$$\text{c. } D_x^3|x| = D_x(D_x^2|x|) = D_x(0) = 0$$

$$\text{d. } D_x^2(|x|^2) = D_x(2x) = 2$$

$$53. \quad \text{a. } D_\theta |\sin \theta| = \frac{|\sin \theta|}{\sin \theta} \cos \theta = \cot \theta |\sin \theta|$$

$$\text{b. } D_\theta |\cos \theta| = \frac{|\cos \theta|}{\cos \theta} (-\sin \theta) = -\tan \theta |\cos \theta|$$

$$\begin{aligned}
 54. \quad a. \quad f(x) &= \sqrt{x+1}; f'(x) = -\frac{1}{2}(x+1)^{-1/2}; a=3 \\
 L(x) &= f(3) + f'(3)(x-3) \\
 &= \sqrt{4} + -\frac{1}{2}(4)^{-1/2}(x-3) \\
 &= 2 - \frac{1}{4}x + \frac{3}{4} \\
 &= -\frac{1}{4}x + \frac{11}{4}
 \end{aligned}$$

$$\begin{aligned}
 b. \quad f(x) &= x \cos x; f'(x) = -x \sin x + \cos x; a=1 \\
 L(x) &= f(1) + f'(1)(x-1) \\
 &= \cos 1 + (-\sin 1 + \cos 1)(x-1) \\
 &= \cos 1 - (\sin 1)x + \sin 1 + (\cos 1)x - \cos 1 \\
 &= (\cos 1 - \sin 1)x + \sin 1 \\
 &\approx -0.3012x + 0.8415
 \end{aligned}$$

3.12 Additional Problem Set

1. $g(t)$ is not differentiable at $t = \alpha$ if

$$\lim_{h \rightarrow 0} \frac{g(\alpha+h) - g(\alpha)}{h} \text{ does not exist.}$$

2. $y = mx + b$; Solve for x :

$$x = \frac{y-b}{m} = g(y)$$

$$\begin{aligned}
 f(g(y)) &= f\left(\frac{y-b}{m}\right) = m\left(\frac{y-b}{m}\right) + b \\
 &= y - b + b = y
 \end{aligned}$$

$$g(f(x)) = g(mx + b) = \frac{mx + b - b}{m} = \frac{mx}{m} = x$$

$$\begin{aligned}
 3. \quad f(f(x)) &= f((1-x^n)^{1/n}) \\
 &= \{1 - [(1-x^n)^{1/n}]^n\}^{1/n} = \{1 - (1-x^n)\}^{1/n} \\
 &= \{x^n\}^{1/n} = x \\
 f'(x) &= \frac{1}{n}(1-x^n)^{\frac{1-n}{n}}(-nx^{n-1}) \\
 &= -x^{n-1}(1-x^n)^{\frac{1-n}{n}} \\
 f'(f(x))f'(x) &= f'\left((1-x^n)^{1/n}\right)f'(x) \\
 &= -[(1-x^n)^{1/n}]^{n-1}\{1 - [(1-x^n)^{1/n}]^n\}^{\frac{1-n}{n}} \\
 &\quad (-1)x^{n-1}(1-x^n)^{\frac{1-n}{n}} \\
 &= (1-x^n)^{\frac{n-1}{n} + \frac{1-n}{n}}\{1 - (1-x^n)\}^{\frac{1-n}{n}}x^{n-1} \\
 &= \{x^n\}^{\frac{1-n}{n}}x^{n-1} = x^{1-n+n-1} = 1
 \end{aligned}$$

$$\begin{aligned}
 4. \quad a. \quad G(F(x)) &= G\left(\sqrt{x^2-7}\right) = \sqrt{\left(\sqrt{x^2-7}\right)^2+7} \\
 &= \sqrt{x^2-7+7} = \sqrt{x^2} = x \text{ for } x \geq \sqrt{7} \\
 F(G(y)) &= F\left(\sqrt{y^2+7}\right) = \sqrt{\left(\sqrt{y^2+7}\right)^2-7} \\
 &= \sqrt{y^2+7-7} = \sqrt{y^2} = y \text{ for } y \geq 0
 \end{aligned}$$

If we think of a function as a rule which tells us what to do to a value for x , then the inverse of that function should undo what the function does. In other words, if we put the output of the function into the inverse function, we should be back where we started from – at x .

$$\begin{aligned}
 b. \quad F'(x) &= \frac{1}{2\sqrt{x^2-7}} \cdot 2x = \frac{x}{\sqrt{x^2-7}} \\
 G'(y) &= \frac{1}{2\sqrt{y^2+7}} \cdot 2y = \frac{y}{\sqrt{y^2+7}}
 \end{aligned}$$

$$\begin{aligned}
 c. \quad F'(G(y))G'(y) &= F'\left(\sqrt{y^2+7}\right)G'(y) \\
 &= \frac{\sqrt{y^2+7}}{\sqrt{\left(\sqrt{y^2+7}\right)^2-7}} \cdot \frac{y}{\sqrt{y^2+7}} \\
 &= \frac{y}{\sqrt{y^2+7-7}} = \frac{y}{\sqrt{y^2}} = 1 \\
 G'(F(x))F'(x) &= G'\left(\sqrt{x^2-7}\right)F'(x) \\
 &= \frac{\sqrt{x^2-7}}{\sqrt{\left(\sqrt{x^2-7}\right)^2+7}} \cdot \frac{x}{\sqrt{x^2-7}} = \frac{x}{\sqrt{x^2+7-7}} \\
 &= \frac{x}{\sqrt{x^2}} = 1
 \end{aligned}$$

5. a. Using the chain rule,

$$D_r[(r+1)^n] = n(r+1)^{n-1}$$

$$\begin{aligned}
 b. \quad (r+1)^n &= C_n^n r^n + C_n^{n-1} r^{n-1} + \dots + C_n^1 r + C_n^0 \\
 \text{so} \\
 D_r[(r+1)^n] &= D_r[C_n^n r^n + C_n^{n-1} r^{n-1} + \dots \\
 &\quad + C_n^1 r + C_n^0] \\
 &= nC_n^n r^{n-1} + (n-1)C_n^{n-1} r^{n-2} + \dots + C_n^1
 \end{aligned}$$

$$(r+1)^{n-1} = C_{n-1}^{n-1} r^{n-1} + C_{n-1}^{n-2} r^{n-2} + \dots + C_{n-1}^1 r + C_{n-1}^0$$

so

$$D_r[(r+1)^n] = n(r+1)^{n-1} \\ = nC_{n-1}^{n-1} r^{n-1} + nC_{n-1}^{n-2} r^{n-2} + nC_{n-1}^1 r + nC_{n-1}^0$$

Equating like powers of r ,

$$nC_n^n = nC_{n-1}^{n-1}, (n-1)C_n^{n-1} = nC_{n-1}^{n-2},$$

$$(n-2)C_n^{n-2} = nC_{n-1}^{n-3}, \dots, C_n^1 = nC_{n-1}^0$$

or

$$mC_n^m = nC_{n-1}^{m-1}$$

c. $C_1^1 = 1$, thus $2 \cdot C_2^2 = 2C_1^1, C_2^2 = C_1^1 = 1$

Similarly, $C_3^3 = C_4^4 = C_n^n = 1$

For $m < n$, then using part b repeatedly,

$$C_n^m = \frac{n}{m} C_{n-1}^{m-1} = \frac{n}{m} \cdot \frac{n-1}{m-1} C_{n-2}^{m-2} \\ = \frac{n \cdot (n-1)}{m \cdot (m-1)} \cdot \frac{n-2}{m-2} C_{n-3}^{m-3} \\ = \dots = \frac{n \cdot (n-1) \dots (n-m+2)}{m \cdot (m-1) \dots 2} C_{n-m+1}^1 \\ = \frac{n \cdot (n-1) \dots (n-m+1)}{m \cdot (m-1) \dots 2 \cdot 1} C_{n-m}^0 \\ = \frac{n \cdot (n-1) \dots (n-m+1)}{1 \cdot 2 \cdot \dots \cdot m}$$

6. a.

x	0.01	0.02	0.03	0.04	0.05
$\sqrt{1+x}$	1.005	1.01	1.0149	1.0198	1.0247
$1 + \frac{x}{2}$	1.005	1.01	1.015	1.02	1.025

The values of $\sqrt{1+x}$ and $1 + \frac{x}{2}$ are very close.

b. $dy = \frac{1}{2\sqrt{1+x}} dx$ At $x = 0$,

$$dy = \frac{1}{2} dx$$

dx	0.01	0.02	0.03	0.04	0.05
dy	0.005	0.01	0.015	0.02	0.025

c. $\alpha = \frac{1}{2}$, error is positive.

$\alpha = -\frac{1}{2}$, error is negative.

d. $dy = \frac{1}{2\sqrt{1+x}} dx$; at $x = 4$, $dy = \frac{1}{2\sqrt{5}} dx$ and $y = \sqrt{5}$.

$$(1+x)^{1/2} \approx \sqrt{5} + \frac{1}{2\sqrt{5}}(x-4) \text{ for } x \text{ near } 4.$$

7. No. The linear approximation is $L(x) = x$ for

both $y = \sin x$ and $y = \tan x$.

For $\sin x$, the error is negative when

$x > 0$ because the approximation lies above the curve; the error is positive when $x < 0$ because the approximation lies below the curve.

For $\tan x$, the error is positive when $x > 0$ because the approximation lies below the curve; the error is negative when $x < 0$ because the approximation lies above the curve.

8. a. $g'(6.0) \approx \frac{60-45}{1} = 15$

$$g'(7.5) \approx \frac{86-68}{1} = 18$$

$$g'(7.25) \approx \frac{77-68}{0.5} = 18$$

b. $g'(7.0) \approx \frac{77-60}{1} = 17$

The rate of change of g' is approximately 2.
 $(\frac{17-15}{1} = 2$ using the above value and the first part of a.)

c. $g'(7.0) \approx 17$ so the tangent line is approximately
 $y - 68 = 17(x - 7)$ or
 $y = 17x - 51$

d. $g(7.1) \approx 17(7.1) - 51 = 69.7$

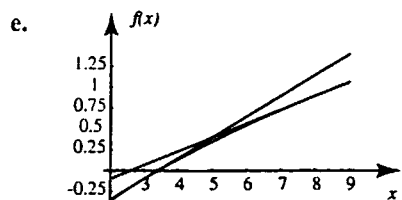
e. $g'(x) = 2x + 3$
 $g'(6) = 15, g'(7.5) = 18, g'(7.25) = 17.5$.
 $g''(x) = 2$, so the rate of change of $g'(x)$ is equal to 2.
 $g'(7.0) = 17$, the tangent line is
 $y = 17x - 51$.
 $g(7.1) \approx 69.7$

9. a. $f(3) = \sqrt{4} - 2.1 = -0.1$,
 $f(8) = \sqrt{9} - 2.1 = 0.9$ so there must be at least one zero of $f(x)$ between $x = 3$ and $x = 8$.

b. $f'(x) = \frac{1}{2\sqrt{1+x}}; f'(8) = \frac{1}{6}$
 $L(x) = 0.9 + \frac{1}{6}(x-8)$ or $L(x) = \frac{1}{6}x - \frac{13}{30}$
 $L(x) = 0$ when $x = \frac{13}{5} = 2.6$ which is less than three, so must be discarded.

c. $f'(3) = \frac{1}{4}$
 $L(x) = -0.1 + \frac{1}{4}(x-3)$ or
 $L(x) = 0.25x - 0.85$
 $L(x) = 0$ when $x = 3.4$

d. $f(x) = 0$ at $x = 3.41$



10. a. $Error(\alpha) = f(\alpha) - L(\alpha)$
 $= f(\alpha) - [c + m(\alpha - \alpha)] = f(\alpha) - c$
 $Error(\alpha) = 0$ implies $f(\alpha) = c$.

b. $|Error(x)| \leq |K(x)(x - \alpha)|$ so

$$\left| \frac{Error(x)}{x - \alpha} \right| \leq |K(x)|$$

Thus, since $\lim_{x \rightarrow \alpha} K(x) = 0$,

$$\lim_{x \rightarrow \alpha} \frac{Error(x)}{x - \alpha} = 0. \text{ But}$$

$$\frac{Error(x)}{x - \alpha} = \frac{f(x) - L(x)}{x - \alpha}$$

$$= \frac{f(x)}{x - \alpha} - \frac{c + m(x - \alpha)}{x - \alpha} = \frac{f(x) - c}{x - \alpha} - m, \text{ so}$$

$$0 = \lim_{x \rightarrow \alpha} \frac{Error(x)}{x - \alpha} = \lim_{x \rightarrow \alpha} \left[\frac{f(x) - c}{x - \alpha} - m \right]$$

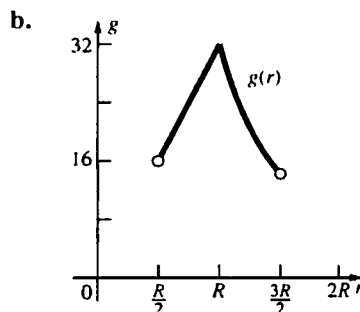
$$= \left[\lim_{x \rightarrow \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} \right] - m = f'(\alpha) - m,$$

hence $f'(\alpha) = m$.

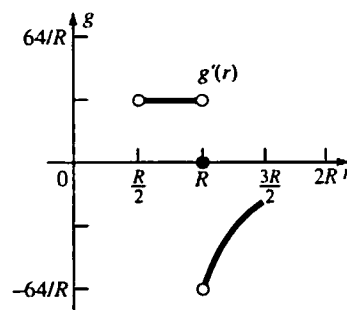
11. a. $\frac{5.04}{93.5} \approx 0.054 \text{ ¢/in.}^2$

b. $dP = P'(A)dA$
 $P'(93.5) = 10.5$ while
 $dA = 8.7 \cdot 11.2 - 93.5 = 3.94 \text{ in.}^2$
 $dP = 10.5 \cdot 3.94 = 41.37 \text{ ¢/pg}$

12. a. $\lim_{r \rightarrow R^+} g(r) = \lim_{r \rightarrow R^+} \frac{GM_1}{r^2} = \frac{GM_1}{R^2} = g$
 $\lim_{r \rightarrow R^-} g(r) = \lim_{r \rightarrow R^-} \frac{GM_1 r}{R^3} = \frac{GM_1}{R^2} = g$
 Thus, $g(r)$ is continuous.



c. $g'(r) = \begin{cases} \frac{-2GM_1}{r^3} & \text{if } r > R \\ 0 & \text{if } r = R \\ \frac{GM_1}{R^3} & \text{if } r < R \end{cases}$



d. No, the derivative is not continuous at $r = R$.

13. Measure r in feet for consistency.

a. $\frac{GM_1}{R^2} = 32 \text{ ft/s}^2$
 $\Delta g = g(3956 \text{ mi}) - g(R)$
 $= \frac{GM_1(20,887,680)}{R^3} - \frac{GM_1}{R^2}$
 $= \frac{GM_1}{R^2} \left(\frac{20,887,680}{20,908,800} - 1 \right)$
 $= 32 \left(-\frac{21,120}{20,908,800} \right)$

$$= -\frac{528}{16,355} \approx -0.0323 \text{ ft/s}^2$$

$$\frac{\Delta g}{g} = \frac{-\frac{528}{16,355}}{32} = -\frac{33}{32,710} \approx -0.0010$$

$$\frac{\Delta g}{g} \times 100 \approx -0.10$$

$$dg \frac{GM_1}{R^3} dr = \frac{32}{R} (-21,120) = -\frac{528}{16,355} = \Delta g$$

The exact and approximate values of the absolute, relative, and percent change in g are the same since $g(r)$ is linear with respect to r for $r < R$.

b. $\Delta g = g(R + 30,000) - g(R)$

$$= \frac{GM_1}{(R + 30,000)^2} - \frac{GM_1}{R^2}$$

$$= \frac{GM_1}{R^2} \left[\frac{R^2}{(R + 30,000)^2} - 1 \right]$$

$$= 32 \left[\frac{R^2 - (R + 30,000)^2}{(R + 30,000)^2} \right]$$

$$= 32 \left(\frac{-60,000R - 900,000,000}{(R + 30,000)^2} \right) \approx -0.0916$$

$$\frac{\Delta g}{g} = \frac{-60,000R - 900,000,000}{(R + 30,000)^2} \approx -0.0029$$

$$\frac{\Delta g}{g} \times 100 \approx -0.29$$

$$dg = -\frac{2GM_1}{R^3} dr = -\frac{2}{R} \left(\frac{GM_1}{R^2} \right) (30,000)$$

$$= \frac{(-60,000)(32)}{20,908,800} \approx -0.0918$$

$$\frac{dg}{g} = \frac{-60,000}{20,908,800} \approx -0.0029$$

$$\frac{dg}{g} \times 100 \approx -0.29$$

c. $\Delta g = g(R + 2000) - g(R)$

$$= \frac{GM_1}{(R + 2000)^2} - \frac{GM_1}{R^2}$$

$$= \frac{GM_1}{R^2} \left[\frac{R^2}{(R + 2000)^2} - 1 \right]$$

$$= 32 \left[\frac{-4000R - 4,000,000}{(R + 2000)^2} \right] \approx -0.00612$$

$$\frac{\Delta g}{g} = \frac{-4000R - 4,000,000}{(R + 2000)^2} \approx -1.913 \times 10^{-4}$$

$$\frac{\Delta g}{g} \times 100 \approx -1.913 \times 10^{-2} = -0.019$$

$$dg = -\frac{2GM_1}{R^3} dr = -\frac{2}{R} (32)(2000)$$

$$= \frac{(-4000)(32)}{20,908,800} \approx -0.00612$$

$$\frac{dg}{g} = \frac{-4000}{20,908,800} \approx -1.913 \times 10^{-4}$$

$$\frac{dg}{g} \times 100 \approx -1.913 \times 10^{-2} = -0.019$$

14. a. The cross-sectional area of the vase is approximately equal to ΔV and the corresponding radius is $r = \sqrt{\Delta V / \pi}$. The table below gives the approximate values for r . The vase becomes slightly narrower as you move above the base, and then gets wider as you near the top.

Depth	V	$A \approx \Delta V$	$r = \sqrt{\Delta V / \pi}$
1	4	4	1.13
2	8	4	1.13
3	11	3	0.98
4	14	3	0.98
5	20	6	1.38
6	28	8	1.60

- b. Near the base, this vase is like the one in part (a), but just above the base it becomes larger. Near the middle of the vase it becomes very narrow. The top of the vase is similar to the one in part (a).

Depth	V	$A \approx \Delta V$	$r = \sqrt{\Delta V / \pi}$
1	4	4	1.13
2	9	5	1.26
3	12	3	0.98
4	14	2	0.80
5	20	6	1.38
6	28	8	1.60

Applications of the Derivative

4.1 Concepts Review

1. continuous; closed
2. extreme
3. endpoints; stationary points; singular points
4. $f'(c) = 0$; $f'(c)$ does not exist

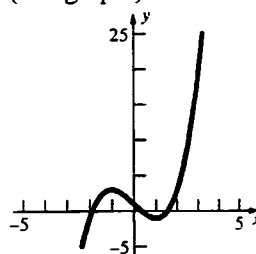
Problem Set 4.1

1. $f'(x) = 2x + 4$; $2x + 4 = 0$ when $x = -2$.
Critical points: $-4, -2, 0$
 $f(-4) = 4, f(-2) = 0, f(0) = 4$
Maximum value = 4, minimum value = 0
2. $h'(x) = 2x + 1$; $2x + 1 = 0$ when $x = -\frac{1}{2}$.
Critical points: $-2, -\frac{1}{2}, 2$
 $h(-2) = 2, h(-\frac{1}{2}) = -\frac{1}{4}, h(2) = 6$
Maximum value = 6, minimum value = $-\frac{1}{4}$
3. $\Psi'(x) = 2x + 3$; $2x + 3 = 0$ when $x = -\frac{3}{2}$.
Critical points: $-2, -\frac{3}{2}, 1$
 $\Psi(-2) = -2, \Psi(-\frac{3}{2}) = -\frac{9}{4}, \Psi(1) = 4$
Maximum value = 4, minimum value = $-\frac{9}{4}$
4. $G'(x) = \frac{1}{5}(6x^2 + 6x - 12) = \frac{6}{5}(x^2 + x - 2)$;
 $x^2 + x - 2 = 0$ when $x = -2, 1$
Critical points: $-3, -2, 1, 3$
 $G(-3) = \frac{9}{5}, G(-2) = 4, G(1) = -\frac{7}{5}, G(3) = 9$

Maximum value = 9,
minimum value = $-\frac{7}{5}$

5. $f'(x) = 3x^2 - 3$; $3x^2 - 3 = 0$ when $x = -1, 1$.
Critical points: $-1, 1$
 $f(-1) = 3, f(1) = -1$
No maximum value, minimum value = -1

(See graph.)



6. $f'(x) = 3x^2 - 3$; $3x^2 - 3 = 0$ when $x = -1, 1$.
Critical points: $-\frac{3}{2}, -1, 1, 3$
 $f(-\frac{3}{2}) = \frac{17}{8}, f(-1) = 3, f(1) = -1, f(3) = 19$
Maximum value = 19, minimum value = -1
7. $h'(r) = -\frac{1}{r^2}$; $h'(r)$ is never 0; $h'(r)$ is not defined when $r = 0$, but $r = 0$ is not in the domain on $[-1, 3]$ since $h(0)$ is not defined.
Critical points: $-1, 3$
Note that $\lim_{x \rightarrow 0^-} h(x) = -\infty$ and $\lim_{x \rightarrow 0^+} h(x) = \infty$.
No maximum value, no minimum value.
8. $g'(x) = -\frac{2x}{(1+x^2)^2}$; $-\frac{2x}{(1+x^2)^2} = 0$ when $x = 0$
Critical points: $-3, 0, 1$
 $g(-3) = \frac{1}{10}, g(0) = 1, g(1) = \frac{1}{2}$
Maximum value = 1, minimum value = $\frac{1}{10}$

Applications of the Derivative

4.1 Concepts Review

1. continuous; closed
2. extreme
3. endpoints; stationary points; singular points
4. $f'(c) = 0$; $f'(c)$ does not exist

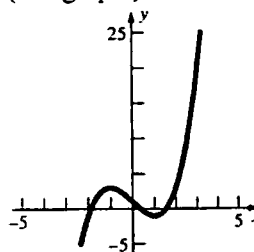
Problem Set 4.1

1. $f'(x) = 2x + 4$; $2x + 4 = 0$ when $x = -2$.
Critical points: $-4, -2, 0$
 $f(-4) = 4, f(-2) = 0, f(0) = 4$
Maximum value = 4, minimum value = 0
2. $h'(x) = 2x + 1$; $2x + 1 = 0$ when $x = -\frac{1}{2}$.
Critical points: $-2, -\frac{1}{2}, 2$
 $h(-2) = 2, h(-\frac{1}{2}) = -\frac{1}{4}, h(2) = 6$
Maximum value = 6, minimum value = $-\frac{1}{4}$
3. $\Psi'(x) = 2x + 3$; $2x + 3 = 0$ when $x = -\frac{3}{2}$.
Critical points: $-2, -\frac{3}{2}, 1$
 $\Psi(-2) = -2, \Psi(-\frac{3}{2}) = -\frac{9}{4}, \Psi(1) = 4$
Maximum value = 4, minimum value = $-\frac{9}{4}$
4. $G'(x) = \frac{1}{5}(6x^2 + 6x - 12) = \frac{6}{5}(x^2 + x - 2)$;
 $x^2 + x - 2 = 0$ when $x = -2, 1$
Critical points: $-3, -2, 1, 3$
 $G(-3) = \frac{9}{5}, G(-2) = 4, G(1) = -\frac{7}{5}, G(3) = 9$

Maximum value = 9,
minimum value = $-\frac{7}{5}$

5. $f'(x) = 3x^2 - 3$; $3x^2 - 3 = 0$ when $x = -1, 1$.
Critical points: $-1, 1$
 $f(-1) = 3, f(1) = -1$
No maximum value, minimum value = -1

(See graph.)



6. $f'(x) = 3x^2 - 3$; $3x^2 - 3 = 0$ when $x = -1, 1$.
Critical points: $-\frac{3}{2}, -1, 1, 3$
 $f(-\frac{3}{2}) = \frac{17}{8}, f(-1) = 3, f(1) = -1, f(3) = 19$
Maximum value = 19, minimum value = -1
7. $h'(r) = -\frac{1}{r^2}$; $h'(r)$ is never 0; $h'(r)$ is not defined when $r = 0$, but $r = 0$ is not in the domain on $[-1, 3]$ since $h(0)$ is not defined.
Critical points: $-1, 3$
Note that $\lim_{x \rightarrow 0^-} h(x) = -\infty$ and $\lim_{x \rightarrow 0^+} h(x) = \infty$.
No maximum value, no minimum value.
8. $g'(x) = -\frac{2x}{(1+x^2)^2}$; $-\frac{2x}{(1+x^2)^2} = 0$ when $x = 0$
Critical points: $-3, 0, 1$
 $g(-3) = \frac{1}{10}, g(0) = 1, g(1) = \frac{1}{2}$
Maximum value = 1, minimum value = $\frac{1}{10}$

$$9. \quad g'(x) = -\frac{2x}{(1+x^2)^2}; \quad -\frac{2x}{(1+x^2)^2} = 0 \text{ when } x = 0.$$

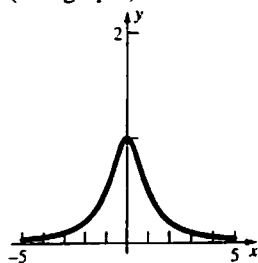
Critical point: 0

$$g(0) = 1$$

As $x \rightarrow \infty$, $g(x) \rightarrow 0^+$; as $x \rightarrow -\infty$, $g(x) \rightarrow 0^+$.

Maximum value = 1, no minimum value

(See graph.)



$$10. \quad f'(x) = \frac{1-x^2}{(1+x^2)^2};$$

$$\frac{1-x^2}{(1+x^2)^2} = 0 \text{ when } x = -1, 1$$

Critical points: -1, 1, 4

$$f(-1) = -\frac{1}{2}, f(1) = \frac{1}{2}, f(4) = \frac{4}{17}$$

$$\text{Maximum value} = \frac{1}{2},$$

$$\text{minimum value} = -\frac{1}{2}$$

$$11. \quad r'(\theta) = \cos \theta; \cos \theta = 0 \text{ when } \theta = \frac{\pi}{2} + k\pi$$

Critical points: $-\frac{\pi}{4}, \frac{\pi}{6}$

$$r\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}, r\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\text{Maximum value} = \frac{\sqrt{3}}{2}, \text{ minimum value} = -\frac{1}{\sqrt{2}}$$

$$12. \quad s'(t) = \cos t + \sin t; \cos t + \sin t = 0 \text{ when}$$

$$\tan t = -1 \text{ or } t = -\frac{\pi}{4} + k\pi.$$

Critical points: 0, $\frac{3\pi}{4}, \pi$

$$s(0) = -1, s\left(\frac{3\pi}{4}\right) = \sqrt{2}, s(\pi) = 1.$$

$$\text{Maximum value} = \sqrt{2},$$

$$\text{minimum value} = -1$$

$$13. \quad a'(x) = \frac{x-1}{|x-1|}; a'(x) \text{ does not exist when } x = 1.$$

Critical points: 0, 1, 3

$$a(0) = 1, a(1) = 0, a(3) = 2$$

Maximum value = 2, minimum value = 0

$$14. \quad f'(s) = \frac{3(3s-2)}{|3s-2|}; f'(s) \text{ does not exist when}$$

$$s = \frac{2}{3}.$$

Critical points: $-1, \frac{2}{3}, 4$

$$f(-1) = 5, f\left(\frac{2}{3}\right) = 0, f(4) = 10$$

Maximum value = 10, minimum value = 0

$$15. \quad g'(x) = \frac{1}{3x^{2/3}}; f'(x) \text{ does not exist when } x = 0.$$

Critical points: -1, 0, 27

$$g(-1) = -1, g(0) = 0, g(27) = 3$$

Maximum value = 3, minimum value = -1

$$16. \quad s'(t) = \frac{2}{5t^{3/5}}; s'(t) \text{ does not exist when } t = 0.$$

Critical points: -1, 0, 32

$$s(-1) = 1, s(0) = 0, s(32) = 4$$

Maximum value = 4, minimum value = 0

$$17. \quad f(x) = x(10-x) = 10x - x^2; f'(x) = 10 - 2x;$$

$$10 - 2x = 0; x = 5$$

Critical points: 0, 5, 10

$f(0) = 0, f(5) = 25, f(10) = 0$; the numbers are $x = 5$ and $10 - x = 5$.

$$18. \quad x \geq x^2 \text{ if } 0 \leq x \leq 1$$

$$f(x) = x - x^2; f'(x) = 1 - 2x;$$

$$f'(x) = 0 \text{ when } x = \frac{1}{2}$$

Critical points: 0, $\frac{1}{2}, 1$

$$f(0) = 0, f(1) = 0, f\left(\frac{1}{2}\right) = \frac{1}{4}; \text{ therefore, } \frac{1}{2}$$

exceeds its square by the maximum amount.

$$19. \quad \text{Let } x \text{ represent the width, then } 100 - x \text{ represents the length.}$$

$$f(x) = x(100-x) = 100x - x^2; f'(x) = 100 - 2x;$$

$$100 - 2x = 0; x = 50$$

Critical points: 0, 50, 100

$$f(0) = f(100) = 0; f(50) = 50(50) = 2500$$

The dimensions should be 50 ft by 50 ft.

$$20. \quad \text{For a rectangle with perimeter } K \text{ and width } x, \text{ the length is } \frac{K}{2} - x. \text{ Then the area is}$$

$$A = x\left(\frac{K}{2} - x\right) = \frac{Kx}{2} - x^2.$$

$$\frac{dA}{dx} = \frac{K}{2} - 2x; \frac{dA}{dx} = 0 \text{ when } x = \frac{K}{4}$$

$$\text{Critical points: } 0, \frac{K}{4}, \frac{K}{2}$$

$$\text{At } x = 0 \text{ or } \frac{K}{2}, A = 0; \text{ at } x = \frac{K}{4}, A = \frac{K^2}{16}.$$

The area is maximized when the width is one fourth of the perimeter, so the rectangle is a square.

21. Let x be the width of the square to be cut out and V the volume of the resulting open box.

$$V = x(24 - 2x)^2 = 4x^3 - 96x^2 + 576x$$

$$\frac{dV}{dx} = 12x^2 - 192x + 576 = 12(x - 12)(x - 4);$$

$$12(x - 12)(x - 4) = 0; x = 12 \text{ or } x = 4.$$

$$\text{Critical points: } 0, 4, 12$$

$$\text{At } x = 0 \text{ or } 12, V = 0; \text{ at } x = 4, V = 1024.$$

The volume of the largest box is 1024 in.³

22. Let x be the length of the cut for the square, so $16 - x$ is the cut for the circle.

$$\text{The area of the square is } \left(\frac{1}{4}x\right)\left(\frac{1}{4}x\right) = \frac{x^2}{16}.$$

The area of the circle is found by finding the radius first: $2\pi r = 16 - x$;

$$r = \frac{16 - x}{2\pi}; A = \pi r^2 = \pi \left(\frac{16 - x}{2\pi}\right)^2 = \frac{(16 - x)^2}{4\pi}.$$

Total area:

$$A = \frac{x^2}{16} + \frac{(16 - x)^2}{4\pi}; \frac{dA}{dx} = \frac{x}{8} + \frac{2(16 - x)(-1)}{4\pi}$$

$$= \frac{x}{8} + \frac{x - 16}{2\pi};$$

$$\frac{dA}{dx} = 0 \text{ when } x = \frac{128}{2\pi + 8} \approx 8.96$$

$$\text{Critical numbers: } 0, 8.96, 16$$

$$\text{At } x = 0, A \approx 20.37; \text{ At } x = 8.96, A \approx 8.96.$$

$$\text{At } x = 16; A = 16.$$

Total area is minimum at $x = 8.96$; the total area is maximum with no cut and the wire bent to form a circle.

23. Let A be the area of the pen.

$$A = x(80 - 2x) = 80x - 2x^2; \frac{dA}{dx} = 80 - 4x;$$

$$80 - 4x = 0; x = 20$$

$$\text{Critical points: } 0, 20, 40.$$

$$\text{At } x = 0 \text{ or } 40, A = 0; \text{ at } x = 20, A = 800.$$

The dimensions are 20 ft by $80 - 2(20) = 40$ ft, with the length along the barn being 40 ft.

24. Let x be the width of each pen, then the length along the barn is $80 - 4x$.

$$A = x(80 - 4x) = 80x - 4x^2; \frac{dA}{dx} = 80 - 8x;$$

$$\frac{dA}{dx} = 0 \text{ when } x = 10.$$

$$\text{Critical points: } 0, 10, 20$$

$$\text{At } x = 0 \text{ or } 20, A = 0; \text{ at } x = 10, A = 400.$$

The area is largest with width 10 ft and length 40 ft.

25. Let A be the area of the pen. The perimeter is $100 + 180 = 280$ ft.

$$y + y - 100 + 2x = 180; y = 140 - x$$

$$A = x(140 - x) = 140x - x^2; \frac{dA}{dx} = 140 - 2x;$$

$$140 - 2x = 0; x = 70$$

Since $0 \leq x \leq 40$, the critical points are 0 and 40.

When $x = 0, A = 0$. When $x = 40, A = 4000$. The dimensions are 40 ft by 100 ft.

26. The perimeter will be $20 + 40 + 80 = 140$ ft, and x is limited by $20 \leq x \leq 30$.

$$2x + 2y - 60 = 80; y = 70 - x$$

$$A = x(70 - x) = 70x - x^2; \frac{dA}{dx} = 70 - 2x;$$

$$70 - 2x = 0; x = 35$$

$$\text{Critical points: } 20, 30$$

$$\text{When } x = 20, A = 1000. \text{ When } x = 30, A = 1200.$$

The pen has dimensions 30 ft by 40 ft.

27. x is limited by $0 \leq x \leq \sqrt{12}$.

$$A = 2x(12 - x^2) = 24x - 2x^3; \frac{dA}{dx} = 24 - 6x^2;$$

$$24 - 6x^2 = 0; x = -2, 2$$

$$\text{Critical points: } 0, 2, \sqrt{12}.$$

$$\text{When } x = 0 \text{ or } \sqrt{12}, A = 0.$$

$$\text{When } x = 2, y = 12 - (2)^2 = 8.$$

The dimensions are $2x = 2(2) = 4$ by 8.

28. Let the x -axis lie on the diameter of the semicircle and the y -axis pass through the middle.

Then the equation $y = \sqrt{r^2 - x^2}$ describes the semicircle. Let (x, y) be the upper-right corner of the rectangle. x is limited by $0 \leq x \leq r$.

$$A = 2xy = 2x\sqrt{r^2 - x^2}$$

$$\frac{dA}{dx} = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2}{\sqrt{r^2 - x^2}}(r^2 - 2x^2)$$

$$\frac{2}{\sqrt{r^2 - x^2}}(r^2 - 2x^2) = 0; x = \frac{r}{\sqrt{2}}$$

$$\text{Critical points: } 0, \frac{r}{\sqrt{2}}, r$$

When $x = 0$ or r , $A = 0$. When $x = \frac{r}{\sqrt{2}}$, $A = r^2$.

$$y = \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} = \frac{r}{\sqrt{2}}$$

The dimensions are $\frac{r}{\sqrt{2}}$ by $\frac{2r}{\sqrt{2}}$.

29. The carrying capacity of the gutter is maximized when the area of the vertical end of the gutter is maximized. The height of the gutter is $3 \sin \theta$. The area is

$$A = 3(3 \sin \theta) + 2\left(\frac{1}{2}\right)(3 \cos \theta)(3 \sin \theta)$$

$$= 9 \sin \theta + 9 \cos \theta \sin \theta.$$

$$\frac{dA}{d\theta} = 9 \cos \theta + 9(-\sin \theta) \sin \theta + 9 \cos \theta \cos \theta$$

$$= 9(\cos \theta - \sin^2 \theta + \cos^2 \theta)$$

$$= 9(2 \cos^2 \theta + \cos \theta - 1)$$

$$2 \cos^2 \theta + \cos \theta - 1 = 0; \cos \theta = -1, \frac{1}{2}; \theta = \pi, \frac{\pi}{3}$$

Since $0 \leq \theta \leq \frac{\pi}{2}$, the critical points are

$$0, \frac{\pi}{3}, \text{ and } \frac{\pi}{2}.$$

When $\theta = 0$, $A = 0$.

$$\text{When } \theta = \frac{\pi}{3}, A = \frac{27\sqrt{3}}{4} \approx 11.7.$$

$$\text{When } \theta = \frac{\pi}{2}, A = 9.$$

The carrying capacity is maximized when $\theta = \frac{\pi}{3}$.

30. The circumference of the top of the tank is the circumference of the circular sheet minus the arc length of the sector,

$20\pi - 10\theta$ meters. The radius of the top of the

tank is $r = \frac{20\pi - 10\theta}{2\pi} = \frac{5}{\pi}(2\pi - \theta)$ meters. The

slant height of the tank is 10 meters, so the height of the tank is

$$h = \sqrt{10^2 - \left(10 - \frac{5\theta}{\pi}\right)^2} = \frac{5}{\pi} \sqrt{4\pi\theta - \theta^2} \text{ meters.}$$

$$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left[\frac{5}{\pi}(2\pi - \theta) \right]^2 \left[\frac{5}{\pi} \sqrt{4\pi\theta - \theta^2} \right]$$

$$= \frac{125}{3\pi^2} (2\pi - \theta)^2 \sqrt{4\pi\theta - \theta^2}$$

$$\frac{dV}{d\theta} = \frac{125}{3\pi^2} \left(2(2\pi - \theta)(-1) \sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^2 \left(\frac{1}{2}\right)(4\pi - 2\theta)}{\sqrt{4\pi\theta - \theta^2}} \right)$$

$$= \frac{125(2\pi - \theta)}{3\pi^2 \sqrt{4\pi\theta - \theta^2}} (3\theta^2 - 12\pi\theta + 4\pi^2):$$

$$\frac{125(2\pi - \theta)}{3\pi^2 \sqrt{4\pi\theta - \theta^2}} (3\theta^2 - 12\pi\theta + 4\pi^2) = 0$$

$$2\pi - \theta = 0 \text{ or } 3\theta^2 - 12\pi\theta + 4\pi^2 = 0$$

$$\theta = 2\pi, \theta = 2\pi - \frac{2\sqrt{6}}{3}\pi, \theta = 2\pi + \frac{2\sqrt{6}}{3}\pi$$

Since $0 < \theta < 2\pi$, the only critical point is

$2\pi - \frac{2\sqrt{6}}{3}\pi$. A graph shows that this maximizes the volume.

31. Let c be the cost of driving the truck in cents. It

takes $\frac{400}{x}$ hours to drive 400 miles.

$$c = \frac{400}{x}(1200) + 400\left(25 + \frac{x}{4}\right)$$

$$= 480,000x^{-1} + 10,000 + 100x$$

$$\frac{dc}{dx} = -480,000x^{-2} + 100;$$

$$-480,000x^{-2} + 100 = 0; x \approx 69$$

Since $40 \leq x \leq 55$, the critical points are 40 and 55.

At $x = 40$, $c = 26,000$.

At $x = 55$, $c \approx 24,227$.

The most economical allowed speed is at 55 mph.

32. Let c be the cost of driving the truck.

$$c = \frac{400}{x}(1200) + 400(40 + 0.05x^{3/2})$$

$$= 480,000x^{-1} + 16,000 + 20x^{3/2}$$

$$\frac{dc}{dx} = -480,000x^{-2} + 30x^{1/2}; \frac{dc}{dx} = 0 \text{ when } x \approx 48$$

Critical points: 40, 48, 55

At $x = 40$, $c \approx \pi 33,060$; At $x = 48$, $c \approx 32,651$.

At $x = 55$, $c \approx 32,885$.

The most economical speed is at approximately 48 mph.

33. Let D be the square of the distance.

$$D = (x - 0)^2 + (y - 4)^2 = x^2 + \left(\frac{x^2}{4} - 4\right)^2$$

$$= \frac{x^4}{16} - x^2 + 16$$

$$\frac{dD}{dx} = \frac{x^3}{4} - 2x; \frac{x^3}{4} - 2x = 0; x(x^2 - 8) = 0$$

$$x = 0, x = \pm 2\sqrt{2}$$

Critical points: $0, 2\sqrt{2}, 2\sqrt{3}$

At $x = 0, y = 0$, and $D = 16$. At $x = 2\sqrt{2}, y = 2$,

and $D = 12$. At $x = 2\sqrt{3}, y = 3$, and $D = 13$.

$$P(2\sqrt{2}, 2), Q(0, 0)$$

34. Note that $\cos t = \frac{h}{r}$, so $h = r \cos t$,

$$\sin t = \frac{1}{r} \sqrt{r^2 - h^2}, \text{ and } \sqrt{r^2 - h^2} = r \sin t$$

$$\text{Area of submerged region} = r^2 - h \sqrt{r^2 - h^2}$$

$$= r^2 - (r \cos t)(r \sin t) = r^2 (1 - \cos t \sin t)$$

A = area of exposed wetted region

$$= \pi r^2 - \pi h^2 - r^2 (1 - \cos t \sin t)$$

$$= r^2 (\pi - \pi \cos^2 t - 1 + \cos t \sin t)$$

$$\frac{dA}{dt} = r^2 (2\pi \cos t \sin t - 1 + \cos^2 t - \sin^2 t)$$

$$= r^2 (2\pi \cos t \sin t - 2 \sin^2 t)$$

$$= 2r^2 \sin t (\pi \cos t - \sin t)$$

Since $0 < t < \pi$, $\frac{dA}{dt} = 0$ only when

$\pi \cos t = \sin t$ or $\tan t = \pi$. In terms of r and h ,

$$\text{this is } \frac{\frac{1}{r} \sqrt{r^2 - h^2}}{\frac{h}{r}} = \pi \text{ or } h = \frac{r}{\sqrt{1 + \pi^2}}.$$

35. Let V be the volume. $y = 4 - x$ and $z = 5 - 2x$.
 x is limited by $x \leq 2.5$.

$$V = x(4 - x)(5 - 2x) = 20x - 13x^2 + 2x^3$$

$$\frac{dV}{dx} = 20 - 26x + 6x^2; 2(3x^2 - 13x + 10) = 0;$$

$$2(3x - 10)(x - 1) = 0;$$

$$x = 1, \frac{10}{3}$$

Critical points: $0, 1, 2.5$

At $x = 0$ or 2.5 , $V = 0$. At $x = 1$, $V = 9$.

Maximum volume when $x = 1, y = 4 - 1 = 3$, and
 $z = 5 - 2(1) = 3$.

36. a. $f'(x) = 3x^2 - 12x + 1; 3x^2 - 12x + 1 = 0$

$$\text{when } x = 2 - \frac{\sqrt{33}}{3} \text{ and } x = 2 + \frac{\sqrt{33}}{3}.$$

$$\text{Critical points: } -1, 2 - \frac{\sqrt{33}}{3}, 2 + \frac{\sqrt{33}}{3}, 5$$

$$f(-1) = -6, f\left(2 - \frac{\sqrt{33}}{3}\right) \approx 2.04,$$

$$f\left(2 + \frac{\sqrt{33}}{3}\right) \approx -26.04, f(5) = -18$$

Maximum value ≈ 2.04 ,

minimum value ≈ -26.04

$$\text{b. } g'(x) = \frac{(x^3 - 6x^2 + x + 2)(3x^2 - 12x + 1)}{|x^3 - 6x^2 + x + 2|};$$

$$g'(x) = 0 \text{ when } x = 2 - \frac{\sqrt{33}}{3} \text{ and}$$

$$x = 2 + \frac{\sqrt{33}}{3}. g'(x) \text{ does not exist when}$$

$$f(x) = 0; \text{ on } [-1, 5], f(x) = 0 \text{ when}$$

$$x \approx -0.4836 \text{ and } x \approx 0.7172$$

$$\text{Critical points: } -1, -0.4836, 2 - \frac{\sqrt{33}}{3},$$

$$0.7172, 2 + \frac{\sqrt{33}}{3}, 5$$

$$g(-1) = 6, g(-0.4836) = 0,$$

$$g\left(2 - \frac{\sqrt{33}}{3}\right) \approx 2.04, g(0.7172) = 0,$$

$$g\left(2 + \frac{\sqrt{33}}{3}\right) \approx 26.04, g(5) = 18$$

Maximum value ≈ 26.04 ,

minimum value = 0

37. a. $f'(x) = x \cos x$; on $[-1, 5]$, $x \cos x = 0$ when

$$x = 0, x = \frac{\pi}{2}, x = \frac{3\pi}{2}$$

$$\text{Critical points: } -1, 0, \frac{\pi}{2}, \frac{3\pi}{2}, 5$$

$$f(-1) \approx 3.38, f(0) = 3, f\left(\frac{\pi}{2}\right) \approx 3.57,$$

$$f\left(\frac{3\pi}{2}\right) \approx -2.71, f(5) \approx -2.51$$

Maximum value ≈ 3.57 ,

minimum value ≈ -2.71

$$\text{b. } g'(x) = \frac{(\cos x + x \sin x + 2)(x \cos x)}{|\cos x + x \sin x + 2|};$$

$$g'(x) = 0 \text{ when } x = 0, x = \frac{\pi}{2}, x = \frac{3\pi}{2}$$

$$g'(x) \text{ does not exist when } f(x) = 0;$$

$$\text{on } [-1, 5], f(x) = 0 \text{ when } x \approx 3.45$$

$$\text{Critical points: } -1, 0, \frac{\pi}{2}, 3.45, \frac{3\pi}{2}, 5$$

$$g(-1) \approx 3.38, g(0) = 3, g\left(\frac{\pi}{2}\right) \approx 3.57,$$

$$g(3.45) = 0, g\left(\frac{3\pi}{2}\right) \approx 2.71,$$

$$g(5) \approx 2.51$$

$$\text{Maximum value} \approx 3.57;$$

$$\text{minimum value} = 0$$

4.2 Concepts Review

1. Increasing; concave up
2. $f'(x) > 0$; $f''(x) < 0$
3. An inflection point
4. $f''(c) = 0$; $f''(c)$ does not exist.

Problem Set 4.2

1. $f'(x) = 3$; $3 > 0$ for all x . $f(x)$ is increasing for all x .
2. $g'(x) = 2x - 1$; $2x - 1 > 0$ when $x > \frac{1}{2}$. $g(x)$ is increasing on $\left[\frac{1}{2}, \infty\right)$ and decreasing on $\left(-\infty, \frac{1}{2}\right]$.
3. $h'(t) = 2t + 2$; $2t + 2 > 0$ when $t > -1$. $h(t)$ is increasing on $[-1, \infty)$ and decreasing on $(-\infty, -1]$.
4. $f'(x) = 3x^2$; $3x^2 > 0$ for $x \neq 0$. $f(x)$ is increasing for all x .
5. $G'(x) = 6x^2 - 18x + 12 = 6(x-2)(x-1)$
Split the x -axis into the intervals $(-\infty, 1)$, $(1, 2)$, $(2, \infty)$.
Test points: $x = 0, \frac{3}{2}, 3$; $G'(0) = 12$, $G'\left(\frac{3}{2}\right) = -\frac{3}{2}$, $G'(3) = 12$
 $G(x)$ is increasing on $(-\infty, 1] \cup [2, \infty)$ and decreasing on $[1, 2]$.
6. $f'(t) = 3t^2 + 6t = 3t(t+2)$
Split the x -axis into the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, \infty)$.
Test points: $t = -3, -1, 1$; $f'(-3) = 9$, $f'(-1) = -3$, $f'(1) = 9$
 $f(t)$ is increasing on $(-\infty, -2] \cup [0, \infty)$ and decreasing on $[-2, 0]$.
7. $h'(z) = z^3 - 2z^2 = z^2(z-2)$
Split the x -axis into the intervals $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$.
Test points: $z = -1, 1, 3$; $h'(-1) = -3$, $h'(1) = -1$, $h'(3) = 9$
 $h(z)$ is increasing on $[2, \infty)$ and decreasing on $(-\infty, 2]$.
8. $f'(x) = \frac{2-x}{x^3}$
Split the x -axis into the intervals $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$.
Test points: $-1, 1, 3$; $f'(-1) = -3$, $f'(1) = 1$, $f'(3) = -\frac{1}{27}$
 $f(x)$ is increasing on $(0, 2]$ and decreasing on $(-\infty, 0) \cup [2, \infty)$.
9. $H'(t) = \cos t$; $H'(t) > 0$ when $0 \leq t < \frac{\pi}{2}$ and $\frac{3\pi}{2} < t \leq 2\pi$.
 $H(t)$ is increasing on $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ and decreasing on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.
10. $R'(\theta) = -2\cos\theta\sin\theta$; $R'(\theta) > 0$ when $\frac{\pi}{2} < \theta < \pi$ and $\frac{3\pi}{2} < \theta < 2\pi$.
 $R(\theta)$ is increasing on $\left[\frac{\pi}{2}, \pi\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ and decreasing on $\left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$.
11. $f''(x) = 2$; $2 > 0$ for all x . $f(x)$ is concave up for all x ; no inflection points.
12. $G''(w) = 2$; $2 > 0$ for all w . $G(w)$ is concave up for all w ; no inflection points.

13. $T''(t) = 18t$; $18t > 0$ when $t > 0$. $T(t)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; $(0, 0)$ is the only inflection point.

14. $f''(z) = 2 - \frac{6}{z^4} = \frac{2}{z^4}(z^4 - 3)$; $z^4 - 3 > 0$ for $z < -\sqrt[4]{3}$ and $z > \sqrt[4]{3}$.
 $f(z)$ is concave up on $(-\infty, -\sqrt[4]{3}) \cup (\sqrt[4]{3}, \infty)$ and concave down on $(-\sqrt[4]{3}, 0) \cup (0, \sqrt[4]{3})$; inflection points are $\left(-\sqrt[4]{3}, \sqrt{3} - \frac{1}{\sqrt{3}}\right)$ and $\left(\sqrt[4]{3}, \sqrt{3} - \frac{1}{\sqrt{3}}\right)$.

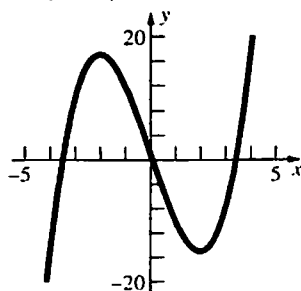
15. $q''(x) = 12x^2 - 36x - 48$; $q''(x) > 0$ when $x < -1$ and $x > 4$.
 $q(x)$ is concave up on $(-\infty, -1) \cup (4, \infty)$ and concave down on $(-1, 4)$; inflection points are $(-1, -19)$ and $(4, -499)$.

16. $f''(x) = 12x^2 + 48x = 12x(x + 4)$; $f''(x) > 0$ when $x < -4$ and $x > 0$.
 $f(x)$ is concave up on $(-\infty, -4) \cup (0, \infty)$ and concave down on $(-4, 0)$; inflection points are $(-4, -258)$ and $(0, -2)$.

17. $F''(x) = 2\sin^2 x - 2\cos^2 x + 4 = 6 - 4\cos^2 x$;
 $6 - 4\cos^2 x > 0$ for all x since $0 \leq \cos^2 x \leq 1$.
 $F(x)$ is concave up for all x ; no inflection points.

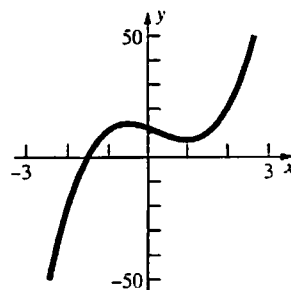
18. $G''(x) = 48 + 24\cos^2 x - 24\sin^2 x$
 $= 24 + 48\cos^2 x$; $24 + 48\cos^2 x > 0$ for all x .
 $G(x)$ is concave up for all x ; no inflection points.

19. $f'(x) = 3x^2 - 12$; $3x^2 - 12 > 0$ when $x < -2$ or $x > 2$.
 $f(x)$ is increasing on $(-\infty, -2] \cup [2, \infty)$ and decreasing on $[-2, 2]$.
 $f''(x) = 6x$; $6x > 0$ when $x > 0$. $f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

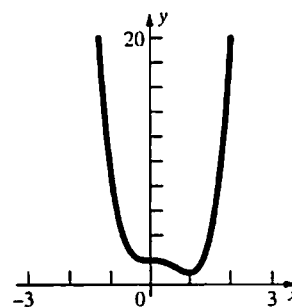


20. $g'(x) = 12x^2 - 6x - 6 = 6(2x + 1)(x - 1)$; $g'(x) > 0$ when $x < -\frac{1}{2}$ or $x > 1$. $g(x)$ is increasing on $\left(-\infty, -\frac{1}{2}\right] \cup [1, \infty)$ and decreasing on $\left[-\frac{1}{2}, 1\right]$.
 $g''(x) = 24x - 6 = 6(4x - 1)$; $g''(x) > 0$ when $x > \frac{1}{4}$.

$g(x)$ is concave up on $\left(\frac{1}{4}, \infty\right)$ and concave down on $\left(-\infty, \frac{1}{4}\right)$.



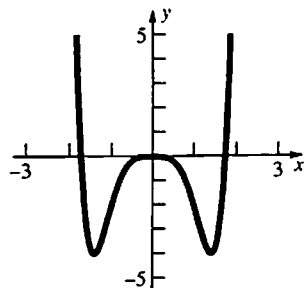
21. $g'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$; $g'(x) > 0$ when $x > 1$. $g(x)$ is increasing on $[1, \infty)$ and decreasing on $(-\infty, 1]$.
 $g''(x) = 36x^2 - 24x = 12x(3x - 2)$; $g''(x) > 0$ when $x < 0$ or $x > \frac{2}{3}$. $g(x)$ is concave up on $(-\infty, 0) \cup \left(\frac{2}{3}, \infty\right)$ and concave down on $\left(0, \frac{2}{3}\right)$.



22. $F'(x) = 6x^5 - 12x^3 = 6x^3(x^2 - 2)$
Split the x -axis into the intervals $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, 0)$, $(0, \sqrt{2})$, $(\sqrt{2}, \infty)$.
Test points: $x = -2, -1, 1, 2$: $F'(-2) = -96$,
 $F'(-1) = 6$, $F'(1) = -6$, $F'(2) = 96$
 $F(x)$ is increasing on $[-\sqrt{2}, 0] \cup [\sqrt{2}, \infty)$ and decreasing on $(-\infty, -\sqrt{2}) \cup [0, \sqrt{2}]$
 $F''(x) = 30x^4 - 36x^2 = 6x^2(5x^2 - 6)$; $5x^2 - 6 > 0$

when $x < -\sqrt{\frac{6}{5}}$ or $x > \sqrt{\frac{6}{5}}$.

$F(x)$ is concave up on $\left(-\infty, -\sqrt{\frac{6}{5}}\right) \cup \left(\sqrt{\frac{6}{5}}, \infty\right)$ and concave down on $\left(-\sqrt{\frac{6}{5}}, \sqrt{\frac{6}{5}}\right)$.



23. $G'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$; $G'(x) > 0$ when $x < -1$ or $x > 1$. $G(x)$ is increasing on $(-\infty, -1] \cup [1, \infty)$ and decreasing on $[-1, 1]$.
 $G''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$;

Split the x -axis into the intervals $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$,

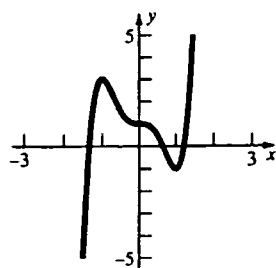
$\left(-\frac{1}{\sqrt{2}}, 0\right)$, $\left(0, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \infty\right)$.

Test points: $x = -1, -\frac{1}{2}, \frac{1}{2}, 1$; $G''(-1) = -30$,

$$G''\left(-\frac{1}{2}\right) = \frac{15}{2}, G''\left(\frac{1}{2}\right) = -\frac{15}{2}, G''(1) = 30.$$

$G(x)$ is concave up on $\left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and

concave down on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)$.



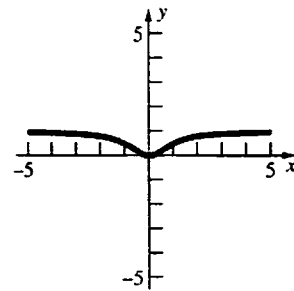
24. $H'(x) = \frac{2x}{(x^2 + 1)^2}$; $H'(x) > 0$ when $x > 0$.

$H(x)$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.

$$H''(x) = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}; H''(x) > 0 \text{ when}$$

$$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}.$$

$H(x)$ is concave up on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and concave down on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$.



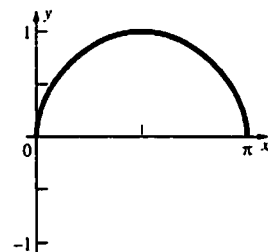
25. $f'(x) = \frac{\cos x}{2\sqrt{\sin x}}$; $f'(x) > 0$ when $0 < x < \frac{\pi}{2}$. $f(x)$

is increasing on $\left[0, \frac{\pi}{2}\right]$ and decreasing on

$\left[\frac{\pi}{2}, \pi\right]$.

$$f''(x) = \frac{-\cos^2 x - 2\sin^2 x}{4\sin^{3/2} x}; f''(x) < 0 \text{ for all } x \text{ in}$$

$(0, \infty)$. $f(x)$ is concave down on $(0, \pi)$.



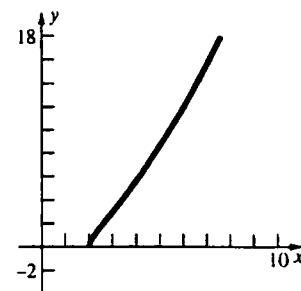
26. $g'(x) = \frac{3x - 4}{2\sqrt{x - 2}}$; $3x - 4 > 0$ when $x > \frac{4}{3}$.

$g(x)$ is increasing on $[2, \infty)$.

$$g''(x) = \frac{3x - 8}{4(x - 2)^{3/2}}; 3x - 8 > 0 \text{ when } x > \frac{8}{3}.$$

$g(x)$ is concave up on $\left(\frac{8}{3}, \infty\right)$ and concave down

on $\left(2, \frac{8}{3}\right)$.



27. $f'(x) = \frac{2-5x}{3x^{1/3}}$; $2-5x > 0$ when $x < \frac{2}{5}$, $f'(x)$

does not exist at $x = 0$.

Split the x -axis into the intervals $(-\infty, 0)$,

$$\left(0, \frac{2}{5}\right), \left(\frac{2}{5}, \infty\right).$$

Test points: $-1, \frac{1}{5}, 1$; $f'(-1) = -\frac{7}{3}$,

$$f'\left(\frac{1}{5}\right) = \frac{\sqrt[3]{5}}{3}, f'(1) = -1.$$

$f(x)$ is increasing on $\left[0, \frac{2}{5}\right]$ and decreasing on

$$(-\infty, 0] \cup \left[\frac{2}{5}, \infty\right).$$

$$f''(x) = \frac{-2(5x+1)}{9x^{4/3}}; -2(5x+1) > 0 \text{ when}$$

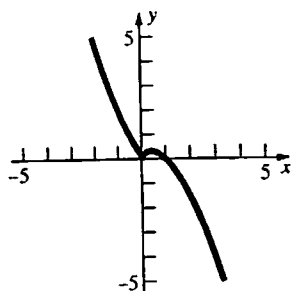
$$x < -\frac{1}{5}, f''(x) \text{ does not exist at } x = 0.$$

Test points: $-1, -\frac{1}{10}, 1$; $f''(-1) = \frac{8}{9}$,

$$f''\left(-\frac{1}{10}\right) = -\frac{10^{4/3}}{9}, f''(1) = -\frac{4}{3}.$$

$f(x)$ is concave up on $\left(-\infty, -\frac{1}{5}\right)$ and concave

down on $\left(-\frac{1}{5}, 0\right) \cup (0, \infty)$.



28. $g'(x) = \frac{4(x+2)}{3x^{2/3}}$; $x+2 > 0$ when $x > -2$, $g'(x)$

does not exist at $x = 0$.

Split the x -axis into the intervals $(-\infty, -2)$,

$(-2, 0)$, $(0, \infty)$.

Test points: $-3, -1, 1$; $g'(-3) = -\frac{4}{3^{5/3}}$,

$$g'(-1) = \frac{4}{3}, g'(1) = 4.$$

$g(x)$ is increasing on $[-2, \infty)$ and decreasing on $(-\infty, -2]$.

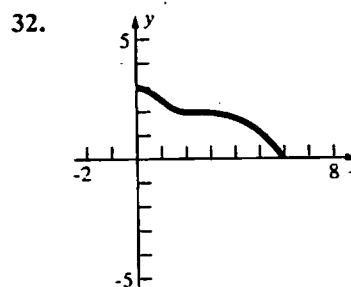
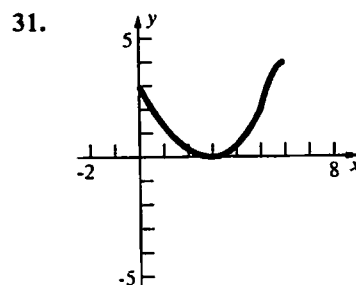
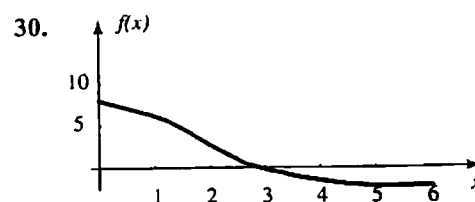
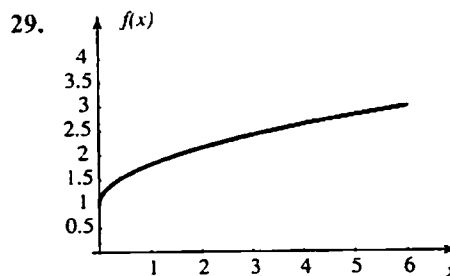
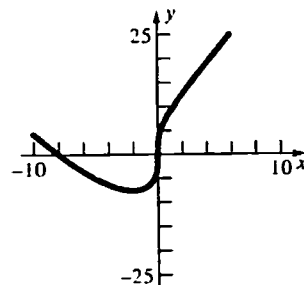
$$g''(x) = \frac{4(x-4)}{9x^{5/3}}; x-4 > 0 \text{ when } x > 4, g''(x)$$

does not exist at $x = 0$.

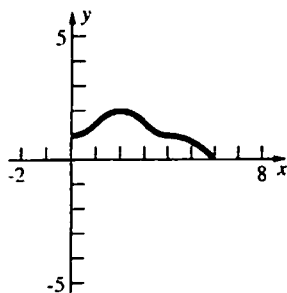
Test points: $-1, 1, 5$; $g''(-1) = \frac{20}{9}$,

$$g''(1) = -\frac{4}{3}, g''(5) = \frac{4}{9(5)^{5/3}}.$$

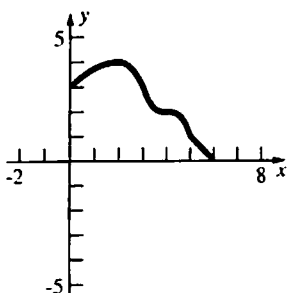
$g(x)$ is concave up on $(-\infty, 0) \cup (4, \infty)$ and concave down on $(0, 4)$.



33.



34.



35. $f(x) = ax^2 + bx + c$; $f'(x) = 2ax + b$;

$f''(x) = 2a$

An inflection point would occur where $f''(x) = 0$, or $2a = 0$. This would only occur when $a = 0$, but if $a = 0$, the equation is not quadratic. Thus, quadratic functions have no points of inflection.

36. $f(x) = ax^3 + bx^2 + cx + d$;

$f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$

An inflection point occurs where $f''(x) = 0$, or $6ax + 2b = 0$.

The function will have an inflection point at

$$x = -\frac{b}{3a}, a \neq 0.$$

37. Suppose that there are points x_1 and x_2 in I where $f'(x_1) > 0$ and $f'(x_2) < 0$. Since f' is continuous on I , the Intermediate Value Theorem says that there is some number c between x_1 and x_2 such that $f'(c) = 0$, which is a contradiction. Thus, either $f'(x) > 0$ for all x in I and f is increasing throughout I or $f'(x) < 0$ for all x in I and f is decreasing throughout I .

38. Since $x^2 + 1 = 0$ has no real solutions, $f'(x)$ exists and is continuous everywhere.
 $x^2 - x + 1 = 0$ has no real solutions. $x^2 - x + 1 > 0$ and $x^2 + 1 > 0$ for all x , so $f'(x) > 0$ for all x .
 Thus f is increasing everywhere.

39. a. Let $f(x) = x^2$ and let $I = [0, a]$, $a > y$.
 $f'(x) = 2x > 0$ on I . Therefore, $f(x)$ is increasing on I , so $f(x) < f(y)$ for $x < y$.

b. Let $f(x) = \sqrt{x}$ and let $I = [0, a]$, $a > y$.
 $f'(x) = \frac{1}{2\sqrt{x}} > 0$ on I . Therefore, $f(x)$ is increasing on I , so $f(x) < f(y)$ for $x < y$.

c. Let $f(x) = \frac{1}{x}$ and let $I = [0, a]$, $a > y$.
 $f'(x) = -\frac{1}{x^2} < 0$ on I . Therefore $f(x)$ is decreasing on I , so $f(x) > f(y)$ for $x < y$.

40. $f'(x) = 3ax^2 + 2bx + c$

In order for $f(x)$ to always be increasing, a , b , and c must meet the condition $3ax^2 + 2bx + c > 0$ for all x . More specifically, $a > 0$ and $b^2 - 3ac < 0$.

41. $f''(x) = \frac{3b - ax}{4x^{5/2}}$. If $(4, 13)$ is an inflection point then $13 = 2a + \frac{b}{2}$ and $\frac{3b - 4a}{4 \cdot 32} = 0$. Solving these equations simultaneously, $a = \frac{39}{8}$ and $b = \frac{13}{2}$.

42. $f(x) = a(x - r_1)(x - r_2)(x - r_3)$
 $f'(x) = a[(x - r_1)(2x - r_2 - r_3) + (x - r_2)(x - r_3)]$
 $f'(x) = a[3x^2 - 2x(r_1 + r_2 + r_3) + r_1r_2 + r_2r_3 + r_1r_3]$
 $f''(x) = a[6x - 2(r_1 + r_2 + r_3)]$
 $a[6x - 2(r_1 + r_2 + r_3)] = 0$
 $6x = 2(r_1 + r_2 + r_3); x = \frac{r_1 + r_2 + r_3}{3}$

43. a. $[f(x) + g(x)]' = f'(x) + g'(x)$.
 Since $f'(x) > 0$ and $g'(x) > 0$ for all x ,
 $f'(x) + g'(x) > 0$ for all x . No additional conditions are needed.

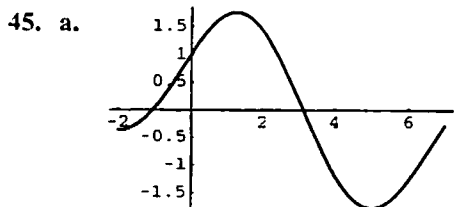
b. $[f(x) \cdot g(x)]' = f(x)g'(x) + f'(x)g(x)$.
 $f(x)g'(x) + f'(x)g(x) > 0$ if
 $f(x) > -\frac{f'(x)}{g'(x)}g(x)$ for all x .

c. $[f(g(x))]' = f'(g(x))g'(x)$.
 Since $f'(x) > 0$ and $g'(x) > 0$ for all x ,
 $f'(g(x))g'(x) > 0$ for all x . No additional conditions are needed.

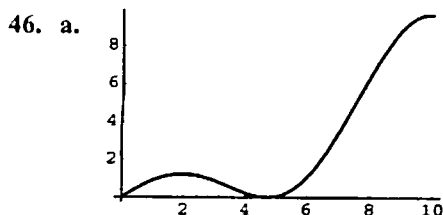
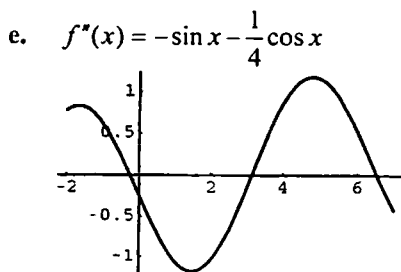
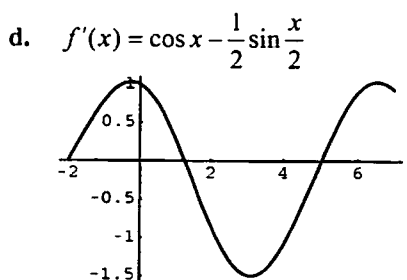
44. a. $[f(x) + g(x)]'' = f''(x) + g''(x)$.
 Since $f''(x) > 0$ and $g'' > 0$ for all x ,
 $f''(x) + g''(x) > 0$ for all x . No additional
 conditions are needed.

- b. $[f(x) \cdot g(x)]' = [f(x)g'(x) + f'(x)g(x)]'$
 $= f(x)g''(x) + f''(x)g(x) + 2f'(x)g'(x)$.
 The additional condition is that
 $f(x)g''(x) + f''(x)g(x) + 2f'(x)g'(x) > 0$
 for all x is needed.

- c. $[f(g(x))]' = [f'(g(x))g'(x)]'$
 $= f'(g(x))g''(x) + f''(g(x))[g'(x)]^2$.
 The additional condition is that
 $f'(g(x)) > -\frac{f''(g(x))[g'(x)]^2}{g''(x)}$ for all x .



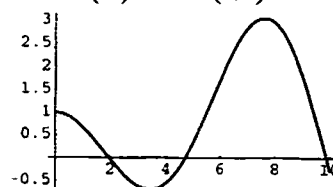
- b. $f'(x) < 0 : (1.3, 5.0)$
 c. $f''(x) < 0 : (-0.25, 3.1) \cup (6.5, 7]$



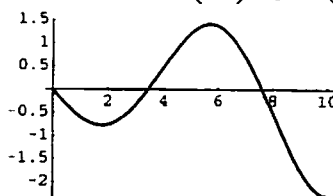
- b. $f'(x) < 0 : (2.0, 4.7) \cup (9.9, 10]$

- c. $f''(x) < 0 : [0, 3.4) \cup (7.6, 10]$

d. $f'(x) = x \left[-\frac{2}{3} \cos\left(\frac{x}{3}\right) \sin\left(\frac{x}{3}\right) \right] + \cos^2\left(\frac{x}{3}\right)$
 $= \cos^2\left(\frac{x}{3}\right) - \frac{x}{3} \sin\left(\frac{2x}{3}\right)$



e. $f''(x) = -\frac{2x}{9} \cos\left(\frac{2x}{3}\right) - \frac{2}{3} \sin\left(\frac{2x}{3}\right)$



47. $f'(x) > 0$ on $(-0.598, 0.680)$

f is increasing on $[-0.598, 0.680]$.

48. $f''(x) < 0$ when $x > 1.63$ in $[-2, 3]$

f is concave down on $(1.63, 3]$.

49. $\frac{dV}{dt} = 2 \text{ in}^3/\text{sec}$

The cup is a portion of a cone with the bottom cut off. If we let x represent the height of the missing cone, we can use similar triangles to show that

$$\frac{x}{3} = \frac{x+5}{3.5}$$

$$3.5x = 3x + 15$$

$$.5x = 15$$

$$x = 30$$

Similar triangles can be used again to show that, at any given time, the radius of the cone at water level is

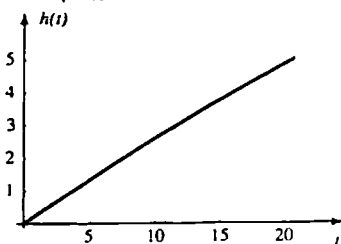
$$r = \frac{h+30}{20}$$

Therefore, the volume of water can be expressed as

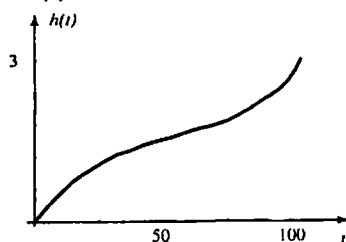
$$V = \frac{\pi(h+30)^3}{1200} - \frac{45\pi}{2}$$

We also know that $V = 2t$ from above. Setting the two volume equations equal to each other and solving for h gives

$$h = \sqrt[3]{\frac{2400}{\pi}t + 27000} - 30.$$

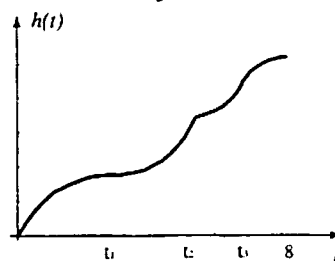


50. The height is always increasing so $h'(t) > 0$. The rate of change of the height decreases for the first 50 minutes and then increases over the next 50 minutes. Thus $h''(t) < 0$ for $0 \leq t \leq 50$ and $h''(t) > 0$ for $50 < t \leq 100$.

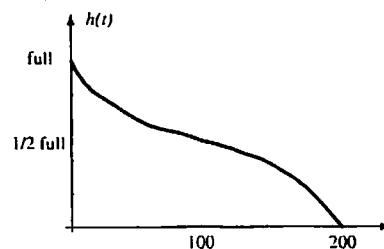


51. $V = 3t$, $0 \leq t \leq 8$. The height is always increasing, so $h'(t) > 0$. The rate of change of the height decreases from time $t = 0$ until time t_1 when the water reaches the middle of the rounded bottom part. The rate of change then increases until time $t_2 = 2t_1$ when the water reaches the neck of the vase. Thus $h''(t) < 0$ for $0 < t < t_1$ and

$h''(t) > 0$ for $t_1 < t < t_2$. For $t_2 < t < t_3$, the rate of change in the height increases until the water reaches the middle of the neck. Then the rate of change decreases until $t = 8$ and the vase is full. Thus, $h''(t) > 0$ for $t_2 < t < t_3$ and $h''(t) < 0$ for $t_3 < t < 8$.



52. $V = 20 - .1t$, $0 \leq t \leq 200$. The height of the water is always decreasing so $h'(t) < 0$. The rate of change in the height increases (the rate is negative, and its absolute value decreases) for the first 100 days and then decreases for the remaining time. Therefore we have $h''(t) > 0$ for $0 < t < 100$, and $h''(t) < 0$ for $100 < t < 200$.



4.3 Concepts Review

1. maximum
2. maximum; minimum
3. maximum
4. local maximum, local minimum, 0

Problem Set 4.3

1. $f'(x) = 3x^2 - 12x = 3x(x - 4)$
Critical points: 0, 4
 $f'(x) > 0$ on $(-\infty, 0)$, $f'(x) < 0$ on $(0, 4)$,
 $f'(x) > 0$ on $(4, \infty)$

$$f''(x) = 6x - 12; f''(0) = -12, f''(4) = 12.$$

Local minimum at $x = 4$; local maximum at $x = 0$

2. $f'(x) = 3x^2 - 12 = 3(x^2 - 4)$
Critical points: -2, 2
 $f'(x) > 0$ on $(-\infty, -2)$, $f'(x) < 0$ on $(-2, 2)$,
 $f'(x) > 0$ on $(2, \infty)$
 $f''(x) = 6x; f''(-2) = -12, f''(2) = 12$
Local minimum at $x = 2$; local maximum at $x = -2$

3. $f'(\theta) = 2 \cos 2\theta; 2 \cos 2\theta \neq 0$ on $\left(0, \frac{\pi}{4}\right)$

No critical points; no local maxima or minima on $\left(0, \frac{\pi}{4}\right)$.

4. $f'(x) = \frac{1}{2} + \cos x; \frac{1}{2} + \cos x = 0$ when $\cos x = -\frac{1}{2}$.

Critical points: $\frac{2\pi}{3}, \frac{4\pi}{3}$

$f'(x) > 0$ on $\left(0, \frac{2\pi}{3}\right)$; $f'(x) < 0$ on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$.

$f'(x) > 0$ on $\left(\frac{4\pi}{3}, 2\pi\right)$

$f''(x) = -\sin x; f''\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}, f''\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Local minimum at $x = \frac{4\pi}{3}$; local maximum at

$x = \frac{2\pi}{3}$.

5. $\Psi'(\theta) = 2 \sin \theta \cos \theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Critical point: 0

$\Psi'(\theta) < 0$ on $\left(-\frac{\pi}{2}, 0\right)$; $\Psi'(\theta) > 0$ on $\left(0, \frac{\pi}{2}\right)$.

$\Psi''(\theta) = 2 \cos^2 \theta - 2 \sin^2 \theta$; $\Psi''(0) = 2$

Local minima at $\theta = 0$

6. $r'(z) = 4z^3$

Critical point: 0

$r'(z) < 0$ on $(-\infty, 0)$;

$r'(z) > 0$ on $(0, \infty)$

$r''(x) = 12x^2$; $r''(0) = 0$; the Second Derivative

Test fails.

Local minimum at $z = 0$; no local maxima

7. $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

Critical points: -1, 1

$f''(x) = 6x$; $f''(-1) = -6$, $f''(1) = 6$

Local minimum value $f(1) = -2$;

local maximum value $f(-1) = 2$

8. $g'(x) = 4x^3 + 2x = 2x(2x^2 + 1)$

Critical point: 0

$g''(x) = 12x^2 + 2$; $g''(0) = 2$

Local minimum value $g(0) = 3$; no local maximum values

9. $H'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$

Critical points: 0, $\frac{3}{2}$

$H''(x) = 12x^2 - 12x = 12x(x - 1)$; $H''(0) = 0$,

$H''\left(\frac{3}{2}\right) = 9$

$H'(x) < 0$ on $(-\infty, 0)$, $H'(x) < 0$ on $\left(0, \frac{3}{2}\right)$

Local minimum value $H\left(\frac{3}{2}\right) = -\frac{27}{16}$; no local

maximum values ($x = 0$ is neither a local minimum nor maximum)

10. $f'(x) = 5(x - 2)^4$

Critical point: 2

$f''(x) = 20(x - 2)^3$; $f''(2) = 0$

$f'(x) > 0$ on $(-\infty, 2)$, $f'(x) > 0$ on $(2, \infty)$

No local minimum or maximum values

11. $g'(t) = -\frac{2}{3(t-2)^{1/3}}$; $g'(t)$ does not exist at $t = 2$.

Critical point: 2

$g'(1) = \frac{2}{3}$, $g'(3) = -\frac{2}{3}$

No local minimum values; local maximum value $g(2) = \pi$.

12. $r'(s) = 3 + \frac{2}{5s^{3/5}} = \frac{15s^{3/5} + 2}{5s^{3/5}}$; $r'(s) = 0$ when

$s = -\left(\frac{2}{15}\right)^{5/3}$, $r'(s)$ does not exist at $s = 0$.

Critical points: $-\left(\frac{2}{15}\right)^{5/3}$, 0

$r''(s) = -\frac{6}{25s^{8/5}}$; $r''\left(-\left(\frac{2}{15}\right)^{5/3}\right) = -\frac{6}{25}\left(\frac{15}{2}\right)^{8/3}$

$r'(s) < 0$ on $\left(-\left(\frac{2}{15}\right)^{5/3}, 0\right)$, $r'(s) > 0$ on $(0, \infty)$

Local minimum value $r(0) = 0$; local maximum value

$r\left(-\left(\frac{2}{15}\right)^{5/3}\right) = -3\left(\frac{2}{15}\right)^{5/3} + \left(\frac{2}{15}\right)^{2/3} = \frac{3}{5}\left(\frac{2}{15}\right)^{2/3}$

13. $f'(t) = 1 + \frac{1}{t^2}$

No critical points

No local minimum or maximum values

14. $f'(x) = \frac{x(x^2 + 8)}{(x^2 + 4)^{3/2}}$

Critical point: 0

$f'(x) < 0$ on $(-\infty, 0)$, $f'(x) > 0$ on $(0, \infty)$

Local minimum value $f(0) = 0$, no local maximum values

$$15. \Lambda'(\theta) = -\frac{1}{1+\sin\theta}; \Lambda'(\theta) \text{ does not exist at}$$

$$\theta = \frac{3\pi}{2}, \text{ but } \Lambda(\theta) \text{ does not exist at that point}$$

either.

No critical points

No local minimum or maximum values

$$16. g'(\theta) = \frac{\sin\theta \cos\theta}{|\sin\theta|}; g'(\theta) = 0 \text{ when } \theta = \frac{\pi}{2}, \frac{3\pi}{2};$$

$g'(\theta)$ does not exist at $\theta = \pi$.

Split the θ -axis into the intervals $\left(0, \frac{\pi}{2}\right)$,

$$\left(\frac{\pi}{2}, \pi\right), \left(\pi, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right).$$

$$\text{Test points: } \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}; g'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},$$

$$g'\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}, g'\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}, g'\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

Local minimum value $g(\pi) = 0$; local maximum

$$\text{values } g\left(\frac{\pi}{2}\right) = 1 \text{ and } g\left(\frac{3\pi}{2}\right) = 1$$

$$17. F'(x) = \frac{3}{\sqrt{x}} - 4; \frac{3}{\sqrt{x}} - 4 = 0 \text{ when } x = \frac{9}{16}$$

$$\text{Critical points: } 0, \frac{9}{16}, 4$$

$$F(0) = 0, F\left(\frac{9}{16}\right) = \frac{9}{4}, F(4) = -4$$

Minimum value $F(4) = -4$; maximum value

$$F\left(\frac{9}{16}\right) = \frac{9}{4}$$

$$18. \text{ From Problem 17, the critical points are } 0 \text{ and } \frac{9}{16}.$$

$$F'(x) > 0 \text{ on } \left(0, \frac{9}{16}\right), F'(x) < 0 \text{ on } \left(\frac{9}{16}, \infty\right)$$

$$F \text{ decreases without bound on } \left(\frac{9}{16}, \infty\right). \text{ No}$$

$$\text{minimum values; maximum value } F\left(\frac{9}{16}\right) = \frac{9}{4}$$

$$19. f'(x) = 64(-1)(\sin x)^{-2} \cos x$$

$$+ 27(-1)(\cos x)^{-2}(-\sin x)$$

$$= -\frac{64 \cos x}{\sin^2 x} + \frac{27 \sin x}{\cos^2 x}$$

$$= \frac{(3 \sin x - 4 \cos x)(9 \sin^2 x + 12 \cos x \sin x + 16 \cos^2 x)}{\sin^2 x \cos^2 x}$$

$$\text{On } \left(0, \frac{\pi}{2}\right), f'(x) = 0 \text{ only where } 3 \sin x = 4 \cos x;$$

$$\tan x = \frac{4}{3};$$

$$x = \tan^{-1} \frac{4}{3} \approx 0.9273$$

Critical point: 0.9273

For $0 < x < 0.9273$, $f'(x) < 0$, while for

$$0.9273 < x < \frac{\pi}{2}, f'(x) > 0$$

$$\text{Minimum value } f\left(\tan^{-1} \frac{4}{3}\right) = \frac{64}{\frac{4}{5}} + \frac{27}{\frac{3}{5}} = 125;$$

no maximum value

$$20. f'(x) = 2x - \frac{2}{x^3}; f'(x) = 0 \text{ when } x = -1, 1 \text{ (only)}$$

$x = 1$ in $(0, \infty)$)

$f'(x) < 0$ on $(0, 1)$, $f'(x) > 0$ on $(1, \infty)$

Minimum value $f(1) = 2$; no maximum value

$$21. g'(x) = 2x + \frac{(8-x)^2(32x) - (16x^2)2(8-x)(-1)}{(8-x)^4}$$

$$= 2x + \frac{256x}{(8-x)^3} = \frac{2x[(8-x)^3 + 128]}{(8-x)^3}$$

For $x > 8$, $g'(x) = 0$ when $(8-x)^3 + 128 = 0$;

$$(8-x)^3 = -128; 8-x = -\sqrt[3]{128};$$

$$x = 8 + 4\sqrt[3]{2} \approx 13.04$$

$$g'(x) < 0 \text{ on } (8, 8 + 4\sqrt[3]{2}),$$

$$g'(x) > 0 \text{ on } (8 + 4\sqrt[3]{2}, \infty)$$

$g(13.04) \approx 277$ is the minimum value

$$22. f'(x) = 2Ax + B; f'(x) = 0 \text{ when } x = -\frac{B}{2A}$$

$f''(x) = 2A > 0$ for all x , so $x = -\frac{B}{2A}$ is a minimum.

$$f\left(-\frac{B}{2A}\right) = A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C$$

$$= \frac{B^2}{4A} - \frac{B^2}{2A} + C = -\frac{B^2}{4A} + C$$

If $B^2 - 4AC \leq 0$, then $\frac{B^2}{4A} \leq C$ (recall $A > 0$), so

$$f\left(-\frac{B}{2A}\right) \geq 0.$$

Since the minimum value of f is nonnegative, $f(x) \geq 0$ for all x .

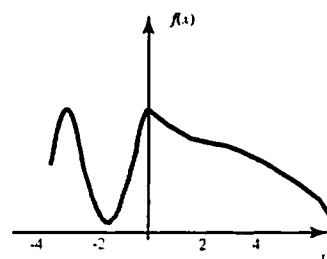
If $f\left(-\frac{B}{2A}\right) \geq 0$, then

$$-\frac{B^2}{4A} + C \geq 0; C \geq \frac{B^2}{4A}; 4AC \geq B^2;$$

$$B^2 - 4AC \leq 0$$

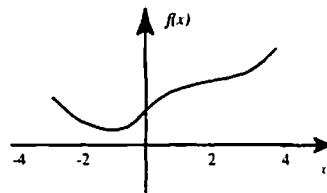
23. $f'(x) = 0$ when $x = 0$ and $x = 1$. On the interval $(-\infty, 0)$ we get $f'(x) < 0$. On $(0, \infty)$, we get $f'(x) > 0$. Thus there is a local min at $x = 0$ but no local max.
24. $f'(x) = 0$ at $x = 1, 2, 3, 4$; $f'(x)$ is negative on $(-\infty, 1) \cup (2, 3) \cup (4, \infty)$ and positive on $(1, 2) \cup (3, 4)$. Thus, the function has a local minimum at $x = 1, 3$ and a local maximum at $x = 2, 4$.
25. $f'(x) = 0$ at $x = 1, 2, 3, 4$; $f'(x)$ is negative on $(3, 4)$ and positive on $(-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (4, \infty)$. Thus, the function has a local minimum at $x = 4$ and a local maximum at $x = 3$.
26. The function is always increasing so there are no local extrema.
27. Critical points: $-2, -1, 2, 3$
Split the x -axis into the intervals $(-\infty, -2)$, $(-2, -1)$, $(-1, 2)$, $(2, 3)$, $(3, \infty)$.
Use test values: $-3, -\frac{3}{2}, 0, \frac{5}{2}$, and 4 to find that
 $f'(x) < 0$ on $(-2, -1) \cup (-1, 2) \cup (2, 3)$ and
 $f'(x) > 0$ on $(-\infty, -2) \cup (3, \infty)$.
Local maximum at $x = -2$; local minimum at $x = 3$.
28. $f'''(c) > 0$ implies that f'' is increasing at c , so f is concave up to the right of c (since $f''(x) > 0$ to the right of c) and concave down to the left of c (since $f''(x) < 0$ to the left of c). Therefore f has a point of inflection at c .

29. Written response (graph)



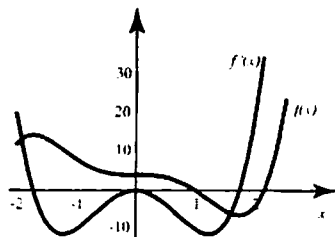
- a. Increasing: $(-\infty, -3] \cup [-1, 0)$
Decreasing: $[-3, -1] \cup (0, \infty)$
- b. Concave up: $(-2, 0) \cup (0, 2)$
Concave down: $(-\infty, -2) \cup (2, \infty)$
- c. Local maximum at $x = -3$; local minimum at $x = -1$
 $f'(0), f''(0)$ do not exist, but $f(x)$ is continuous. $x = 0$ is a critical point.
 $f'(0) > 0$ on $[-1, 0]$, and $f'(0) < 0$ on $(0, \infty)$.
Thus, at $x = 0$ there is a local maximum.
- d. Inflection points at $x = -2, 2$

30. Written response (graph)



- a. Increasing: $[-1, \infty)$
Decreasing: $(-\infty, -1]$
- b. Concave up: $(-2, 0) \cup (2, \infty)$
Concave down: $(0, 2)$
- c. No local maximum; local minimum at $x = -1$
- d. Inflection points at $x = 0, 2$
31. $f'(x) = 5x^4 - 15x^2$
 $f''(x) = -30x + 20x^3$
Solving $f'(x) = 0$ gives critical numbers of $x = 0, \pm\sqrt{3}$. Using the second derivative we get that $f''(0) = 0$, $f''(\sqrt{3}) > 0$, and $f''(-\sqrt{3}) < 0$. The second derivative fails at $x = 0$ (there is actually a point of inflection there) but indicates that we have a local maximum $f(-\sqrt{3}) \approx 14.392$ and a local minimum $f(\sqrt{3}) \approx -6.392$ (also the global minimum). The global maximum,

$f(2.5) \approx 23.531$, on $[-2, 2.5]$ occurs at the right endpoint.



32. $f'(x) = \cos x + \frac{1}{2} \sin\left(\frac{x}{2}\right)$

$$f''(x) = -\frac{1}{4} \cos\left(\frac{x}{2}\right) - \sin x$$

Zooming in on the graph of the first derivative gives critical numbers of $x \approx -5.01, -1.27, 2.01, 4.28$.

Using the second derivative we get that

$$f''(-5.01) < 0; f''(-1.27) > 0;$$

$$f''(2.01) < 0; f''(4.28) > 0.$$

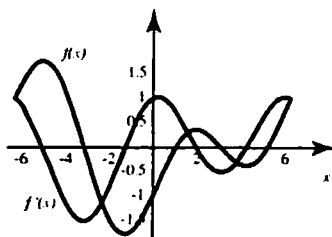
So we have a local maximum of

$$f(-5.01) \approx 1.76 \text{ (also the global max) and}$$

$$f(2.01) \approx 0.369. \text{ We also have a local minimum of}$$

$$f(-1.27) \approx -1.76 \text{ (the global minimum) and}$$

$$f(4.28) \approx -0.369.$$



33. $f'(x) = (x^3 - x) \sec^2 x + (3x^2 - 1) \tan x$

$$f''(x) = 6x \tan x$$

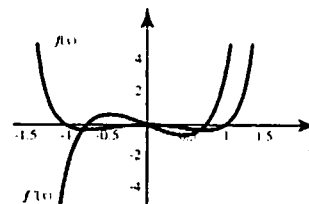
$$+ 2(\sec^2 x)(3x^2 - 1 + x(x^2 - 1) \tan x)$$

Graphing the first derivative and zooming in gives critical numbers of $x = 0$ and

$x \approx -0.745, 0.745$. Using the second derivative

gives

$f''(-0.745) > 0; f''(0) < 0; f''(0.745) > 0$. So have a local maximum of $f(0) = -2$ and local minimums of $f(-0.745) = f(0.745) \approx -0.306$ (also the global minimum). There is no global maximum since the graph blows up at the endpoints.



34. $f'(x) = \frac{-2x(x^4 - 1)}{(x^4 + x^2 + 1)^2}$

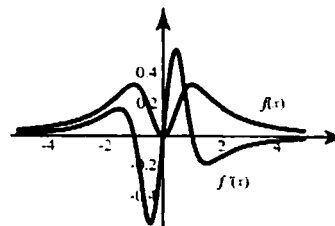
$$f''(x) = \frac{6x^8 - 2x^6 - 24x^4 - 6x^2 + 2}{(x^4 + x^2 + 1)^3}$$

Solving the equation $f'(x) = 0$ yields critical numbers of $x = -1, 0, 1$. Using the second derivative we see that

$$f''(-1) < 0; f''(0) > 0; f''(1) < 0. \text{ Therefore, we}$$

have a local minimum of $f(0) = 0$ (also the global min). We have two local maximums

$$\text{of } f(-1) = f(1) = \frac{1}{3} \text{ (also the global maximum).}$$



4.4 Concepts Review

1. $0 < x < \infty$

2. $2x + \frac{200}{x}$

3. $3x + \frac{300}{x}$

4. minimum; maximum

Problem Set 4.4

1. Let x be one number, y be the other, and Q be the sum of the squares.

$$xy = -16$$

$$y = -\frac{16}{x}$$

The possible values for x are in $(-\infty, 0)$ or $(0, \infty)$.

$$Q = x^2 + y^2 = x^2 + \frac{256}{x^2}$$

$$\frac{dQ}{dx} = 2x - \frac{512}{x^3}$$

$$2x - \frac{512}{x^3} = 0$$

$$x^4 = 256$$

$$x = \pm 4$$

The critical points are $-4, 4$.

$$\frac{dQ}{dx} < 0 \text{ on } (-\infty, -4) \text{ and } (0, 4). \quad \frac{dQ}{dx} > 0 \text{ on}$$

$$(-4, 0) \text{ and } (4, \infty).$$

When $x = -4, y = 4$ and when $x = 4, y = -4$. The two numbers are -4 and 4 .

2. Let x be the number.

$$Q = \sqrt{x} - 8x$$

x will be in the interval $(0, \infty)$.

$$\frac{dQ}{dx} = \frac{1}{2}x^{-1/2} - 8$$

$$\frac{1}{2}x^{-1/2} - 8 = 0$$

$$x^{-1/2} = 16$$

$$x = \frac{1}{256}$$

$$\frac{dQ}{dx} > 0 \text{ on } \left(0, \frac{1}{256}\right) \text{ and } \frac{dQ}{dx} < 0 \text{ on } \left(\frac{1}{256}, \infty\right).$$

Q attains its maximum value at $x = \frac{1}{256}$.

3. Let x be the number.

$$Q = \sqrt[4]{x} - 2x$$

x will be in the interval $(0, \infty)$.

$$\frac{dQ}{dx} = \frac{1}{4}x^{-3/4} - 2$$

$$\frac{1}{4}x^{-3/4} - 2 = 0$$

$$x^{-3/4} = 8$$

$$x = \frac{1}{16}$$

$$\frac{dQ}{dx} > 0 \text{ on } \left(0, \frac{1}{16}\right) \text{ and } \frac{dQ}{dx} < 0 \text{ on } \left(\frac{1}{16}, \infty\right)$$

Q attains its maximum value at $x = \frac{1}{16}$.

4. Let x be one number, y be the other, and Q be the sum of the squares.

$$xy = -12$$

$$y = -\frac{12}{x}$$

The possible values for x are in $(-\infty, 0)$ or $(0, \infty)$.

$$Q = x^2 + y^2 = x^2 + \frac{144}{x^2}$$

$$\frac{dQ}{dx} = 2x - \frac{288}{x^3}$$

$$2x - \frac{288}{x^3} = 0$$

$$x^4 = 144$$

$$x = \pm 2\sqrt{3}$$

The critical points are $-2\sqrt{3}, 2\sqrt{3}$.

$$\frac{dQ}{dx} < 0 \text{ on } (-\infty, -2\sqrt{3}) \text{ and } (0, 2\sqrt{3}).$$

$$\frac{dQ}{dx} > 0 \text{ on } (-2\sqrt{3}, 0) \text{ and } (2\sqrt{3}, \infty).$$

When $x = -2\sqrt{3}, y = 2\sqrt{3}$ and when

$$x = 2\sqrt{3}, y = -2\sqrt{3}.$$

The two numbers are $-2\sqrt{3}$ and $2\sqrt{3}$.

5. Let Q be the square of the distance between (x, y) and $(0, 5)$.

$$Q = (x-0)^2 + (y-5)^2 = x^2 + (x^2-5)^2$$

$$= x^4 - 9x^2 + 25$$

$$\frac{dQ}{dx} = 4x^3 - 18x$$

$$4x^3 - 18x = 0$$

$$2x(2x^2 - 9) = 0$$

$$x = 0, \pm \frac{3}{\sqrt{2}}$$

$$\frac{dQ}{dx} < 0 \text{ on } \left(-\infty, -\frac{3}{\sqrt{2}}\right) \text{ and } \left(0, \frac{3}{\sqrt{2}}\right).$$

$$\frac{dQ}{dx} > 0 \text{ on } \left(-\frac{3}{\sqrt{2}}, 0\right) \text{ and } \left(\frac{3}{\sqrt{2}}, \infty\right).$$

$$\text{When } x = -\frac{3}{\sqrt{2}}, y = \frac{9}{2} \text{ and when } x = \frac{3}{\sqrt{2}},$$

$$y = \frac{9}{2}.$$

The points are $\left(-\frac{3}{\sqrt{2}}, \frac{9}{2}\right)$ and $\left(\frac{3}{\sqrt{2}}, \frac{9}{2}\right)$.

6. Let Q be the square of the distance between (x, y) and $(10, 0)$.

$$Q = (x-10)^2 + (y-0)^2 = (2y^2 - 10)^2 + y^2$$

$$= 4y^4 - 39y^2 + 100$$

$$\frac{dQ}{dy} = 16y^3 - 78y$$

$$16y^3 - 78y = 0$$

$$2y(8y^2 - 39) = 0$$

$$y = 0, \pm \frac{\sqrt{39}}{2\sqrt{2}}$$

$$\frac{dQ}{dy} < 0 \text{ on } \left(-\infty, -\frac{\sqrt{39}}{2\sqrt{2}}\right) \text{ and } \left(0, \frac{\sqrt{39}}{2\sqrt{2}}\right).$$

$$\frac{dQ}{dy} > 0 \text{ on } \left(-\frac{\sqrt{39}}{2\sqrt{2}}, 0\right) \text{ and } \left(\frac{\sqrt{39}}{2\sqrt{2}}, \infty\right).$$

$$\text{When } y = -\frac{\sqrt{39}}{2\sqrt{2}}, x = \frac{39}{4} \text{ and when}$$

$$y = \frac{\sqrt{39}}{2\sqrt{2}}, x = \frac{39}{4}.$$

$$\text{The points are } \left(\frac{39}{4}, -\frac{\sqrt{39}}{2\sqrt{2}}\right) \text{ and } \left(\frac{39}{4}, \frac{\sqrt{39}}{2\sqrt{2}}\right).$$

7. $xy = 900$; $y = \frac{900}{x}$

The possible values for x are in $(0, \infty)$.

$$Q = 4x + 3y = 4x + 3\left(\frac{900}{x}\right) = 4x + \frac{2700}{x}$$

$$\frac{dQ}{dx} = 4 - \frac{2700}{x^2}$$

$$4 - \frac{2700}{x^2} = 0$$

$$x^2 = 675$$

$$x = \pm 15\sqrt{3}$$

$x = 15\sqrt{3}$ is the only critical point in $(0, \infty)$.

$$\frac{dQ}{dx} < 0 \text{ on } (0, 15\sqrt{3}) \text{ and}$$

$$\frac{dQ}{dx} > 0 \text{ on } (15\sqrt{3}, \infty).$$

$$\text{When } x = 15\sqrt{3}, y = \frac{900}{15\sqrt{3}} = 20\sqrt{3}.$$

Q has a minimum when $x = 15\sqrt{3} \approx 25.98$ ft and $y = 20\sqrt{3} \approx 34.64$ ft.

8. $xy = 300$; $y = \frac{300}{x}$

The possible values for x are in $(0, \infty)$.

$$Q = 6x + 4y = 6x + \frac{1200}{x}$$

$$\frac{dQ}{dx} = 6 - \frac{1200}{x^2}$$

$$6 - \frac{1200}{x^2} = 0$$

$$x^2 = 200$$

$$x = \pm 10\sqrt{2}$$

$x = 10\sqrt{2}$ is the only critical point in $(0, \infty)$.

$$\frac{dQ}{dx} < 0 \text{ on } (0, 10\sqrt{2}) \text{ and } \frac{dQ}{dx} > 0 \text{ on } (10\sqrt{2}, \infty)$$

$$\text{When } x = 10\sqrt{2}, y = \frac{300}{10\sqrt{2}} = 15\sqrt{2}.$$

Q has a minimum when $x = 10\sqrt{2} \approx 14.14$ ft and $y = 15\sqrt{2} \approx 21.21$ ft.

9. $xy = 300$; $y = \frac{300}{x}$

The possible values for x are in $(0, \infty)$.

$$Q = 3(6x + 2y) + 2(2y) = 18x + 10y = 18x + \frac{3000}{x}$$

$$\frac{dQ}{dx} = 18 - \frac{3000}{x^2}$$

$$18 - \frac{3000}{x^2} = 0$$

$$x^2 = \frac{500}{3}$$

$$x = \pm \frac{10\sqrt{5}}{\sqrt{3}}$$

$x = \frac{10\sqrt{5}}{\sqrt{3}}$ is the only critical point in $(0, \infty)$.

$$\frac{dQ}{dx} < 0 \text{ on } \left(0, \frac{10\sqrt{5}}{\sqrt{3}}\right) \text{ and}$$

$$\frac{dQ}{dx} > 0 \text{ on } \left(\frac{10\sqrt{5}}{\sqrt{3}}, \infty\right).$$

$$\text{When } x = \frac{10\sqrt{5}}{\sqrt{3}}, y = \frac{300}{\frac{10\sqrt{5}}{\sqrt{3}}} = 6\sqrt{15}$$

Q has a minimum when $x = \frac{10\sqrt{5}}{\sqrt{3}} \approx 12.91$ ft and

$y = 6\sqrt{15} \approx 23.24$ ft.

10. $xy = 900$; $y = \frac{900}{x}$

The possible values for x are in $(0, \infty)$.

$$Q = 6x + 4y = 6x + \frac{3600}{x}$$

$$\frac{dQ}{dx} = 6 - \frac{3600}{x^2}$$

$$6 - \frac{3600}{x^2} = 0$$

$$x^2 = 600$$

$$x = \pm 10\sqrt{6}$$

$x = 10\sqrt{6}$ is the only critical point in $(0, \infty)$.

$$\frac{dQ}{dx} < 0 \text{ on } (0, 10\sqrt{6}) \text{ and } \frac{dQ}{dx} > 0 \text{ on } (10\sqrt{6}, \infty).$$

$$\text{When } x = 10\sqrt{6}, y = \frac{900}{10\sqrt{6}} = 15\sqrt{6}$$

Q has a minimum when $x = 10\sqrt{6} \approx 24.49$ ft and $y = 15\sqrt{6} \approx 36.74$.

$$\text{It appears that } \frac{x}{y} = \frac{2}{3}.$$

Suppose that each pen has area A .

$$xy = A; y = \frac{A}{x}$$

The possible values for x are in $(0, \infty)$.

$$Q = 6x + 4y = 6x + \frac{4A}{x}$$

$$\frac{dQ}{dx} = 6 - \frac{4A}{x^2}$$

$$6 - \frac{4A}{x^2} = 0$$

$$x^2 = \frac{2A}{3}$$

$$x = \pm \sqrt{\frac{2A}{3}}$$

$$x = \sqrt{\frac{2A}{3}} \text{ is the only critical point on } (0, \infty).$$

$$\frac{dQ}{dx} < 0 \text{ on } \left(0, \sqrt{\frac{2A}{3}}\right) \text{ and}$$

$$\frac{dQ}{dx} > 0 \text{ on } \left(\sqrt{\frac{2A}{3}}, \infty\right).$$

$$\text{When } x = \sqrt{\frac{2A}{3}}, y = \frac{A}{\sqrt{\frac{2A}{3}}} = \sqrt{\frac{3A}{2}}$$

$$\frac{x}{y} = \frac{\sqrt{\frac{2A}{3}}}{\sqrt{\frac{3A}{2}}} = \frac{2}{3}$$

11. We are interested in the global extrema for the distance of the object from the observer. We obtain the same extrema by considering the squared distance

$$D(x) = (x - 2.6656)^2 + (42 + x - .08x^2)^2$$

The first derivative is given by

$$D'(x) = \frac{16}{625}x^3 - \frac{12}{25}x^2 - \frac{236}{25}x - \frac{49168}{625}$$

This function can be plotted on a graphing calculator and then analyzed with the TRACE and ZOOM features. Doing so yields critical numbers of $x \approx 6.8267, 28.0$. The second derivative is

$$D''(x) = \frac{48}{625}x^2 - \frac{24}{25}x - \frac{236}{25}$$

which gives $D''(6.8267) < 0$ and $D''(28) > 0$. So we have a local maximum at $x \approx 6.827$ and a local minimum at $x \approx 28.0$. Thus, the object is at the point $(28, 7.28)$ when it is closest to the observer and at the point $(6.8267, 45.098)$ when it is farthest from the observer.

12. Let x be the distance from I_1 .

$$Q = \frac{kI_1}{x^2} + \frac{kI_2}{(s-x)^2}$$

$$\frac{dQ}{dx} = \frac{-2kI_1}{x^3} + \frac{2kI_2}{(s-x)^3}$$

$$-\frac{2kI_1}{x^3} + \frac{2kI_2}{(s-x)^3} = 0; \frac{x^3}{(s-x)^3} = \frac{I_1}{I_2};$$

$$x = \frac{s\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}$$

$$\frac{d^2Q}{dx^2} = \frac{6kI_1}{x^4} + \frac{6kI_2}{(s-x)^4} > 0, \text{ so this point}$$

minimizes the sum.

13. Let x be the distance from P to where the woman lands the boat. She must row a distance of

$\sqrt{x^2 + 4}$ miles and walk $10 - x$ miles. This will

take her $T(x) = \frac{\sqrt{x^2 + 4}}{3} + \frac{10 - x}{4}$ hours;

$$0 \leq x \leq 10. T'(x) = \frac{x}{3\sqrt{x^2 + 4}} - \frac{1}{4}; T'(x) = 0$$

$$\text{when } x = \frac{6}{\sqrt{7}}.$$

$$T(0) = \frac{19}{6} \text{ hr} = 3 \text{ hr}, 10 \text{ min};$$

$$T\left(\frac{6}{\sqrt{7}}\right) = \frac{15 + \sqrt{7}}{6} \approx 2.94 \text{ hr},$$

$$T(10) = \frac{\sqrt{104}}{3} \approx 3.40 \text{ hr}$$

She should land the boat $\frac{6}{\sqrt{7}} \approx 2.27$ mi down the shore from P .

$$14. T(x) = \frac{\sqrt{x^2 + 4}}{3} + \frac{10 - x}{50}, 0 \leq x \leq 10.$$

$$T'(x) = \frac{x}{3\sqrt{x^2 + 4}} - \frac{1}{50}; T'(x) = 0 \text{ when}$$

$$x = \frac{6}{\sqrt{2491}}$$

$$T(0) = \frac{13}{15} \approx 0.867 \text{ hr}; T\left(\frac{6}{\sqrt{2491}}\right) \approx 0.865 \text{ hr};$$

$$T(10) \approx 3.399 \text{ hr}$$

She should land the boat $\frac{6}{\sqrt{2491}} \approx 0.12$ mi down the shore from P .

$$15. T(x) = \frac{\sqrt{x^2 + 4}}{20} + \frac{10 - x}{4}, 0 \leq x \leq 10.$$

$$T'(x) = \frac{x}{20\sqrt{x^2 + 4}} - \frac{1}{4}; T'(x) = 0 \text{ has no solution.}$$

$$T(0) = \frac{2}{20} + \frac{10}{4} = \frac{13}{5} \text{ hr} = 2 \text{ hr, } 36 \text{ min}$$

$$T(10) = \frac{\sqrt{104}}{20} \approx 0.5 \text{ hr}$$

She should take the boat all the way to town.

16. Let x be the length of cable on land, $0 \leq x \leq L$.

Let C be the cost.

$$C = a\sqrt{(L-x)^2 + w^2} + bx$$

$$\frac{dC}{dx} = -\frac{a(L-x)}{\sqrt{(L-x)^2 + w^2}} + b$$

$$-\frac{a(L-x)}{\sqrt{(L-x)^2 + w^2}} + b = 0 \text{ when}$$

$$b^2[(L-x)^2 + w^2] = a^2(L-x)^2$$

$$(a^2 - b^2)(L-x)^2 = b^2w^2$$

$$x = L - \frac{bw}{\sqrt{a^2 - b^2}} \text{ ft on land;}$$

$$\frac{aw}{\sqrt{a^2 - b^2}} \text{ ft under water}$$

$$\frac{d^2C}{dx^2} = \frac{aw^2}{[(L-x)^2 + w^2]^{3/2}} > 0 \text{ for all } x, \text{ so this minimizes the cost.}$$

17. Let the coordinates of the first ship at 7:00 a.m. be $(0, 0)$. Thus, the coordinates of the second ship at 7:00 a.m. are $(-60, 0)$. Let t be the time in hours since 7:00 a.m. The coordinates of the first and second ships at t are $(-20t, 0)$ and $(-60 + 15\sqrt{2}t, -15\sqrt{2}t)$ respectively. Let D be the square of the distances at t .

$$D = (-20t + 60 - 15\sqrt{2}t)^2 + (0 + 15\sqrt{2}t)^2$$

$$= (1300 + 600\sqrt{2})t^2 - (2400 + 1800\sqrt{2})t + 3600$$

$$\frac{dD}{dt} = 2(1300 + 600\sqrt{2})t - (2400 + 1800\sqrt{2})$$

$$2(1300 + 600\sqrt{2})t - (2400 + 1800\sqrt{2}) = 0 \text{ when}$$

$$t = \frac{12 + 9\sqrt{2}}{13 + 6\sqrt{2}} \approx 1.15 \text{ hrs or } 1 \text{ hr, } 9 \text{ min}$$

$$D \text{ is the minimum at } t = \frac{12 + 9\sqrt{2}}{13 + 6\sqrt{2}} \text{ since } \frac{d^2D}{dt^2} > 0$$

for all t .

The ships are closest at 8:09 A.M.

18. Write y in terms of x : $y = \frac{b}{a}\sqrt{a^2 - x^2}$ (positive

square root since the point is in the first quadrant). Compute the slope of the tangent line:

$$y' = -\frac{bx}{a\sqrt{a^2 - x^2}}.$$

Find the y -intercept, y_0 , of the tangent line through the point (x, y) :

$$\frac{y_0 - y}{0 - x} = -\frac{bx}{a\sqrt{a^2 - x^2}}$$

$$y_0 = \frac{bx^2}{a\sqrt{a^2 - x^2}} + y = \frac{bx^2}{a\sqrt{a^2 - x^2}} + \frac{b}{a}\sqrt{a^2 - x^2} = \frac{ab}{\sqrt{a^2 - x^2}}$$

Find the x -intercept, x_0 , of the tangent line through the point (x, y) :

$$\frac{y - 0}{x - x_0} = -\frac{bx}{a\sqrt{a^2 - x^2}}$$

$$x_0 = \frac{ay\sqrt{a^2 - x^2}}{bx} + x = \frac{a^2 - x^2}{x} + x = \frac{a^2}{x}$$

Compute the Area A of the resulting triangle and maximize:

$$A = \frac{1}{2}x_0y_0 = \frac{a^3b}{2x\sqrt{a^2 - x^2}} = \frac{a^3b}{2}\left(x\sqrt{a^2 - x^2}\right)^{-1}$$

$$\frac{dA}{dx} = -\frac{a^3b}{2}\left(x\sqrt{a^2 - x^2}\right)^{-2}\left(\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}\right)$$

$$= \frac{a^3 b}{2x^2(a^2 - x^2)^{3/2}}(2x^2 - a^2)$$

$$\frac{a^3 b}{2x^2(a^2 - x^2)^{3/2}}(2x^2 - a^2) = 0 \text{ when}$$

$$x = \frac{a}{\sqrt{2}}; y = \frac{b}{a} \sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2} = \frac{b}{\sqrt{2}}$$

$$y' = -\frac{b\left(\frac{a}{\sqrt{2}}\right)}{a\sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2}} = -\frac{b}{a}$$

Note that $\frac{dA}{dx} < 0$ on $\left(0, \frac{a}{\sqrt{2}}\right)$ and

$\frac{dA}{dx} > 0$ on $\left(\frac{a}{\sqrt{2}}, a\right)$, so A is a minimum at

$x = \frac{a}{\sqrt{2}}$. Then the equation of the tangent line is

$$y = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right) + \frac{b}{\sqrt{2}} \text{ or } bx + ay - ab\sqrt{2} = 0.$$

19. Let x be the radius of the base of the cylinder and h the height.

$$V = \pi x^2 h; r^2 = x^2 + \left(\frac{h}{2}\right)^2; x^2 = r^2 - \frac{h^2}{4}$$

$$V = \pi\left(r^2 - \frac{h^2}{4}\right)h = \pi hr^2 - \frac{\pi h^3}{4}$$

$$\frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4}; V' = 0 \text{ when } h = \pm \frac{2\sqrt{3}r}{3}$$

Since $\frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$, the volume is maximized

when $h = \frac{2\sqrt{3}r}{3}$.

$$\begin{aligned} V &= \pi\left(\frac{2\sqrt{3}}{3}r\right)r^2 - \frac{\pi\left(\frac{2\sqrt{3}}{3}r\right)^3}{4} \\ &= \frac{2\pi\sqrt{3}}{3}r^3 - \frac{2\pi\sqrt{3}}{9}r^3 = \frac{4\pi\sqrt{3}}{9}r^3 \end{aligned}$$

20. Let r be the radius of the circle, x the length of the rectangle, and y the width of the rectangle.

$$P = 2x + 2y; r^2 = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2; r^2 = \frac{x^2}{4} + \frac{y^2}{4};$$

$$y = \sqrt{4r^2 - x^2}; P = 2x + 2\sqrt{4r^2 - x^2}$$

$$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4r^2 - x^2}};$$

$$2 - \frac{2x}{\sqrt{4r^2 - x^2}} = 0; 2\sqrt{4r^2 - x^2} = 2x;$$

$$16r^2 - 4x^2 = 4x^2; x = \pm\sqrt{2}r$$

$$\frac{d^2P}{dx^2} = -\frac{8r^2}{(4r^2 - x^2)^{3/2}} < 0 \text{ when } x = \sqrt{2}r;$$

$$y = \sqrt{4r^2 - 2r^2} = \sqrt{2}r$$

The rectangle with maximum perimeter is a square with side length $\sqrt{2}r$.

21. Let x be the radius of the cylinder, r the radius of the sphere, and h the height of the cylinder.

$$A = 2\pi xh; r^2 = x^2 + \frac{h^2}{4}; x = \sqrt{r^2 - \frac{h^2}{4}}$$

$$A = 2\pi\sqrt{r^2 - \frac{h^2}{4}}h = 2\pi\sqrt{h^2r^2 - \frac{h^4}{4}}$$

$$\frac{dA}{dh} = \frac{\pi(2r^2h - h^3)}{\sqrt{h^2r^2 - \frac{h^4}{4}}}; A' = 0 \text{ when } h = 0, \pm\sqrt{2}r$$

$$\frac{dA}{dh} > 0 \text{ on } (0, \sqrt{2}r) \text{ and } \frac{dA}{dh} < 0 \text{ on } (\sqrt{2}r, 2r),$$

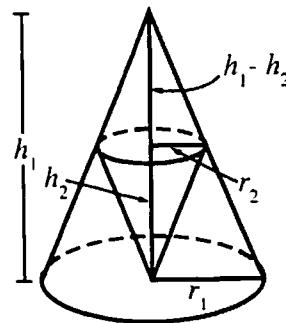
so A is a maximum when $h = \sqrt{2}r$.

The dimensions are $h = \sqrt{2}r, x = \frac{r}{\sqrt{2}}$.

22. Let r_1 and h_1 be the radius and altitude of the outer cone; r_2 and h_2 the radius and altitude of the inner cone.

$$V_1 = \frac{1}{3}\pi r_1^2 h_1 \text{ is fixed. } r_1 = \sqrt{\frac{3V_1}{\pi h_1}}$$

By similar triangles $\frac{h_1 - h_2}{h_1} = \frac{r_2}{r_1}$ (see figure).



$$r_2 = r_1\left(1 - \frac{h_2}{h_1}\right) = \sqrt{\frac{3V_1}{\pi h_1}}\left(1 - \frac{h_2}{h_1}\right)$$

$$V_2 = \frac{1}{3}\pi r_2^2 h_2 = \frac{1}{3}\pi\left[\sqrt{\frac{3V_1}{\pi h_1}}\left(1 - \frac{h_2}{h_1}\right)\right]^2 h_2$$

$$= \frac{\pi}{3} \cdot \frac{3V_1 h_2}{\pi h_1} \left(1 - \frac{h_2}{h_1}\right)^2 = V_1 \frac{h_2}{h_1} \left(1 - \frac{h_2}{h_1}\right)^2$$

Let $k = \frac{h_2}{h_1}$, the ratio of the altitudes of the cones,

$$\text{then } V_2 = V_1 k(1-k)^2.$$

$$\frac{dV_2}{dk} = V_1(1-k)^2 - 2kV_1(1-k) = V_1(1-k)(1-3k)$$

$$0 < k < 1 \text{ so } \frac{dV_2}{dk} = 0 \text{ when } k = \frac{1}{3}.$$

$$\frac{d^2V_2}{dk^2} = V_1(6k-4); \frac{d^2V_2}{dk^2} < 0 \text{ when } k = \frac{1}{3}$$

The altitude of the inner cone must be $\frac{1}{3}$ the altitude of the outer cone.

23. Let x be the length of a side of the square, so

$$\frac{100-4x}{3} \text{ is the side of the triangle, } 0 \leq x \leq 25$$

$$A = x^2 + \frac{1}{2} \left(\frac{100-4x}{3} \right) \frac{\sqrt{3}}{2} \left(\frac{100-4x}{3} \right)$$

$$= x^2 + \frac{\sqrt{3}}{4} \left(\frac{10,000 - 800x + 16x^2}{9} \right)$$

$$\frac{dA}{dx} = 2x - \frac{200\sqrt{3}}{9} + \frac{8\sqrt{3}}{9}x; A' = 0 \text{ when } x \approx 10.87$$

Critical points: $x = 0, 10.87, 25$

At $x = 0, A \approx 481$; at $x = 10.87, A \approx 272$; at $x = 25, A = 625$.

- For minimum area, the cut should be approximately $4(10.87) = 43.48$ cm from one end and the shorter length should be bent to form the square.
- For maximum area, the wire should not be cut, it should be bent to form a square.

24. Let x be the length of the sides of the base, y be the height of the box, and k be the cost per square inch of the material in the sides of the box.

$$V = x^2 y;$$

$$\text{The cost is } C = 1.2kx^2 + 1.5ky^2 + 4kxy$$

$$= 2.7kx^2 + 4kx \left(\frac{V}{x^2} \right) = 2.7kx^2 + \frac{4kV}{x}$$

$$\frac{dC}{dx} = 5.4kx - \frac{4kV}{x^2}; \frac{dC}{dx} = 0 \text{ when } x \approx 0.905\sqrt[3]{V}$$

$$y \approx \frac{V}{(0.905\sqrt[3]{V})^2} \approx 1.22\sqrt[3]{V}$$

25. Let r be the radius of the cylinder and h the height of the cylinder.

$$V = \pi r^2 h + \frac{2}{3} \pi r^3; h = \frac{V - \frac{2}{3} \pi r^3}{\pi r^2} = \frac{V}{\pi r^2} - \frac{2}{3} r$$

Let k be the cost per square foot of the cylindrical wall. The cost is

$$C = k(2\pi r h) + 2k(2\pi r^2)$$

$$= k \left(2\pi r \left(\frac{V}{\pi r^2} - \frac{2}{3} r \right) + 4\pi r^2 \right) = k \left(\frac{2V}{r} + \frac{8\pi r^2}{3} \right)$$

$$\frac{dC}{dr} = k \left(-\frac{2V}{r^2} + \frac{16\pi r}{3} \right); k \left(-\frac{2V}{r^2} + \frac{16\pi r}{3} \right) = 0$$

$$\text{when } r^3 = \frac{3V}{8\pi}, r = \frac{1}{2} \left(\frac{3V}{\pi} \right)^{1/3}$$

$$h = \frac{4V}{\pi \left(\frac{3V}{\pi} \right)^{2/3}} - \frac{1}{3} \left(\frac{3V}{\pi} \right)^{1/3} = \left(\frac{3V}{\pi} \right)^{1/3}$$

For a given volume V , the height of the cylinder is

$$\left(\frac{3V}{\pi} \right)^{1/3} \text{ and the radius is } \frac{1}{2} \left(\frac{3V}{\pi} \right)^{1/3}.$$

$$26. \frac{dx}{dt} = 2 \cos 2t - 2\sqrt{3} \sin 2t;$$

$$\frac{dx}{dt} = 0 \text{ when } \tan 2t = \frac{1}{\sqrt{3}};$$

$$2t = \frac{\pi}{6} + \pi n \text{ for any integer } n$$

$$t = \frac{\pi}{12} + \frac{\pi}{2} n$$

$$\text{When } t = \frac{\pi}{12} + \frac{\pi}{2} n,$$

$$|x| = \left| \sin \left(\frac{\pi}{6} + \pi n \right) + \sqrt{3} \cos \left(\frac{\pi}{6} + \pi n \right) \right|$$

$$= \left| \sin \frac{\pi}{6} \cos \pi n + \cos \frac{\pi}{6} \sin \pi n \right|$$

$$+ \sqrt{3} \left(\cos \frac{\pi}{6} \cos \pi n - \sin \frac{\pi}{6} \sin \pi n \right)$$

$$= \left| (-1)^n \frac{1}{2} + (-1)^n \frac{3}{2} \right| = 2.$$

The farthest the weight gets from the origin is 2 units.

$$27. A = \frac{r^2 \theta}{2}; \theta = \frac{2A}{r^2}$$

The perimeter is

$$Q = 2r + r\theta = 2r + \frac{2Ar}{r^2} = 2r + \frac{2A}{r}$$

$$\frac{dQ}{dr} = 2 - \frac{2A}{r^2}; Q = 0 \text{ when } r = \sqrt{A}$$

$$\theta = \frac{2A}{(\sqrt{A})^2} = 2$$

$$\frac{d^2Q}{dr^2} = \frac{4A}{r^3} > 0, \text{ so this minimizes the perimeter.}$$

28. The distance from the fence to the base of the

ladder is $\frac{h}{\tan \theta}$.

The length of the ladder is x .

$$\cos \theta = \frac{\frac{h}{\tan \theta} + w}{x}; x \cos \theta = \frac{h}{\tan \theta} + w;$$

$$x = \frac{h}{\sin \theta} + \frac{w}{\cos \theta}$$

$$\frac{dx}{d\theta} = -\frac{h \cos \theta}{\sin^2 \theta} + \frac{w \sin \theta}{\cos^2 \theta}; \frac{w \sin^3 \theta - h \cos^3 \theta}{\sin^2 \theta \cos^2 \theta} = 0$$

$$\text{when } \tan^3 \theta = \frac{h}{w}$$

$$\theta = \tan^{-1} \sqrt[3]{\frac{h}{w}}$$

$$\tan \theta = \frac{\sqrt[3]{h}}{\sqrt[3]{w}}; \sin \theta = \frac{\sqrt[3]{h}}{\sqrt{h^{2/3} + w^{2/3}}},$$

$$\cos \theta = \frac{\sqrt[3]{w}}{\sqrt{h^{2/3} + w^{2/3}}}$$

$$x = h \left(\frac{\sqrt{h^{2/3} + w^{2/3}}}{\sqrt[3]{h}} \right) + w \left(\frac{\sqrt{h^{2/3} + w^{2/3}}}{\sqrt[3]{w}} \right)$$

$$= (h^{2/3} + w^{2/3})^{3/2}$$

29. Let d_1 be the distance that the light travels in medium 1 and let d_2 be the distance the light travels in medium 2.

$$d_1 = \sqrt{a^2 + x^2}, d_2 = \sqrt{b^2 + (d-x)^2}$$

The time that it takes the light to travel from A to B is

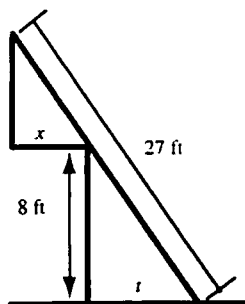
$$t = \frac{d_1}{c_1} + \frac{d_2}{c_2} = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c_2}$$

$$\begin{aligned} \frac{dt}{dx} &= \frac{x}{c_1 \sqrt{a^2 + x^2}} - \frac{d-x}{c_2 \sqrt{b^2 + (d-x)^2}} \\ &= \frac{\sin \theta_1}{c_1} - \frac{\sin \theta_2}{c_2} \end{aligned}$$

$$\frac{dt}{dx} = 0 \text{ implies } \frac{\sin \theta_1}{c_1} = \frac{\sin \theta_2}{c_2}.$$

30. From Problem 29, since there is no change in the medium, $\sin \theta_1 = \sin \theta_2$ where θ_1 and θ_2 are measured from the perpendicular to the mirror. θ_1 and θ_2 are both less than 90° so $\sin \theta_1 = \sin \theta_2$ implies that $\theta_1 = \theta_2$. Thus $90^\circ - \theta_1 = 90^\circ - \theta_2$ or the angle of incidence equals the angle of reflection.

31. Consider the following sketch.



$$\text{By similar triangles, } \frac{x}{27 - \sqrt{t^2 + 64}} = \frac{t}{\sqrt{t^2 + 64}}.$$

$$x = \frac{27t}{\sqrt{t^2 + 64}} - t$$

$$\frac{dx}{dt} = \frac{27\sqrt{t^2 + 64} - \frac{27t^2}{\sqrt{t^2 + 64}}}{t^2 + 64} - 1 = \frac{1728}{(t^2 + 64)^{3/2}} - 1$$

$$\frac{1728}{(t^2 + 64)^{3/2}} - 1 = 0 \text{ when } t = 4\sqrt{5}$$

$$\frac{d^2x}{dt^2} = \frac{-5184t}{(t^2 + 64)^{5/2}}; \left. \frac{d^2x}{dt^2} \right|_{t=4\sqrt{5}} < 0$$

Therefore

$$x = \frac{27(4\sqrt{5})}{\sqrt{(4\sqrt{5})^2 + 64}} - 4\sqrt{5} = 5\sqrt{5} \approx 11.18 \text{ ft is the}$$

maximum horizontal overhang.

32. Let x be the length of the edges of the cube. The surface area of the cube is $6x^2$ so $0 \leq x \leq \frac{1}{\sqrt{6}}$.

The surface area of the sphere is $4\pi r^2$, so

$$6x^2 + 4\pi r^2 = 1, r = \sqrt{\frac{1-6x^2}{4\pi}}$$

$$V = x^3 + \frac{4}{3}\pi r^3 = x^3 + \frac{1}{6\sqrt{\pi}}(1-6x^2)^{3/2}$$

$$\frac{dV}{dx} = 3x^2 - \frac{3}{\sqrt{\pi}}x\sqrt{1-6x^2} = 3x \left(x - \sqrt{\frac{1-6x^2}{\pi}} \right)$$

$$\frac{dV}{dx} = 0 \text{ when } x = 0, \frac{1}{\sqrt{6+\pi}}$$

$$V(0) = \frac{1}{6\sqrt{\pi}} \approx 0.094 \text{ m}^3.$$

$$V\left(\frac{1}{\sqrt{6+\pi}}\right) = (6+\pi)^{-3/2} + \frac{1}{6\sqrt{\pi}}\left(1 - \frac{6}{6+\pi}\right)^{3/2}$$

$$= \left(1 + \frac{\pi}{6}\right)(6+\pi)^{-3/2} = \frac{1}{6\sqrt{6+\pi}} \approx 0.055 \text{ m}^3$$

For maximum volume: no cube, a sphere of radius

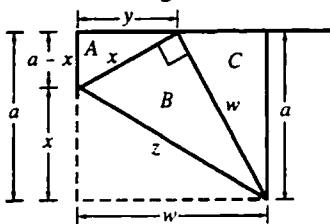
$$\frac{1}{2\sqrt{\pi}} \approx 0.282 \text{ meters.}$$

For minimum volume: cube with sides of length

$$\frac{1}{\sqrt{6+\pi}} \approx 0.331 \text{ meters,}$$

$$\text{sphere of radius } \frac{1}{2\sqrt{6+\pi}} \approx 0.165 \text{ meters}$$

33. Consider the figure below.



$$\text{a. } y = \sqrt{x^2 - (a-x)^2} = \sqrt{2ax - a^2}$$

$$\text{Area of } A = A = \frac{1}{2}(a-x)y$$

$$= \frac{1}{2}(a-x)\sqrt{2ax - a^2}$$

$$\frac{dA}{dx} = -\frac{1}{2}\sqrt{2ax - a^2} + \frac{\frac{1}{2}(a-x)(\frac{1}{2})(2a)}{\sqrt{2ax - a^2}}$$

$$= \frac{a^2 - \frac{3}{2}ax}{\sqrt{2ax - a^2}}$$

$$\frac{a^2 - \frac{3}{2}ax}{\sqrt{2ax - a^2}} = 0 \text{ when } x = \frac{2a}{3}.$$

$$\frac{dA}{dx} > 0 \text{ on } \left(\frac{a}{2}, \frac{2a}{3}\right) \text{ and } \frac{dA}{dx} < 0 \text{ on } \left(\frac{2a}{3}, a\right),$$

$$\text{so } x = \frac{2a}{3} \text{ maximizes the area of triangle } A.$$

b. Triangle A is similar to triangle C , so

$$w = \frac{ax}{y} = \frac{ax}{\sqrt{2ax - a^2}}$$

$$\text{Area of } B = B = \frac{1}{2}xw = \frac{ax^2}{2\sqrt{2ax - a^2}}$$

$$\frac{dB}{dx} = \frac{a}{2} \left(\frac{2x\sqrt{2ax - a^2} - x^2 \frac{a}{\sqrt{2ax - a^2}}}{2ax - a^2} \right)$$

$$= \frac{a}{2} \left(\frac{2x(2ax - a^2) - ax^2}{(2ax - a^2)^{3/2}} \right) = \frac{a}{2} \left(\frac{3ax^2 - 2xa^2}{(2ax - a^2)^{3/2}} \right)$$

$$\frac{a^2}{2} \left(\frac{3x^2 - 2xa}{(2ax - a^2)^{3/2}} \right) = 0 \text{ when } x = 0, \frac{2a}{3}$$

$$\text{Since } x = 0 \text{ is not possible, } x = \frac{2a}{3}.$$

$$\frac{dB}{dx} < 0 \text{ on } \left(\frac{a}{2}, \frac{2a}{3}\right) \text{ and } \frac{dB}{dx} > 0 \text{ on } \left(\frac{2a}{3}, a\right),$$

$$\text{so } x = \frac{2a}{3} \text{ minimizes the area of triangle } B.$$

$$\text{c. } z = \sqrt{x^2 + w^2} = \sqrt{x^2 + \frac{a^2 x^2}{2ax - a^2}}$$

$$= \sqrt{\frac{2ax^3}{2ax - a^2}}$$

$$\frac{dz}{dx} = \frac{1}{2} \sqrt{\frac{2ax - a^2}{2ax^3}} \left(\frac{6ax^2(2ax - a^2) - 2ax^3(2a)}{(2ax - a^2)^2} \right)$$

$$= \frac{4a^2 x^3 - 3a^3 x^2}{\sqrt{2ax^3(2ax - a^2)^3}}$$

$$\frac{dz}{dx} = 0 \text{ when } x = 0, \frac{3a}{4}$$

$$x = \frac{3a}{4}$$

$$\frac{dz}{dx} < 0 \text{ on } \left(\frac{a}{2}, \frac{3a}{4}\right) \text{ and } \frac{dz}{dx} > 0 \text{ on } \left(\frac{3a}{4}, a\right),$$

$$\text{so } x = \frac{3a}{4} \text{ minimizes length } z.$$

34. Let $2x$ be the length of a bar and $2y$ be the width of a bar.

$$x = a \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = a\left(\frac{1}{\sqrt{2}} \cos \frac{\theta}{2} + \frac{1}{\sqrt{2}} \sin \frac{\theta}{2}\right) = \frac{a}{\sqrt{2}}\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2}\right)$$

$$y = a \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = a\left(\frac{1}{\sqrt{2}} \cos \frac{\theta}{2} - \frac{1}{\sqrt{2}} \sin \frac{\theta}{2}\right) = \frac{a}{\sqrt{2}}\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)$$

Compute the area A of the cross and maximize.

$$A = 2(2x)(2y) - (2y)^2$$

$$\begin{aligned} &= 8\left[\frac{a}{\sqrt{2}}\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2}\right)\right]\left[\frac{a}{\sqrt{2}}\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)\right] - 4\left[\frac{a}{\sqrt{2}}\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)\right]^2 \\ &= 4a^2\left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}\right) - 2a^2\left(1 - 2\cos \frac{\theta}{2} \sin \frac{\theta}{2}\right) = 4a^2 \cos \theta - 2a^2(1 - \sin \theta) \end{aligned}$$

$$\frac{dA}{d\theta} = -4a^2 \sin \theta + 2a^2 \cos \theta$$

$$-4a^2 \sin \theta + 2a^2 \cos \theta = 0 \text{ when } \tan \theta = \frac{1}{2}$$

$$\sin \theta = \frac{1}{\sqrt{5}}, \cos \theta = \frac{2}{\sqrt{5}}$$

$$\frac{d^2 A}{d\theta^2} < 0 \text{ when } \tan \theta = \frac{1}{2}, \text{ so this maximizes the area.}$$

$$A = 4a^2\left(\frac{2}{\sqrt{5}}\right) - 2a^2\left(1 - \frac{1}{\sqrt{5}}\right) = \frac{10a^2}{\sqrt{5}} - 2a^2 = 2a^2(\sqrt{5} - 1)$$

35. a. $L'(\theta) = 15(9 + 25 - 30 \cos \theta)^{-1/2} \sin \theta = 15(34 - 30 \cos \theta)^{-1/2} \sin \theta$

$$L''(\theta) = -\frac{15}{2}(34 - 30 \cos \theta)^{-3/2}(30 \sin \theta) \sin \theta + 15(34 - 30 \cos \theta)^{-1/2} \cos \theta$$

$$= -225(34 - 30 \cos \theta)^{-3/2} \sin^2 \theta + 15(34 - 30 \cos \theta)^{-1/2} \cos \theta$$

$$= 15(34 - 30 \cos \theta)^{-3/2}[-15 \sin^2 \theta + (34 - 30 \cos \theta) \cos \theta]$$

$$= 15(34 - 30 \cos \theta)^{-3/2}[-15 \sin^2 \theta + 34 \cos \theta - 30 \cos^2 \theta]$$

$$= 15(34 - 30 \cos \theta)^{-3/2}[-15 + 34 \cos \theta - 15 \cos^2 \theta]$$

$$= -15(34 - 30 \cos \theta)^{-3/2}[15 \cos^2 \theta - 34 \cos \theta + 15]$$

$$L'' = 0 \text{ when } \cos \theta = \frac{34 \pm \sqrt{(34)^2 - 4(15)(15)}}{2(15)} = \frac{5}{3}, \frac{3}{5}$$

$$\theta = \cos^{-1}\left(\frac{3}{5}\right)$$

$$L'\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = 15\left(9 + 25 - 30\left(\frac{3}{5}\right)\right)^{-1/2}\left(\frac{4}{5}\right) = 3$$

$$L\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = \left(9 + 25 - 30\left(\frac{3}{5}\right)\right)^{1/2} = 4$$

$\phi = 90^\circ$ since the resulting triangle is a 3-4-5 right triangle.

- b. $L'(\theta) = 65(25 + 169 - 130 \cos \theta)^{-1/2} \sin \theta = 65(194 - 130 \cos \theta)^{-1/2} \sin \theta$

$$L''(\theta) = -\frac{65}{2}(194 - 130 \cos \theta)^{-3/2}(130 \sin \theta) \sin \theta + 65(194 - 130 \cos \theta)^{-1/2} \cos \theta$$

$$= -4225(194 - 130 \cos \theta)^{-3/2} \sin^2 \theta + 65(194 - 130 \cos \theta)^{-1/2} \cos \theta$$

$$\begin{aligned}
&= 65(194 - 130 \cos \theta)^{-3/2} [-65 \sin^2 \theta + (194 - 130 \cos \theta) \cos \theta] \\
&= 65(194 - 130 \cos \theta)^{-3/2} [-65 \sin^2 \theta + 194 \cos \theta - 130 \cos^2 \theta] \\
&= 65(194 - 130 \cos \theta)^{-3/2} [-65 \cos^2 \theta + 194 \cos \theta - 65] \\
&= -65(194 - 130 \cos \theta)^{-3/2} [65 \cos^2 \theta - 194 \cos \theta + 65]
\end{aligned}$$

$$L'' = 0 \text{ when } \cos \theta = \frac{194 \pm \sqrt{(194)^2 - 4(65)(65)}}{2(65)} = \frac{13}{5}, \frac{5}{13}$$

$$\theta = \cos^{-1} \left(\frac{5}{13} \right)$$

$$L' \left(\cos^{-1} \left(\frac{5}{13} \right) \right) = 65 \left(25 + 169 - 130 \left(\frac{5}{13} \right) \right)^{1/2} \left(\frac{12}{13} \right) = 5$$

$$L \left(\cos^{-1} \left(\frac{5}{13} \right) \right) = \left(25 + 169 - 130 \left(\frac{5}{13} \right) \right)^{1/2} = 12$$

$\phi = 90^\circ$ since the resulting triangle is a 5-12-13 right triangle.

c. When the tips are separating most rapidly, $\phi = 90^\circ$, $L = \sqrt{m^2 - h^2}$, $L' = h$

$$d. \quad L'(\theta) = hm(h^2 + m^2 - 2hm \cos \theta)^{-1/2} \sin \theta$$

$$L''(\theta) = -h^2 m^2 (h^2 + m^2 - 2hm \cos \theta)^{-3/2} \sin^2 \theta + hm(h^2 + m^2 - 2hm \cos \theta)^{-1/2} \cos \theta$$

$$= hm(h^2 + m^2 - 2hm \cos \theta)^{-3/2} [-hm \sin^2 \theta + (h^2 + m^2) \cos \theta - 2hm \cos^2 \theta]$$

$$= hm(h^2 + m^2 - 2hm \cos \theta)^{-3/2} [-hm \cos^2 \theta + (h^2 + m^2) \cos \theta - hm]$$

$$= -hm(h^2 + m^2 - 2hm \cos \theta)^{-3/2} [hm \cos^2 \theta - (h^2 + m^2) \cos \theta + hm]$$

$$L'' = 0 \text{ when } hm \cos^2 \theta - (h^2 + m^2) \cos \theta + hm = 0$$

$$(h \cos \theta - m)(m \cos \theta - h) = 0$$

$$\cos \theta = \frac{m}{h}, \frac{h}{m}$$

$$\text{Since } h < m, \cos \theta = \frac{h}{m} \text{ so } \theta = \cos^{-1} \left(\frac{h}{m} \right).$$

$$L' \left(\cos^{-1} \left(\frac{h}{m} \right) \right) = hm \left(h^2 + m^2 - 2hm \left(\frac{h}{m} \right) \right)^{-1/2} \frac{\sqrt{m^2 - h^2}}{m} = hm(m^2 - h^2)^{-1/2} \frac{\sqrt{m^2 - h^2}}{m} = h$$

$$L \left(\cos^{-1} \left(\frac{h}{m} \right) \right) = \left(h^2 + m^2 - 2hm \left(\frac{h}{m} \right) \right)^{1/2} = \sqrt{m^2 - h^2}$$

$$\text{Since } h^2 + L^2 = m^2, \phi = 90^\circ.$$

36. Following the same reasoning as in Problem 11, we are interested in finding the global extrema for the distance of the object from the observer. We will obtain the same results by considering the squared distance instead. The squared distance can be expressed as

$$D(x) = (x - 2)^2 + \left(100 + x - \frac{1}{10}x^2 \right)^2$$

The first and second derivatives are given by

$$D'(x) = \frac{1}{25}x^3 - \frac{3}{5}x^2 - 36x + 196 \text{ and}$$

$$D''(x) = \frac{3}{25}(x^2 - 10x - 300)$$

Using a computer package, we can solve the equation $D'(x) = 0$ to find the critical points. The critical points are $x \approx 5.1538, 36.148$. Using the second derivative we see that

$$D''(5.1538) \approx -38.9972 \text{ (max) and}$$

$$D''(36.148) \approx 77.4237 \text{ (min)}$$

Therefore, the position of the object closest to the observer is $\approx (36.148, 5.48)$ while the position of the object farthest from the person is $\approx (5.1538, 102.5)$.

(Remember to go back to the original equation for the path of the object once you find the critical points.)

37. Here we are interested in minimizing the distance between the earth and the asteroid. Using the coordinates P and Q for the two bodies, we can use the distance formula to obtain a suitable equation. However, for simplicity, we will minimize the squared distance to find the critical points. The squared distance between the objects is given by

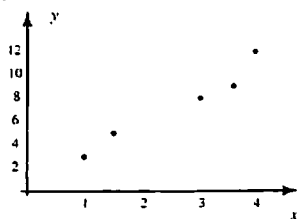
$$D(t) = (93 \cos(2\pi t) - 60 \cos[2\pi(1.5t - 1)])^2 + (93 \sin(2\pi t) - 120 \sin[2\pi(1.5t - 1)])^2$$

The first derivative is

$$D'(t) \approx -34359[\cos(2\pi t)][\sin(9.48761t)] + [\cos(9.48761t)][(204932 \sin(9.48761t) - 141643 \sin(2\pi t))]$$

Plotting the function and its derivative reveal a periodic relationship due to the orbiting of the objects. Careful examination of the graphs reveals that there is indeed a minimum squared distance (and hence a minimum distance) that occurs only once. The critical value for this occurrence is $t \approx 13.8279$. This value gives a distance between the objects of ≈ 0.047851 million miles = 47,851 miles

38. a.



- b. There are only a few data points, but they do seem fairly linear.
- c. The data values can be entered into most scientific calculators to utilize the Least Squares Regression feature. Alternately one could use the formulas for the slope and intercept provided in the text. The resulting line should be $y = 0.56852 + 2.6074x$
- d. Using the result from c., the predicted number of surface imperfections on a sheet with area

2.0 square feet is

$$y = 0.56852 + 2.6074(2.0) = 5.7833 \approx 6$$

since we can't have partial imperfections

$$\begin{aligned} 39. \quad a. \quad \frac{dS}{db} &= \frac{d}{db} \sum_{i=1}^n [y_i - (5 + bx_i)]^2 \\ &= \sum_{i=1}^n \frac{d}{db} [y_i - (5 + bx_i)]^2 \\ &= \sum_{i=1}^n 2(y_i - 5 - bx_i)(-x_i) \\ &= 2 \left[\sum_{i=1}^n (-x_i y_i + 5x_i + bx_i^2) \right] \\ &= -2 \sum_{i=1}^n x_i y_i + 10 \sum_{i=1}^n x_i + 2b \sum_{i=1}^n x_i^2 \end{aligned}$$

Setting $\frac{dS}{db} = 0$ gives

$$\begin{aligned} 0 &= -2 \sum_{i=1}^n x_i y_i + 10 \sum_{i=1}^n x_i + 2b \sum_{i=1}^n x_i^2 \\ 0 &= -\sum_{i=1}^n x_i y_i + 5 \sum_{i=1}^n x_i + b \sum_{i=1}^n x_i^2 \\ b \sum_{i=1}^n x_i^2 &= \sum_{i=1}^n x_i y_i - 5 \sum_{i=1}^n x_i \\ b &= \frac{\sum_{i=1}^n x_i y_i - 5 \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2} \end{aligned}$$

You should check that this is indeed the value of b that minimizes the sum. Taking the second derivative yields

$$\frac{d^2 S}{db^2} = 2 \sum_{i=1}^n x_i^2$$

which is always positive (unless all the x values are zero). Therefore, the value for b above does minimize the sum as required.

- b. Using the formula from a., we get that

$$b = \frac{(2037) - 5(52)}{590} \approx 3.0119$$

- c. The Least Squares Regression line is $y = 5 + 3.0119x$

Using this line, the predicted total number of labor hours to produce a lot of 15 brass bookcases is

$$\begin{aligned} y &= 5 + 3.0119(15) \\ &\approx 50.179 \text{ hours} \end{aligned}$$

4.5 Concepts Review

1. continuous
2. number of units sold; price per unit
3. fixed; variable
4. marginal revenue; marginal cost

Problem Set 4.5

1. Profit is $P(y) = ny - 10n$

$$= \frac{100y}{y-10} + 20y(100-y) - \frac{1000}{y-10} - 200(100-y)$$

$$= 20(-y^2 + 110y - 995)$$

$$\frac{dP}{dy} = 20(-2y + 110) = 40(55 - y)$$

$$P'(y) = 0 \text{ when } y = 55$$

$$P''(y) = -40 \text{ so the maximum profit is when the items are sold at \$55 each.}$$

2. $C(x) = 7000 + 100x$

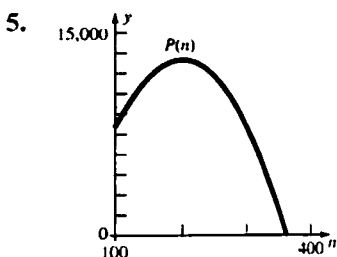
3. $n = 100 + 10 \frac{250 - p(n)}{5}$ so $p(n) = 300 - \frac{n}{2}$

$$R(n) = np(n) = 300n - \frac{n^2}{2}$$

4. $P(n) = R(n) - C(n)$

$$= 300n - \frac{n^2}{2} - (7000 + 100n)$$

$$= -7000 + 200n - \frac{n^2}{2}$$



Estimate $n \approx 200$

$$P'(n) = 200 - n; 200 - n = 0 \text{ when } n = 200.$$

$$P''(n) = -1, \text{ so profit is maximum at } n = 200.$$

6. $\frac{C(x)}{x} = \frac{100}{x} + 3.002 - 0.0001x$

$$\text{When } x = 1600, \frac{C(x)}{x} = 2.9045 \text{ or \$2.90 per}$$

unit.

$$\frac{dC}{dx} = 3.002 - 0.0002x$$

$$C'(1600) = 2.682 \text{ or \$2.68}$$

7. $\frac{C(n)}{n} = \frac{1000}{n} + \frac{n}{1200}$

$$\text{When } n = 800, \frac{C(n)}{n} \approx 1.9167 \text{ or \$1.92 per unit.}$$

$$\frac{dC}{dn} = \frac{n}{600}$$

$$C'(800) \approx 1.333 \text{ or \$1.33}$$

8. a. $\frac{dC}{dx} = 33 - 18x + 3x^2$

$$\frac{d^2C}{dx^2} = -18 + 6x; \frac{d^2C}{dx^2} = 0 \text{ when } x = 3$$

$$\frac{d^2C}{dx^2} < 0 \text{ on } (0, 3), \frac{d^2C}{dx^2} > 0 \text{ on } (3, \infty)$$

Thus, the marginal cost is a minimum when $x = 3$ or 300 units.

- b. $33 - 18(3) + 3(3)^2 = 6$

9. a. $R(x) = xp(x) = 20x + 4x^2 - \frac{x^3}{3}$

$$\frac{dR}{dx} = 20 + 8x - x^2$$

- b. Increasing when $\frac{dR}{dx} > 0$

$$20 + 8x - x^2 > 0 \text{ on } [0, 10]$$

Total revenue is increasing if $0 \leq x \leq 10$.

- c. $\frac{d^2R}{dx^2} = 8 - 2x; \frac{d^2R}{dx^2} = 0 \text{ when } x = 4$

$$\frac{d^3R}{dx^3} = -2; \frac{dR}{dx} \text{ is maximum at } x = 4.$$

10. $R(x) = x \left(182 - \frac{x}{36} \right)^{1/2}$

$$\frac{dR}{dx} = x \frac{1}{2} \left(182 - \frac{x}{36} \right)^{-1/2} \left(-\frac{1}{36} \right) + \left(182 - \frac{x}{36} \right)^{1/2}$$

$$= \left(182 - \frac{x}{36} \right)^{-1/2} \left(182 - \frac{x}{24} \right)$$

$$\frac{dR}{dx} = 0 \text{ when } x = 4368$$

$$x_1 = 4368; R(4368) \approx 34,021.83$$

$$\text{At } x_1, \frac{dR}{dx} = 0.$$

$$11. R(x) = \frac{800x}{x+3} - 3x$$

$$\frac{dR}{dx} = \frac{(x+3)(800) - 800x}{(x+3)^2} - 3 = \frac{2400}{(x+3)^2} - 3;$$

$$\frac{dR}{dx} = 0 \text{ when } x = 20\sqrt{2} - 3 \approx 25$$

$$x_1 = 25; R(25) \approx 639.29$$

$$\text{At } x_1, \frac{dR}{dx} = 0.$$

$$12. p(x) = 12 - (0.20)\frac{(x-400)}{10} = 20 - 0.02x$$

$$R(x) = 20x - 0.02x^2$$

$$\frac{dR}{dx} = 20 - 0.04x; \frac{dR}{dx} = 0 \text{ when } x = 500$$

Total revenue is maximized at $x_1 = 500$.

$$13. x = 4000 + \frac{6-p(x)}{0.15}(250);$$

$$p(x) = 6 - (0.15)\frac{(x-4000)}{250} = 8.4 - 0.0006x$$

$$R(x) = 8.4x - 0.0006x^2$$

$$\frac{dR}{dx} = 8.4 - 0.0012x; \frac{dR}{dx} = 0 \text{ when } x = 7000$$

Revenue is maximum when

$$p(x) = 8.4 - 0.0006(7000) = \$4.20 \text{ per yard.}$$

$$14. a. x = 500 + \frac{20-p(x)}{0.50}(50);$$

$$p(x) = 20 - (0.50)\frac{(x-500)}{50} = 25 - 0.01x$$

$$b. R(x) = 25x - 0.01x^2;$$

$$P(x) = R(x) - C(x)$$

$$= (25x - 0.01x^2) - (4200 + 5.10x + 0.0001x^2)$$

$$= -4200 + 19.9x - 0.0101x^2$$

$$\frac{dP}{dx} = 19.9 - 0.0202x; \frac{dP}{dx} = 0 \text{ when } x \approx 985$$

$$\frac{d^2P}{dx^2} = -0.0202; \text{ profit is maximized at}$$

$$x = 985.$$

$$c. p(985) = 25 - 0.01(985) = 15.15$$

$$d. \frac{dp}{dx} = -0.01$$

$$15. a. C(x) = \begin{cases} 6000 + 1.40x & \text{if } 0 \leq x \leq 4500 \\ 6000 + 1.60x & \text{if } 4500 < x \end{cases}$$

$$b. x = 4000 + \frac{7-p(x)}{0.10}(100);$$

$$p(x) = 7 - (0.10)\frac{x-4000}{100}$$

$$p(x) = 11 - 0.001x$$

$$c. R(x) = 11x - 0.001x^2$$

For $0 \leq x \leq 4500$,

$$P(x) = (11x - 0.001x^2) - (6000 + 1.40x)$$

$$= -6000 + 9.6x - 0.001x^2$$

$$\frac{dP}{dx} = 9.6 - 0.002x; \frac{dP}{dx} = 0 \text{ when } x = 4800;$$

this is not in the interval $[0, 4500]$.

The critical numbers are 0 and 4500.

$$P(0) = -6000 \text{ and } P(4500) = 16,950$$

For $4500 < x$,

$$P(x) = (11x - 0.001x^2) - (6000 + 1.60x)$$

$$= -6000 + 9.4x - 0.001x^2$$

$$\frac{dP}{dx} = 9.4 - 0.002x; \frac{dP}{dx} = 0 \text{ when } x = 4700$$

$$P(4700) = 16,090$$

Therefore, the number of units for maximum profit is $x = 4500$.

$$16. R(x) = 10x - 0.001x^2; 0 \leq x \leq 300$$

$$P(x) = (10x - 0.001x^2) - (200 + 4x - 0.01x^2)$$

$$= -200 + 6x + 0.009x^2$$

$$\frac{dP}{dx} = 6 + 0.018x; \frac{dP}{dx} = 0 \text{ when } x \approx -333$$

Critical numbers: $x = 0, 300$; $P(0) = -200$;

$P(300) = 2410$; Maximum profit is \$2410 at $x = 300$.

$$17. C(x) = \begin{cases} 200 + 4x - 0.01x^2 & \text{if } 0 \leq x \leq 300 \\ 800 + 3x - 0.01x^2 & \text{if } 300 < x \leq 450 \end{cases}$$

$$P(x) = \begin{cases} -200 + 6x + 0.009x^2 & \text{if } 0 \leq x \leq 300 \\ -800 + 7x + 0.009x^2 & \text{if } 300 < x \leq 450 \end{cases}$$

There are no stationary points on the interval $[0, 300]$. On $[300, 450]$:

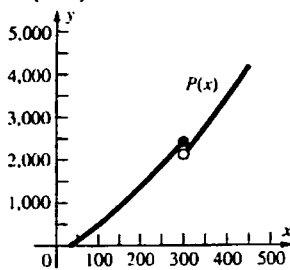
$$\frac{dP}{dx} = 7 + 0.018x; \frac{dP}{dx} = 0 \text{ when } x \approx -389$$

The critical numbers are 0, 300, 450.

$$P(0) = -200, P(300) = 2410, P(450) = 4172.5$$

Monthly profit is maximized at $x = 450$.

$$P(450) = 4172.50$$



18. The number of lots ordered per year is $\frac{1000}{x}$. Let

C represent the inventory cost.

$$C = \frac{1000}{x}(200 + 3x) + 20\left(\frac{x}{2}\right)$$

$$= \frac{200,000}{x} + 3000 + 10x$$

$$\frac{dC}{dx} = -\frac{200,000}{x^2} + 10; -\frac{200,000}{x^2} + 10 = 0 \text{ when}$$

$$x = \sqrt{20,000} \approx 141.42$$

$$\frac{d^2C}{dx^2} = \frac{400,000}{x^3}; \frac{d^2C}{dx^2} > 0 \text{ at } x = \sqrt{20,000}$$

Since the lot size must be a whole number

$$x = 141 \text{ or } x = 142.$$

$$C(141) \approx 5828.44, C(142) \approx 5828.45$$

$$x = 141$$

19. The number of lots ordered per year is $\frac{N}{x}$. Let C represent the inventory cost.

$$C = \frac{N}{x}(F + Bx) + \frac{x}{2}A = \frac{FN}{x} + BN + \frac{Ax}{2}$$

$$\frac{dC}{dx} = -\frac{FN}{x^2} + \frac{A}{2}; -\frac{FN}{x^2} + \frac{A}{2} = 0 \text{ when}$$

$$x = \sqrt{\frac{2FN}{A}}$$

20. If N is increased to $4N$,

$$x = \sqrt{\frac{2F(4N)}{A}} = 2\sqrt{\frac{2FN}{A}}$$

Therefore, the ideal lot size will double.

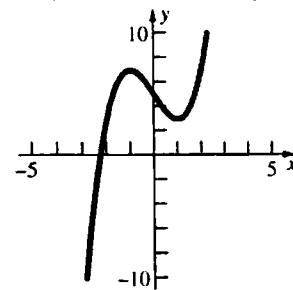
4.6 Concepts Review

1. $f(x)$; $-f(x)$
2. decreasing; concave up
3. $x = -1, x = 2, x = 3; y = 1$
4. polynomial; rational.

Problem Set 4.6

1. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
Neither an even nor an odd function.
 y -intercept: 5; x -intercept: ≈ -2.3
 $f'(x) = 3x^2 - 3; 3x^2 - 3 = 0$ when $x = -1, 1$
Critical points: $-1, 1$
 $f'(x) > 0$ when $x < -1$ or $x > 1$
 $f(x)$ is increasing on $(-\infty, -1] \cup [1, \infty)$ and decreasing on $[-1, 1]$.
Local minimum $f(1) = 3$;
local maximum $f(-1) = 7$
 $f''(x) = 6x; f''(x) > 0$ when $x > 0$.
 $f(x)$ is concave up on $(0, \infty)$ and concave down

on $(-\infty, 0)$; inflection point $(0, 5)$.

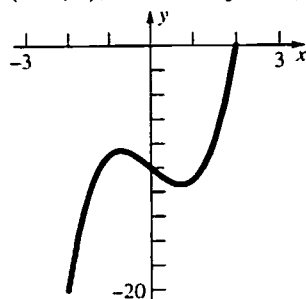


2. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
Neither an even nor an odd function.
 y -intercept: -10 ; x -intercept: 2
 $f'(x) = 6x^2 - 3 = 3(2x^2 - 1); 2x^2 - 1 = 0$ when
 $x = -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$
Critical points: $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$
 $f'(x) > 0$ when $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$
 $f(x)$ is increasing on $(-\infty, -\frac{1}{\sqrt{2}}) \cup [\frac{1}{\sqrt{2}}, \infty)$ and decreasing on $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$.

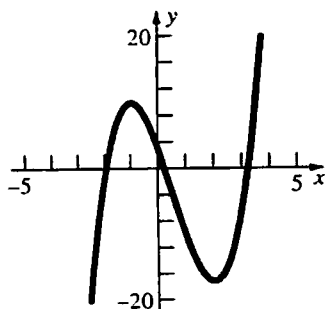
Local minimum $f\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2} - 10 \approx -11.4$

Local maximum $f\left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2} - 10 \approx -8.6$

$f''(x) = 12x$; $f''(x) > 0$ when $x > 0$. $f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; inflection point $(0, -10)$.



3. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
 Neither an even nor an odd function.
 y-intercept: 3; x-intercepts: $\approx -2.0, 0.2, 3.2$
 $f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$;
 $f'(x) = 0$ when $x = -1, 2$
 Critical points: $-1, 2$
 $f'(x) > 0$ when $x < -1$ or $x > 2$
 $f(x)$ is increasing on $(-\infty, -1] \cup [2, \infty)$ and decreasing on $[-1, 2]$.
 Local minimum $f(2) = -17$;
 local maximum $f(-1) = 10$
 $f''(x) = 12x - 6 = 6(2x-1)$;
 $f''(x) > 0$ when $x > \frac{1}{2}$.
 $f(x)$ is concave up on $\left(\frac{1}{2}, \infty\right)$ and concave down on $\left(-\infty, \frac{1}{2}\right)$; inflection point: $\left(\frac{1}{2}, -\frac{7}{2}\right)$

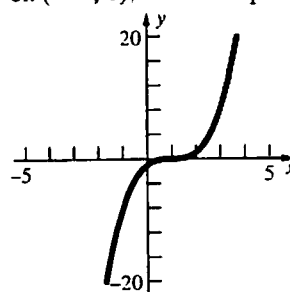


4. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
 Neither an even nor an odd function
 y-intercept: -1 ; x-intercept: 1
 $f'(x) = 3(x-1)^2$; $f'(x) = 0$ when $x = 1$
 Critical point: 1
 $f'(x) > 0$ for all $x \neq 1$
 $f(x)$ is increasing on $(-\infty, \infty)$

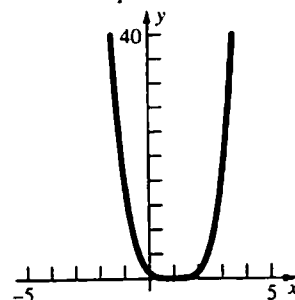
No local minima or maxima

$f''(x) = 6(x-1)$; $f''(x) > 0$ when $x > 1$.

$f(x)$ is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$; inflection point $(1, 0)$



5. Domain: $(-\infty, \infty)$; range: $[0, \infty)$
 Neither an even nor an odd function.
 y-intercept: 1 ; x-intercept: 1
 $G'(x) = 4(x-1)^3$; $G'(x) = 0$ when $x = 1$
 Critical point: 1
 $G'(x) > 0$ for $x > 1$
 $G(x)$ is increasing on $[1, \infty)$ and decreasing on $(-\infty, 1]$.
 Global minimum $f(1) = 0$; no local maxima
 $G''(x) = 12(x-1)^2$; $G''(x) > 0$ for all $x \neq 1$
 $G(x)$ is concave up on $(-\infty, 1) \cup (1, \infty)$; no inflection points



6. Domain: $(-\infty, \infty)$; range: $\left[-\frac{1}{4}, \infty\right)$
 $H(-t) = (-t)^2[(-t)^2 - 1] = t^2(t^2 - 1) = H(t)$; even function; symmetric with respect to the y-axis.
 y-intercept: 0 ; t-intercepts: $-1, 0, 1$
 $H'(t) = 4t^3 - 2t = 2t(2t^2 - 1)$; $H'(t) = 0$ when
 $t = -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

Critical points: $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

$H'(t) > 0$ for $-\frac{1}{\sqrt{2}} < t < 0$ or $\frac{1}{\sqrt{2}} < t$.

$H(t)$ is increasing on $\left[-\frac{1}{\sqrt{2}}, 0\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$ and

decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[0, \frac{1}{\sqrt{2}}\right]$

Global minima $f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$, $f\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$;

Local maximum $f(0) = 0$

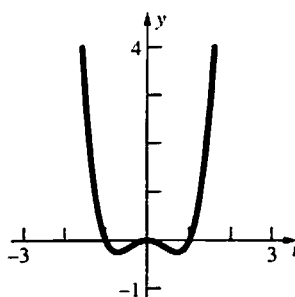
$H''(t) = 12t^2 - 2 = 2(6t^2 - 1)$; $H'' > 0$ when

$t < -\frac{1}{\sqrt{6}}$ or $t > \frac{1}{\sqrt{6}}$

$H(t)$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{6}}\right) \cup \left(\frac{1}{\sqrt{6}}, \infty\right)$

and concave down on $\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$; inflection

points $H\left(-\frac{1}{\sqrt{6}}, -\frac{5}{36}\right)$ and $H\left(\frac{1}{\sqrt{6}}, \frac{5}{36}\right)$



7. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

Neither an even nor an odd function.

y-intercept: 10; x-intercept: $1 - 1^{1/3} \approx -1.2$

$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$; $f'(x) = 0$ when

$x = 1$.

Critical point: 1

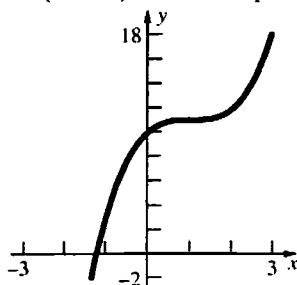
$f'(x) > 0$ for all $x \neq 1$.

$f(x)$ is increasing on $(-\infty, \infty)$ and decreasing nowhere.

No local maxima or minima

$f''(x) = 6x - 6 = 6(x-1)$; $f''(x) > 0$ when $x > 1$.

$f(x)$ is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$; inflection point $(1, 11)$



8. Domain: $(-\infty, \infty)$; range: $\left[-\frac{16}{3}, \infty\right)$

$$F(-s) = \frac{4(-s)^4 - 8(-s)^2 - 12}{3} = \frac{4s^4 - 8s^2 - 12}{3}$$

$= F(s)$; even function; symmetric with respect to the y-axis

y-intercept: -4; s-intercepts: $-\sqrt{3}, \sqrt{3}$

$$F'(s) = \frac{16}{3}s^3 - \frac{16}{3}s = \frac{16}{3}s(s^2 - 1); F'(s) = 0$$

when $s = -1, 0, 1$.

Critical points: -1, 0, 1

$F'(s) > 0$ when $-1 < s < 0$ or $s > 1$.

$F(s)$ is increasing on $[-1, 0] \cup [1, \infty)$ and decreasing on $(-\infty, -1] \cup [0, 1]$

Global minima $F(-1) = -\frac{16}{3}$, $F(1) = -\frac{16}{3}$; local

maximum $F(0) = -4$

$$F''(s) = 16s^2 - \frac{16}{3} = 16\left(s^2 - \frac{1}{3}\right); F''(s) > 0$$

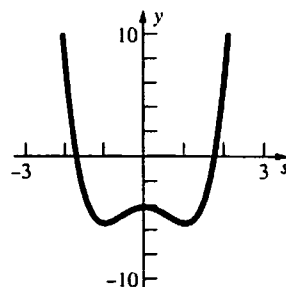
when $s < -\frac{1}{\sqrt{3}}$ or $s > \frac{1}{\sqrt{3}}$

$F(s)$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$

and concave down on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$;

inflection points

$$F\left(-\frac{1}{\sqrt{3}}, -\frac{128}{27}\right), F\left(\frac{1}{\sqrt{3}}, -\frac{128}{27}\right)$$



9. Domain: $(-\infty, -1) \cup (-1, \infty)$;

range: $(-\infty, 1) \cup (1, \infty)$

Neither an even nor an odd function

y-intercept: 0; x-intercept: 0

$$g'(x) = \frac{1}{(x+1)^2}; g'(x) \text{ is never } 0.$$

No critical points

$g'(x) > 0$ for all $x \neq -1$.

$g(x)$ is increasing on $(-\infty, -1) \cup (-1, \infty)$.

No local minima or maxima

$$g''(x) = -\frac{2}{(x+1)^3}; g''(x) > 0 \text{ when } x < -1.$$

$g(x)$ is concave up on $(-\infty, -1)$ and concave down on $(-1, \infty)$; no inflection points (-1 is not in the domain of g).

$$\lim_{x \rightarrow \infty} \frac{x}{x+1} = \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x}} = 1;$$

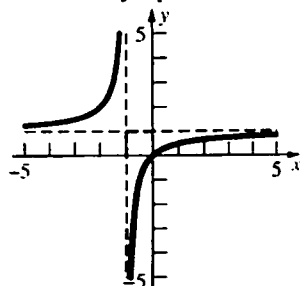
$$\lim_{x \rightarrow -\infty} \frac{x}{x+1} = \lim_{x \rightarrow -\infty} \frac{1}{1 + \frac{1}{x}} = 1;$$

horizontal asymptote: $y = 1$

$$\text{As } x \rightarrow -1^-, x+1 \rightarrow 0^- \text{ so } \lim_{x \rightarrow -1^-} \frac{x}{x+1} = \infty;$$

$$\text{as } x \rightarrow -1^+, x+1 \rightarrow 0^+ \text{ so } \lim_{x \rightarrow -1^+} \frac{x}{x+1} = -\infty;$$

vertical asymptote: $x = -1$



10. Domain: $(-\infty, 0) \cup (0, \infty)$;

range: $(-\infty, -4\pi] \cup [0, \infty)$

Neither an even nor an odd function

No y -intercept; s -intercept: π

$$g'(s) = \frac{s^2 - \pi^2}{s^2}; g'(s) = 0 \text{ when } s = -\pi, \pi$$

Critical points: $-\pi, \pi$

$g'(s) > 0$ when $s < -\pi$ or $s > \pi$

$g(s)$ is increasing on $(-\infty, -\pi] \cup [\pi, \infty)$ and

decreasing on $[-\pi, 0) \cup (0, \pi]$.

Local minimum $g(\pi) = 0$;

local maximum $g(-\pi) = -4\pi$

$$g''(s) = \frac{2\pi^2}{s^3}; g''(s) > 0 \text{ when } s > 0$$

$g(s)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; no inflection points (0 is not in the domain of $g(s)$).

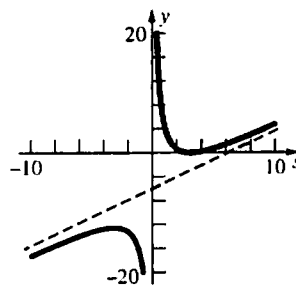
$$g(s) = s - 2\pi + \frac{\pi^2}{s}; y = s - 2\pi \text{ is an oblique}$$

asymptote.

$$\text{As } s \rightarrow 0^-, (s - \pi)^2 \rightarrow \pi^2, \text{ so } \lim_{s \rightarrow 0^-} g(s) = -\infty;$$

$$\text{as } s \rightarrow 0^+, (s - \pi)^2 \rightarrow \pi^2, \text{ so } \lim_{s \rightarrow 0^+} g(s) = \infty;$$

$s = 0$ is a vertical asymptote.



11. Domain: $(-\infty, \infty)$; range: $\left[-\frac{1}{4}, \frac{1}{4}\right]$

$$f(-x) = \frac{-x}{(-x)^2 + 4} = -\frac{x}{x^2 + 4} = -f(x); \text{ odd}$$

function; symmetric with respect to the origin.

y -intercept: 0; x -intercept: 0

$$f'(x) = \frac{4 - x^2}{(x^2 + 4)^2}; f'(x) = 0 \text{ when } x = -2, 2$$

Critical points: $-2, 2$

$f'(x) > 0$ for $-2 < x < 2$

$f(x)$ is increasing on $[-2, 2]$ and decreasing on $(-\infty, -2] \cup [2, \infty)$.

Global minimum $f(-2) = -\frac{1}{4}$; global maximum

$$f(2) = \frac{1}{4}$$

$$f''(x) = \frac{2x(x^2 - 12)}{(x^2 + 4)^3}; f''(x) > 0 \text{ when}$$

$$-2\sqrt{3} < x < 0 \text{ or } x > 2\sqrt{3}$$

$f(x)$ is concave up on $(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)$ and

concave down on $(-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3})$;

inflection points $\left(-2\sqrt{3}, -\frac{\sqrt{3}}{8}\right), (0, 0),$

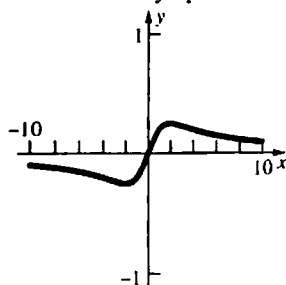
$$\left(2\sqrt{3}, \frac{\sqrt{3}}{8}\right)$$

$$\lim_{x \rightarrow \infty} \frac{x}{x^2 + 4} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1 + \frac{4}{x^2}} = 0;$$

$$\lim_{x \rightarrow -\infty} \frac{x}{x^2 + 4} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x}}{1 + \frac{4}{x^2}} = 0;$$

$y = 0$ is a horizontal asymptote.

No vertical asymptotes



12. Domain: $(-\infty, \infty)$; range: $[0, 1)$

$$\Lambda(-\theta) = \frac{(-\theta)^2}{(-\theta)^2 + 1} = \frac{\theta^2}{\theta^2 + 1} = \Lambda(\theta); \text{ even}$$

function; symmetric with respect to the y-axis.

y-intercept: 0; θ -intercept: 0

$$\Lambda'(\theta) = \frac{2\theta}{(\theta^2 + 1)^2}; \Lambda'(\theta) = 0 \text{ when } \theta = 0$$

Critical point: 0

$$\Lambda'(\theta) > 0 \text{ when } \theta > 0$$

$\Lambda(\theta)$ is increasing on $[0, \infty)$ and

decreasing on $(-\infty, 0]$.

Global minimum $\Lambda(0) = 0$; no local maxima

$$\Lambda''(\theta) = \frac{2(1 - 3\theta^2)}{(\theta^2 + 1)^3}; \Lambda''(\theta) > 0 \text{ when}$$

$$-\frac{1}{\sqrt{3}} < \theta < \frac{1}{\sqrt{3}}$$

$\Lambda(\theta)$ is concave up on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and

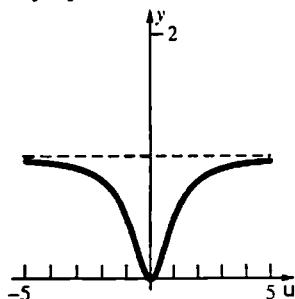
concave down on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$;

inflection points $\left(-\frac{1}{\sqrt{3}}, \frac{1}{4}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$

$$\lim_{\theta \rightarrow \infty} \frac{\theta^2}{\theta^2 + 1} = \lim_{\theta \rightarrow \infty} \frac{1}{1 + \frac{1}{\theta^2}} = 1;$$

$$\lim_{\theta \rightarrow -\infty} \frac{\theta^2}{\theta^2 + 1} = \lim_{\theta \rightarrow -\infty} \frac{1}{1 + \frac{1}{\theta^2}} = 1;$$

$y = 1$ is a horizontal asymptote. No vertical asymptotes



13. Domain: $(-\infty, 1) \cup (1, \infty)$;

range $(-\infty, 1) \cup (1, \infty)$

Neither an even nor an odd function

y-intercept: 0; x-intercept: 0

$$h(x) = -\frac{1}{(x-1)^2}; h'(x) \text{ is never } 0.$$

No critical points

$$h'(x) < 0 \text{ for all } x \neq 1.$$

$h(x)$ is increasing nowhere and

decreasing on $(-\infty, 1) \cup (1, \infty)$.

No local maxima or minima

$$h''(x) = \frac{2}{(x-1)^3}; h''(x) > 0 \text{ when } x > 1$$

$h'(x)$ is concave up on $(1, \infty)$ and concave

down on $(-\infty, 1)$; no inflection points (1 is not in the domain of $h(x)$)

$$\lim_{x \rightarrow \infty} \frac{x}{x-1} = \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{1}{x}} = 1;$$

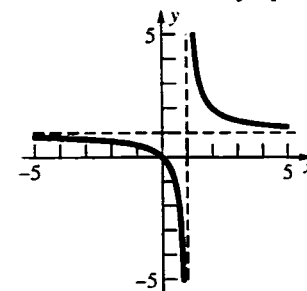
$$\lim_{x \rightarrow -\infty} \frac{x}{x-1} = \lim_{x \rightarrow -\infty} \frac{1}{1 - \frac{1}{x}} = 1;$$

$y = 1$ is a horizontal asymptote.

$$\text{As } x \rightarrow 1^-, x-1 \rightarrow 0^- \text{ so } \lim_{x \rightarrow 1^-} \frac{x}{x-1} = -\infty;$$

$$\text{as } x \rightarrow 1^+, x-1 \rightarrow 0^+ \text{ so } \lim_{x \rightarrow 1^+} \frac{x}{x-1} = \infty;$$

$x = 1$ is a vertical asymptote.



14. Domain: $(-\infty, \infty)$

Range: $0 < P \leq 1$

Even function since

$$P(-x) = \frac{1}{(-x)^2 + 1} = \frac{1}{x^2 + 1} = P(x)$$

so the function is symmetric with respect to the y-axis.

y-intercept: $y = 1$

x-intercept: none

$$P'(x) = \frac{-2x}{(x^2 + 1)^2}; P'(x) \text{ is } 0 \text{ when } x = 0.$$

critical point: $x = 0$

$P'(x) > 0$ when $x < 0$ so $P(x)$ is increasing on

$(-\infty, 0]$ and decreasing on $[0, \infty)$. Global maximum $P(0) = 1$; no local minima.

$$P''(x) = \frac{6x^2 - 2}{(x^2 + 1)^3}$$

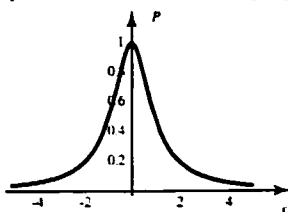
$P''(x) > 0$ on $(-\infty, -1/\sqrt{3}) \cup (1/\sqrt{3}, \infty)$ (concave up) and $P''(x) < 0$ on $(-1/\sqrt{3}, 1/\sqrt{3})$ (concave down).

$$\text{Inflection points: } \left(\pm \frac{1}{\sqrt{3}}, \frac{3}{4} \right)$$

No vertical asymptotes.

$$\lim_{x \rightarrow \infty} P(x) = 0; \lim_{x \rightarrow -\infty} P(x) = 0$$

$y = 0$ is a horizontal asymptote.



15. Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$;

range: $(-\infty, \infty)$

Neither an even nor an odd function

y-intercept: $-\frac{3}{2}$; x-intercepts: 1, 3

$$f'(x) = \frac{3x^2 - 10x + 11}{(x+1)^2(x-2)^2}; f'(x) \text{ is never } 0.$$

No critical points

$$f'(x) > 0 \text{ for all } x \neq -1, 2$$

$f(x)$ is increasing on

$$(-\infty, -1) \cup (-1, 2) \cup (2, \infty).$$

No local minima or maxima

$$f''(x) = \frac{-6x^3 + 30x^2 - 66x + 42}{(x+1)^3(x-2)^3}; f''(x) > 0 \text{ when}$$

$$x < -1 \text{ or } 1 < x < 2$$

$f(x)$ is concave up on $(-\infty, -1) \cup (1, 2)$ and concave down on $(-1, 1) \cup (2, \infty)$;

inflection point $f(1) = 0$

$$\lim_{x \rightarrow \infty} \frac{(x-1)(x-3)}{(x+1)(x-2)} = \lim_{x \rightarrow \infty} \frac{x^2 - 4x + 3}{x^2 - x - 2}$$

$$= \lim_{x \rightarrow \infty} \frac{1 - \frac{4}{x} + \frac{3}{x^2}}{1 - \frac{1}{x} - \frac{2}{x^2}} = 1;$$

$$\lim_{x \rightarrow -\infty} \frac{(x-1)(x-3)}{(x+1)(x-2)} = \lim_{x \rightarrow -\infty} \frac{1 - \frac{4}{x} + \frac{3}{x^2}}{1 - \frac{1}{x} - \frac{2}{x^2}} = 1;$$

$y = 1$ is a horizontal asymptote.

As $x \rightarrow -1^-$, $x-1 \rightarrow -2$, $x-3 \rightarrow -4$,

$x-2 \rightarrow -3$, and $x+1 \rightarrow 0^-$ so $\lim_{x \rightarrow -1^-} f(x) = \infty$;

as $x \rightarrow -1^+$, $x-1 \rightarrow -2$, $x-3 \rightarrow -4$,

$x-2 \rightarrow -3$, and $x+1 \rightarrow 0^+$, so

$$\lim_{x \rightarrow -1^+} f(x) = -\infty$$

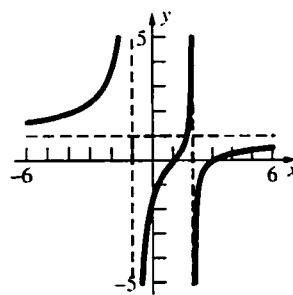
As $x \rightarrow 2^-$, $x-1 \rightarrow 1$, $x-3 \rightarrow -1$, $x+1 \rightarrow 3$, and

$x-2 \rightarrow 0^-$, so $\lim_{x \rightarrow 2^-} f(x) = \infty$; as

$x \rightarrow 2^+$, $x-1 \rightarrow 1$, $x-3 \rightarrow -1$, $x+1 \rightarrow 3$, and

$x-2 \rightarrow 0^+$, so $\lim_{x \rightarrow 2^+} f(x) = -\infty$

$x = -1$ and $x = 2$ are vertical asymptotes.



16. Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, -2] \cup [2, \infty)$

Odd function since

$$w(-z) = \frac{(-z)^2 + 1}{-z} = -\frac{z^2 + 1}{z} = -w(z); \text{ symmetric}$$

with respect to the origin.

y-intercept: none

x-intercept: none

$$w'(z) = 1 - \frac{1}{z^2}; w'(z) = 0 \text{ when } z = \pm 1.$$

critical points: $z = \pm 1$. $w'(z) > 0$ on

$(-\infty, -1) \cup (1, \infty)$ so the function is increasing on

$(-\infty, -1] \cup [1, \infty)$. The function is decreasing on $[-1, 0) \cup (0, 1)$.

local maximum $w(1) = 2$ and local minimum

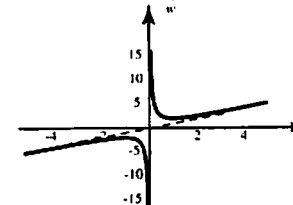
$w(-1) = -2$. No global extrema.

$$w''(z) = \frac{2}{z^3} > 0 \text{ when } z > 0. \text{ Concave up on}$$

$(0, \infty)$ and concave down on $(-\infty, 0)$.

No horizontal asymptote; $x = 0$ is a vertical asymptote; the line $y = z$ is an oblique (or slant) asymptote.

No inflection points.



17. Domain: $(-\infty, 1) \cup (1, \infty)$

Range: $(-\infty, \infty)$

Neither even nor odd function.

y-intercept: $y = 6$; x-intercept: $x = -3, 2$

$$g'(x) = \frac{x^2 - 2x + 5}{(x-1)^2}; g'(x) \text{ is never zero. No}$$

critical points.

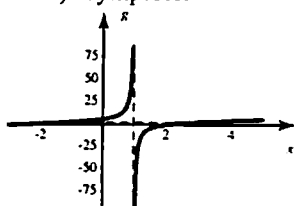
$g'(x) > 0$ over the entire domain so the function is always increasing. No local extrema.

$$f''(x) = \frac{-8}{(x-1)^3}; f''(x) > 0 \text{ when}$$

$x < 1$ (concave up) and $f''(x) < 0$ when

$x > 1$ (concave down); no inflection points.

No horizontal asymptote; $x = 1$ is a vertical asymptote; the line $y = x + 2$ is an oblique (or slant) asymptote.



18. Domain: $(-\infty, \infty)$; range: $[0, \infty)$

$$f(-x) = |-x|^3 = |x|^3 = f(x); \text{ even function;}$$

symmetric with respect to the y-axis.

y-intercept: 0; x-intercept: 0

$$f'(x) = 3|x|^2 \left(\frac{x}{|x|} \right) = 3x|x|; f'(x) = 0 \text{ when } x = 0$$

Critical point: 0

$$f'(x) > 0 \text{ when } x > 0$$

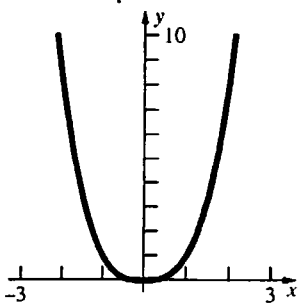
$f(x)$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.

Global minimum $f(0) = 0$; no local maxima

$$f''(x) = 3|x| + \frac{3x^2}{|x|} = 6|x| \text{ as } x^2 = |x|^2;$$

$$f''(x) > 0 \text{ when } x \neq 0$$

$f(x)$ is concave up on $(-\infty, 0) \cup (0, \infty)$; no inflection points



19. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

$$R(-z) = -z|-z| = -z|z| = -R(z); \text{ odd function;}$$

symmetric with respect to the origin.

y-intercept: 0; z-intercept: 0

$$R'(z) = |z| + \frac{z^2}{|z|} = 2|z| \text{ since } z^2 = |z|^2 \text{ for all } z;$$

$$R'(z) = 0 \text{ when } z = 0$$

Critical point: 0

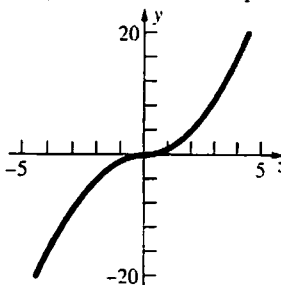
$$R'(z) > 0 \text{ when } z \neq 0$$

$R(z)$ is increasing on $(-\infty, \infty)$ and decreasing nowhere.

No local minima or maxima

$$R''(z) = \frac{2z}{|z|}; R''(z) > 0 \text{ when } z > 0.$$

$R(z)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; inflection point $(0, 0)$.



20. Domain: $(-\infty, \infty)$; range: $[0, \infty)$

$$H(-q) = (-q)^2 |-q| = q^2 |q| = H(q); \text{ even}$$

function; symmetric with respect to the y-axis.

y-intercept: 0; q-intercept: 0

$$H'(q) = 2q|q| + \frac{q^3}{|q|} = \frac{3q^3}{|q|} = 3q|q| \text{ as } |q|^2 = q^2$$

for all q ; $H'(q) = 0$ when $q = 0$

Critical point: 0

$$H'(q) > 0 \text{ when } q > 0$$

$H(q)$ is increasing on $[0, \infty)$ and

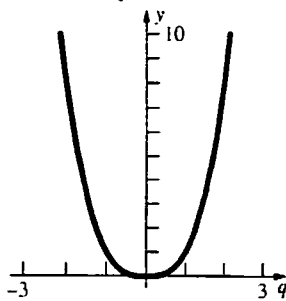
decreasing on $(-\infty, 0]$.

Global minimum $H(0) = 0$; no local maxima

$$H''(q) = 3|q| + \frac{3q^2}{|q|} = 6|q|; H''(q) > 0 \text{ when}$$

$q \neq 0$.

$H(q)$ is concave up on $(-\infty, 0) \cup (0, \infty)$; no inflection points.



21. Domain: $(-\infty, \infty)$; range: $[0, \infty)$

Neither an even nor an odd function.

Note that for $x \leq 0$, $|x| = -x$ so $|x| + x = 0$, while

for $x > 0$, $|x| = x$ so $\frac{|x| + x}{2} = x$.

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 3x^2 + 2x & \text{if } x > 0 \end{cases}$$

y-intercept: 0; x-intercepts: $(-\infty, 0]$

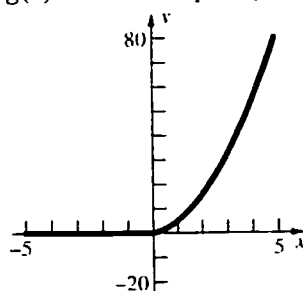
$$g'(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 6x + 2 & \text{if } x > 0 \end{cases}$$

No critical points for $x > 0$.

$g(x)$ is increasing on $[0, \infty)$ and decreasing nowhere.

$$g''(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 6 & \text{if } x > 0 \end{cases}$$

$g(x)$ is concave up on $(0, \infty)$; no inflection points



22. Domain: $(-\infty, \infty)$; range: $[0, \infty)$

Neither an even nor an odd function. Note that

for $x < 0$, $|x| = -x$ so $\frac{|x| - x}{2} = -x$, while for

$x \geq 0$, $|x| = x$ so $\frac{|x| - x}{2} = 0$.

$$h(x) = \begin{cases} -x^3 + x^2 - 6x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

y-intercept: 0; x-intercepts: $[0, \infty)$

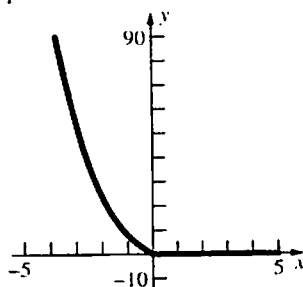
$$h'(x) = \begin{cases} -3x^2 + 2x - 6 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

No critical points for $x < 0$

$h(x)$ is increasing nowhere and decreasing on $(-\infty, 0]$.

$$h''(x) = \begin{cases} -6x + 2 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

$h(x)$ is concave up on $(-\infty, 0)$; no inflection points



23. Domain: $(-\infty, \infty)$; range: $[0, 1]$

$f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$; even function; symmetric with respect to the y-axis.
y-intercept: 0; x-intercepts: $k\pi$ where k is any integer.

$$f'(x) = \frac{\sin x}{|\sin x|} \cos x; f'(x) = 0 \text{ when } x = \frac{\pi}{2} + k\pi$$

and $f'(x)$ does not exist when $x = k\pi$, where k is any integer.

Critical points: $\frac{k\pi}{2}$, where k is any integer;

$f'(x) > 0$ when $\sin x$ and $\cos x$ are either both positive or both negative.

$f(x)$ is increasing on $\left[k\pi, k\pi + \frac{\pi}{2}\right]$ and decreasing

on $\left[k\pi + \frac{\pi}{2}, (k+1)\pi\right]$ where k is any integer.

Global minima $f(k\pi) = 0$; global maxima

$$f\left(k\pi + \frac{\pi}{2}\right) = 1, \text{ where } k \text{ is any integer.}$$

$$f''(x) = \frac{\cos^2 x}{|\sin x|} - \frac{\sin^2 x}{|\sin x|}$$

$$+ \sin x \cos x \left(-\frac{1}{|\sin x|^2} \right) \left(\frac{\sin x}{|\sin x|} \right) (\cos x)$$

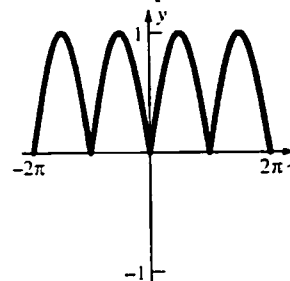
$$= \frac{\cos^2 x}{|\sin x|} - \frac{\sin^2 x}{|\sin x|} - \frac{\cos^2 x}{|\sin x|}$$

$$= -\frac{\sin^2 x}{|\sin x|} = -|\sin x|$$

$f''(x) < 0$ when $x \neq k\pi$, k any integer

$f(x)$ is never concave up and concave down on $(k\pi, (k+1)\pi)$ where k is any integer.

No inflection points



24. Domain: $[2k\pi, (2k+1)\pi]$ where k is any integer; range: $[0, 1]$

Neither an even nor an odd function

y-intercept: 0; x-intercepts: $k\pi$, where k is any integer.

$$f'(x) = \frac{\cos x}{2\sqrt{\sin x}}; f'(x) = 0 \text{ when } x = 2k\pi + \frac{\pi}{2}$$

while $f'(x)$ does not exist when $x = k\pi$, k any integer.

Critical points: $k\pi, 2k\pi + \frac{\pi}{2}$ where k is any integer

$$f'(x) > 0 \text{ when } 2k\pi < x < 2k\pi + \frac{\pi}{2}$$

$f(x)$ is increasing on $\left[2k\pi, 2k\pi + \frac{\pi}{2}\right]$ and

decreasing on $\left[2k\pi + \frac{\pi}{2}, (2k+1)\pi\right]$, k any

integer.

Global minima $f(k\pi) = 0$; global maxima

$$f\left(2k\pi + \frac{\pi}{2}\right) = 1, \text{ } k \text{ any integer}$$

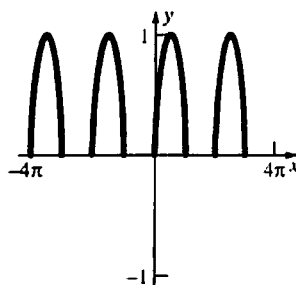
$$f''(x) = \frac{-\cos^2 x - 2\sin^2 x}{4\sin^{3/2} x} = \frac{-1 - \sin^2 x}{4\sin^{3/2} x}$$

$$= -\frac{1 + \sin^2 x}{4\sin^{3/2} x};$$

$$f''(x) < 0 \text{ for all } x.$$

$f(x)$ is concave down on $(2k\pi, (2k+1)\pi)$;

no inflection points



25. Domain: $(-\infty, \infty)$

Range: $[0, 1]$

Even function since

$$h(-t) = \cos^2(-t) = \cos^2 t = h(t)$$

so the function is symmetric with respect to the y-axis.

$$y\text{-intercept: } y = 1; t\text{-intercepts: } x = \frac{\pi}{2} + k\pi$$

where k is any integer.

$$h'(t) = -2\cos t \sin t; h'(t) = 0 \text{ at } t = \frac{k\pi}{2}.$$

$$\text{Critical points: } t = \frac{k\pi}{2}$$

$$h'(t) > 0 \text{ when } k\pi + \frac{\pi}{2} < t < (k+1)\pi. \text{ The}$$

function is increasing on the intervals

$\left[k\pi + (\pi/2), (k+1)\pi\right]$ and decreasing on the

intervals $\left[k\pi, k\pi + (\pi/2)\right]$.

Global maxima $h(k\pi) = 1$

$$\text{Global minima } h\left(\frac{\pi}{2} + k\pi\right) = 0$$

$$h''(t) = 2\sin^2 t - 2\cos^2 t = -2(\cos 2t)$$

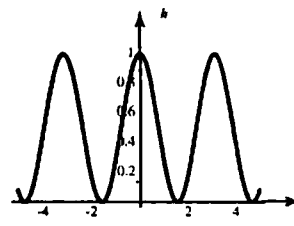
$$h''(t) < 0 \text{ on } \left(k\pi - \frac{\pi}{4}, k\pi + \frac{\pi}{4}\right) \text{ so } h \text{ is concave}$$

$$\text{down, and } h''(t) > 0 \text{ on } \left(k\pi + \frac{\pi}{4}, k\pi + \frac{3\pi}{4}\right) \text{ so } h$$

is concave up.

$$\text{Inflection points: } \left(\frac{k\pi}{2} + \frac{\pi}{4}, \frac{1}{2}\right)$$

No vertical asymptotes; no horizontal asymptotes.



26. Domain: all reals except $t = \frac{\pi}{2} + k\pi$

Range: $[0, \infty)$

$$y\text{-intercepts: } y = 0; t\text{-intercepts: } t = k\pi$$

where k is any integer.

Even function since

$$g(-t) = \tan^2(-t) = (-\tan t)^2 = \tan^2 t$$

so the function is symmetric with respect to the y-axis.

$$g'(t) = 2 \sec^2 t \tan t = \frac{2 \sin t}{\cos^3 t}; \quad g'(t) = 0 \text{ when}$$

$$t = k\pi.$$

Critical points: $k\pi$

$g(t)$ is increasing on $\left[k\pi, k\pi + \frac{\pi}{2}\right)$ and

decreasing on $\left(k\pi - \frac{\pi}{2}, k\pi\right]$.

Global minima $g(k\pi) = 0$; no local maxima

$$\begin{aligned} g'(t) &= 2 \frac{\cos^4 t + \sin t(3) \cos^2 t \sin t}{\cos^6 t} = 2 \frac{\cos^2 t + 3 \sin^2 t}{\cos^4 t} \\ &= 2 \frac{1 + 2 \sin^2 t}{\cos^4 t} > 0 \end{aligned}$$

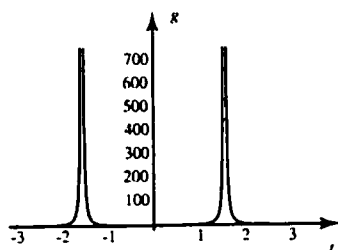
over the entire domain. Thus the function is

concave up on $\left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$; no

inflection points.

No horizontal asymptotes; $t = \frac{\pi}{2} + k\pi$ are

vertical asymptotes.



27. Domain: $\approx (-\infty, 0.44) \cup (0.44, \infty)$;

range: $(-\infty, \infty)$

Neither an even nor an odd function

y-intercept: 0; x-intercepts: 0, ≈ 0.24

$$f'(x) = \frac{74.6092x^3 - 58.2013x^2 + 7.82109x}{(7.126x - 3.141)^2};$$

$$f'(x) = 0 \text{ when } x = 0, \approx 0.17, \approx 0.61$$

Critical points: 0, ≈ 0.17 , ≈ 0.61

$$f'(x) > 0 \text{ when } 0 < x < 0.17 \text{ or } 0.61 < x$$

$f(x)$ is increasing on $\approx [0, 0.17] \cup [0.61, \infty)$

and decreasing on

$$(-\infty, 0] \cup [0.17, 0.44) \cup (0.44, 0.61]$$

Local minima $f(0) = 0$; $f(0.61) \approx 0.60$; local

maximum $f(0.17) \approx 0.01$

$$f''(x) = \frac{531.665x^3 - 703.043x^2 + 309.887x - 24.566}{(7.126x - 3.141)^3};$$

$$f''(x) > 0 \text{ when } x < 0.10 \text{ or } x > 0.44$$

$f(x)$ is concave up on $(-\infty, 0.10) \cup (0.44, \infty)$

and concave down on $(0.10, 0.44)$;

inflection point $\approx (0.10, 0.003)$

$$\lim_{x \rightarrow \infty} \frac{5.235x^3 - 1.245x^2}{7.126x - 3.141} = \lim_{x \rightarrow \infty} \frac{5.235x^2 - 1.245x}{7.126 - \frac{3.141}{x}} = \infty$$

so $f(x)$ does not have a horizontal asymptote.

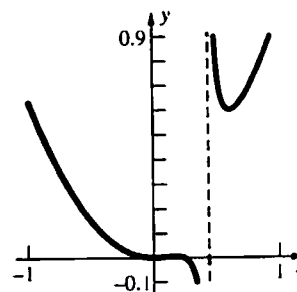
As $x \rightarrow 0.44^-$, $5.235x^3 - 1.245x^2 \rightarrow 0.20$ while

$$7.126x - 3.141 \rightarrow 0^-, \text{ so } \lim_{x \rightarrow 0.44^-} f(x) = -\infty;$$

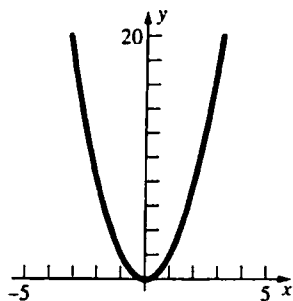
as $x \rightarrow 0.44^+$, $5.235x^3 - 1.245x^2 \rightarrow 0.20$ while

$$7.126x - 3.141 \rightarrow 0^+, \text{ so } \lim_{x \rightarrow 0.44^+} f(x) = \infty;$$

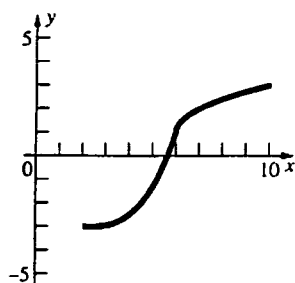
$x \approx 0.44$ is a vertical asymptote of $f(x)$.



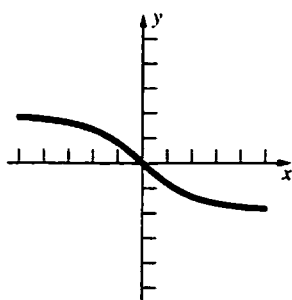
28.



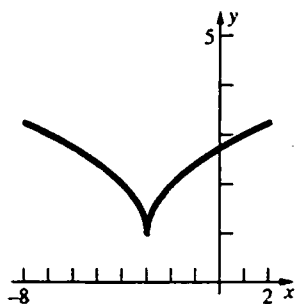
29.



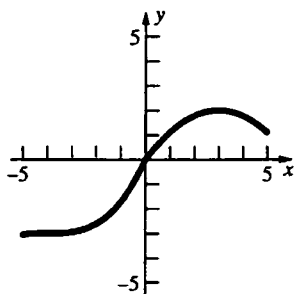
30.



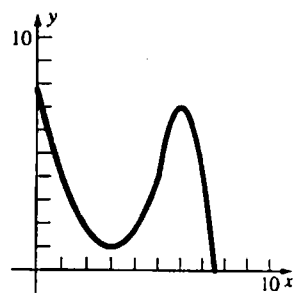
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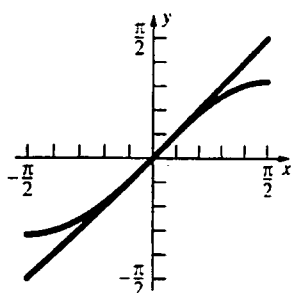
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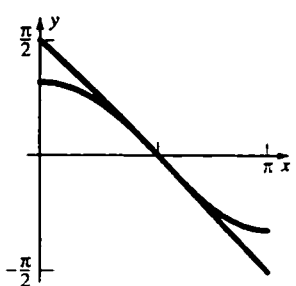
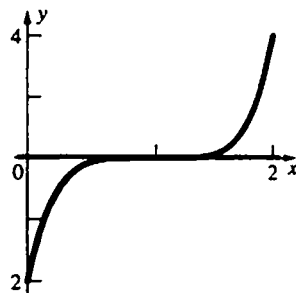
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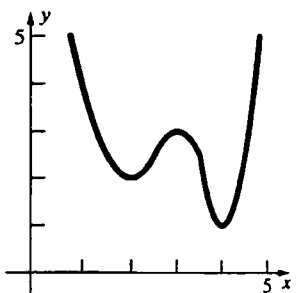
34.

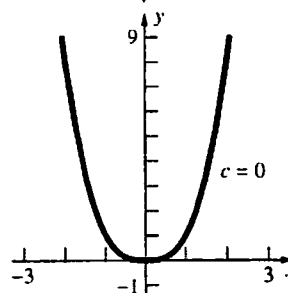
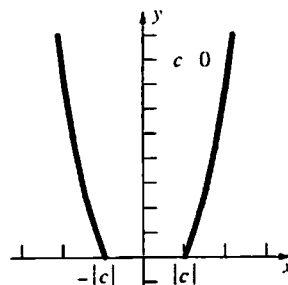
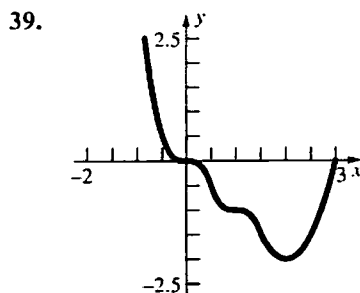
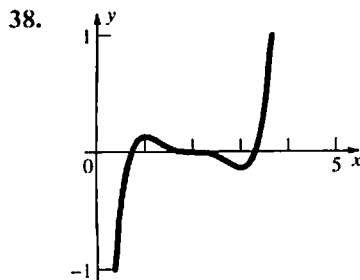


35.

36. $y' = 5(x-1)^4$; $y'' = 20(x-1)^3$; $y''(x) > 0$ When $x > 1$; inflection point (1, 3)At $x = 1$, $y' = 0$, so the linear approximation is a horizontal line.

37.





40. Let $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$ and $f''(x) = 2a$. An inflection point occurs where $f''(x)$ changes from positive to negative, but $2a$ is either always positive or always negative, so $f(x)$ does not have any inflection points. ($f''(x) = 0$ only when $a = 0$, but then $f(x)$ is not a quadratic curve.)

41. Let $f(x) = ax^3 + bx^2 + cx + d$, then $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$. As long as $a \neq 0$, $f''(x)$ will be positive on one side of $x = \frac{b}{3a}$ and negative on the other side. $x = \frac{b}{3a}$ is the only inflection point.

42. Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, then $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$ and $f''(x) = 12ax^2 + 6bx + 2c = 2(6ax^2 + 3bx + c)$. Inflection points can only occur when $f''(x)$ changes sign from positive to negative and $f''(x) = 0$. $f''(x)$ has at most 2 zeros, thus $f(x)$ has at most 2 inflection points.

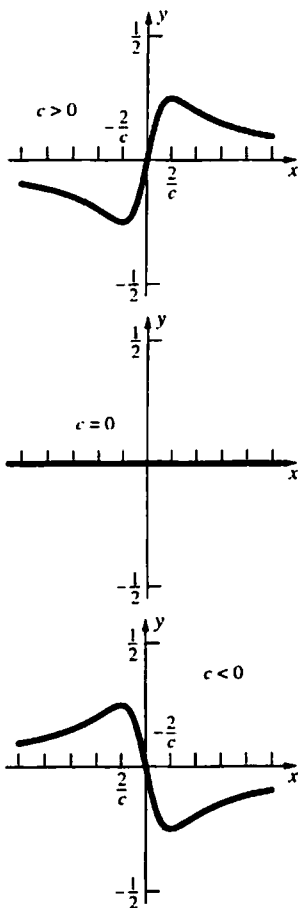
43. Since the c term is squared, the only difference occurs when $c = 0$. When $c = 0$, $y = x^2\sqrt{x^2} = |x|^3$ which has domain $(-\infty, \infty)$ and range $[0, \infty)$. When $c \neq 0$, $y = x^2\sqrt{x^2 - c^2}$ has domain $(-\infty, -|c|] \cup [|c|, \infty)$ and range $[0, \infty)$.

44. Let $f(x) = \frac{cx}{4 + (cx)^2} = \frac{cx}{4 + c^2x^2}$
 $f'(x) = \frac{c(4 - c^2x^2)}{(4 + c^2x^2)^2}$; $f'(x) = 0$ when $x = \pm \frac{2}{c}$
 unless $c = 0$, in which case $f(x) = 0$ and $f'(x) = 0$.

If $c > 0$, $f(x)$ is increasing on $\left[-\frac{2}{c}, \frac{2}{c}\right]$ and decreasing on $\left(-\infty, -\frac{2}{c}\right) \cup \left[\frac{2}{c}, \infty\right)$, thus, $f(x)$ has a global minimum at $f\left(-\frac{2}{c}\right) = -\frac{1}{4}$ and a global maximum of $f\left(\frac{2}{c}\right) = \frac{1}{4}$.

If $c < 0$ $f(x)$ is increasing on $\left(-\infty, \frac{2}{c}\right) \cup \left[-\frac{2}{c}, \infty\right)$ and decreasing on $\left[\frac{2}{c}, -\frac{2}{c}\right]$. Thus, $f(x)$ has a global minimum at $f\left(-\frac{2}{c}\right) = -\frac{1}{4}$ and a global maximum at $f\left(\frac{2}{c}\right) = \frac{1}{4}$.

$f''(x) = \frac{2c^3x(c^2x^2 - 12)}{(4 + c^2x^2)^3}$, so $f(x)$ has inflection points at $x = 0, \pm \frac{2\sqrt{3}}{c}$, $c \neq 0$



45. Let $f(x) = \frac{1}{(cx^2 - 4)^2 + cx^2}$, then

$$f'(x) = \frac{2cx(7 - 2cx^2)}{[(cx^2 - 4)^2 + cx^2]^2};$$

If $c > 0$, $f'(x) = 0$ when $x = 0, \pm\sqrt{\frac{7}{2c}}$.

If $c < 0$, $f'(x) = 0$ when $x = 0$.

Note that $f(x) = \frac{1}{16}$ (a horizontal line) if $c = 0$.

If $c > 0$, $f'(x) > 0$ when $x < -\sqrt{\frac{7}{2c}}$ and

$0 < x < \sqrt{\frac{7}{2c}}$, so $f(x)$ is increasing on

$\left(-\infty, -\sqrt{\frac{7}{2c}}\right] \cup \left[0, \sqrt{\frac{7}{2c}}\right]$ and decreasing on

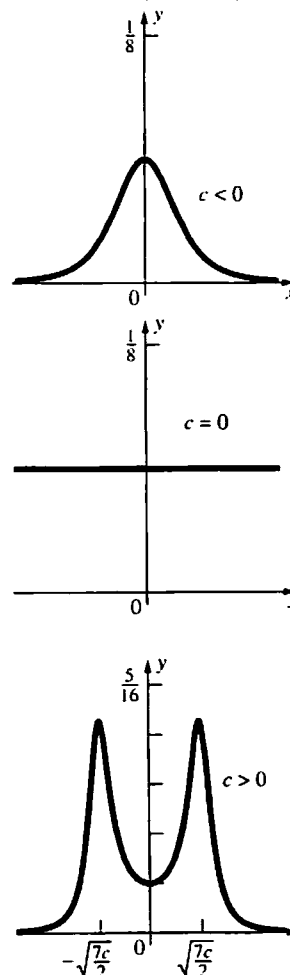
$\left[-\sqrt{\frac{7}{2c}}, 0\right] \cup \left[\sqrt{\frac{7}{2c}}, \infty\right)$. Thus, $f(x)$ has local

maxima $f\left(-\sqrt{\frac{7}{2c}}\right) = \frac{4}{15}$, $f\left(\sqrt{\frac{7}{2c}}\right) = \frac{4}{15}$ and

local minimum $f(0) = \frac{1}{16}$. If $c < 0$, $f'(x) > 0$

when $x < 0$, so $f(x)$ is increasing on $(-\infty, 0]$ and

decreasing on $[0, \infty)$. Thus, $f(x)$ has a local maximum $f(0) = \frac{1}{16}$. Note that $f(x) > 0$ and has horizontal asymptote $y = 0$.



46. Let $f(x) = \frac{1}{x^2 + 4x + c}$. By the quadratic formula, $x^2 + 4x + c = 0$ when $x = -2 \pm \sqrt{4 - c}$. Thus $f(x)$ has vertical asymptote(s) at $x = -2 \pm \sqrt{4 - c}$ when $c \leq 4$.

$$f'(x) = \frac{-2x - 4}{(x^2 + 4x + c)^2}; f'(x) = 0 \text{ when } x = -2,$$

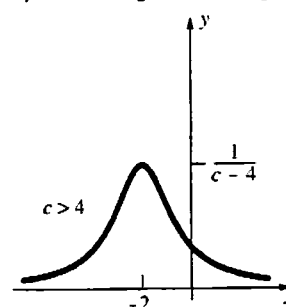
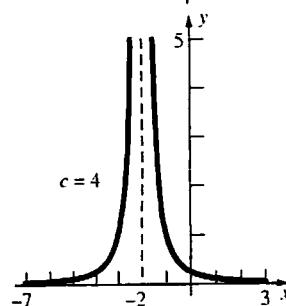
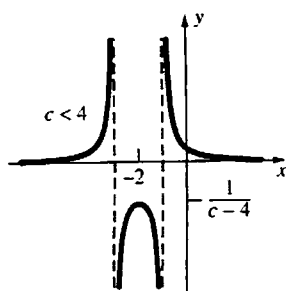
unless $c = 4$ since then $x = -2$ is a vertical asymptote.

For $c \neq 4$, $f'(x) > 0$ when $x < -2$, so $f(x)$ is increasing on $(-\infty, -2]$ and decreasing on $[-2, \infty)$ (with the asymptotes excluded). Thus

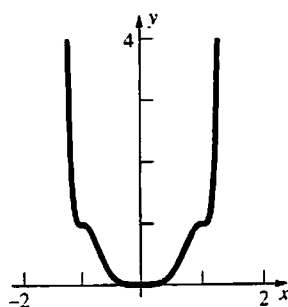
$f(x)$ has a local maximum at $f(-2) = \frac{1}{c - 4}$. For

$c = 4$, $f'(x) = -\frac{2}{(x + 2)^3}$ so $f(x)$ is increasing on

$(-\infty, -2)$ and decreasing on $(-2, \infty)$.



47.



Justification:

$$f(1) = g(1) = 1$$

$$f(-x) = g((-x)^4) = g(x^4) = f(x)$$

f is an even function: symmetric with respect to the y -axis.

$$f'(x) = g'(x^4)4x^3$$

$$f'(x) > 0 \text{ for } x \text{ on } (0, 1) \cup (1, \infty)$$

$$f'(x) < 0 \text{ for } x \text{ on } (-\infty, -1) \cup (-1, 0)$$

$$f'(x) = 0 \text{ for } x = -1, 0, 1 \text{ since } f' \text{ is continuous.}$$

$$f''(x) = g''(x^4)16x^6 + g'(x)12x^2$$

$$f''(x) = 0 \text{ for } x = -1, 0, 1$$

$$f''(x) > 0 \text{ for } x \text{ on } (0, x_0) \cup (1, \infty)$$

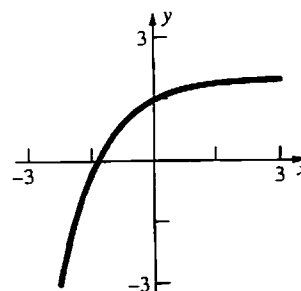
$$f''(x) < 0 \text{ for } x \text{ on } (x_0, 1)$$

Where x_0 is a root of $f''(x) = 0$ (assume that there is only one root on $(0, 1)$).

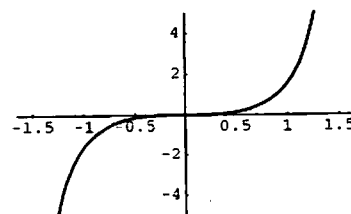
48. Suppose $H'''(1) < 0$, then $H''(x)$ is decreasing in a neighborhood around $x = 1$. Thus, $H''(x) > 0$ to the left of 1 and $H''(x) < 0$ to the right of 1, so $H(x)$ is concave up to the left of 1 and concave down to the right of 1. Suppose $H'''(1) > 0$, then $H''(x)$ is increasing in a neighborhood around $x = 1$. Thus, $H''(x) < 0$ to the left of 1 and $H''(x) > 0$ to the right of 1, so $H(x)$ is concave up to the right of 1 and concave down to the left of 1. In either case, $H(x)$ has a point of inflection at $x = 1$.

49. a. Not possible; $F'(x) > 0$ means that $F(x)$ is increasing. $F''(x) > 0$ means that the rate at which $F(x)$ is increasing never slows down. Thus the values of F must eventually become positive.
- b. Not possible; If $F(x)$ is concave down for all x , then $F(x)$ cannot always be positive.

c.

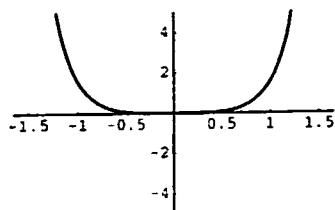


50. a.



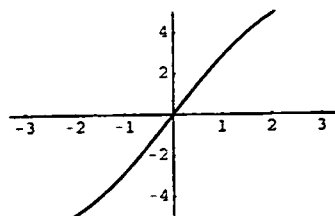
No global extrema; inflection point at $(0, 0)$

b.



No global maximum: global minimum at $(0, 0)$; no inflection points

c.



Global minimum

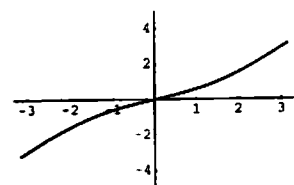
$$f(-\pi) = -2\pi + \sin(-\pi) = -2\pi \approx -6.3;$$

global maximum

$$f(\pi) = 2\pi + \sin \pi = 2\pi \approx 6.3;$$

inflection point at $(0, 0)$

d.



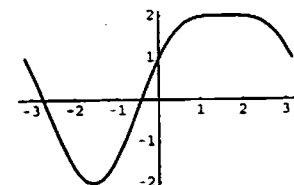
Global minimum

$$f(-\pi) = -\pi - \frac{\sin(-\pi)}{2} = -\pi \approx 3.1; \text{ global}$$

$$\text{maximum } f(\pi) = \pi + \frac{\sin \pi}{2} = \pi \approx 3.1;$$

inflection point at $(0, 0)$.

51. a.



$$f'(x) = 2 \cos x - 2 \cos x \sin x \\ = 2 \cos x(1 - \sin x);$$

$$f'(x) = 0 \text{ when } x = -\frac{\pi}{2}, \frac{\pi}{2}$$

$$f''(x) = -2 \sin x - 2 \cos^2 x + 2 \sin^2 x \\ = 4 \sin^2 x - 2 \sin x - 2; f''(x) = 0 \text{ when}$$

$$\sin x = -\frac{1}{2} \text{ or } \sin x = 1 \text{ which occur when}$$

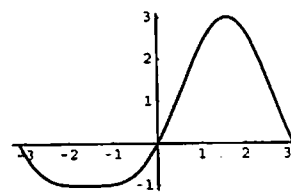
$$x = -\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{2}$$

$$\text{Global minimum } f\left(-\frac{\pi}{2}\right) = -2; \text{ global}$$

$$\text{maximum } f\left(\frac{\pi}{2}\right) = 2; \text{ inflection points}$$

$$f\left(-\frac{\pi}{6}\right) = -\frac{1}{4}, f\left(-\frac{5\pi}{6}\right) = -\frac{1}{4}$$

b.



$$f'(x) = 2 \cos x + 2 \sin x \cos x \\ = 2 \cos x(1 + \sin x); f'(x) = 0 \text{ when}$$

$$x = -\frac{\pi}{2}, \frac{\pi}{2}$$

$$f''(x) = -2 \sin x + 2 \cos^2 x - 2 \sin^2 x \\ = -4 \sin^2 x - 2 \sin x + 2; f''(x) = 0 \text{ when}$$

$$\sin x = -1 \text{ or } \sin x = \frac{1}{2} \text{ which occur when}$$

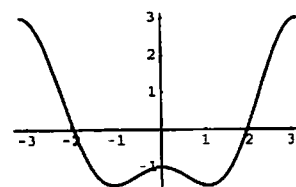
$$x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\text{Global minimum } f\left(-\frac{\pi}{2}\right) = -1; \text{ global}$$

$$\text{maximum } f\left(\frac{\pi}{2}\right) = 3; \text{ inflection points}$$

$$f\left(\frac{\pi}{6}\right) = \frac{5}{4}, f\left(\frac{5\pi}{6}\right) = \frac{5}{4}.$$

c.



$$f'(x) = -2 \sin 2x + 2 \sin x \\ = -4 \sin x \cos x + 2 \sin x = 2 \sin x(1 - 2 \cos x);$$

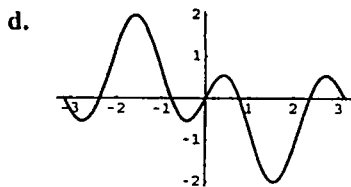
$$f'(x) = 0 \text{ when } x = -\pi, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \pi$$

$$f''(x) = -4 \cos 2x + 2 \cos x; f''(x) = 0 \text{ when} \\ x \approx -2.2, -0.6, 0.6, 2.2$$

$$\text{Global minimum } f\left(-\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right) = -1.5;$$

$$\text{Global maximum } f(-\pi) = f(\pi) = 3;$$

$$\text{inflection points } f(-2.2) \approx 0.87, \\ f(-0.6) \approx -1.29, f(0.6) \approx -1.29, \\ f(2.2) \approx 0.87$$



$$f'(x) = 3\cos 3x - \cos x; f'(x) = 0 \text{ when}$$

$3\cos 3x = \cos x$ which occurs when

$$x = -\frac{\pi}{2}, \frac{\pi}{2} \text{ and when}$$

$$x \approx -2.7, -0.4, 0.4, 2.7$$

$f''(x) = -9\sin 3x + \sin x$ which occurs when

$$x = -\pi, 0, \pi \text{ and when}$$

$$x \approx -2.13, -1.02, 1.02, 2.13$$

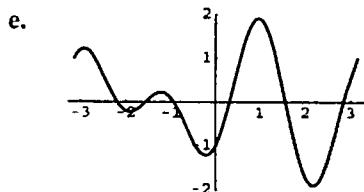
$$\text{Global minimum } f\left(\frac{\pi}{2}\right) = -2;$$

$$\text{global maximum } f\left(-\frac{\pi}{2}\right) = 2;$$

$$\text{inflection points } f(-2.13) \approx 0.7,$$

$$f(-1.02) \approx 0.8; f(0) = 0, f(1.02) \approx -0.8,$$

$$f(2.13) \approx -0.7$$



$$f'(x) = 2\cos 2x + 3\sin 3x$$

Using the graphs, $f(x)$ has a global minimum

at $f(2.17) \approx -1.9$ and a global maximum at

$f(0.97) \approx 1.9$

$$f''(x) = -4\sin 2x + 9\cos 3x; f''(x) = 0 \text{ when}$$

$$x = -\frac{\pi}{2}, \frac{\pi}{2} \text{ and when}$$

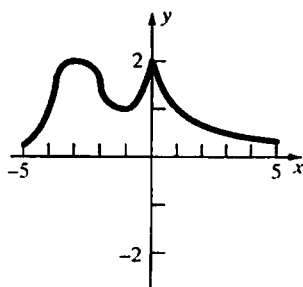
$$x \approx -2.47, -0.67, 0.41, 2.73.$$

Inflection points $f(-2.47) \approx 0.54,$

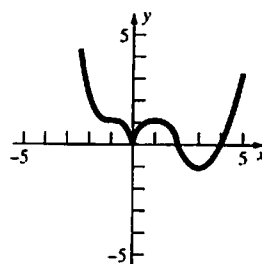
$$f\left(-\frac{\pi}{2}\right) = 0, f(-0.67) \approx -0.55,$$

$$f(0.41) \approx 0.40, f\left(\frac{\pi}{2}\right) = 0, f(2.73) \approx -0.40$$

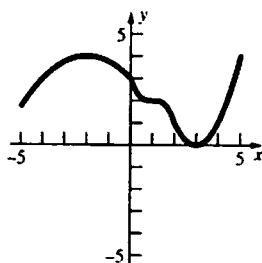
52.



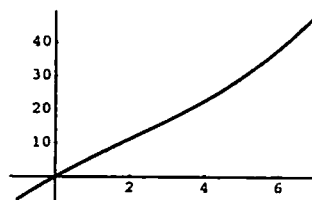
53.



54.



55. a.



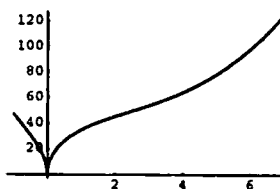
$$f'(x) = \frac{2x^2 - 9x + 40}{\sqrt{x^2 - 6x + 40}}; f'(x) \text{ is never } 0,$$

and always positive, so $f(x)$ is increasing for all x . Thus, on $[-1, 7]$, the global minimum is $f(-1) \approx -6.9$ and the global maximum is $f(7) \approx 48.0$.

$$f''(x) = \frac{2x^3 - 18x^2 + 147x - 240}{(x^2 - 6x + 40)^{3/2}}; f''(x) = 0$$

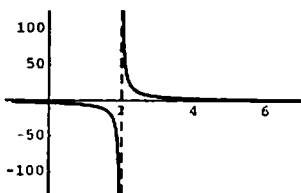
when $x \approx 2.02$; inflection point $f(2.02) \approx 11.4$

b.

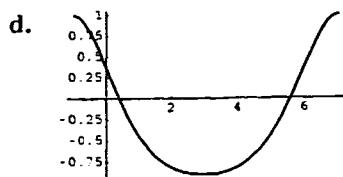


Global minimum $f(0) = 0$; global maximum $f(7) \approx 124.4$; inflection point at $x \approx 2.34$, $f(2.34) \approx 48.09$

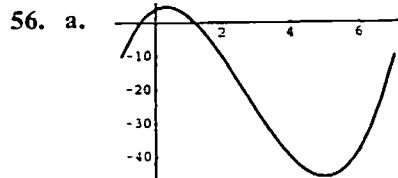
c.



No global minimum or maximum; no inflection points



Global minimum $f(3) \approx -0.9$;
global maximum $f(-1) \approx 1.0$ or $f(7) \approx 1.0$;
Inflection points at $x \approx 0.05$ and $x \approx 5.9$,
 $f(0.05) \approx 0.3$, $f(5.9) \approx 0.3$.



$$f'(x) = 3x^2 - 16x + 5; f'(x) = 0 \text{ when}$$

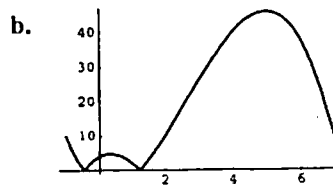
$$x = \frac{1}{3}, 5.$$

$$\text{Global minimum } f(5) = -46;$$

$$\text{global maximum } f\left(\frac{1}{3}\right) \approx 4.8$$

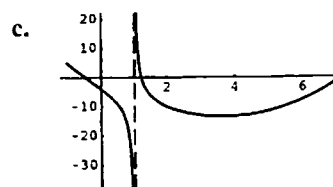
$$f''(x) = 6x - 16; f''(x) = 0 \text{ when } x = \frac{8}{3};$$

$$\text{inflection point } f\left(\frac{8}{3}\right) \approx -20.6$$

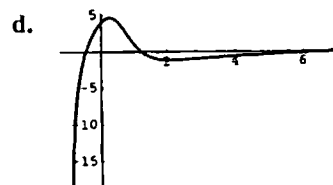


Global minimum when $x \approx -0.5$ and
 $x \approx 1.2$, $f(-0.5) \approx 0$, $f(1.2) \approx 0$;
global maximum $f(5) = 46$

$$\text{Inflection point } f\left(\frac{8}{3}\right) \approx 20.6$$



No global minimum or maximum;
inflection point at
 $x \approx -0.26$, $f(-0.26) \approx -1.7$



No global minimum, global maximum when
 $x \approx 0.26$, $f(0.26) \approx 4.7$
Inflection points when $x \approx 0.75$ and
 $x \approx 3.15$, $f(0.75) \approx 2.6$, $f(3.15) \approx -0.88$

4.7 Concepts Review

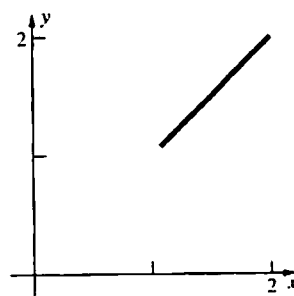
1. continuous; (a, b) : $f(b) - f(a) = f'(c)(b - a)$
2. $f'(0)$ does not exist.
3. $F(x) = G(x) + C$
4. $x^4 + C$

Problem Set 4.7

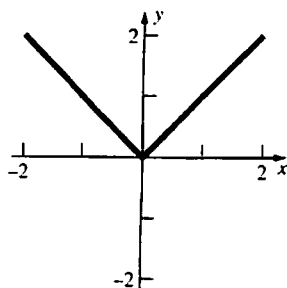
$$1. f'(x) = \frac{x}{|x|}$$

$$\frac{f(2) - f(1)}{2 - 1} = \frac{2 - 1}{1} = 1$$

$$\frac{c}{|c|} = 1 \text{ for all } c > 0, \text{ hence for all } c \text{ in } (1, 2)$$



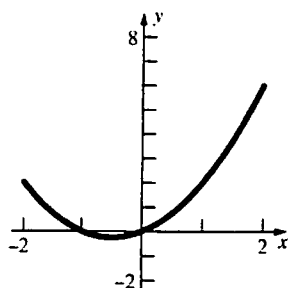
2. The Mean Value Theorem does not apply because $g'(0)$ does not exist.



3. $f'(x) = 2x + 1$

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{6 - 2}{4} = 1$$

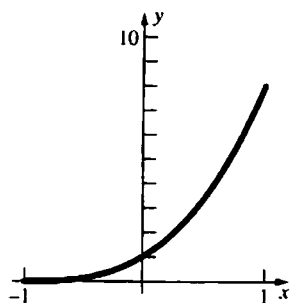
$2c + 1 = 1$ when $c = 0$



4. $g'(x) = 3(x+1)^2$

$$\frac{g(1) - g(-1)}{1 - (-1)} = \frac{8 - 0}{2} = 4$$

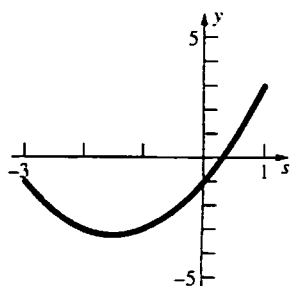
$3(c+1)^2 = 4$ when $c = -1 + \frac{2}{\sqrt{3}} \approx 0.15$



5. $H'(s) = 2s + 3$

$$\frac{H(1) - H(-3)}{1 - (-3)} = \frac{3 - (-1)}{4} = 1$$

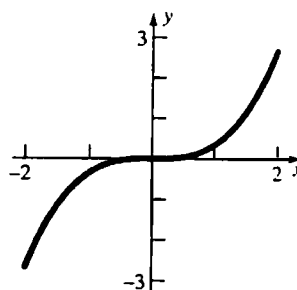
$2c + 3 = 1$ when $c = -1$



6. $F'(x) = x^2$

$$\frac{F(2) - F(-2)}{2 - (-2)} = \frac{\frac{8}{3} - (-\frac{8}{3})}{4} = \frac{4}{3}$$

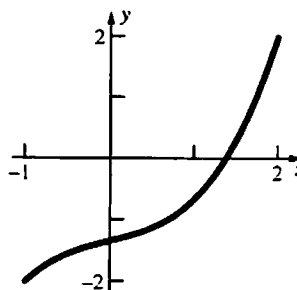
$c^2 = \frac{4}{3}$ when $c = \pm \frac{2}{\sqrt{3}} \approx \pm 1.15$



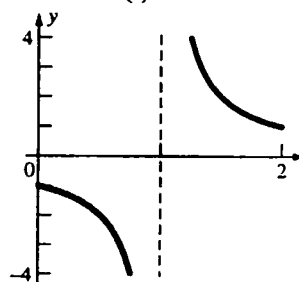
7. $f'(z) = \frac{1}{3}(3z^2 + 1) = z^2 + \frac{1}{3}$

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{2 - (-2)}{3} = \frac{4}{3}$$

$c^2 + \frac{1}{3} = \frac{4}{3}$ when $c = -1, 1$, but -1 is not in $(-1, 2)$ so $c = 1$ is the only solution.



8. The Mean Value Theorem does not apply because $F(t)$ is not continuous at $t = 1$.

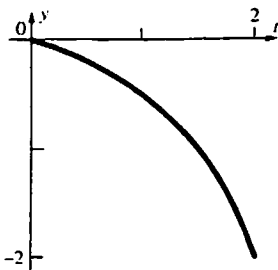


$$9. h'(x) = -\frac{3}{(x-3)^2}$$

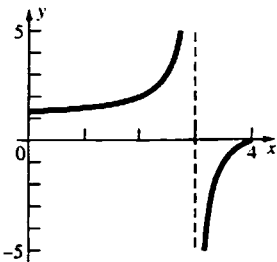
$$\frac{h(2) - h(0)}{2 - 0} = \frac{-2 - 0}{2} = -1$$

$$-\frac{3}{(c-3)^2} = -1 \text{ when } c = 3 \pm \sqrt{3},$$

$$c = 3 - \sqrt{3} \approx 1.27 \quad (3 + \sqrt{3} \text{ is not in } (0, 2).)$$



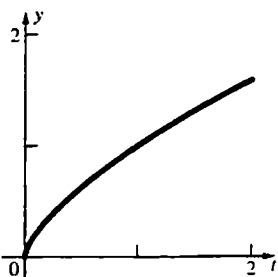
10. The Mean Value Theorem does not apply because $f(x)$ is not continuous at $x = 3$.



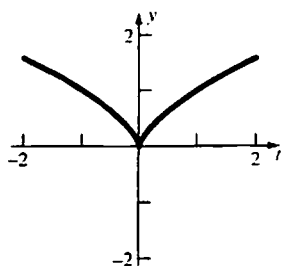
$$11. h'(t) = \frac{2}{3t^{1/3}}$$

$$\frac{h(2) - h(0)}{2 - 0} = \frac{2^{2/3} - 0}{2} = 2^{-1/3}$$

$$\frac{2}{3c^{1/3}} = 2^{-1/3} \text{ when } c = \frac{16}{27} \approx 0.59$$



12. The Mean Value Theorem does not apply because $h'(0)$ does not exist.

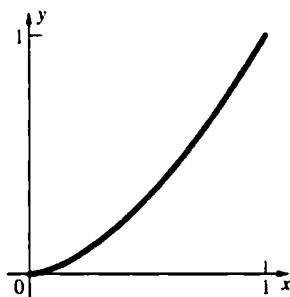


$$13. g'(x) = \frac{5}{3}x^{2/3}$$

$$\frac{g(1) - g(0)}{1 - 0} = \frac{1 - 0}{1} = 1$$

$$\frac{5}{3}c^{2/3} = 1 \text{ when } c = \pm \left(\frac{3}{5}\right)^{3/2},$$

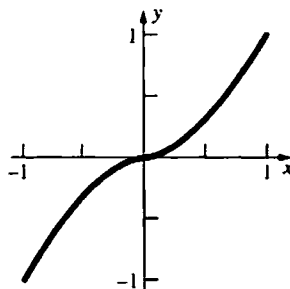
$$c = \left(\frac{3}{5}\right)^{3/2} \approx 0.46, \quad \left(-\left(\frac{3}{5}\right)^{3/2} \text{ is not in } (0, 1).\right)$$



$$14. g'(x) = \frac{5}{3}x^{2/3}$$

$$\frac{g(1) - g(-1)}{1 - (-1)} = \frac{1 - (-1)}{2} = 1$$

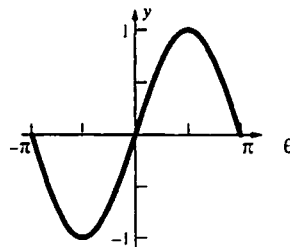
$$\frac{5}{3}c^{2/3} = 1 \text{ when } c = \pm \left(\frac{3}{5}\right)^{3/2} \approx \pm 0.46$$



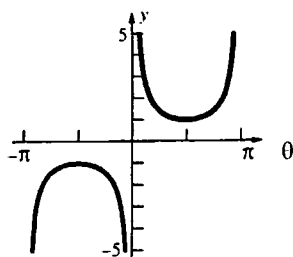
$$15. S'(\theta) = \cos \theta$$

$$\frac{S(\pi) - S(-\pi)}{\pi - (-\pi)} = \frac{0 - 0}{2\pi} = 0$$

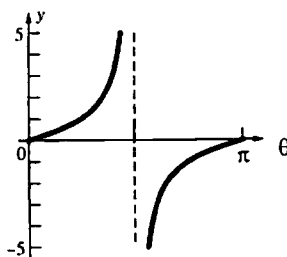
$$\cos c = 0 \text{ when } c = \pm \frac{\pi}{2}.$$



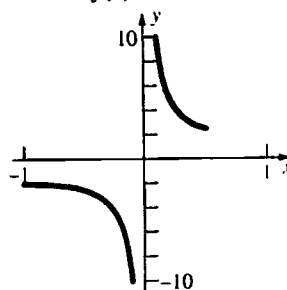
16. The Mean Value Theorem does not apply because $C(\theta)$ is not continuous at $\theta = -\pi, 0, \pi$.



17. The Mean Value Theorem does not apply because $T(\theta)$ is not continuous at $\theta = \frac{\pi}{2}$.



18. The Mean Value Theorem does not apply because $f(x)$ is not continuous at $x = 0$.

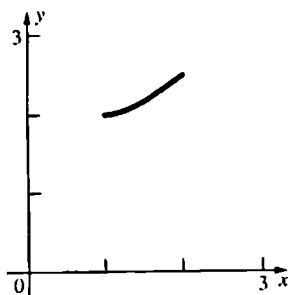


19. $f'(x) = 1 - \frac{1}{x^2}$

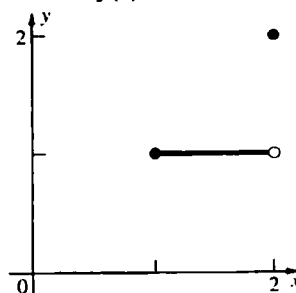
$$\frac{f(2) - f(1)}{2 - 1} = \frac{\frac{5}{2} - 2}{1} = \frac{1}{2}$$

$$1 - \frac{1}{c^2} = \frac{1}{2} \text{ when } c = \pm\sqrt{2}, \quad c = \sqrt{2} \approx 1.41$$

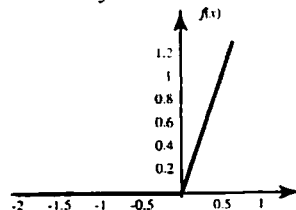
($c = -\sqrt{2}$ is not in $(1, 2)$.)



20. The Mean Value Theorem does not apply because $f(x)$ is not continuous at $x = 2$.



21. The Mean Value Theorem does not apply because f is not differentiable at $x = 0$.



22. By the Mean Value Theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ for some } c \text{ in } (a, b).$$

$$\text{Since } f(b) = f(a), \quad \frac{0}{b - a} = f'(c); \quad f'(c) = 0.$$

23. $\frac{f(8) - f(0)}{8 - 0} = -\frac{1}{4}$

There are three values for c such that

$$f'(c) = -\frac{1}{4}.$$

They are approximately 1.5, 3.75, and 7.

24. $f'(x) = 2\alpha x + \beta$

$$\frac{f(b) - f(a)}{b - a} = \frac{1}{b - a} [\alpha(b^2 - a^2) + \beta(b - a)]$$

$$= \alpha(a + b) + \beta$$

$2\alpha c + \beta = \alpha(a + b) + \beta$ when $c = \frac{a + b}{2}$ which is the midpoint of $[a, b]$.

25. By the Monotonicity Theorem, f is increasing on the intervals (a, x_0) and (x_0, b) .

To show that $f(x_0) > f(x)$ for x in (a, x_0) , consider f on the interval (a, x_0) .

f satisfies the conditions of the Mean Value Theorem on the interval $[x, x_0]$ for x in (a, x_0) .

So for some c in (x, x_0) ,

$$f(x_0) - f(x) = f'(c)(x_0 - x).$$

Because

$$f'(c) > 0 \text{ and } x_0 - x > 0, \quad f(x_0) - f(x) > 0.$$

so $f(x_0) > f(x)$.

Similar reasoning shows that
 $f(x) > f(x_0)$ for x in (x_0, b) .
 Therefore, f is increasing on (a, b) .

26. a. $f'(x) = 3x^2 > 0$ except at $x = 0$ in $(-\infty, \infty)$.
 $f(x) = x^3$ is increasing on $(-\infty, \infty)$ by
 Problem 25.

- b. $f'(x) = 5x^4 > 0$ except at $x = 0$ in $(-\infty, \infty)$.
 $f(x) = x^5$ is increasing on $(-\infty, \infty)$ by
 Problem 25.

- c. $f'(x) = \begin{cases} 3x^2 & x \leq 0 \\ 1 & x > 0 \end{cases} > 0$ except at $x = 0$ in
 $(-\infty, \infty)$.
 $f(x) = \begin{cases} x^3 & x \leq 0 \\ x & x > 0 \end{cases}$ is increasing on
 $(-\infty, \infty)$ by Problem 25.

27. $s(t)$ is defined in any interval not containing $t = 0$.
 $s'(c) = -\frac{1}{c^2} < 0$ for all $c \neq 0$. For any a, b with
 $a < b$ and both either positive or negative, the
 Mean Value Theorem says
 $s(b) - s(a) = s'(c)(b - a)$ for some c in (a, b) .
 Since $a < b$, $b - a > 0$ while $s'(c) < 0$, hence
 $s(b) - s(a) < 0$, or $s(b) < s(a)$.
 Thus, $s(t)$ is decreasing on any interval not
 containing $t = 0$.

28. $s'(c) = -\frac{2}{c^3} < 0$ for all $c > 0$. If $0 < a < b$, the
 Mean Value Theorem says
 $s(b) - s(a) = s'(c)(b - a)$ for some c in (a, b) .
 Since $a < b$, $b - a > 0$ while $s'(c) < 0$, hence
 $s(b) - s(a) < 0$, or $s(b) < s(a)$. Thus, $s(t)$ is
 decreasing on any interval to the right of the
 origin.

29. $F'(x) = 0$ and $G(x) = 0$; $G'(x) = 0$.

By Theorem B,
 $F(x) = G(x) + C$, so $F(x) = 0 + C = C$.

30. $F(x) = \cos^2 x + \sin^2 x$; $F(0) = 1^2 + 0^2 = 1$
 $F'(x) = 2 \cos x(-\sin x) + 2 \sin x(\cos x) = 0$
 By Problem 29, $F(x) = C$ for all x .
 Since $F(0) = 1$, $C = 1$, so $\sin^2 x + \cos^2 x = 1$ for
 all x .

31. Let $G(x) = Dx$; $F'(x) = D$ and $G'(x) = D$.
 By Theorem B, $F(x) = G(x) + C$; $F(x) = Dx + C$.

32. $F'(x) = 5$; $F(0) = 4$
 $F(x) = 5x + C$ by Problem 31.
 $F(0) = 4$ so $C = 4$.
 $F(x) = 5x + 4$

33. Since $f(a)$ and $f(b)$ have opposite signs, 0 is
 between $f(a)$ and $f(b)$. $f(x)$ is continuous on $[a, b]$,
 since it has a derivative. Thus, by the
 Intermediate Value Theorem, there is at least one
 point c ,
 $a < c < b$ with $f(c) = 0$.
 Suppose there are two points, c and c' , $c < c'$ in
 (a, b) with $f(c) = f(c') = 0$. Then by Rolle's
 Theorem, there is at least one number d in (c, c')
 with $f'(d) = 0$. This contradicts the given
 information that $f'(x) \neq 0$ for all x in $[a, b]$, thus
 there cannot be more than one x in $[a, b]$ where
 $f(x) = 0$.

34. $f'(x) = 6x^2 - 18x = 6x(x - 3)$; $f'(x) = 0$ when
 $x = 0$ or $x = 3$.
 $f(-1) = -10$, $f(0) = 1$ so, by Problem 33, $f(x) = 0$
 has exactly one solution on $(-1, 0)$.
 $f(0) = 1$, $f(1) = -6$ so, by Problem 33, $f(x) = 0$ has
 exactly one solution on $(0, 1)$.
 $f(4) = -15$, $f(5) = 26$ so, by Problem 33, $f(x) = 0$
 has exactly one solution on $(4, 5)$.

35. Suppose there is more than one zero between
 successive distinct zeros of f' . That is, there are
 a and b such that $f(a) = f(b) = 0$ with a and b
 between successive distinct zeros of f' . Then by
 Rolle's Theorem, there is a c between a and b
 such that $f'(c) = 0$. This contradicts the
 supposition that a and b lie between successive
 distinct zeros.

36. Let x_1, x_2 , and x_3 be the three values such that
 $g(x_1) = g(x_2) = g(x_3) = 0$ and
 $a \leq x_1 < x_2 < x_3 \leq b$. By applying Rolle's
 Theorem (see Problem 22) there is at least one
 number x_4 in (x_1, x_2) and one number x_5 in
 (x_2, x_3) such that $g'(x_4) = g'(x_5) = 0$. Then by
 applying Rolle's Theorem to $g'(x)$, there is at
 least one number x_6 in (x_4, x_5) such that
 $g''(x_6) = 0$.

37. $f(x)$ is a polynomial function so it is continuous
 on $[0, 4]$ and $f''(x)$ exists for all x on $(0, 4)$.
 $f(1) = f(2) = f(3) = 0$, so by Problem 36, there are
 at least two values of x in $[0, 4]$ where $f'(x) = 0$
 and at least one value of x in $[0, 4]$ where
 $f''(x) = 0$.

38. By applying the Mean Value Theorem and taking the absolute value of both sides,

$$\frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} = |f'(c)|, \text{ for some } c \text{ in } (x_1, x_2).$$

Since $|f'(x)| \leq M$ for all x in (a, b) ,

$$\frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} \leq M; |f(x_2) - f(x_1)| \leq M|x_2 - x_1|.$$

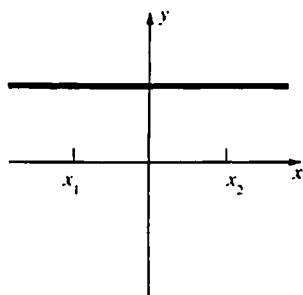
39. $f'(x) = 2 \cos 2x$; $|f'(x)| \leq 2$

$$\frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} = |f'(c)|; \frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} \leq 2$$

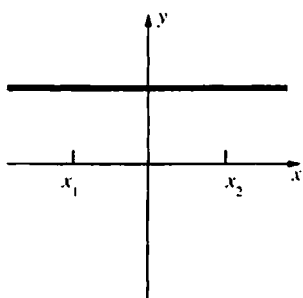
$$|f(x_2) - f(x_1)| \leq 2|x_2 - x_1|;$$

$$|\sin 2x_2 - \sin 2x_1| \leq 2|x_2 - x_1|$$

40. a.



- b.



41. Suppose $f'(x) \geq 0$. Let a and b lie in the interior of I such that $b > a$. By the Mean Value Theorem, there is a point c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}; \frac{f(b) - f(a)}{b - a} \geq 0.$$

Since $a < b$, $f(b) \geq f(a)$, so f is nondecreasing.

Suppose $f'(x) \leq 0$. Let a and b lie in the interior of I such that $b > a$. By the Mean Value Theorem, there is a point c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}; \frac{f(b) - f(a)}{b - a} \leq 0. \text{ Since}$$

$a < b$, $f(a) \geq f(b)$, so f is nonincreasing.

42. $[f^2(x)]' = 2f(x)f'(x)$

Because $f(x) \geq 0$ and $f'(x) \geq 0$ on I , $[f^2(x)]' \geq 0$ on I .

As a consequence of the Mean Value Theorem,

$$f^2(x_2) - f^2(x_1) \geq 0 \text{ for all } x_2 > x_1 \text{ on } I.$$

Therefore f^2 is nondecreasing.

43. Let $f(x) = h(x) - g(x)$.

$$f'(x) = h'(x) - g'(x); f'(x) \geq 0 \text{ for all } x \text{ in}$$

(a, b) since $g'(x) \leq h'(x)$ for all x in (a, b) , so f is nondecreasing on (a, b) by Problem 41. Thus

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2);$$

$$h(x_1) - g(x_1) \leq h(x_2) - g(x_2);$$

$$g(x_2) - g(x_1) \leq h(x_2) - h(x_1) \text{ for all } x_1 \text{ and } x_2 \text{ in } (a, b).$$

44. Let $f(x) = \sqrt{x}$ so $f'(x) = \frac{1}{2\sqrt{x}}$. Apply the Mean Value Theorem to f on the interval $[x, x+2]$ for $x > 0$.

$$\text{Thus } \sqrt{x+2} - \sqrt{x} = \frac{1}{2\sqrt{c}}(2) = \frac{1}{\sqrt{c}} \text{ for some } c \text{ in } (x, x+2).$$

$$\text{Observe } \frac{1}{\sqrt{x+2}} < \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{x}}.$$

$$\text{Thus as } x \rightarrow \infty, \frac{1}{\sqrt{c}} \rightarrow 0.$$

$$\text{Therefore } \lim_{x \rightarrow \infty} (\sqrt{x+2} - \sqrt{x}) = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{c}} = 0.$$

45. Let $f(x) = \sin x$. $f'(x) = \cos x$, so

$$|f'(x)| = |\cos x| \leq 1 \text{ for all } x.$$

By the Mean Value Theorem,

$$\frac{f(x) - f(y)}{x - y} = f'(c) \text{ for some } c \text{ in } (x, y).$$

$$\text{Thus, } \frac{|f(x) - f(y)|}{|x - y|} = |f'(c)| \leq 1;$$

$$|\sin x - \sin y| \leq |x - y|.$$

46. Let d be the difference in distance between horse A and horse B as a function of time t .

Then d' is the difference in speeds.

Let t_0 and t_1 be the start and finish times of the race.

$$d(t_0) = d(t_1) = 0$$

By the Mean Value Theorem,

$$\frac{d(t_1) - d(t_0)}{t_1 - t_0} = d'(c) \text{ for some } c \text{ in } (t_0, t_1).$$

$$\text{Therefore } d'(c) = 0 \text{ for some } c \text{ in } (t_0, t_1).$$

47. Let s be the difference in speeds between horse A and horse B as function of time t .
Then s' is the difference in accelerations.
Let t_2 be the time in Problem 46 at which the horses had the same speeds and let t_1 be the finish time of the race.
 $s(t_2) = s(t_1) = 0$
By the Mean Value Theorem,
 $\frac{s(t_1) - s(t_2)}{t_1 - t_2} = s'(c)$ for some c in (t_2, t_1) .
Therefore $s'(c) = 0$ for some c in (t_2, t_1) .

48. Suppose $x > c$. Then by the Mean Value Theorem,
 $f(x) - f(c) = f'(a)(x - c)$ for some a in (c, x) .
Since f is concave up, $f'' > 0$ and by the Monotonicity Theorem f' is increasing.
Therefore $f'(a) > f'(c)$ and
 $f(x) - f(c) = f'(a)(x - c) > f'(c)(x - c)$
 $f(x) > f(c) + f'(c)(x - c)$, $x > c$
Suppose $x < c$. Then by the Mean Value Theorem,
 $f(c) - f(x) = f'(a)(c - x)$ for some a in (x, c) .
Since f is concave up, $f'' > 0$, and by the Monotonicity Theorem f' is increasing.
Therefore, $f'(c) > f'(a)$ and
 $f(c) - f(x) = f'(a)(c - x) < f'(c)(c - x)$
 $-f(x) < -f(c) + f'(c)(c - x)$
 $f(x) > f(c) - f'(c)(c - x)$
 $f(x) > f(c) + f'(c)(x - c)$, $x < c$
Therefore $f(x) > f(c) + f'(c)(x - c)$, $x \neq c$.

49. If $|f(y) - f(x)| \leq M(y - x)^2$, then
 $\frac{|f(y) - f(x)|}{|y - x|} \leq M|y - x|$. By the Mean Value Theorem, there is some c between x and y such that $\frac{|f(y) - f(x)|}{|y - x|} = |f'(c)| \leq M|y - x|$. Since the given inequality is true for all x and y , we can choose x and y so close together that $M|y - x|$ is arbitrarily close to 0. Thus, in order for the last inequality to be true, $f'(c)$ must be 0, hence $f(x)$ is a constant function.

50. $f(x) = x^{1/3}$ on $[0, a]$ or $[-a, 0]$ where a is any positive number. $f'(0)$ does not exist, but $f(x)$ has a vertical tangent line at $x = 0$.

51. Let $f(t)$ be the distance traveled at time t .

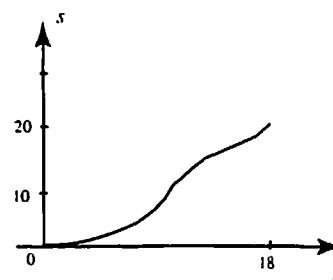
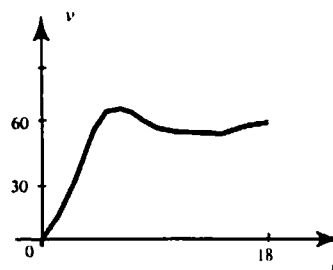
$$\frac{f(2) - f(0)}{2 - 0} = \frac{112 - 0}{2} = 56$$

By the Mean Value Theorem, there is a time c such that $f'(c) = 56$.

At some time during the trip, Johnny must have gone 56 miles per hour.

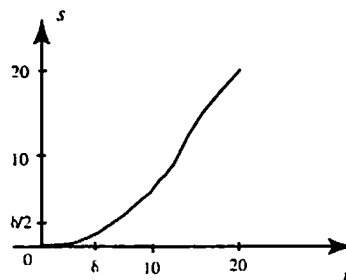
52. s is differentiable with $s(0) = 0$ and $s(18) = 20$ so we can apply the Mean Value Theorem. There exists a c in the interval $(0, 18)$ such that

$$\begin{aligned} v(c) = s'(c) &= \frac{(20 - 0)}{(18 - 0)} \\ &\approx 1.11 \text{ miles per minute} \\ &\approx 66.67 \text{ miles per hour} \end{aligned}$$



53. Since the car is stationary at $t = 0$, and since v is continuous, there exists a δ such that $v(t) < \frac{1}{2}$ for all t in the interval $[0, \delta]$. $v(t)$ is therefore less than $\frac{1}{2}$ and $s(\delta) < \delta \cdot \frac{1}{2} = \frac{\delta}{2}$. By the Mean Value Theorem, there exists a c in the interval $(\delta, 20)$ such that

$$\begin{aligned}
 v(c) = s'(c) &= \frac{\left(20 - \frac{\delta}{2}\right)}{(20 - \delta)} \\
 &> \frac{20 - \delta}{20 - \delta} \\
 &= 1 \text{ mile per minute} \\
 &= 60 \text{ miles per hour}
 \end{aligned}$$



4.8 Chapter Review

Concepts Test

1. True: Max-Min Existence Theorem
2. True: Since c is an interior point and f is differentiable ($f'(c)$ exists), by the Critical Point Theorem, c is a stationary point ($f'(c) = 0$).
3. True: For example, let $f(x) = \sin x$.
4. False: $f(x) = x^{1/3}$ is continuous and increasing for all x , but $f'(x)$ does not exist at $x = 0$.
5. True: $f'(x) = 18x^5 + 16x^3 + 4x$;
 $f''(x) = 90x^4 + 48x^2 + 4$, which is greater than zero for all x .
6. False: For example, $f(x) = x^3$ is increasing on $[-1, 1]$ but $f'(0) = 0$.
7. True: When $f'(x) > 0$, $f(x)$ is increasing.
8. False: If $f''(c) = 0$, c is a candidate, but not necessarily an inflection point. For example, if $f(x) = x^4$, $f''(0) = 0$ but $x = 0$ is not an inflection point.
9. True: $f(x) = ax^2 + bx + c$;
 $f'(x) = 2ax + b$; $f''(x) = 2a$
10. True: If $f(x)$ is increasing for all x in $[a, b]$, the maximum occurs at b .
11. False: $\tan^2 x$ has a minimum value of 0. This occurs whenever $x = k\pi$ where k is an integer.
12. True: $\lim_{x \rightarrow \frac{\pi}{2}^-} (2x^3 + x) = \infty$ while
 $\lim_{x \rightarrow -\frac{\pi}{2}^+} (2x^3 + x) = -\infty$
13. True: $\lim_{x \rightarrow \frac{\pi}{2}} (2x^3 + x + \tan x) = \infty$ while
 $\lim_{x \rightarrow -\frac{\pi}{2}} (2x^3 + x + \tan x) = -\infty$.
14. False: At $x = 3$ there is a removable discontinuity.
15. True: $\lim_{x \rightarrow \infty} \frac{x^2 + 1}{1 - x^2} = \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x^2}}{\frac{1}{x^2} - 1}$
 $= \frac{1}{-1} = -1$ and
 $\lim_{x \rightarrow -\infty} \frac{x^2 + 1}{1 - x^2} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x^2}}{\frac{1}{x^2} - 1}$
 $= \frac{1}{-1} = -1$.
16. True: $\frac{3x^2 + 2x + \sin x}{x} - (3x + 2) = \frac{\sin x}{x}$;
 $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$ and $\lim_{x \rightarrow -\infty} \frac{\sin x}{x} = 0$.
17. True: The function is differentiable on $(0, 2)$.
18. False: $f'(x) = \frac{x}{|x|}$ so $f'(0)$ does not exist.
19. False: There are two points: $x = -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}$.

20. True: Let $g(x) = D$ where D is any number. Then $g'(x) = 0$ and so, by Theorem B of Section 4.7, $f(x) = g(x) + C = D + C$, which is a constant, for all x in (a, b) .
21. False: For example if $f(x) = x^4$, $f'(0) = f''(0) = 0$ but f has a minimum at $x = 0$.
22. True: $\frac{dy}{dx} = \cos x$; $\frac{d^2y}{dx^2} = -\sin x$; $-\sin x = 0$ has infinitely many solutions.
23. True: $A = xy = K$; $y = \frac{K}{x}$
 $P = 2x + \frac{2K}{x}$; $\frac{dP}{dx} = 2 - \frac{2K}{x^2}$; $\frac{dP}{dx} = 0$
 when $x = \sqrt{K}$, $y = \sqrt{K}$
24. True: By the Mean Value Theorem, the derivative must be zero between each pair of distinct x -intercepts.
25. True: If $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ for $x_1 < x_2$,
 $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, so $f + g$ is increasing.
26. False: Let $f(x) = g(x) = 2x$, $f'(x) > 0$ and $g'(x) > 0$ for all x , but $f(x)g(x) = 4x^2$ is decreasing on $(-\infty, 0)$.
27. True: Since $f''(x) > 0$, $f'(x)$ is increasing for $x \geq 0$. Therefore, $f'(x) > 0$ for x in $[0, \infty)$, so $f(x)$ is increasing.
28. False: If $f(3) = 4$, the Mean Value Theorem requires that at some point c in $[0, 3]$,
 $f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{4 - 1}{3 - 0} = 1$ which does not contradict that $f'(x) \leq 2$ for all x in $[0, 3]$.
29. True: If the function is nondecreasing, $f'(x)$ must be greater than or equal to zero, and if $f'(x) \geq 0$, f is nondecreasing. This can be seen using the Mean Value Theorem.
30. True: However, if the constant is 0, the functions are the same.
31. False: For example, let $f(x) = e^x$.
 $\lim_{x \rightarrow -\infty} e^x = 0$. so $y = 0$ is a horizontal asymptote.
32. True: If $f(c)$ is a global maximum then $f(c)$ is the maximum value of f on $(a, b) \leftrightarrow S$ where (a, b) is any interval containing c and S is the domain of f . Hence, $f(c)$ is a local maximum value.
33. True: $f'(x) = 3ax^2 + 2bx + c$; $f'(x) = 0$
 when $x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$ by the Quadratic Formula. $f''(x) = 6ax + 2b$ so
 $f''\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right) = \pm 2\sqrt{b^2 - 3ac}$.
 Thus, if $b^2 - 3ac > 0$, one critical point is a local maximum and the other is a local minimum.
 (If $b^2 - 3ac = 0$ the only critical point is an inflection point while if $b^2 - 3ac < 0$ there are no critical points.)
 On an open interval, no local maxima can come from endpoints, so there can be at most one local maximum in an open interval.
34. True: $f'(x) = a \neq 0$ so $f(x)$ has no local minima or maxima. On an open interval, no local minima or maxima can come from endpoints, so $f(x)$ has no local minima.

Sample Test Problems

- $f'(x) = 2x - 2$; $2x - 2 = 0$ when $x = 1$.
 Critical points: 0, 1, 4
 $f(0) = 0$, $f(1) = -1$, $f(4) = 8$
 Global minimum $f(1) = -1$;
 global maximum $f(4) = 8$
- $f'(t) = -\frac{1}{t^2}$; $-\frac{1}{t^2}$ is never 0.
 Critical points: 1, 4
 $f(1) = 1$, $f(4) = \frac{1}{4}$
 Global minimum $f(4) = \frac{1}{4}$;
 global maximum $f(1) = 1$.

3. $f'(z) = -\frac{2}{z^3}; -\frac{2}{z^3}$ is never 0.

Critical points: $-2, -\frac{1}{2}$

$f(-2) = \frac{1}{4}, f\left(-\frac{1}{2}\right) = 4$

Global minimum $f(-2) = \frac{1}{4}$;

global maximum $f\left(-\frac{1}{2}\right) = 4$.

4. $f'(x) = -\frac{2}{x^3}; -\frac{2}{x^3}$ is never 0.

Critical point: -2

$f(-2) = \frac{1}{4}$

$f'(x) > 0$ for $x < 0$, so f is increasing.

Global minimum $f(-2) = \frac{1}{4}$; no global maximum.

5. $f'(x) = \frac{x}{|x|}$; $f'(x)$ does not exist at $x = 0$.

Critical points: $-\frac{1}{2}, 0, 1$

$f\left(-\frac{1}{2}\right) = \frac{1}{2}, f(0) = 0, f(1) = 1$

Global minimum $f(0) = 0$;
global maximum $f(1) = 1$

6. $f'(s) = 1 + \frac{s}{|s|}$; $f'(s)$ does not exist when $s = 0$.

For $s < 0$, $|s| = -s$ so $f(s) = s - s = 0$ and
 $f'(s) = 1 - 1 = 0$.

Critical points: 1 and all s in $[-1, 0]$

$f(1) = 2, f(s) = 0$ for s in $[-1, 0]$

Global minimum $f(s) = 0, -1 \leq s \leq 0$;

global maximum $f(1) = 2$.

7. $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$; $f'(x) = 0$
when $x = 0, 1$

Critical points: $-2, 0, 1, 3$

$f(-2) = 80, f(0) = 0, f(1) = -1, f(3) = 135$

Global minimum $f(1) = -1$;

global maximum $f(3) = 135$

8. $f'(u) = \frac{u(7u-12)}{3(u-2)^{2/3}}$; $f'(u) = 0$ when $u = 0, \frac{12}{7}$

$f'(2)$ does not exist.

Critical points: $-1, 0, \frac{12}{7}, 2, 3$

$f(-1) = \sqrt[3]{-3} \approx -1.44, f(0) = 0,$

$f\left(\frac{12}{7}\right) = \frac{144}{49}\sqrt[3]{-\frac{2}{7}} \approx -1.94, f(2) = 0, f(3) = 9$

Global minimum $f\left(\frac{12}{7}\right) \approx -1.94$;

global maximum $f(3) = 9$

9. $f'(x) = 10x^4 - 20x^3 = 10x^3(x-2)$;

$f'(x) = 0$ when $x = 0, 2$

Critical points: $-1, 0, 2, 3$

$f(-1) = 0, f(0) = 7, f(2) = -9, f(3) = 88$

Global minimum $f(2) = -9$;

global maximum $f(3) = 88$

10. $f'(x) = 3(x-1)^2(x+2)^2 + 2(x-1)^3(x+2)$

$= (x-1)^2(x+2)(5x+4)$; $f'(x) = 0$ when

$x = -2, -\frac{4}{5}, 1$

Critical points: $-2, -\frac{4}{5}, 1, 2$

$f(-2) = 0, f\left(-\frac{4}{5}\right) = -\frac{26,244}{3125} \approx -8.40,$

$f(1) = 0, f(2) = 16$

Global minimum $f\left(-\frac{4}{5}\right) \approx -8.40$;

global maximum $f(2) = 16$

11. $f'(\theta) = \cos \theta$; $f'(\theta) = 0$ when $\theta = \frac{\pi}{2}$ in

$\left[\frac{\pi}{4}, \frac{4\pi}{3}\right]$

Critical points: $\frac{\pi}{4}, \frac{\pi}{2}, \frac{4\pi}{3}$

$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.71, f\left(\frac{\pi}{2}\right) = 1,$

$f\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \approx -0.87$

Global minimum $f\left(\frac{4\pi}{3}\right) \approx -0.87$;

global maximum $f\left(\frac{\pi}{2}\right) = 1$

12. $f'(\theta) = 2\sin \theta \cos \theta - \cos \theta = \cos \theta(2\sin \theta - 1)$;

$f'(\theta) = 0$ when $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ in $[0, \pi]$

Critical points: $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

$f(0) = 0, f\left(\frac{\pi}{6}\right) = -\frac{1}{4}, f\left(\frac{\pi}{2}\right) = 0,$

$$f\left(\frac{5\pi}{6}\right) = -\frac{1}{4}, f(\pi) = 0$$

$$\text{Global minimum } f\left(\frac{\pi}{6}\right) = -\frac{1}{4} \text{ or } f\left(\frac{5\pi}{6}\right) = -\frac{1}{4};$$

$$\text{global maximum } f(0) = 0, f\left(\frac{\pi}{2}\right) = 0, \text{ or}$$

$$f(\pi) = 0$$

$$13. f'(x) = 3 - 2x; f'(x) > 0 \text{ when } x < \frac{3}{2}.$$

$$f''(x) = -2; f''(x) \text{ is always negative.}$$

$$f(x) \text{ is increasing on } \left(-\infty, \frac{3}{2}\right) \text{ and concave down on } (-\infty, \infty).$$

$$14. f'(x) = 9x^8; f'(x) > 0 \text{ for all } x \neq 0.$$

$$f''(x) = 72x^7; f''(x) < 0 \text{ when } x < 0.$$

$$f(x) \text{ is increasing on } (-\infty, \infty) \text{ and concave down on } (-\infty, 0).$$

$$15. f'(x) = 3x^2 - 3 = 3(x^2 - 1); f'(x) > 0 \text{ when } x < -1 \text{ or } x > 1.$$

$$f''(x) = 6x; f''(x) < 0 \text{ when } x < 0.$$

$$f(x) \text{ is increasing on } (-\infty, -1] \cup [1, \infty) \text{ and concave down on } (-\infty, 0).$$

$$16. f'(x) = -6x^2 - 6x + 12 = -6(x+2)(x-1);$$

$$f'(x) > 0 \text{ when } -2 < x < 1.$$

$$f''(x) = -12x - 6 = -6(2x+1); f''(x) < 0 \text{ when}$$

$$x > -\frac{1}{2}.$$

$$f(x) \text{ is increasing on } [-2, 1] \text{ and concave down on } \left(-\frac{1}{2}, \infty\right).$$

$$17. f'(x) = 4x^3 - 20x^4 = 4x^3(1-5x); f'(x) > 0$$

$$\text{when } 0 < x < \frac{1}{5}.$$

$$f''(x) = 12x^2 - 80x^3 = 4x^2(3-20x); f''(x) < 0$$

$$\text{when } x > \frac{3}{20}.$$

$$f(x) \text{ is increasing on } \left[0, \frac{1}{5}\right] \text{ and concave down on}$$

$$\left(\frac{3}{20}, \infty\right).$$

$$18. f'(x) = 3x^2 - 6x^4 = 3x^2(1-2x^2); f'(x) > 0$$

$$\text{when } -\frac{1}{\sqrt{2}} < x < 0 \text{ and } 0 < x < \frac{1}{\sqrt{2}}.$$

$$f''(x) = 6x - 24x^3 = 6x(1-4x^2); f''(x) < 0 \text{ when}$$

$$-\frac{1}{2} < x < 0 \text{ or } x > \frac{1}{2}.$$

$$f(x) \text{ is increasing on } \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \text{ and concave}$$

$$\text{down on } \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right).$$

$$19. f'(x) = 3x^2 - 4x^3 = x^2(3-4x); f'(x) > 0 \text{ when}$$

$$x < \frac{3}{4}.$$

$$f''(x) = 6x - 12x^2 = 6x(1-2x); f''(x) < 0 \text{ when}$$

$$x < 0 \text{ or } x > \frac{1}{2}.$$

$$f(x) \text{ is increasing on } \left(-\infty, \frac{3}{4}\right] \text{ and concave down}$$

$$\text{on } (-\infty, 0) \cup \left(\frac{1}{2}, \infty\right).$$

$$20. g'(t) = 3t^2 - \frac{1}{t^2}; g'(t) > 0 \text{ when } 3t^2 > \frac{1}{t^2} \text{ or}$$

$$t^4 > \frac{1}{3}, \text{ so } t < -\frac{1}{3^{1/4}} \text{ or } t > \frac{1}{3^{1/4}}.$$

$$g'(t) \text{ is increasing on } \left(-\infty, -\frac{1}{3^{1/4}}\right] \cup \left[\frac{1}{3^{1/4}}, \infty\right)$$

$$\text{and decreasing on } \left[-\frac{1}{3^{1/4}}, 0\right) \cup \left(0, \frac{1}{3^{1/4}}\right].$$

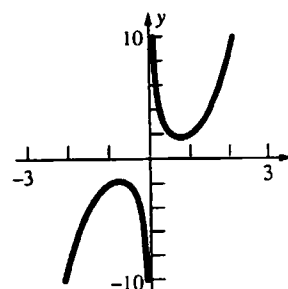
$$\text{Local minimum } g\left(\frac{1}{3^{1/4}}\right) = \frac{1}{3^{3/4}} + 3^{1/4} \approx 1.75;$$

$$\text{local maximum}$$

$$g\left(-\frac{1}{3^{1/4}}\right) = -\frac{1}{3^{3/4}} - 3^{1/4} \approx -1.75$$

$$g''(t) = 6t + \frac{2}{t^3}; g''(t) > 0 \text{ when } t > 0. g(t) \text{ has no}$$

$$\text{inflection point since } g(0) \text{ does not exist.}$$



21. $f'(x) = 2x(x-4) + x^2 = 3x^2 - 8x = x(3x-8)$;

$f'(x) > 0$ when $x < 0$ or $x > \frac{8}{3}$

$f(x)$ is increasing on $(-\infty, 0] \cup [\frac{8}{3}, \infty)$ and

decreasing on $[0, \frac{8}{3}]$

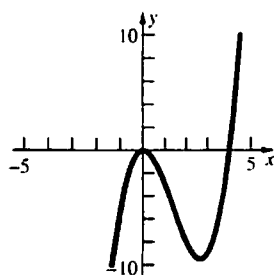
Local minimum $f(\frac{8}{3}) = -\frac{256}{27} \approx -9.48$;

local maximum $f(0) = 0$

$f''(x) = 6x - 8$; $f''(x) > 0$ when $x > \frac{4}{3}$.

$f(x)$ is concave up on $(\frac{4}{3}, \infty)$ and concave down

on $(-\infty, \frac{4}{3})$; inflection point $(\frac{4}{3}, -\frac{128}{27})$



22. $f'(x) = -\frac{8x}{(x^2+1)^2}$; $f'(x) = 0$ when $x = 0$.

$f''(x) = \frac{8(3x^2-1)}{(x^2+1)^3}$; $f''(0) = -8$, so $f(0) = 6$ is a

local maximum. $f'(x) > 0$ for $x < 0$ and

$f'(x) < 0$ for $x > 0$ so

$f(0) = 6$ is a global maximum value. $f(x)$ has no minimum value.

23. $f'(x) = 4x^3 - 2$; $f'(x) = 0$ when $x = \frac{1}{\sqrt[3]{2}}$.

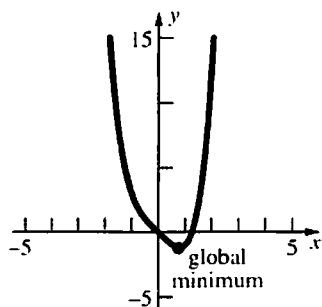
$f''(x) = 12x^2$; $f''(x) = 0$ when $x = 0$.

$f''(\frac{1}{\sqrt[3]{2}}) = \frac{12}{2^{2/3}} > 0$, so

$f(\frac{1}{\sqrt[3]{2}}) = \frac{1}{2^{4/3}} - \frac{2}{2^{1/3}} = -\frac{3}{2^{4/3}}$ is a global minimum.

$f''(x) > 0$ for all $x \neq 0$; no inflection points

No horizontal or vertical asymptotes



24. $f'(x) = 2(x^2-1)(2x) = 4x(x^2-1) = 4x^3 - 4x$;

$f'(x) = 0$ when $x = -1, 0, 1$.

$f''(x) = 12x^2 - 4 = 4(3x^2 - 1)$; $f''(x) = 0$ when

$x = \pm \frac{1}{\sqrt{3}}$.

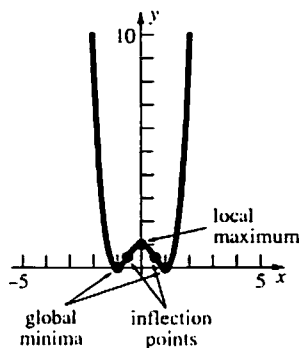
$f''(-1) = 8$, $f''(0) = -4$, $f''(1) = 8$

Global minima $f(-1) = 0$, $f(1) = 0$;

local maximum $f(0) = 1$

Inflection points $(\pm \frac{1}{\sqrt{3}}, \frac{4}{9})$

No horizontal or vertical asymptotes



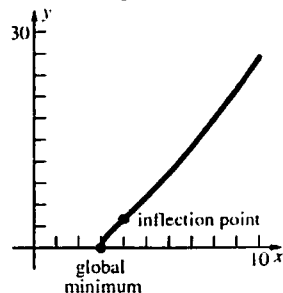
25. $f'(x) = \frac{3x-6}{2\sqrt{x-3}}$; $f'(x) = 0$ when $x = 2$, but $x = 2$

is not in the domain of $f(x)$. $f'(x)$ does not exist when $x = 3$.

$f''(x) = \frac{3(x-4)}{4(x-3)^{3/2}}$; $f''(x) = 0$ when $x = 4$.

Global minimum $f(3) = 0$; no local maxima

Inflection point $(4, 4)$



26. $f'(x) = -\frac{1}{(x-3)^2}$; $f'(x) < 0$ for all $x \neq 3$.

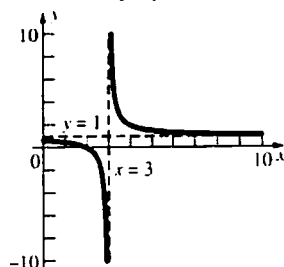
$f''(x) = \frac{2}{(x-3)^3}$; $f''(x) > 0$ when $x > 3$.

No local minima or maxima
No inflection points

$$\lim_{x \rightarrow \infty} \frac{x-2}{x-3} = \lim_{x \rightarrow \infty} \frac{1-\frac{2}{x}}{1-\frac{3}{x}} = 1$$

Horizontal asymptote $y = 1$

Vertical asymptote $x = 3$



27. $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$; $f'(x) = 0$ when $x = 0, 1$.

$f''(x) = 36x^2 - 24x = 12x(3x-2)$; $f''(x) = 0$

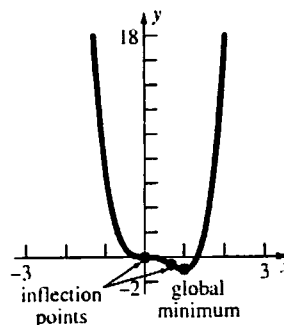
when $x = 0, \frac{2}{3}$.

$f''(1) = 12$, so $f(1) = -1$ is a minimum.

Global minimum $f(1) = -1$; no local maxima

Inflection points $(0, 0), (\frac{2}{3}, -\frac{16}{27})$

No horizontal or vertical asymptotes.



28. $f'(x) = 1 + \frac{1}{x^2}$; $f'(x) > 0$ for all $x \neq 0$.

$f''(x) = -\frac{2}{x^3}$; $f''(x) > 0$ when $x < 0$ and

$f''(x) < 0$ when $x > 0$.

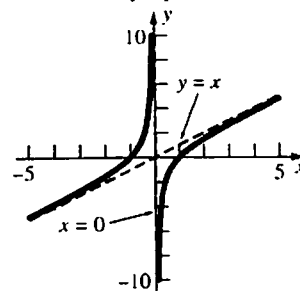
No local minima or maxima
No inflection points

$f(x) = x - \frac{1}{x}$, so

$$\lim_{x \rightarrow \infty} [f(x) - x] = \lim_{x \rightarrow \infty} \left(-\frac{1}{x}\right) = 0 \text{ and } y = x \text{ is an}$$

oblique asymptote.

Vertical asymptote $x = 0$



29. $f'(x) = 3 + \frac{1}{x^2}$; $f'(x) > 0$ for all $x \neq 0$.

$f''(x) = -\frac{2}{x^3}$; $f''(x) > 0$ when $x < 0$ and

$f''(x) < 0$ when $x > 0$

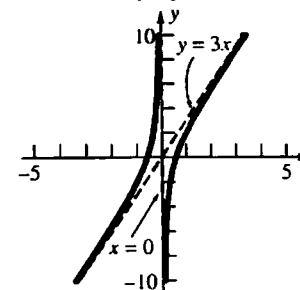
No local minima or maxima
No inflection points

$f(x) = 3x - \frac{1}{x}$, so

$$\lim_{x \rightarrow \infty} [f(x) - 3x] = \lim_{x \rightarrow \infty} \left(-\frac{1}{x}\right) = 0 \text{ and } y = 3x \text{ is an}$$

oblique asymptote.

Vertical asymptote $x = 0$



30. $f'(x) = -\frac{4}{(x+1)^3}$; $f'(x) > 0$ when $x < -1$ and

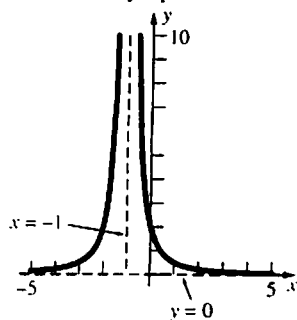
$f'(x) < 0$ when $x > -1$.

$f''(x) = \frac{12}{(x+1)^4}$; $f''(x) > 0$ for all $x \neq -1$.

No local minima or maxima
No inflection points

$\lim_{x \rightarrow \infty} f(x) = 0, \lim_{x \rightarrow -\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote.

Vertical asymptote $x = -1$



31. $f'(x) = -\sin x - \cos x$; $f'(x) = 0$ when

$$x = -\frac{\pi}{4}, \frac{3\pi}{4}$$

$f''(x) = -\cos x + \sin x$; $f''(x) = 0$ when

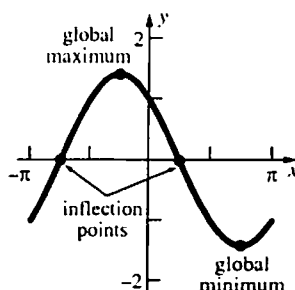
$$x = -\frac{3\pi}{4}, \frac{\pi}{4}$$

$$f''\left(-\frac{\pi}{4}\right) = -\sqrt{2}, f''\left(\frac{3\pi}{4}\right) = \sqrt{2}$$

Global minimum $f\left(\frac{3\pi}{4}\right) = -\sqrt{2}$;

global maximum $f\left(-\frac{\pi}{4}\right) = \sqrt{2}$

Inflection points $\left(-\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right)$



32. $f'(x) = \cos x - \sec^2 x$; $f'(x) = 0$ when $x = 0$

$$f''(x) = -\sin x - 2\sec^2 x \tan x$$

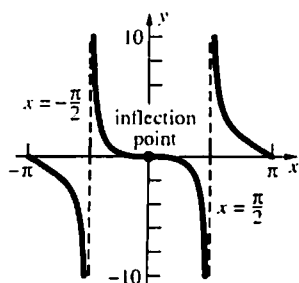
$$= -\sin x(1 + 2\sec^3 x)$$

$f''(x) = 0$ when $x = 0$

No local minima or maxima

Inflection point $f(0) = 0$

Vertical asymptotes $x = -\frac{\pi}{2}, \frac{\pi}{2}$



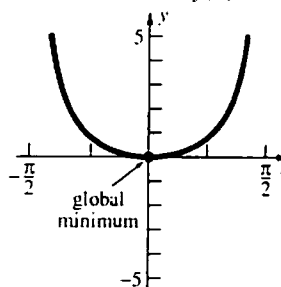
33. $f'(x) = x \sec^2 x + \tan x$; $f'(x) = 0$ when $x = 0$

$$f''(x) = 2 \sec^2 x(1 + x \tan x); f''(x) \text{ is never 0 on}$$

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

$$f''(0) = 2$$

Global minimum $f(0) = 0$

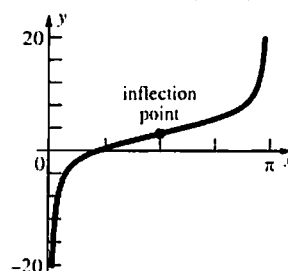


34. $f'(x) = 2 + \csc^2 x$; $f'(x) > 0$ on $(0, \pi)$

$$f''(x) = -2 \cot x \csc^2 x; f''(x) = 0 \text{ when}$$

$$x = \frac{\pi}{2}; f''(x) > 0 \text{ on } \left(\frac{\pi}{2}, \pi\right)$$

Inflection point $\left(\frac{\pi}{2}, \pi\right)$



35. $f'(x) = \cos x - 2 \cos x \sin x = \cos x(1 - 2 \sin x)$;

$$f'(x) = 0 \text{ when } x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

$$f''(x) = -\sin x + 2 \sin^2 x - 2 \cos^2 x; f''(x) = 0$$

$$\text{when } x \approx -2.51, -0.63, 1.00, 2.14$$

$$f''\left(-\frac{\pi}{2}\right) = 3, f''\left(\frac{\pi}{6}\right) = -\frac{3}{2}, f''\left(\frac{\pi}{2}\right) = 1,$$

$$f''\left(\frac{5\pi}{6}\right) = -\frac{3}{2}$$

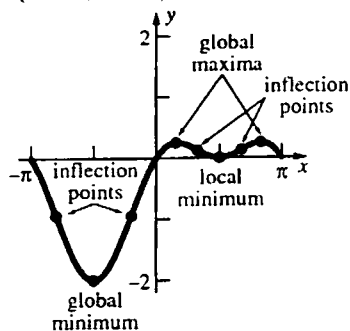
Global minimum $f\left(-\frac{\pi}{2}\right) = -2,$

local minimum $f\left(\frac{\pi}{2}\right) = 0;$

global maxima $f\left(\frac{\pi}{6}\right) = \frac{1}{4}, f\left(\frac{5\pi}{6}\right) = \frac{1}{4}$

Inflection points $(-2.51, -0.94),$

$(-0.63, -0.94), (1.00, 0.13), (2.14, 0.13)$



36. $f'(x) = -2\sin x - 2\cos x$; $f'(x) = 0$ when

$$x = -\frac{\pi}{4}, \frac{3\pi}{4}.$$

$f''(x) = -2\cos x + 2\sin x$; $f''(x) = 0$ when

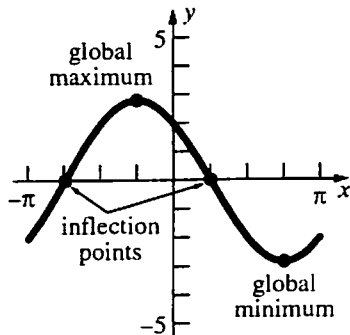
$$x = -\frac{3\pi}{4}, \frac{\pi}{4}.$$

$$f''\left(-\frac{\pi}{4}\right) = -2\sqrt{2}, f''\left(\frac{3\pi}{4}\right) = 2\sqrt{2}$$

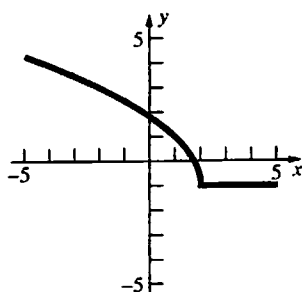
Global minimum $f\left(\frac{3\pi}{4}\right) = -2\sqrt{2}$;

global maximum $f\left(-\frac{\pi}{4}\right) = 2\sqrt{2}$

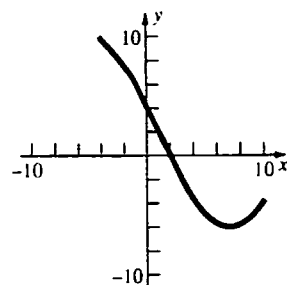
Inflection points $\left(-\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right)$



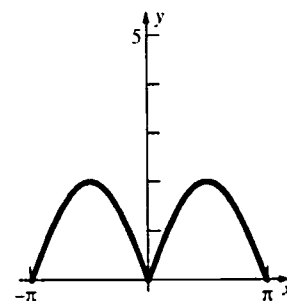
37.



38.



39.



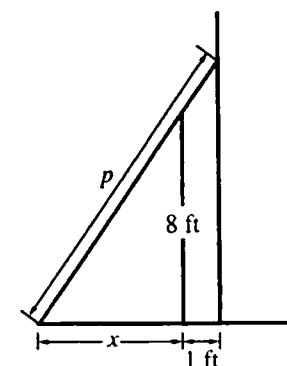
40. Let x be the length of a turned up side and let l be the (fixed) length of the sheet of metal.

$$V = x(16 - 2x)l = 16xl - 2x^2l$$

$$\frac{dV}{dx} = 16l - 4xl; V' = 0 \text{ when } x = 4$$

$$\frac{d^2V}{dx^2} = -4l; 4 \text{ inches should be turned up for each side.}$$

41. Let p be the length of the plank and let x be the distance from the fence to where the plank touches the ground. See the figure below.



By properties of similar triangles,

$$\frac{p}{x+1} = \frac{\sqrt{x^2 + 64}}{x}$$

$$p = \left(1 + \frac{1}{x}\right)\sqrt{x^2 + 64}$$

Minimize p :

$$\frac{dp}{dx} = -\frac{1}{x^2}\sqrt{x^2 + 64} + \left(1 + \frac{1}{x}\right)\frac{x}{\sqrt{x^2 + 64}}$$

$$\begin{aligned}
 &= \frac{1}{x^2 \sqrt{x^2 + 64}} \left(-(x^2 + 64) + \left(1 + \frac{1}{x}\right)x^3 \right) \\
 &= \frac{x^3 - 64}{x^2 \sqrt{x^2 + 64}} \\
 &\frac{x^3 - 64}{x^2 \sqrt{x^2 + 64}} = 0; x = 4 \\
 &\frac{dp}{dx} < 0 \text{ if } x < 4, \frac{dp}{dx} > 0 \text{ if } x > 4 \\
 &\text{When } x = 4, p = \left(1 + \frac{1}{4}\right)\sqrt{16 + 64} \approx 11.18 \text{ ft.}
 \end{aligned}$$

42. Let x be the width and y the height of a page.
 $A = xy$. Because of the margins,
 $(y - 4)(x - 3) = 27$ or $y = \frac{27}{x - 3} + 4$

$$\begin{aligned}
 A &= \frac{27x}{x - 3} + 4x; \\
 \frac{dA}{dx} &= \frac{(x - 3)(27) - 27x}{(x - 3)^2} + 4 = -\frac{81}{(x - 3)^2} + 4 \\
 \frac{dA}{dx} &= 0 \text{ when } x = -\frac{3}{2}, \frac{15}{2} \\
 \frac{d^2A}{dx^2} &= \frac{162}{(x - 3)^3}; \frac{d^2A}{dx^2} > 0 \text{ when } x = \frac{15}{2} \\
 x &= \frac{15}{2}; y = 10
 \end{aligned}$$

43. $\frac{1}{2}\pi r^2 h = 128\pi$

$$h = \frac{256}{r^2}$$

Let S be the surface area of the trough.

$$S = \pi r^2 + \pi r h = \pi r^2 + \frac{256\pi}{r}$$

$$\frac{dS}{dr} = 2\pi r - \frac{256\pi}{r^2}$$

$$2\pi r - \frac{256\pi}{r^2} = 0; r^3 = 128, r = 4\sqrt[3]{2}$$

Since $\frac{d^2S}{dr^2} > 0$ when $r = 4\sqrt[3]{2}$, $r = 4\sqrt[3]{2}$

minimizes S .

$$h = \frac{256}{(4\sqrt[3]{2})^2} = 8\sqrt[3]{2}$$

44. $f'(x) = \begin{cases} \frac{x}{2} + \frac{3}{2} & \text{if } -2 < x < 0 \\ -\frac{x+2}{3} & \text{if } 0 < x < 2 \end{cases}$

$$\frac{x}{2} + \frac{3}{2} = 0; x = -3, \text{ which is not in the domain.}$$

$$-\frac{x+2}{3} = 0; x = -2, \text{ which is not in the domain.}$$

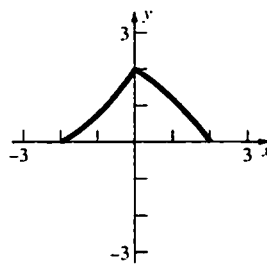
Critical points: $x = -2, 0, 2$

$$f(-2) = 0, f(0) = 2, f(2) = 0$$

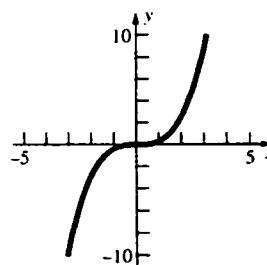
Minima $f(-2) = 0, f(2) = 0$, maximum $f(0) = 2$.

$$f''(x) = \begin{cases} \frac{1}{2} & \text{if } -2 < x < 0 \\ -\frac{1}{3} & \text{if } 0 < x < 2 \end{cases}$$

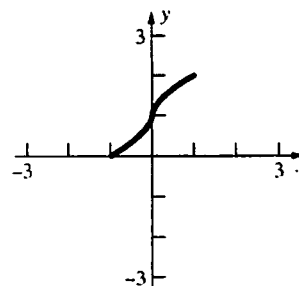
Concave up on $(-2, 0)$, concave down on $(0, 2)$



45. a. $f'(x) = x^2$
 $\frac{f(3) - f(-3)}{3 - (-3)} = \frac{9 + 9}{6} = 3$
 $c^2 = 3; c = -\sqrt{3}, \sqrt{3}$



- b. The Mean Value Theorem does not apply because $F'(0)$ does not exist.

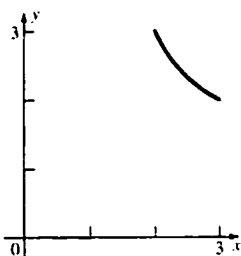


$$c. \quad g'(x) = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$\frac{g(3)-g(2)}{3-2} = \frac{2-3}{1} = -1$$

$$\frac{-2}{(c-1)^2} = -1; c = 1 \pm \sqrt{2}$$

Only $c = 1 + \sqrt{2}$ is in the interval $(2, 3)$.



$$46. \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 24x - 3$$

$$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24; 12(x^2 - 3x + 2) = 0 \text{ when}$$

$$x = 1, 2$$

Inflection points: $x = 1, y = 5$

and $x = 2, y = 11$

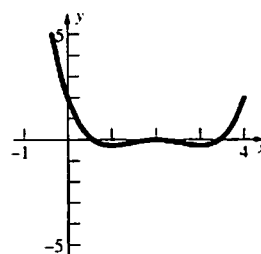
$$\text{Slope at } x = 1: \left. \frac{dy}{dx} \right|_{x=1} = 7$$

Tangent line: $y - 5 = 7(x - 1); y = 7x - 2$

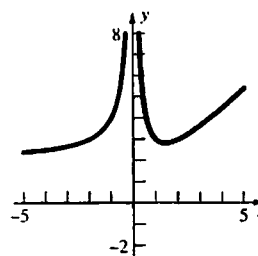
$$\text{Slope at } x = 2: \left. \frac{dy}{dx} \right|_{x=2} = 5$$

Tangent line: $y - 11 = 5(x - 2); y = 5x + 1$

47.



48.



4.9 Additional Problem Set

$$1. \quad a. \quad ab \leq \left(\frac{a+b}{2} \right)^2 = \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{a^2}{4} + \frac{1}{2}ab + \frac{b^2}{4}$$

This is true if

$$0 \leq \frac{a^2}{4} - \frac{1}{2}ab + \frac{b^2}{4} = \left(\frac{a}{2} - \frac{b}{2} \right)^2 = \left(\frac{a-b}{2} \right)^2$$

Since a square can never be negative, this is always true.

$$b. \quad F(b) = \frac{a^2 + 2ab + b^2}{4b}$$

As $b \rightarrow 0^+$, $a^2 + 2ab + b^2 \rightarrow a^2$ while

$4b \rightarrow 0^+$, thus $\lim_{b \rightarrow 0^+} F(b) = \infty$ which is not

close to a .

$$\lim_{b \rightarrow \infty} \frac{a^2 + 2ab + b^2}{4b} = \lim_{b \rightarrow \infty} \frac{\frac{a^2}{b} + 2a + b}{4} = \infty,$$

so when b is very large,

$F(b)$ is not close to a .

$$F'(b) = \frac{2(a+b)(4b) - 4(a+b)^2}{16b^2}$$

$$= \frac{4b^2 - 4a^2}{16b^2} = \frac{b^2 - a^2}{4b^2};$$

$F'(b) = 0$ when $b^2 = a^2$ or $b = a$ since a and b are both positive.

$$F(a) = \frac{(a+a)^2}{4a} = \frac{4a^2}{4a} = a$$

Thus $a \leq \frac{(a+b)^2}{4b}$ for all $b > 0$ or

$$ab \leq \frac{(a+b)^2}{4} \text{ which leads to } \sqrt{ab} \leq \frac{a+b}{2}.$$

$$c. \quad \text{Let } F(b) = \frac{1}{b} \left(\frac{a+b+c}{3} \right)^3 = \frac{(a+b+c)^3}{27b}$$

$$F'(b) = \frac{3(a+b+c)^2(27b) - 27(a+b+c)^3}{27^2 b^2}$$

$$= \frac{(a+b+c)^2[3b-(a+b+c)]}{27b^2}$$

$$= \frac{(a+b+c)^2(2b-a-c)}{27b^2};$$

$$F'(b) = 0 \text{ when } b = \frac{a+c}{2}.$$

$$F\left(\frac{a+c}{2}\right) = \frac{2}{a+c} \cdot \left(\frac{a+c}{3} + \frac{a+c}{6}\right)^3$$

$$= \frac{2}{a+c} \left(\frac{3(a+c)}{6}\right)^3 = \frac{2}{a+c} \left(\frac{a+c}{2}\right)^3 = \left(\frac{a+c}{2}\right)^2$$

$$\text{Thus } \left(\frac{a+c}{2}\right)^2 \leq \frac{1}{b} \left(\frac{a+b+c}{3}\right)^3 \text{ for all } b > 0.$$

$$\text{From Problem 1b, } ac \leq \left(\frac{a+c}{2}\right)^2, \text{ thus}$$

$$ac \leq \frac{1}{b} \left(\frac{a+b+c}{3}\right)^3 \text{ or } abc \leq \left(\frac{a+b+c}{3}\right)^3$$

which gives the desired result

$$(abc)^{1/3} \leq \frac{a+b+c}{3}.$$

2. Let $a = lw$, $b = lh$, and $c = hw$. then

$S = 2(a+b+c)$ while $V^2 = abc$. By problem 1c,

$$(abc)^{1/3} \leq \frac{a+b+c}{3} \text{ so } (V^2)^{1/3} \leq \frac{2(a+b+c)}{2 \cdot 3} = \frac{S}{6}.$$

In problem 1c, the minimum occurs, hence equality

holds, when $b = \frac{a+c}{2}$. In the result used from

Problem 1b, equality holds when $c = a$, thus

$$b = \frac{a+a}{2} = a, \text{ so } a = b = c. \text{ For the boxes, this}$$

means $l = w = h$, so the box is a cube.

3. (Proof by induction.)

Suppose that the result holds for all m with $1 \leq m \leq N$. Now let $n = N+1$, and

$$F(x_1) = \frac{1}{x_1} \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right)^n = \frac{(x_1 + x_2 + \dots + x_n)^n}{n^n x_1}$$

$$F'(x_1) = \frac{n(x_1 + x_2 + \dots + x_n)^{n-1}(n^n x_1) - n^n (x_1 + x_2 + \dots + x_n)^n}{n^n x_1^2} = \frac{(x_1 + x_2 + \dots + x_n)^{n-1} [nx_1 - (x_1 + x_2 + \dots + x_n)]}{x_1^2}$$

$$= \frac{(x_1 + x_2 + \dots + x_n)^{n-1} [(n-1)x_1 - x_2 - \dots - x_n]}{x_1^2}$$

$$F'(x_1) = 0 \text{ when } x_1 = \frac{x_2 + x_3 + \dots + x_n}{n-1}$$

$$F\left(\frac{x_2 + x_3 + \dots + x_n}{n-1}\right) = \frac{n-1}{x_2 + x_3 + \dots + x_n} \left(\frac{x_2 + x_3 + \dots + x_n}{n(n-1)} + \frac{x_2 + x_3 + \dots + x_n}{n} \right)^n$$

$$= \frac{n-1}{x_2 + x_3 + \dots + x_n} \left(\frac{x_2 + x_3 + \dots + x_n}{n-1} \right)^n = \left(\frac{x_2 + x_3 + \dots + x_n}{n-1} \right)^{n-1}$$

$$\text{Thus, } \left(\frac{x_2 + x_3 + \dots + x_n}{n-1} \right)^{n-1} \leq \frac{1}{x_1} \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right)^n \text{ for all } x_1 > 0.$$

$$\text{But, by assumption, } (x_2 x_3 \dots x_n)^{\frac{1}{n-1}} \leq \frac{x_2 + x_3 + \dots + x_n}{n-1} \text{ with equality holding when } x_2 = x_3 = \dots = x_n.$$

$$\text{Thus, } x_2 x_3 \dots x_n \leq \left(\frac{x_2 + x_3 + \dots + x_n}{n-1} \right)^{n-1} \leq \frac{1}{x_1} \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right)^n, \text{ and combining the first and last terms}$$

$$(x_1 x_2 \dots x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \dots + x_n}{n} \text{ with equality holding when } x_2 = x_3 = \dots = x_n \text{ and}$$

$$x_1 = \frac{x_2 + x_3 + \dots + x_n}{n-1} = \frac{(n-1)x_2}{n-1} = x_2 \text{ or } x_1 = x_2 = \dots = x_n.$$

The case where $n=1$ is clear, $n=2$ and $n=3$ were proved in Problems 1b and 1c, so the relation holds for any set of positive x_j .

4. a. Square both sides of the given inequality to

$$\text{get } \left(\frac{x+y}{2}\right)^2 \leq \frac{x^2+y^2}{2}$$

$$\frac{x^2}{4} + \frac{xy}{2} + \frac{y^2}{4} \leq \frac{x^2}{2} + \frac{y^2}{2}$$

$$0 \leq \frac{x^2}{4} - \frac{xy}{2} + \frac{y^2}{4} = \left(\frac{x-y}{2}\right)^2$$

which is always true since squares are never negative.

- b. From Problem 4a, $\frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}}$ so it

remains to show that $\frac{2xy}{x+y} \leq \frac{x+y}{2}$. This is

$$\text{equivalent to } 4xy \leq (x+y)^2 = x^2 + 2xy + y^2$$

or $0 \leq x^2 - 2xy + y^2 = (x-y)^2$ which is true since squares are never negative.

$$5. P_{ns}(p) = \frac{n!}{s!(n-s)!} p^s (1-p)^{n-s}$$

$$P'_{ns}(p) = \frac{n!}{s!(n-s)!} [sp^{s-1}(1-p)^{n-s} - (n-s)p^s(1-p)^{n-s-1}]$$

$$= \frac{n!}{s!(n-s)!} p^{s-1}(1-p)^{n-s-1} [s(1-p) - (n-s)p]$$

$$= \frac{n!}{s!(n-s)!} p^{s-1}(1-p)^{n-s-1} (s-np)$$

$$P'_{ns}(p) = 0 \text{ when } p = 0, \frac{s}{n}, 1$$

$$P_{ns}(0) = P_{ns}(1) = 0$$

$$P_{ns}\left(\frac{s}{n}\right) = \frac{n!}{s!(n-s)!} \left(\frac{s}{n}\right)^s \left(1 - \frac{s}{n}\right)^{n-s} > 0,$$

thus P_{ns} is a maximum when $p = \frac{s}{n}$.

6. The ellipse has equation

$$y = \pm \sqrt{b^2 - \frac{b^2 x^2}{a^2}} = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

Let $(x, y) = \left(x, \frac{b}{a} \sqrt{a^2 - x^2}\right)$ be the upper right-

hand corner of the rectangle (use a and b positive). Then the dimensions of the rectangle

are $2x$ by $\frac{2b}{a} \sqrt{a^2 - x^2}$ and the area is

$$A(x) = \frac{4bx}{a} \sqrt{a^2 - x^2}.$$

$$A'(x) = \frac{4b}{a} \sqrt{a^2 - x^2} - \frac{4bx^2}{a\sqrt{a^2 - x^2}} = \frac{4b(a^2 - 2x^2)}{a\sqrt{a^2 - x^2}};$$

$A'(x) = 0$ when $x = \frac{a}{\sqrt{2}}$, so the corner is at

$\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$. The corners of the rectangle are at

$$\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right), \left(-\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right), \left(-\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}\right),$$

$$\left(\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}\right).$$

7. If the end of the cylinder has radius r and h is the height of the cylinder, the surface area is

$$A = 2\pi r^2 + 2\pi rh \text{ so } h = \frac{A}{2\pi r} - r.$$

The volume is

$$V = \pi r^2 h = \pi r^2 \left(\frac{A}{2\pi r} - r\right) = \frac{Ar}{2} - \pi r^3.$$

$$V'(r) = \frac{A}{2} - 3\pi r^2; V'(r) = 0 \text{ when } r = \sqrt{\frac{A}{6\pi}},$$

$V''(r) = -6\pi r$, so the volume is maximum when

$$r = \sqrt{\frac{A}{6\pi}}.$$

$$h = \frac{A}{2\pi r} - r = 2\sqrt{\frac{A}{6\pi}} = 2r$$

8. If the rectangle has length l and width w , the diagonal is $d = \sqrt{l^2 + w^2}$, so $l = \sqrt{d^2 - w^2}$. The area is $A = lw = w\sqrt{d^2 - w^2}$.

$$A'(w) = \sqrt{d^2 - w^2} - \frac{w^2}{\sqrt{d^2 - w^2}} = \frac{d^2 - 2w^2}{\sqrt{d^2 - w^2}};$$

$A'(w) = 0$ when $w = \frac{d}{\sqrt{2}}$ and so

$$l = \sqrt{d^2 - \frac{d^2}{2}} = \frac{d}{\sqrt{2}}. A'(w) > 0 \text{ on } \left(0, \frac{d}{\sqrt{2}}\right) \text{ and}$$

$A'(w) < 0$ on $\left(\frac{d}{\sqrt{2}}, d\right)$. Maximum area is for a square.

5.1 Concepts Review

$$1. \quad rx^{r-1}; \frac{x^{r+1}}{r+1} + C, r \neq -1$$

$$2. \quad r[f(x)]^{r-1} f'(x); [f(x)]^r f'(x)$$

$$3. \quad u = x^4 + 3x^2 + 1, du = (4x^3 + 6x)dx \\ \int (x^4 + 3x^2 + 1)^8 (4x^3 + 6x)dx = \int u^8 du \\ = \frac{u^9}{9} + C = \frac{(x^4 + 3x^2 + 1)^9}{9} + C$$

$$4. \quad c_1 \int f(x)dx + c_2 \int g(x)dx$$

Problem Set 5.1

$$1. \quad \int 5dx = 5x + C$$

$$2. \quad \int (x-4)dx = \int xdx - 4 \int 1dx \\ = \frac{x^2}{2} - 4x + C$$

$$3. \quad \int (x^2 + \pi)dx = \int x^2 dx + \pi \int 1dx = \frac{x^3}{3} + \pi x + C$$

$$4. \quad \int (3x^2 + \sqrt{3})dx = 3 \int x^2 dx + \sqrt{3} \int 1dx \\ = 3 \frac{x^3}{3} + \sqrt{3}x + C = x^3 + \sqrt{3}x + C$$

$$5. \quad \int x^{5/4} dx = \frac{x^{9/4}}{\frac{9}{4}} + C = \frac{4}{9} x^{9/4} + C$$

$$6. \quad \int 3x^{2/3} dx = 3 \int x^{2/3} dx = 3 \left(\frac{x^{5/3}}{\frac{5}{3}} + C_1 \right) \\ = \frac{9}{5} x^{5/3} + C$$

$$7. \quad \int \frac{1}{\sqrt[3]{x^2}} dx = \int x^{-2/3} dx = 3x^{1/3} + C = 3\sqrt[3]{x} + C$$

$$8. \quad \int 7x^{-3/4} dx = 7 \int x^{-3/4} dx = 7(4x^{1/4} + C_1) \\ = 28x^{1/4} + C$$

$$9. \quad \int (x^2 - x)dx = \int x^2 dx - \int x dx = \frac{x^3}{3} - \frac{x^2}{2} + C$$

$$10. \quad \int (3x^2 - \pi x)dx = 3 \int x^2 dx - \pi \int x dx \\ = 3 \left(\frac{x^3}{3} + C_1 \right) - \pi \left(\frac{x^2}{2} + C_2 \right) \\ = x^3 - \frac{\pi x^2}{2} + C$$

$$11. \quad \int (4x^5 - x^3)dx = 4 \int x^5 dx - \int x^3 dx \\ = 4 \left(\frac{x^6}{6} + C_1 \right) - \left(\frac{x^4}{4} + C_2 \right) \\ = \frac{2x^6}{3} - \frac{x^4}{4} + C$$

$$12. \quad \int (x^{100} + x^{99})dx = \int x^{100} dx + \int x^{99} dx \\ = \frac{x^{101}}{101} + \frac{x^{100}}{100} + C$$

$$13. \quad \int (27x^7 + 3x^5 - 45x^3 + \sqrt{2}x)dx \\ = 27 \int x^7 dx + 3 \int x^5 dx - 45 \int x^3 dx + \sqrt{2} \int x dx \\ = \frac{27x^8}{8} + \frac{x^6}{2} - \frac{45x^4}{4} + \frac{\sqrt{2}x^2}{2} + C$$

$$14. \quad \int [x^2(x^3 + 5x^2 - 3x + \sqrt{3})]dx \\ = \int (x^5 + 5x^4 - 3x^3 + \sqrt{3}x^2)dx \\ = \int x^5 dx + 5 \int x^4 dx - 3 \int x^3 dx + \sqrt{3} \int x^2 dx \\ = \frac{x^6}{6} + x^5 - \frac{3x^4}{4} + \frac{\sqrt{3}x^3}{3} + C$$

$$\begin{aligned}
 15. \quad \int \left(\frac{3}{x^2} - \frac{2}{x^3} \right) dx &= \int (3x^{-2} - 2x^{-3}) dx \\
 &= 3 \int x^{-2} dx - 2 \int x^{-3} dx \\
 &= \frac{3x^{-1}}{-1} - \frac{2x^{-2}}{-2} + C \\
 &= -\frac{3}{x} + \frac{1}{x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \int \left(\frac{\sqrt{2x}}{x} + \frac{3}{x^5} \right) dx &= \int (\sqrt{2} x^{-1/2} + 3x^{-5}) dx \\
 &= \frac{\sqrt{2} x^{1/2}}{\frac{1}{2}} + \frac{3x^{-4}}{-4} + C \\
 &= 2\sqrt{2x} - \frac{3}{4x^4} + C
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \int \frac{4x^6 + 3x^4}{x^3} dx &= \int (4x^3 + 3x) dx \\
 &= 4 \int x^3 dx + 3 \int x dx \\
 &= x^4 + \frac{3x^2}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \int \frac{x^6 - x}{x^3} dx &= \int (x^3 - x^{-2}) dx \\
 &= \int x^3 dx - \int x^{-2} dx = \frac{x^4}{4} - \frac{x^{-1}}{-1} + C \\
 &= \frac{x^4}{4} + \frac{1}{x} + C
 \end{aligned}$$

$$19. \quad \int (x^2 + x) dx = \int x^2 dx + \int x dx = \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$\begin{aligned}
 20. \quad \int (x^3 + \sqrt{x}) dx &= \int x^3 dx + \int x^{1/2} dx \\
 &= \frac{x^4}{4} + \frac{x^{3/2}}{\frac{3}{2}} + C = \frac{x^4}{4} + \frac{2\sqrt{x^3}}{3} + C
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \text{Let } u = x + 1; \text{ then } du &= dx. \\
 \int (x+1)^2 dx &= \int u^2 du = \frac{u^3}{3} + C = \frac{(x+1)^3}{3} + C
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \int (z + \sqrt{2} z)^2 dz &= \int [(1 + \sqrt{2})z]^2 dz \\
 &= (1 + \sqrt{2})^2 \int z^2 dz = \frac{(1 + \sqrt{2})^2 z^3}{3} + C
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \int \frac{(z^2 + 1)^2}{\sqrt{z}} dz &= \int \frac{z^4 + 2z^2 + 1}{\sqrt{z}} dz \\
 &= \int z^{7/2} dz + 2 \int z^{3/2} dz + \int z^{-1/2} dz \\
 &= \frac{2}{9} z^{9/2} + \frac{4}{5} z^{5/2} + 2z^{1/2} + C
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \int \frac{s(s+1)^2}{\sqrt{s}} ds &= \int \frac{s^3 + 2s^2 + s}{\sqrt{s}} ds \\
 &= \int s^{5/2} ds + 2 \int s^{3/2} ds + \int s^{1/2} ds \\
 &= \frac{2s^{7/2}}{7} + \frac{4s^{5/2}}{5} + \frac{2s^{3/2}}{3} + C
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \int (\sin \theta - \cos \theta) d\theta &= \int \sin \theta d\theta - \int \cos \theta d\theta \\
 &= -\cos \theta - \sin \theta + C
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \int (t^2 - 2 \cos t) dt &= \int t^2 dt - 2 \int \cos t dt \\
 &= \frac{t^3}{3} - 2 \sin t + C
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \text{Let } g(x) &= \sqrt{2}x + 1; \text{ then } g'(x) = \sqrt{2}. \\
 \int (\sqrt{2}x + 1)^3 \sqrt{2} dx &= \int [g(x)]^3 g'(x) dx \\
 &= \frac{[g(x)]^4}{4} + C = \frac{(\sqrt{2}x + 1)^4}{4} + C
 \end{aligned}$$

$$\begin{aligned}
 28. \quad \text{Let } g(x) &= \pi x^3 + 1; \text{ then } g'(x) = 3\pi x^2. \\
 \int (\pi x^3 + 1)^4 3\pi x^2 dx &= \int [g(x)]^4 g'(x) dx \\
 &= \frac{[g(x)]^5}{5} + C = \frac{(\pi x^3 + 1)^5}{5} + C
 \end{aligned}$$

$$\begin{aligned}
 29. \quad \text{Let } u &= 5x^3 + 3x - 8; \text{ then } du = (15x^2 + 3) dx. \\
 \int (5x^2 + 1)(5x^3 + 3x - 8)^6 dx &= \int \frac{1}{3} (15x^2 + 3)(5x^3 + 3x - 8)^6 dx \\
 &= \frac{1}{3} \int u^6 du = \frac{1}{3} \left(\frac{u^7}{7} + C_1 \right) \\
 &= \frac{(5x^3 + 3x - 8)^7}{21} + C
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \text{Let } u &= 5x^3 + 3x - 2; \text{ then } du = (15x^2 + 3) dx. \\
 \int (5x^2 + 1)\sqrt{5x^3 + 3x - 2} dx &= \int \frac{1}{3} (15x^2 + 3)\sqrt{5x^3 + 3x - 2} dx \\
 &= \frac{1}{3} \int u^{1/2} du = \frac{1}{3} \left(\frac{2}{3} u^{3/2} + C_1 \right)
 \end{aligned}$$

$$= \frac{2}{9}(5x^3 + 3x - 2)^{3/2} + C$$

$$= \frac{2}{9}\sqrt{(5x^3 + 3x - 2)^3} + C$$

31. Let $u = 2t^2 - 11$; then $du = 4t \, dt$.

$$\int 3t\sqrt{2t^2 - 11} \, dt = \int \frac{3}{4}(4t)(2t^2 - 11)^{1/3} \, dt$$

$$= \frac{3}{4} \int u^{1/3} \, du = \frac{3}{4} \left(\frac{3}{4} u^{4/3} + C_1 \right)$$

$$= \frac{9}{16} (2t^2 - 11)^{4/3} + C$$

$$= \frac{9}{16} \sqrt[3]{(2t^2 - 11)^4} + C$$

32. Let $u = 2y^2 + 5$; then $du = 4y \, dy$

$$\int \frac{3y}{\sqrt{2y^2 + 5}} \, dy = \int \frac{3}{4}(4y)(2y^2 + 5)^{-1/2} \, dy$$

$$= \frac{3}{4} \int u^{-1/2} \, du = \frac{3}{4} (2u^{1/2} + C_1)$$

$$= \frac{3}{2} \sqrt{2y^2 + 5} + C$$

33. $f'(x) = \int (3x + 1) \, dx = \frac{3}{2}x^2 + x + C_1$

$$f(x) = \int \left(\frac{3}{2}x^2 + x + C_1 \right) \, dx$$

$$= \frac{1}{2}x^3 + \frac{1}{2}x^2 + C_1x + C_2$$

34. $f'(x) = \int (-2x + 3) \, dx = -x^2 + 3x + C_1$

$$f(x) = \int (-x^2 + 3x + C_1) \, dx$$

$$= -\frac{1}{3}x^3 + \frac{3}{2}x^2 + C_1x + C_2$$

35. $f'(x) = \int x^{1/2} \, dx = \frac{2}{3}x^{3/2} + C_1$

$$f(x) = \int \left(\frac{2}{3}x^{3/2} + C_1 \right) \, dx$$

$$= \frac{4}{15}x^{5/2} + C_1x + C_2$$

36. $f'(x) = \int x^{4/3} \, dx = \frac{3}{7}x^{7/3} + C_1$

$$f(x) = \int \left(\frac{3}{7}x^{7/3} + C_1 \right) \, dx = \frac{9}{70}x^{10/3} + C_1x + C_2$$

37. $f''(x) = x + x^{-3}$

$$f'(x) = \int (x + x^{-3}) \, dx = \frac{x^2}{2} - \frac{x^{-2}}{2} + C_1$$

$$f(x) = \int \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2} + C_1 \right) \, dx$$

$$= \frac{1}{6}x^3 + \frac{1}{2}x^{-1} + C_1x + C_2$$

$$= \frac{1}{6}x^3 + \frac{1}{2x} + C_1x + C_2$$

38. $f'(x) = 2 \int (x+1)^{1/3} \, dx = \frac{3}{2}(x+1)^{4/3} + C_1$

$$f(x) = \int \left[\frac{3}{2}(x+1)^{4/3} + C_1 \right] \, dx$$

$$= \frac{9}{14}(x+1)^{7/3} + C_1x + C_2$$

39. The Product Rule for derivatives says

$$\frac{d}{dx}[f(x)g(x) + C] = f(x)g'(x) + f'(x)g(x).$$

Thus,

$$\int [f(x)g'(x) + f'(x)g(x)] \, dx = f(x)g(x) + C.$$

40. The Quotient Rule for derivatives says

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} + C \right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Thus, $\int \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \, dx = \frac{f(x)}{g(x)} + C.$

41. Let $f(x) = x^2$, $g(x) = \sqrt{x-1}$.

$$f'(x) = 2x, \quad g'(x) = \frac{1}{2\sqrt{x-1}}$$

$$\int \left[\frac{x^2}{2\sqrt{x-1}} + 2x\sqrt{x-1} \right] \, dx$$

$$= \int [f(x)g'(x) + f'(x)g(x)] \, dx = f(x)g(x) + C$$

$$= x^2\sqrt{x-1} + C$$

42. Let $f(x) = x^3$, $g(x) = (2x+5)^{-1/2}$.

$$f'(x) = 3x^2, \quad g'(x) = -(2x+5)^{-3/2}$$

$$= -\frac{1}{(2x+5)^{3/2}}$$

$$\int \left[\frac{-x^3}{(2x+5)^{3/2}} + \frac{3x^2}{\sqrt{2x+5}} \right] \, dx$$

$$= \int [f(x)g'(x) + g(x)f'(x)] \, dx$$

$$= f(x)g(x) + C = x^3(2x+5)^{-1/2} + C$$

$$= \frac{x^3}{\sqrt{2x+5}} + C$$

43. $\int f''(x)dx = \int \frac{d}{dx} f'(x)dx = f'(x) + C$

$$f'(x) = \sqrt{x^3+1} + \frac{3x^3}{2\sqrt{x^3+1}} = \frac{5x^3+2}{2\sqrt{x^3+1}} \text{ so}$$

$$\int f''(x)dx = \frac{5x^3+2}{2\sqrt{x^3+1}} + C.$$

44. $\frac{d}{dx} \left(\frac{f(x)}{\sqrt{g(x)}} + C \right)$

$$= \frac{\sqrt{g(x)}f'(x) - f(x)\frac{1}{2}[g(x)]^{-1/2}g'(x)}{g(x)}$$

$$= \frac{2g(x)f'(x) - f(x)g'(x)}{2[g(x)]^{3/2}}$$

Thus,

$$\int \frac{2g(x)f'(x) - f(x)g'(x)}{2[g(x)]^{3/2}} = \frac{f(x)}{\sqrt{g(x)}} + C$$

45. The Product Rule for derivatives says that

$$\frac{d}{dx}[f^m(x)g^n(x) + C]$$

$$= f^m(x)[g^n(x)]' + [f^m(x)]'g^n(x)$$

$$= f^m(x)[ng^{n-1}(x)g'(x)] + [mf^{m-1}(x)f'(x)]g^n(x)$$

$$= f^{m-1}(x)g^{n-1}(x)[nf(x)g'(x) + mg(x)f'(x)].$$

Thus,

$$\int f^{m-1}(x)g^{n-1}(x)[nf(x)g'(x) + mg(x)f'(x)]dx$$

$$= f^m(x)g^n(x) + C.$$

46. Let $u = \sin[(x^2+1)^4]$;

then $du = \cos[(x^2+1)^4]4(x^2+1)^3(2x)dx$.

$$du = 8x \cos[(x^2+1)^4](x^2+1)^3 dx$$

$$\int \sin^3[(x^2+1)^4] \cos[(x^2+1)^4](x^2+1)^3 x dx$$

$$= \int u^3 \cdot \frac{1}{8} du = \frac{1}{8} \int u^3 du = \frac{1}{8} \left(\frac{u^4}{4} + C_1 \right)$$

$$= \frac{\sin^4[(x^2+1)^4]}{32} + C$$

47. If $x \geq 0$, then $|x| = x$ and $\int |x|dx = \frac{1}{2}x^2 + C$.

If $x < 0$, then $|x| = -x$ and $\int |x|dx = -\frac{1}{2}x^2 + C$.

$$\int |x|dx = \begin{cases} \frac{1}{2}x^2 + C & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 + C & \text{if } x < 0 \end{cases}$$

48. Using $\sin^2 \frac{u}{2} = \frac{1 - \cos u}{2}$.

$$\int \sin^2 x dx = \int \frac{1 - \cos 2x}{2} dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

49. Different software may produce different, but equivalent answers. These answers were produced by Mathematica.

a. $\int 6 \sin(3(x-2))dx = -2 \cos(3(x-2)) + C$

b. $\int \sin^3\left(\frac{x}{6}\right)dx = \frac{1}{2} \cos\left(\frac{x}{2}\right) - \frac{9}{2} \cos\left(\frac{x}{6}\right) + C$

c. $\int (x^2 \cos 2x + x \sin 2x)dx = \frac{x^2 \sin 2x}{2} + C$

50. a. $F_1(x) = \int (x \sin x)dx = \sin x - x \cos x + C_1$

$$F_2(x) = \int (\sin x - x \cos x + C_1)dx$$

$$= -2 \cos x - x \sin x + C_1x + C_2$$

$$F_3(x) = \int (-2 \cos x - x \sin x + C_1x + C_2)dx$$

$$= x \cos x - 3 \sin x + \frac{1}{2}C_1x^2 + C_2x + C_3$$

$$F_4(x) = \int (x \cos x - 3 \sin x + \frac{1}{2}C_1x^2 + C_2x + C_3)dx$$

$$= x \sin x + 4 \cos x + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

b. $F_{16}(x) = x \sin x + 16 \cos x + \sum_{n=1}^{16} \frac{C_n x^{16-n}}{(16-n)!}$

5.2 Concepts Review

1. differential equation
2. function

3. separate variables

4. $-32t + v_0; -16t^2 + v_0t + s_0$

Problem Set 5.2

$$1. \frac{dy}{dx} = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} + \frac{x}{y} = \frac{-x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = 0$$

$$2. \frac{dy}{dx} = C$$

$$-x \frac{dy}{dx} + y = -Cx + Cx = 0$$

$$3. \frac{dy}{dx} = C_1 \cos x - C_2 \sin x;$$

$$\frac{d^2 y}{dx^2} = -C_1 \sin x - C_2 \cos x$$

$$\frac{d^2 y}{dx^2} + y$$

$$= (-C_1 \sin x - C_2 \cos x) + (C_1 \sin x + C_2 \cos x) = 0$$

$$4. \text{ For } y = \sin(x+C), \frac{dy}{dx} = \cos(x+C)$$

$$\left(\frac{dy}{dx}\right)^2 + y^2 = \cos^2(x+C) + \sin^2(x+C) = 1$$

$$\text{For } y = \pm 1, \frac{dy}{dx} = 0.$$

$$\left(\frac{dy}{dx}\right)^2 + y^2 = 0^2 + (\pm 1)^2 = 1$$

$$5. \frac{dy}{dx} = x^2 + 1$$

$$dy = (x^2 + 1) dx$$

$$\int dy = \int (x^2 + 1) dx$$

$$y + C_1 = \frac{x^3}{3} + x + C_2$$

$$y = \frac{x^3}{3} + x + C$$

$$\text{At } x = 1, y = 1:$$

$$1 = \frac{1}{3} + 1 + C; C = -\frac{1}{3}$$

$$y = \frac{x^3}{3} + x - \frac{1}{3}$$

$$6. \frac{dy}{dx} = x^{-3} + 2$$

$$dy = (x^{-3} + 2) dx$$

$$\int dy = \int (x^{-3} + 2) dx$$

$$y + C_1 = -\frac{x^{-2}}{2} + 2x + C_2$$

$$y = -\frac{1}{2x^2} + 2x + C$$

$$\text{At } x = 1, y = 3:$$

$$3 = -\frac{1}{2} + 2 + C; C = \frac{3}{2}$$

$$y = -\frac{1}{2x^2} + 2x + \frac{3}{2}$$

$$7. \frac{dy}{dx} = \frac{x}{y}$$

$$\int y dy = \int x dx$$

$$\frac{y^2}{2} + C_1 = \frac{x^2}{2} + C_2$$

$$y^2 = x^2 + C$$

$$y = \pm \sqrt{x^2 + C}$$

$$\text{At } x = 1, y = 1:$$

$$1 = \pm \sqrt{1 + C}; C = 0 \text{ and the square root is positive.}$$

$$y = \sqrt{x^2} \text{ or } y = |x|$$

$$8. \frac{dy}{dx} = \sqrt{\frac{x}{y}}$$

$$\int \sqrt{y} dy = \int \sqrt{x} dx$$

$$\frac{2}{3} y^{3/2} + C_1 = \frac{2}{3} x^{3/2} + C_2$$

$$y^{3/2} = x^{3/2} + C$$

$$y = (x^{3/2} + C)^{2/3}$$

$$\text{At } x = 1, y = 4:$$

$$4 = (1 + C)^{2/3}; C = 7$$

$$y = (x^{3/2} + 7)^{2/3}$$

$$9. \frac{dz}{dt} = t^2 z^2$$

$$\int z^{-2} dz = \int t^2 dt$$

$$-z^{-1} + C_1 = \frac{t^3}{3} + C_2$$

$$\frac{1}{z} = -\frac{t^3}{3} + C_3 = \frac{C - t^3}{3}$$

$$z = \frac{3}{C - t^3}$$

$$\text{At } t = 1, z = \frac{1}{3}:$$

$$\frac{1}{3} = \frac{3}{C-1}; C-1=9; C=10$$

$$z = \frac{3}{10-t^3}$$

$$10. \frac{dy}{dt} = y^4$$

$$\int y^{-4} dy = \int dt$$

$$-\frac{1}{3y^3} + C_1 = t + C_2$$

$$y = -\frac{1}{\sqrt[3]{3t+C}}$$

$$\text{At } t=0, y=1:$$

$$C = -1$$

$$y = -\frac{1}{\sqrt[3]{3t-1}}$$

$$11. \frac{ds}{dt} = 16t^2 + 4t - 1$$

$$\int ds = \int (16t^2 + 4t - 1) dt$$

$$s + C_1 = \frac{16}{3}t^3 + 2t^2 - t + C_2$$

$$s = \frac{16}{3}t^3 + 2t^2 - t + C$$

$$\text{At } t=0, s=100:$$

$$C = 100$$

$$s = \frac{16}{3}t^3 + 2t^2 - t + 100$$

$$12. \frac{du}{dt} = u^3(t^3 - t)$$

$$\int u^{-3} du = \int (t^3 - t) dt$$

$$-\frac{1}{2u^2} + C_1 = \frac{t^4}{4} - \frac{t^2}{2} + C_2$$

$$u^{-2} = t^2 - \frac{t^4}{2} + C$$

$$u = \left(t^2 - \frac{t^4}{2} + C \right)^{-1/2}$$

$$\text{At } t=0, u=4:$$

$$4 = C^{-1/2}; C = \frac{1}{16}$$

$$u = \left(t^2 - \frac{t^4}{2} + \frac{1}{16} \right)^{-1/2}$$

$$13. \frac{dy}{dx} = (2x+1)^4$$

$$y = \int (2x+1)^4 dx = \frac{1}{2} \int (2x+1)^4 2 dx$$

$$= \frac{1}{2} \frac{(2x+1)^5}{5} + C = \frac{(2x+1)^5}{10} + C$$

$$\text{At } x=0, y=6:$$

$$6 = \frac{1}{10} + C; C = \frac{59}{10}$$

$$y = \frac{(2x+1)^5}{10} + \frac{59}{10} = \frac{(2x+1)^5 + 59}{10}$$

$$14. \frac{dy}{dx} = -y^2 x(x^2 + 2)^4$$

$$-\int y^{-2} dy = \frac{1}{2} \int 2x(x^2 + 2)^4 dx$$

$$\frac{1}{y} + C_1 = \frac{1}{2} \frac{(x^2 + 2)^5}{5} + C_2$$

$$\frac{1}{y} = \frac{(x^2 + 2)^5 + C}{10}$$

$$y = \frac{10}{(x^2 + 2)^5 + C}$$

$$\text{At } x=0, y=1:$$

$$1 = \frac{10}{32 + C}; C = 10 - 32 = -22$$

$$y = \frac{10}{(x^2 + 2)^5 - 22}$$

$$15. \frac{dy}{dx} = 3x$$

$$y = \int 3x dx = \frac{3}{2}x^2 + C$$

$$\text{At } (1, 2):$$

$$2 = \frac{3}{2} + C$$

$$C = \frac{1}{2}$$

$$y = \frac{3}{2}x^2 + \frac{1}{2} = \frac{3x^2 + 1}{2}$$

$$16. \frac{dy}{dx} = 3y^2$$

$$\int y^{-2} dy = 3 \int dx$$

$$-\frac{1}{y} + C_1 = 3x + C_2$$

$$\frac{1}{y} = -3x + C$$

$$y = \frac{1}{C-3x}$$

At (1, 2):

$$2 = \frac{1}{C-3}$$

$$C = \frac{7}{2}$$

$$y = \frac{1}{\frac{7}{2}-3x} = \frac{2}{7-6x}$$

$$17. \quad v = \int t \, dt = \frac{t^2}{2} + v_0$$

$$v = \frac{t^2}{2} + 3$$

$$s = \int \left(\frac{t^2}{2} + 3 \right) dt = \frac{t^3}{6} + 3t + s_0$$

$$s = \frac{t^3}{6} + 3t + 0 = \frac{t^3}{6} + 3t$$

At $t = 2$:

$$v = 5 \text{ cm/s}$$

$$s = \frac{22}{3} \text{ cm}$$

$$18. \quad v = \int (1+t)^{-4} dt = -\frac{1}{3(1+t)^3} + C$$

$$v_0 = 0:0 = -\frac{1}{3(1+0)^3} + C; C = \frac{1}{3}$$

$$v = -\frac{1}{3(1+t)^3} + \frac{1}{3}$$

$$s = \int \left(-\frac{1}{3(1+t)^3} + \frac{1}{3} \right) dt = \frac{1}{6(1+t)^2} + \frac{1}{3}t + C$$

$$s_0 = 10:10 = \frac{1}{6(1+0)^2} + \frac{1}{3}(0) + C; C = \frac{59}{6}$$

$$s = \frac{1}{6(1+t)^2} + \frac{1}{3}t + \frac{59}{6}$$

At $t = 2$:

$$v = -\frac{1}{81} + \frac{1}{3} = \frac{26}{81} \text{ cm/s}$$

$$s = \frac{1}{54} + \frac{2}{3} + \frac{59}{6} = \frac{284}{27} \text{ cm}$$

$$19. \quad v = \int (2t+1)^{1/3} dt = \frac{1}{2} \int (2t+1)^{1/3} 2dt$$

$$= \frac{3}{8}(2t+1)^{4/3} + C_1$$

$$v_0 = 0:0 = \frac{3}{8} + C_1; C_1 = -\frac{3}{8}$$

$$v = \frac{3}{8}(2t+1)^{4/3} - \frac{3}{8}$$

$$s = \frac{3}{8} \int (2t+1)^{4/3} dt - \frac{3}{8} \int 1 dt$$

$$= \frac{3}{16} \int (2t+1)^{4/3} 2dt - \frac{3}{8} \int 1 dt$$

$$= \frac{9}{112}(2t+1)^{7/3} - \frac{3}{8}t + C_2$$

$$s_0 = 10:10 = \frac{9}{112} + C_2; C_2 = \frac{1111}{112}$$

$$s = \frac{9}{112}(2t+1)^{7/3} - \frac{3}{8}t + \frac{1111}{112}$$

$$\text{At } t = 2: v = \frac{3}{8}(5)^{4/3} - \frac{3}{8} \approx 2.83$$

$$s = \frac{9}{112}(5)^{7/3} - \frac{6}{8} + \frac{1111}{112} \approx 12.6$$

$$20. \quad v = \int (3t+1)^{-3} dt = \frac{1}{3} \int (3t+1)^{-3} 3dt$$

$$= -\frac{1}{6}(3t+1)^{-2} + C_1$$

$$v_0 = 4:4 = -\frac{1}{6} + C_1; C_1 = \frac{25}{6}$$

$$v = -\frac{1}{6}(3t+1)^{-2} + \frac{25}{6}$$

$$s = -\frac{1}{6} \int (3t+1)^{-2} dt + \int \frac{25}{6} dt$$

$$= -\frac{1}{18} \int (3t+1)^{-2} 3dt + \frac{25}{6} \int 1 dt$$

$$= \frac{1}{18}(3t+1)^{-1} + \frac{25}{6}t + C_2$$

$$s_0 = 0:0 = \frac{1}{18} + C_2; C_2 = -\frac{1}{18}$$

$$s = \frac{1}{18}(3t+1)^{-1} + \frac{25}{6}t - \frac{1}{18}$$

$$\text{At } t = 2: v = -\frac{1}{6}(7)^{-2} + \frac{25}{6} \approx 4.16$$

$$s = \frac{1}{18}(7)^{-1} + \frac{25}{3} - \frac{1}{18} \approx 8.29$$

$$21. \quad v = -32t + 96,$$

$$s = -16t^2 + 96t + s_0 = -16t^2 + 96t$$

$$v = 0 \text{ at } t = 3$$

$$\text{At } t = 3, s = -16(3^2) + 96(3) = 144 \text{ ft}$$

$$22. a = \frac{dv}{dt} = k$$

$$v = \int k dt = kt + v_0 = \frac{ds}{dt};$$

$$s = \int (kt + v_0) dt = \frac{k}{2} t^2 + v_0 t + s_0 = \frac{k}{2} t^2 + v_0 t$$

$$v = 0 \text{ when } t = -\frac{v_0}{k}. \text{ Then}$$

$$s = \frac{k}{2} \left(-\frac{v_0}{k} \right)^2 + \left(-\frac{v_0}{k} \right) = -\frac{v_0^2}{2k}.$$

$$23. \frac{dv}{dt} = -5.28$$

$$\int dv = -\int 5.28 dt$$

$$v = \frac{ds}{dt} = -5.28t + v_0 = -5.28t + 56$$

$$\int ds = \int (-5.28t + 56) dt$$

$$s = -2.64t^2 + 56t + s_0 = -2.64t^2 + 56t + 1000$$

$$\text{When } t = 4.5, v = 32.24 \text{ ft/s and } s = 1198.54 \text{ ft}$$

$$24. v = 0 \text{ when } t = \frac{-56}{-5.28} \approx 10.6061. \text{ Then}$$

$$s \approx -2.64(10.6061)^2 + 56(10.6061) + 1000$$

$$\approx 1296.97 \text{ ft}$$

$$25. \frac{dV}{dt} = -kS$$

$$\text{Since } V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2,$$

$$4\pi r^2 \frac{dr}{dt} = -k4\pi r^2 \text{ so } \frac{dr}{dt} = -k.$$

$$\int dr = -\int k dt$$

$$r = -kt + C$$

$$2 = -k(0) + C \text{ and } 0.5 = -k(10) + C, \text{ so}$$

$$C = 2 \text{ and } k = \frac{3}{20}. \text{ Then, } r = -\frac{3}{20}t + 2.$$

$$26. \text{ Solving } v = -136 = -32t \text{ yields } t = \frac{17}{4}.$$

$$\text{Then } s = 0 = -16\left(\frac{17}{4}\right)^2 + (0)\left(\frac{17}{4}\right) + s_0, \text{ so}$$

$$s_0 = 289 \text{ ft.}$$

$$27. v_{\text{esc}} = \sqrt{2gR}$$

$$\text{For the Moon, } v_{\text{esc}} \approx \sqrt{2(0.165)(32)(1080 \cdot 5280)}$$

$$\approx 7760 \text{ ft/s} \approx 1.470 \text{ mi/s.}$$

$$\text{For Venus, } v_{\text{esc}} \approx \sqrt{2(0.85)(32)(3800 \cdot 5280)}$$

$$\approx 33,038 \text{ ft/s} \approx 6.257 \text{ mi/s.}$$

$$\text{For Jupiter, } v_{\text{esc}} \approx 194,369 \text{ ft/s} \approx 36.812 \text{ mi/s.}$$

$$\text{For the Sun, } v_{\text{esc}} \approx 2,021,752 \text{ ft/s}$$

$$\approx 382.908 \text{ mi/s.}$$

$$28. v_0 = 60 \text{ mi/h} = 88 \text{ ft/s}$$

$$v = 0 = -11t + 88; t = 8 \text{ sec}$$

$$29. a = \frac{dv}{dt} = \frac{\Delta v}{\Delta t} = \frac{60 - 45}{10} = 1.5 \text{ mi/h/s} = 2.2 \text{ ft/s}^2$$

$$30. 75 = \frac{8}{2}(3.75)^2 + v_0(3.75) + 0; v_0 = 5 \text{ ft/s}$$

$$31. \text{ For the first 10 s, } a = \frac{dv}{dt} = 6t, v = 3t^2, \text{ and}$$

$$s = t^3. \text{ So } v(10) = 300 \text{ and } s(10) = 1000. \text{ After}$$

$$10 \text{ s, } a = \frac{dv}{dt} = -10, v = -10(t - 10) + 300, \text{ and}$$

$$s = -5(t - 10)^2 + 300(t - 10) + 1000. v = 0 \text{ at}$$

$$t = 40, \text{ at which time } s = 5500 \text{ m.}$$

$$32. \text{ a. After accelerating for 8 seconds, the velocity is } 8 \cdot 3 = 24 \text{ m/s.}$$

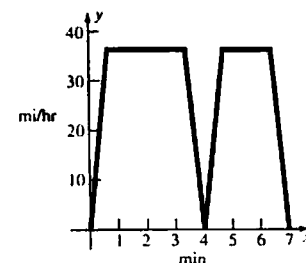
$$\text{b. Since acceleration and deceleration are constant, the average velocity during those times is}$$

$$\frac{24}{2} = 12 \text{ m/s. Solve } 0 = -4t + 24 \text{ to get the time}$$

$$\text{spent decelerating. } t = \frac{24}{4} = 6 \text{ s;}$$

$$d = (12)(8) + (24)(100) + (12)(6) = 2568 \text{ m.}$$

$$33. \text{ a.}$$



$$\text{b. Since the trip that involves 1 min more travel time at speed } v_m \text{ is 0.6 mi longer,}$$

$$v_m = 0.6 \text{ mi/min}$$

$$= 36 \text{ mi/h.}$$

$$\text{c. From part b, } v_m = 0.6 \text{ mi/min. Note that the average speed during acceleration and}$$

$$\text{deceleration is } \frac{v_m}{2} = 0.3 \text{ mi/min. Let } t \text{ be the time spent between stop C and stop D at the}$$

$$\text{constant speed } v_m, \text{ so}$$

$$0.6t + 0.3(4 - t) = 2 \text{ miles. Therefore,}$$

$$t = 2\frac{2}{3} \text{ min and the time spent accelerating}$$

$$\text{is } \frac{4 - 2\frac{2}{3}}{2} = \frac{2}{3} \text{ min.}$$

$$a = \frac{0.6 - 0}{\frac{2}{3}} = 0.9 \text{ mi/min}^2.$$

34. For the balloon, $\frac{dh}{dt} = 4$, so $h(t) = 4t + C_1$. Set $t = 0$ at the time when Victoria threw the ball, and height 0 at the ground, then $h(t) = 4t + 64$. The height of the ball is given by $s(t) = -16t^2 + v_0t$, since $s_0 = 0$. The maximum height of the ball is when $t = \frac{v_0}{32}$, since then $s'(t) = 0$. At this time
- $$h(t) = s(t) \text{ or } 4\left(\frac{v_0}{32}\right) + 64 = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right).$$
- Solve this for v_0 to get $v_0 \approx 68.125$ feet per second.

35. a. $\frac{dV}{dt} = C_1\sqrt{h}$ where h is the depth of the water. Here, $V = \pi r^2 h = 100h$, so $h = \frac{V}{100}$. Hence $\frac{dV}{dt} = C_1 \frac{\sqrt{V}}{10}$, $V(0) = 1600$, $V(40) = 0$.
- b. $\int 10V^{-1/2} dV = \int C_1 dt$; $20\sqrt{V} = C_1t + C_2$;
 $V(0) = 1600$: $C_2 = 20 \cdot 40 = 800$;
 $V(40) = 0$: $C_1 = -\frac{800}{40} = -20$
 $V(t) = \frac{1}{400}(-20t + 800)^2$

c. $V(10) = \frac{1}{400}(-200 + 800)^2 = 900 \text{ cm}^3$

36. a. $\frac{dP}{dt} = C_1\sqrt[3]{P}$, $P(0) = 1000$, $P(10) = 1700$
 where t is the number of years since 1970.
- b. $\int P^{-1/3} dP = \int C_1 dt$; $\frac{3}{2}P^{2/3} = C_1t + C_2$
 $P(0) = 1000$: $C_2 = \frac{3}{2} \cdot 1000^{2/3} = 150$
 $P(10) = 1700$: $C_1 = \frac{\frac{3}{2} \cdot 1700^{2/3} - 150}{10} \approx 6.3660$
 $P = (4.2440t + 100)^{3/2}$
- c. $4000 = (4.2440t + 100)^{3/2}$
 $t = \frac{4000^{2/3} - 100}{4.2440} \approx 35.812$
 $t \approx 36$ years, so the population will reach 4000 by 2016.

37. Initially, $v = -32t$ and $s = -16t^2 + 16$. $s = 0$ when $t = 1$. Later, the ball falls 9 ft in a time given by $0 = -16t^2 + 9$, or $\frac{3}{4}$ s, and on impact has a velocity of $-32\left(\frac{3}{4}\right) = -24$ ft/s. By symmetry, 24 ft/s must be the velocity right after the first bounce. So
- a. $v(t) = \begin{cases} -32t & \text{for } 0 \leq t \leq 1 \\ -32(t-1) + 24 & \text{for } 1 < t \leq 2.5 \end{cases}$
- b. $9 = -16t^2 + 16 \Rightarrow t \approx 0.66$ sec; s also equals 9 at the apex of the first rebound at $t = 1.75$ sec.

5.3 Concepts Review

- $2 \cdot \frac{5(6)}{2} = 30$; $2(5) = 10$
- $3(9) - 2(7) = 13$; $9 + 4(10) = 49$
- $1 - \frac{1}{10} = 0.9$
- $2 \cdot \frac{6(7)}{2} - \frac{6(7)(13)}{6} = -49$

Problem Set 5.3

- $\sum_{k=1}^6 (k-1) = \sum_{k=1}^6 k - \sum_{k=1}^6 1 = \frac{6(7)}{2} - 6(1) = 15$
- $\sum_{i=1}^6 i^2 = \frac{6(7)(13)}{6} = 91$

$$\begin{aligned}
 3. \quad \sum_{k=1}^7 \frac{1}{k+1} &= \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} \\
 &+ \frac{1}{4+1} + \frac{1}{5+1} + \frac{1}{6+1} + \frac{1}{7+1} \\
 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{1443}{840} = \frac{481}{280}
 \end{aligned}$$

$$4. \quad \sum_{l=3}^8 (l+1)^2 = 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 = 271$$

$$\begin{aligned}
 5. \quad \sum_{m=1}^8 (-1)^m 2^{m-2} &= (-1)^1 2^{-1} + (-1)^2 2^0 + (-1)^3 2^1 \\
 &+ (-1)^4 2^2 + (-1)^5 2^3 + (-1)^6 2^4 \\
 &+ (-1)^7 2^5 + (-1)^8 2^6 \\
 &= -\frac{1}{2} + 1 - 2 + 4 - 8 + 16 - 32 + 64 = \frac{85}{2}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \sum_{k=3}^7 \frac{(-1)^k 2^k}{(k+1)} \\
 &= \frac{(-1)^3 2^3}{4} + \frac{(-1)^4 2^4}{5} \\
 &+ \frac{(-1)^5 2^5}{6} + \frac{(-1)^6 2^6}{7} + \frac{(-1)^7 2^7}{8} = -\frac{1154}{105}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \sum_{n=1}^6 n \cos(n\pi) &= \cos \pi + 2 \cos 2\pi + 3 \cos 3\pi \\
 &+ 4 \cos 4\pi + 5 \cos 5\pi + 6 \cos 6\pi \\
 &= -1 + 2 - 3 + 4 - 5 + 6 = 3
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \sum_{k=-1}^6 k \sin\left(\frac{k\pi}{2}\right) \\
 &= -\sin\left(-\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) + 2 \sin(\pi) \\
 &+ 3 \sin\left(\frac{3\pi}{2}\right) + 4 \sin(2\pi) + 5 \sin\left(\frac{5\pi}{2}\right) + 6 \sin(3\pi) \\
 &= 1 + 1 + 0 - 3 + 0 + 5 + 0 = 4
 \end{aligned}$$

$$9. \quad 1 + 2 + 3 + \cdots + 41 = \sum_{i=1}^{41} i$$

$$10. \quad 2 + 4 + 6 + 8 + \cdots + 50 = \sum_{i=1}^{25} 2i$$

$$11. \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100} = \sum_{i=1}^{100} \frac{1}{i}$$

$$12. \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{100} = \sum_{i=1}^{100} \frac{(-1)^{i+1}}{i}$$

$$13. \quad a_1 + a_3 + a_5 + a_7 + \cdots + a_{99} = \sum_{i=1}^{50} a_{2i-1}$$

$$14. \quad b_{-1} + b_1 + b_3 + b_5 + \cdots + b_{101} = \sum_{i=1}^{502} b_{2i-3}$$

$$15. \quad f(c_1) + f(c_2) + \cdots + f(c_n) = \sum_{i=1}^n f(c_i)$$

$$\begin{aligned}
 16. \quad f(w_1)\Delta x + f(w_2)\Delta x + \cdots + f(w_n)\Delta x \\
 = \sum_{i=1}^n f(w_i)\Delta x
 \end{aligned}$$

$$17. \quad \sum_{i=1}^{10} (a_i + b_i) = \sum_{i=1}^{10} a_i + \sum_{i=1}^{10} b_i = 40 + 50 = 90$$

$$\begin{aligned}
 18. \quad \sum_{n=1}^{10} (3a_n + 2b_n) &= 3 \sum_{n=1}^{10} a_n + 2 \sum_{n=1}^{10} b_n \\
 &= 3(40) + 2(50) = 220
 \end{aligned}$$

$$\begin{aligned}
 19. \quad \sum_{p=0}^9 (a_{p+1} - b_{p+1}) &= \sum_{p=1}^{10} a_p - \sum_{p=1}^{10} b_p = 40 - 50 \\
 &= -10
 \end{aligned}$$

$$\begin{aligned}
 20. \quad \sum_{q=1}^{10} (a_q - b_q - q) &= \sum_{q=1}^{10} a_q - \sum_{q=1}^{10} b_q - \sum_{q=1}^{10} q \\
 &= 40 - 50 - \frac{10(11)}{2} = -65
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \sum_{k=1}^{40} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
 &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{40} - \frac{1}{41} \right) \\
 &= 1 - \frac{1}{41} = \frac{40}{41}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \sum_{k=1}^{10} (2^k - 2^{k-1}) \\
 &= (2 - 1) + (4 - 2) + (8 - 4) + \cdots + (1024 - 512) \\
 &= -1 + 1024 = 1023
 \end{aligned}$$

$$\begin{aligned}
 23. \quad & \sum_{k=3}^{20} \left(\frac{1}{(k+1)^2} - \frac{1}{k^2} \right) \\
 &= \left(\frac{1}{4^2} - \frac{1}{3^2} \right) + \left(\frac{1}{5^2} - \frac{1}{4^2} \right) + \cdots + \left(\frac{1}{21^2} - \frac{1}{20^2} \right) \\
 &= -\frac{1}{3^2} + \frac{1}{21^2} = -\frac{49}{441} + \frac{1}{441} = -\frac{48}{441} = -\frac{16}{147}
 \end{aligned}$$

$$\begin{aligned}
 24. \quad & \sum_{k=3}^{m+1} (a_k - a_{k-1}) \\
 &= (a_3 - a_2) + (a_4 - a_3) + (a_5 - a_4) + \cdots \\
 &+ (a_m - a_{m-1}) + (a_{m+1} - a_m) = -a_2 + a_{m+1}
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & \sum_{i=1}^{100} (3i - 2) = 3 \sum_{i=1}^{100} i - \sum_{i=1}^{100} 2 = 3(5050) - 2(100) \\
 &= 14,950
 \end{aligned}$$

$$\begin{aligned}
 26. \quad & \sum_{i=1}^{10} [(i-1)(4i+3)] = \sum_{i=1}^{10} (4i^2 - i - 3) \\
 &= 4 \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} i - \sum_{i=1}^{10} 3 = 4(385) - 55 - 3(10) \\
 &= 1455
 \end{aligned}$$

$$\begin{aligned}
 27. \quad & \sum_{k=1}^{10} (k^3 - k^2) = \sum_{k=1}^{10} k^3 - \sum_{k=1}^{10} k^2 = 3025 - 385 \\
 &= 2640
 \end{aligned}$$

$$\begin{aligned}
 28. \quad & \sum_{k=1}^{10} 5k^2(k+4) = \sum_{k=1}^{10} (5k^3 + 20k^2) \\
 &= 5 \sum_{k=1}^{10} k^3 + 20 \sum_{k=1}^{10} k^2 = 5(3025) + 20(385) \\
 &= 22,825
 \end{aligned}$$

$$\begin{aligned}
 29. \quad & \sum_{i=1}^n (2i^2 - 3i + 1) = 2 \sum_{i=1}^n i^2 - 3 \sum_{i=1}^n i + \sum_{i=1}^n 1 \\
 &= \frac{2n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + n \\
 &= \frac{2n^3 + 3n^2 + n}{3} - \frac{3n^2 + 3n}{2} + n = \frac{4n^3 - 3n^2 - n}{6}
 \end{aligned}$$

$$\begin{aligned}
 30. \quad & \sum_{i=1}^n (2i-3)^2 = \sum_{i=1}^n (4i^2 - 12i + 9) \\
 &= 4 \sum_{i=1}^n i^2 - 12 \sum_{i=1}^n i + \sum_{i=1}^n 9 \\
 &= \frac{4n(n+1)(2n+1)}{6} - \frac{12n(n+1)}{2} + 9n \\
 &= \frac{4n^3 - 12n^2 + 11n}{3}
 \end{aligned}$$

$$31. \quad \sum_{i=3}^{19} i(i-2) = \sum_{k=1}^{17} (k+2)k$$

$$32. \quad \sum_{k=5}^{14} k 2^{k-4} = \sum_{i=1}^{10} (i+4)2^i$$

$$33. \quad \sum_{k=0}^{10} \frac{k}{k+1} = \sum_{i=1}^{11} \frac{i-1}{i}$$

$$34. \quad \sum_{k=4}^{13} (k-3) \sin \left(\frac{\pi}{k-3} \right) = \sum_{i=1}^{10} i \sin \left(\frac{\pi}{i} \right)$$

$$35. \quad \sum_{i=1}^{10} \frac{3i}{5} \cdot \frac{1}{5} = \frac{3}{25} \sum_{i=1}^{10} i = \frac{33}{5}$$

$$\begin{aligned}
 36. \quad & S - rS = a + ar + ar^2 + \cdots + ar^n \\
 & \quad - (ar + ar^2 + \cdots + ar^n + ar^{n+1}) \\
 &= a - ar^{n+1} \\
 &= S(1-r); S = \frac{a - ar^{n+1}}{1-r}
 \end{aligned}$$

$$\begin{aligned}
 37. \quad \text{a.} \quad & \sum_{k=0}^{10} \left(\frac{1}{2} \right)^k = \frac{1 - \left(\frac{1}{2} \right)^{11}}{\frac{1}{2}} = 2 - \left(\frac{1}{2} \right)^{10}, \text{ so} \\
 & \sum_{k=1}^{10} \left(\frac{1}{2} \right)^k = 1 - \left(\frac{1}{2} \right)^{10}.
 \end{aligned}$$

$$\begin{aligned}
 \text{b.} \quad & \sum_{k=0}^{10} 2^k = \frac{1 - 2^{11}}{-1} = 2^{11} - 1, \text{ so} \\
 & \sum_{k=1}^{10} 2^k = 2^{11} - 2.
 \end{aligned}$$

$$\begin{aligned}
 38. \quad & S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\
 & + S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\
 & 2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \\
 & 2S = n(n+1) \\
 & S = \frac{n(n+1)}{2}
 \end{aligned}$$

$$\begin{aligned}
 39. \quad S &= a + (a+d) + (a+2d) + \cdots + [a+(n-2)d] + [a+(n-1)d] + (a+nd) \\
 + S &= (a+nd) + [a+(n-1)d] + [a+(n-2)d] + \cdots + (a+2d) + (a+d) + a \\
 2S &= (2a+nd) + (2a+nd) + (2a+nd) + \cdots + (2a+nd) + (2a+nd) + (2a+nd) \\
 2S &= (n+1)(2a+nd) \\
 S &= \frac{(n+1)(2a+nd)}{2}
 \end{aligned}$$

$$40. \quad \sum_{k=1}^1 \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} = 1 \geq 1; \quad \sum_{k=1}^2 \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} \approx 1.707 \geq \sqrt{2} \approx 1.414$$

Assume that $\sum_{k=1}^N \frac{1}{\sqrt{k}} \geq \sqrt{N}$ for all $N \leq n$.

$$\text{Then } \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^n \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n^2+n+1}}{\sqrt{n+1}} \geq \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$$

$$41. \quad \bar{x} = \frac{1}{7}(2+5+7+8+9+10+14) = \frac{55}{7} \approx 7.86$$

$$\begin{aligned}
 s^2 &= \frac{1}{7} \left[\left(2 - \frac{55}{7}\right)^2 + \left(5 - \frac{55}{7}\right)^2 + \left(7 - \frac{55}{7}\right)^2 + \left(8 - \frac{55}{7}\right)^2 \right. \\
 &\quad \left. + \left(9 - \frac{55}{7}\right)^2 + \left(10 - \frac{55}{7}\right)^2 + \left(14 - \frac{55}{7}\right)^2 \right] \approx 12.4
 \end{aligned}$$

$$42. \quad \text{a. } \bar{x} = 1, s^2 = 0$$

$$\text{b. } \bar{x} = 1001, s^2 = 0$$

$$\begin{aligned}
 \text{c. } \bar{x} &= 2 \\
 s^2 &= \frac{1}{3} \left[(1-2)^2 + (2-2)^2 + (3-2)^2 \right] \\
 &= \frac{1}{3} \left[(-1)^2 + 0^2 + 1^2 \right] \\
 &= \frac{1}{3}(2) = \frac{2}{3}
 \end{aligned}$$

$$\text{d. } \bar{x} = 1,000,002$$

$$s^2 = \frac{1}{3} \left[(-1)^2 + 0^2 + 1^2 \right] = \frac{2}{3}$$

$$43. \quad \text{a. } \sum_{i=1}^n (x_i - \bar{x}) = \sum_{i=1}^n x_i - \sum_{i=1}^n \bar{x} = n\bar{x} - n\bar{x} = 0$$

$$\begin{aligned}
 \text{b. } s^2 &= \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^n (x_i^2 - 2\bar{x}x_i + \bar{x}^2) \\
 &= \frac{1}{n} \sum_{i=1}^n x_i^2 - \frac{2\bar{x}}{n} \sum_{i=1}^n x_i + \frac{1}{n} \sum_{i=1}^n \bar{x}^2
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{n} \sum_{i=1}^n x_i^2 - \frac{2\bar{x}}{n} (n\bar{x}) + \frac{1}{n} (n\bar{x}^2) \\
 &= \left(\frac{1}{n} \sum_{i=1}^n x_i^2 \right) - 2\bar{x}^2 + \bar{x}^2 = \left(\frac{1}{n} \sum_{i=1}^n x_i^2 \right) - \bar{x}^2
 \end{aligned}$$

44. The variance of n identical numbers is 0. Let c be the constant. Then

$$\bar{x} = \frac{1}{n} \left[(c-c)^2 + (c-c)^2 + \cdots + (c-c)^2 \right] = 0$$

45. Let $S(c) = \sum_{i=1}^n (x_i - c)^2$. Then

$$\begin{aligned}
 S'(c) &= \frac{d}{dc} \sum_{i=1}^n (x_i - c)^2 \\
 &= \sum_{i=1}^n \frac{d}{dc} (x_i - c)^2 \\
 &= \sum_{i=1}^n 2(x_i - c)(-1) \\
 &= -2 \sum_{i=1}^n x_i + 2nc
 \end{aligned}$$

$$S''(c) = 2n$$

Set $S'(c) = 0$ and solve for c :

$$-2 \sum_{i=1}^n x_i + 2nc = 0$$

$$c = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

Since $S''(\bar{x}) = 2n > 0$ we know that $\bar{x}$ minimizes $S(c)$.

$$\begin{aligned}
46. \quad \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) &= \sum_{i=1}^n (x_i y_i - x_i \bar{y} - y_i \bar{x} + \bar{x} \bar{y}) \\
&= \sum_{i=1}^n x_i y_i - \bar{y} \sum_{i=1}^n x_i - \bar{x} \sum_{i=1}^n y_i + n \bar{x} \bar{y} \\
&= \sum_{i=1}^n x_i y_i - \bar{y} n \bar{x} - \bar{x} n \bar{y} + n \bar{x} \bar{y} \\
&= \sum_{i=1}^n x_i y_i - n \bar{x} \bar{y}
\end{aligned}$$

$$\begin{aligned}
47. \quad (i+1)^4 - i^4 &= 4i^3 + 6i^2 + 4i + 1 \\
\sum_{i=1}^n [(i+1)^4 - i^4] &= \sum_{i=1}^n (4i^3 + 6i^2 + 4i + 1) \\
(n+1)^4 - 1^4 &= 4 \sum_{i=1}^n i^3 + 6 \sum_{i=1}^n i^2 + 4 \sum_{i=1}^n i + \sum_{i=1}^n 1 \\
n^4 + 4n^3 + 6n^2 + 4n &= 4 \sum_{i=1}^n i^3 + 6 \frac{n(n+1)(2n+1)}{6} \\
&\quad + 4 \frac{n(n+1)}{2} + n
\end{aligned}$$

Solving for $\sum_{i=1}^n i^3$ gives

$$\begin{aligned}
4 \sum_{i=1}^n i^3 &= n^4 + 4n^3 + 6n^2 + 4n - (2n^3 + 3n^2 + n) \\
&\quad - (2n^2 + 2n) - n
\end{aligned}$$

$$\begin{aligned}
4 \sum_{i=1}^n i^3 &= n^4 + 2n^3 + n^2 \\
\sum_{i=1}^n i^3 &= \frac{n^4 + 2n^3 + n^2}{4} = \left[\frac{n(n+1)}{2} \right]^2
\end{aligned}$$

$$\begin{aligned}
48. \quad (i+1)^5 - i^5 &= 5i^4 + 10i^3 + 10i^2 + 5i + 1 \\
\sum_{i=1}^n [(i+1)^5 - i^5] &= 5 \sum_{i=1}^n i^4 + 10 \sum_{i=1}^n i^3 + 10 \sum_{i=1}^n i^2 + 5 \sum_{i=1}^n i + \sum_{i=1}^n 1 \\
(n+1)^5 - 1^5 &= 5 \sum_{i=1}^n i^4 + 10 \frac{n^2(n+1)^2}{4} \\
&\quad + 10 \frac{n(n+1)(2n+1)}{6} + 5 \frac{n(n+1)}{2} + n \\
n^5 + 5n^4 + 10n^3 + 10n^2 + 5n &= 5 \sum_{i=1}^n i^4 + \frac{5}{2} n^2 (n+1)^2 \\
&\quad + \frac{10}{3} n(n+1)(2n+1) + \frac{5}{2} n(n+1) + n
\end{aligned}$$

Solving for $\sum_{i=1}^n i^4$ yields

$$\begin{aligned}\sum_{i=1}^n i^4 &= \frac{1}{5} \left[n^5 + \frac{5}{2} n^4 + \frac{5}{3} n^3 - \frac{1}{6} n \right] \\ &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\end{aligned}$$

49.
$$\begin{aligned}\sum_{i=1}^n (a_i t + b_i)^2 &= \sum_{i=1}^n (a_i^2 t^2 + 2a_i b_i t + b_i^2) \\ &= t^2 \sum_{i=1}^n a_i^2 + 2t \sum_{i=1}^n a_i b_i + \sum_{i=1}^n b_i^2 \\ &= At^2 + 2Bt + C\end{aligned}$$

Because $\sum_{i=1}^n (a_i t + b_i)^2 \geq 0$, $At^2 + 2Bt + C \geq 0$.

50. Since $At^2 + 2Bt + C \geq 0$ for all t

$$(2B)^2 - 4AC \leq 0$$

$$4B^2 - 4AC \leq 0$$

$$B^2 - AC \leq 0$$

$$B^2 \leq AC$$

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right)$$

51.
$$\begin{aligned}\sum_{i=1}^n (a_i - a_{i-1}) b_{i-1} &= \sum_{i=1}^n a_i b_{i-1} - \sum_{i=1}^n a_{i-1} b_{i-1} \\ &= \sum_{i=1}^n a_i b_{i-1} - \sum_{i=0}^{n-1} a_i b_i \\ &= \sum_{i=1}^n a_i b_{i-1} - \sum_{i=1}^n a_i b_i + a_n b_n - a_0 b_0 \\ &= a_n b_n - a_0 b_0 + \sum_{i=1}^n a_i (b_{i-1} - b_i) \\ &= a_n b_n - a_0 b_0 - \sum_{i=1}^n a_i (b_i - b_{i-1})\end{aligned}$$

52.
$$\begin{aligned}\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{n+1}\end{aligned}$$

53. Suppose we have a $(n+1) \times n$ grid. Shade in

$n+1-k$ boxes in the k th column. There are n columns, and the shaded area is $1+2+\cdots+n$.

The shaded area is also half the area of the grid or $\frac{n(n+1)}{2}$. Thus, $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Suppose we have a square grid with sides of

length $1+2+\cdots+n = \frac{n(n+1)}{2}$. From the diagram

the area is $1^3 + 2^3 + \cdots + n^3$ or $\left[\frac{n(n+1)}{2} \right]^2$. Thus,

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2.$$

54. a. The number of gifts given on the n th day is

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

The total number of gifts is $\sum_{i=1}^{12} \frac{i(i+1)}{2} = 364$.

b. For n days, the total number of gifts is

$$\begin{aligned}\sum_{i=1}^n \frac{i(i+1)}{2} &= \sum_{i=1}^n \frac{i^2}{2} + \sum_{i=1}^n \frac{i}{2} = \frac{1}{2} \sum_{i=1}^n i^2 + \frac{1}{2} \sum_{i=1}^n i \\ &= \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} \right] + \frac{1}{2} \left[\frac{n(n+1)}{2} \right]\end{aligned}$$

$$= \frac{1}{4}n(n+1)\left(\frac{2n+1}{3}+1\right) = \frac{1}{12}n(n+1)(2n+4)$$

$$= \frac{1}{6}n(n+1)(n+2)$$

55. The bottom layer contains $10 \cdot 16 = 160$ oranges, the next layer contains $9 \cdot 15 = 135$ oranges, the third layer contains $8 \cdot 14 = 112$ oranges, and so on, up to the top layer, which contains $1 \cdot 7 = 7$ oranges. The stack contains

$$1 \cdot 7 + 2 \cdot 8 + \dots + 9 \cdot 15 + 10 \cdot 16$$

$$= \sum_{i=1}^{10} i(6+i)$$

$$= \sum_{i=1}^{10} i(16-10+i) = 715 \text{ oranges.}$$

If the bottom layer is 50 oranges by 60 oranges, the stack contains

$$\sum_{i=1}^{50} i(60-50+i) = \sum_{i=1}^{50} i(10+i) = 55,675.$$

For a general stack whose base is m rows of n oranges with $m \leq n$, the stack contains

$$\sum_{i=1}^m i(n-m+i) = (n-m) \sum_{i=1}^m i + \sum_{i=1}^m i^2$$

$$= (n-m) \frac{m(m+1)}{2} + \frac{m(m+1)(2m+1)}{6}$$

$$= \frac{m(m+1)(3n-m+1)}{6}$$

5.4 Concepts Review

- 16
- inscribed; circumscribed
- $\frac{1}{2}(4)(4) = 8; 8 + \frac{1}{2}(2)(2) = 10$
- 6

Problem Set 5.4

$$1. A = \frac{1}{2} \left[1 + \frac{3}{2} + 2 + \frac{5}{2} \right] = \frac{7}{2}$$

$$2. A = \frac{1}{4} \left[1 + \frac{5}{4} + \frac{3}{2} + \frac{7}{4} + 2 + \frac{9}{4} + \frac{5}{2} + \frac{11}{4} \right] = \frac{15}{4}$$

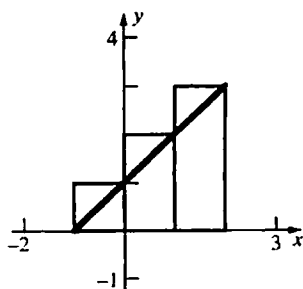
$$3. A = \frac{1}{2} \left[\frac{3}{2} + 2 + \frac{5}{2} + 3 \right] = \frac{9}{2}$$

$$4. A = \frac{1}{4} \left[\frac{5}{4} + \frac{3}{2} + \frac{7}{4} + 2 + \frac{9}{4} + \frac{5}{2} + \frac{11}{4} + 3 \right] = \frac{17}{4}$$

$$5. A = \frac{1}{2} \left[\left(\frac{1}{2} \cdot 0^2 + 1 \right) + \left(\frac{1}{2} \cdot \left(\frac{1}{2} \right)^2 + 1 \right) + \left(\frac{1}{2} \cdot 1^2 + 1 \right) + \left(\frac{1}{2} \cdot \left(\frac{3}{2} \right)^2 + 1 \right) \right] = \frac{1}{2} \left(1 + \frac{9}{8} + \frac{3}{2} + \frac{17}{8} \right) = \frac{23}{8}$$

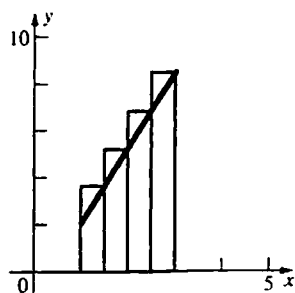
$$6. A = \frac{1}{2} \left[\left(\frac{1}{2} \cdot \left(\frac{1}{2} \right)^2 + 1 \right) + \left(\frac{1}{2} \cdot 1^2 + 1 \right) + \left(\frac{1}{2} \cdot \left(\frac{3}{2} \right)^2 + 1 \right) + \left(\frac{1}{2} \cdot 2^2 + 1 \right) \right] = \frac{1}{2} \left(\frac{9}{8} + \frac{3}{2} + \frac{17}{8} + 3 \right) = \frac{31}{8}$$

7.



$$A = 1(1 + 2 + 3) = 6$$

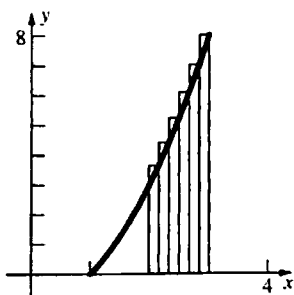
8.



$$A = \frac{1}{2} \left[\left(3 \cdot \frac{3}{2} - 1 \right) + (3 \cdot 2 - 1) + \left(3 \cdot \frac{5}{2} - 1 \right) + (3 \cdot 3 - 1) \right]$$

$$= \frac{1}{2} \left(\frac{7}{2} + 5 + \frac{13}{2} + 8 \right) = \frac{23}{2}$$

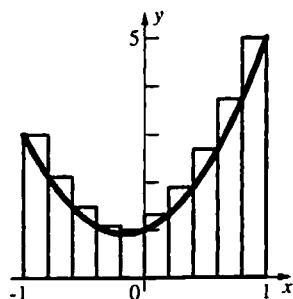
9.



$$A = \frac{1}{6} \left[\left(\left(\frac{13}{6} \right)^2 - 1 \right) + \left(\left(\frac{7}{3} \right)^2 - 1 \right) + \left(\left(\frac{5}{2} \right)^2 - 1 \right) + \left(\left(\frac{8}{3} \right)^2 - 1 \right) + \left(\left(\frac{17}{6} \right)^2 - 1 \right) + (3^2 - 1) \right]$$

$$= \frac{1}{6} \left(\frac{133}{36} + \frac{40}{9} + \frac{21}{4} + \frac{55}{9} + \frac{253}{36} + 8 \right) = \frac{1243}{216}$$

10.



$$A = \frac{1}{5} \left[(3(-1)^2 + (-1) + 1) + \left(3 \left(-\frac{4}{5} \right)^2 + \left(-\frac{4}{5} \right) + 1 \right) + \left(3 \left(-\frac{3}{5} \right)^2 + \left(-\frac{3}{5} \right) + 1 \right) + \left(3 \left(-\frac{2}{5} \right)^2 + \left(-\frac{2}{5} \right) + 1 \right) + (3(0)^2 + 0 + 1) \right.$$

$$\left. + \left(3 \left(\frac{1}{5} \right)^2 + \frac{1}{5} + 1 \right) + \left(3 \left(\frac{2}{5} \right)^2 + \frac{2}{5} + 1 \right) + \left(3 \left(\frac{3}{5} \right)^2 + \frac{3}{5} + 1 \right) + \left(3 \left(\frac{4}{5} \right)^2 + \frac{4}{5} + 1 \right) + (3(1)^2 + 1 + 1) \right]$$

$$= \frac{1}{5} [3 + 2.12 + 1.48 + 1.08 + 1 + 1.32 + 1.88 + 2.68 + 3.72 + 5] = 4.656$$

$$11. \Delta x = \frac{1}{n}, x_i = \frac{i}{n}$$

$$f(x_i)\Delta x = \left(\frac{i}{n} + 2 \right) \left(\frac{1}{n} \right) = \frac{i}{n^2} + \frac{2}{n}$$

$$A(S_n) = \left[\left(\frac{1}{n^2} + \frac{2}{n} \right) + \left(\frac{2}{n^2} + \frac{2}{n} \right) + \cdots + \left(\frac{n}{n^2} + \frac{2}{n} \right) \right] = \frac{1}{n^2} (1+2+3+\cdots+n) + 2 = \frac{n(n+1)}{2n^2} + 2 = \frac{1}{2n} + \frac{5}{2}$$

$$\lim_{n \rightarrow \infty} A(S_n) = \lim_{n \rightarrow \infty} \left(\frac{1}{2n} + \frac{5}{2} \right) = \frac{5}{2}$$

12. $\Delta x = \frac{1}{n}, x_i = \frac{i}{n}$

$$f(x_i) \Delta x = \left[\frac{1}{2} \cdot \left(\frac{i}{n} \right)^2 + 1 \right] \left(\frac{1}{n} \right) = \frac{i^2}{2n^3} + \frac{1}{n}$$

$$A(S_n) = \left[\left(\frac{1^2}{2n^3} + \frac{1}{n} \right) + \left(\frac{2^2}{2n^3} + \frac{1}{n} \right) + \cdots + \left(\frac{n^2}{2n^3} + \frac{1}{n} \right) \right] = \frac{1}{2n^3} (1^2 + 2^2 + 3^2 + \cdots + n^2) + 1$$

$$= \frac{1}{2n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] + 1 = \frac{1}{12} \left[\frac{2n^3 + 3n^2 + n}{n^3} \right] + 1 = \frac{1}{12} \left[2 + \frac{3}{n} + \frac{1}{n^2} \right] + 1$$

$$\lim_{n \rightarrow \infty} A(S_n) = \lim_{n \rightarrow \infty} \left[\frac{1}{12} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) + 1 \right] = \frac{7}{6}$$

13. $\Delta x = \frac{2}{n}, x_i = -1 + \frac{2i}{n}$

$$f(x_i) \Delta x = \left[2 \left(-1 + \frac{2i}{n} \right) + 2 \right] \left(\frac{2}{n} \right) = \frac{8i}{n^2}$$

$$A(S_n) = \left[\left(\frac{8}{n^2} \right) + \left(\frac{16}{n^2} \right) + \cdots + \left(\frac{8n}{n^2} \right) \right]$$

$$= \frac{8}{n^2} (1+2+3+\cdots+n) = \frac{8}{n^2} \left[\frac{n(n+1)}{2} \right]$$

$$= 4 \left[\frac{n^2 + n}{n^2} \right] = 4 + \frac{4}{n}$$

$$\lim_{n \rightarrow \infty} A(S_n) = \lim_{n \rightarrow \infty} \left(4 + \frac{4}{n} \right) = 4$$

14. First, consider $a = 0$ and $b = 2$.

$$\Delta x = \frac{2}{n}, x_i = \frac{2i}{n}$$

$$f(x_i) \Delta x = \left(\frac{2i}{n} \right)^2 \left(\frac{2}{n} \right) = \frac{8i^2}{n^3}$$

$$A(S_n) = \left[\left(\frac{8}{n^3} \right) + \left(\frac{8(2^2)}{n^3} \right) + \cdots + \left(\frac{8n^2}{n^3} \right) \right]$$

$$= \frac{8}{n^3} (1^2 + 2^2 + \cdots + n^2) = \frac{8}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$= \frac{4}{3} \left[\frac{2n^3 + 3n^2 + n}{n^3} \right] = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

$$\lim_{n \rightarrow \infty} A(S_n) = \lim_{n \rightarrow \infty} \left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} \right) = \frac{8}{3}$$

By symmetry, $A = 2 \left(\frac{8}{3} \right) = \frac{16}{3}$.

15. $\Delta x = \frac{1}{n}, x_i = \frac{i}{n}$

$$f(x_i) \Delta x = \left(\frac{i}{n} \right)^3 \left(\frac{1}{n} \right) = \frac{i^3}{n^4}$$

$$A(S_n) = \left[\frac{1}{n^4} (1^3) + \frac{1}{n^4} (2^3) + \cdots + \frac{1}{n^4} (n^3) \right]$$

$$= \frac{1}{n^4} (1^3 + 2^3 + \cdots + n^3) = \frac{1}{n^4} \left[\frac{n(n+1)}{2} \right]^2$$

$$= \frac{1}{n^4} \left[\frac{n^4 + 2n^3 + n^2}{4} \right] = \frac{1}{4} \left[1 + \frac{2}{n} + \frac{1}{n^2} \right]$$

$$\lim_{n \rightarrow \infty} A(S_n) = \lim_{n \rightarrow \infty} \frac{1}{4} \left[1 + \frac{2}{n} + \frac{1}{n^2} \right] = \frac{1}{4}$$

16. $\Delta x = \frac{1}{n}, x_i = \frac{i}{n}$

$$f(x_i) \Delta x = \left[\left(\frac{i}{n} \right)^3 + \frac{i}{n} \right] \left(\frac{1}{n} \right) = \frac{i^3}{n^4} + \frac{i}{n^2}$$

$$A(S_n) = \frac{1}{n^4} (1^3 + 2^3 + \cdots + n^3) + \frac{1}{n^2} (1 + 2 + \cdots + n)$$

$$= \frac{1}{n^4} \left[\frac{n(n+1)}{2} \right]^2 + \frac{1}{n^2} \left[\frac{n(n+1)}{2} \right]$$

$$= \frac{n^2 + 2n + 1}{4n^2} + \frac{n^2 + n}{2n^2} = \frac{3n^2 + 4n + 1}{4n^2} = \frac{3}{4} + \frac{1}{n} + \frac{1}{4n^2}$$

$$\lim_{n \rightarrow \infty} A(S_n) = \frac{3}{4}$$

$$\begin{aligned} 17. \quad f(t_i)\Delta t &= \left[\frac{i}{n} + 2 \right] \frac{1}{n} = \frac{i}{n^2} + \frac{2}{n} \\ A(S_n) &= \sum_{i=1}^n \left(\frac{i}{n^2} + \frac{2}{n} \right) = \frac{1}{n^2} \sum_{i=1}^n i + \sum_{i=1}^n \frac{2}{n} \\ &= \frac{1}{n^2} \left[\frac{n(n+1)}{2} \right] + 2 \\ &= \left[\frac{n^2 + n}{2n^2} \right] + 2 \\ &= \left(\frac{1}{2} + \frac{1}{2n} \right) + 2 \end{aligned}$$

$$\lim_{n \rightarrow \infty} A(S_n) = \frac{1}{2} + 2 = \frac{5}{2}$$

The object traveled $2\frac{1}{2}$ ft.

$$\begin{aligned} 18. \quad f(t_i)\Delta t &= \left[\frac{1}{2} \left(\frac{i}{n} \right)^2 + 1 \right] \frac{1}{n} = \frac{i^2}{2n^3} + \frac{1}{n} \\ A(S_n) &= \sum_{i=1}^n \left(\frac{i^2}{2n^3} + \frac{1}{n} \right) = \frac{1}{2n^3} \sum_{i=1}^n i^2 + \sum_{i=1}^n \frac{1}{n} \\ &= \frac{1}{2n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] + 1 = \frac{1}{12} \left[2 + \frac{3}{n} + \frac{1}{n^2} \right] + 1 \\ \lim_{n \rightarrow \infty} A(S_n) &= \frac{1}{12} (2) + 1 = \frac{7}{6} \approx 1.17 \end{aligned}$$

The subject traveled about 1.17 feet.

$$\begin{aligned} 19. \quad a. \quad f(x_i)\Delta x &= \left(\frac{ib}{n} \right)^2 \left(\frac{b}{n} \right) = \frac{b^3 i^2}{n^3} \\ A_0^b &= \frac{b^3}{n^3} \sum_{i=1}^n i^2 = \frac{b^3}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] \\ &= \frac{b^3}{6} \left[2 + \frac{3}{n} + \frac{1}{n^2} \right] \\ \lim_{n \rightarrow \infty} A_0^b &= \frac{2b^3}{6} = \frac{b^3}{3} \end{aligned}$$

b. Since $a \geq 0$, $A_0^b = A_0^a + A_a^b$, or

$$A_a^b = A_0^b - A_0^a = \frac{b^3}{3} - \frac{a^3}{3}.$$

$$20. \quad A_3^5 = \frac{5^3}{3} - \frac{3^3}{3} = \frac{98}{3} \approx 32.7$$

The object traveled about 32.7 m.

$$21. \quad a. \quad A_0^5 = \frac{5^3}{3} = \frac{125}{3}$$

$$b. \quad A_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{63}{3} = 21$$

$$c. \quad A_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{117}{3} = 39$$

$$22. \quad a. \quad \Delta x = \frac{b}{n}, x_i = \frac{bi}{n}$$

$$f(x_i)\Delta x = \left(\frac{bi}{n} \right)^m \left(\frac{b}{n} \right) = \frac{b^{m+1} i^m}{n^{m+1}}$$

$$A(S_n) = \frac{b^{m+1}}{n^{m+1}} \sum_{i=1}^n i^m$$

$$= \frac{b^{m+1}}{n^{m+1}} \left[\frac{n^{m+1}}{m+1} + C_n \right]$$

$$= \frac{b^{m+1}}{m+1} + \frac{b^{m+1} C_n}{n^{m+1}}$$

$$A_0^b(x^m) = \lim_{n \rightarrow \infty} A(S_n) = \frac{b^{m+1}}{m+1}$$

$\lim_{n \rightarrow \infty} \frac{C_n}{n^{m+1}} = 0$ since C_n is a polynomial in n of degree m .

b. Notice that $A_a^b(x^m) = A_0^b(x^m) - A_0^a(x^m)$.

Thus, using part a, $A_a^b(x^m) = \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}$.

$$23. \quad a. \quad A_0^2(x^3) = \frac{2^{3+1}}{3+1} = 4$$

$$b. \quad A_1^2(x^3) = \frac{2^{3+1}}{3+1} - \frac{1^{3+1}}{3+1} = 4 - \frac{1}{4} = \frac{15}{4}$$

$$\begin{aligned} c. \quad A_1^2(x^5) &= \frac{2^{5+1}}{5+1} - \frac{1^{5+1}}{5+1} = \frac{32}{6} - \frac{1}{6} = \frac{63}{6} \\ &= \frac{21}{2} = 10.5 \end{aligned}$$

$$d. \quad A_0^2(x^9) = \frac{2^{9+1}}{9+1} = \frac{1024}{10} = 102.4$$

24. Inscribed:

Consider an isosceles triangle formed by one side of the polygon and the center of the circle. The

angle at the center is $\frac{2\pi}{n}$. The length of the base

is $2r \sin \frac{\pi}{n}$. The height is $r \cos \frac{\pi}{n}$. Thus the area

of the triangle is $r^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} = \frac{1}{2} r^2 \sin \frac{2\pi}{n}$.

$$A_n = n \left(\frac{1}{2} r^2 \sin \frac{2\pi}{n} \right) = \frac{1}{2} n r^2 \sin \frac{2\pi}{n}$$

Circumscribed:

Consider an isosceles triangle formed by one side of the polygon and the center of the circle. The

angle at the center is $\frac{2\pi}{n}$. The length of the base

is $2r \tan \frac{\pi}{n}$. The height is r . Thus the area of the

triangle is $r^2 \tan \frac{\pi}{n}$.

$$B_n = n \left(r^2 \tan \frac{\pi}{n} \right) = n r^2 \tan \frac{\pi}{n}$$

$$\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} \frac{1}{2} n r^2 \sin \frac{2\pi}{n} = \lim_{n \rightarrow \infty} \pi r^2 \left(\frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \right)$$

$$= \pi r^2$$

$$\lim_{n \rightarrow \infty} B_n = \lim_{n \rightarrow \infty} n r^2 \tan \frac{\pi}{n} = \lim_{n \rightarrow \infty} \frac{\pi r^2}{\cos \frac{\pi}{n}} \left(\frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} \right)$$

$$= \pi r^2$$

5.5 Concepts Review

1. Riemann sum

3. $A_{\text{up}} - A_{\text{down}}$

2. definite integral: $\int_a^b f(x) dx$

$$4. 8 - \frac{1}{2} = \frac{15}{2}$$

Problem Set 5.5

$$1. R_p = f(2)(2.5 - 1) + f(3)(3.5 - 2.5) + f(4.5)(5 - 3.5) = 4(1.5) + 3(1) + (-2.25)(1.5) = 5.625$$

$$2. R_p = f(0.5)(0.7 - 0) + f(1.5)(1.7 - 0.7) + f(2)(2.7 - 1.7) + f(3.5)(4 - 2.7) \\ = 1.25(0.7) + (-0.75)(1) + (-1)(1) + 1.25(1.3) = 0.75$$

$$3. R_p = \sum_{i=1}^5 f(\bar{x}_i) \Delta x_i = f(3)(3.75 - 3) + f(4)(4.25 - 3.75) + f(4.75)(5.5 - 4.25) + f(6)(6 - 5.5) + f(6.5)(7 - 6) \\ = 2(0.75) + 3(0.5) + 3.75(1.25) + 5(0.5) + 5.5(1) = 15.6875$$

$$4. R_p = \sum_{i=1}^4 f(\bar{x}_i) \Delta x_i = f(-2)(-1.3 + 3) + f(-0.5)(0 + 1.3) + f(0)(0.9 - 0) + f(2)(2 - 0.9) \\ = 4(1.7) + 3.25(1.3) + 3(0.9) + 2(1.1) = 15.925$$

$$5. R_p = \sum_{i=1}^8 f(\bar{x}_i) \Delta x_i = [f(-1.75) + f(-1.25) + f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)](0.5) \\ = [-0.21875 - 0.46875 - 0.46875 - 0.21875 + 0.28125 + 1.03125 + 2.03125 + 3.28125](0.5) = 2.625$$

$$6. R_p = \sum_{i=1}^6 f(\bar{x}_i) \Delta x_i = [f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)](0.5) \\ = [1.5 + 5 + 14.5 + 33 + 63.5 + 109](0.5) = 113.25$$

$$7. \int_1^3 x^3 dx$$

$$8. \int_0^2 (x+1)^3 dx$$

$$9. \int_{-1}^1 \frac{x^2}{1+x} dx$$

$$10. \int_0^{\pi} (\sin x)^2 dx$$

$$11. \Delta x = \frac{2}{n}, \bar{x}_i = \frac{2i}{n}$$

$$f(\bar{x}_i) = \bar{x}_i + 1 = \frac{2i}{n} + 1$$

$$\begin{aligned} \sum_{i=1}^n f(\bar{x}_i) \Delta x &= \sum_{i=1}^n \left[1 + i \left(\frac{2}{n} \right) \right] \frac{2}{n} \\ &= \frac{2}{n} \sum_{i=1}^n 1 + \frac{4}{n^2} \sum_{i=1}^n i = \frac{2}{n}(n) + \frac{4}{n^2} \left[\frac{n(n+1)}{2} \right] \\ &= 2 + 2 \left(1 + \frac{1}{n} \right) \end{aligned}$$

$$\int_0^2 (x+1) dx = \lim_{n \rightarrow \infty} \left[2 + 2 \left(1 + \frac{1}{n} \right) \right] = 4$$

$$12. \Delta x = \frac{2}{n}, \bar{x}_i = \frac{2i}{n}$$

$$f(\bar{x}_i) = \left(\frac{2i}{n} \right)^2 + 1 = \frac{4i^2}{n^2} + 1$$

$$\begin{aligned} \sum_{i=1}^n f(\bar{x}_i) \Delta x &= \sum_{i=1}^n \left[1 + i^2 \left(\frac{4}{n^2} \right) \right] \frac{2}{n} \\ &= \frac{2}{n} \sum_{i=1}^n 1 + \frac{8}{n^3} \sum_{i=1}^n i^2 = \frac{2}{n}(n) + \frac{8}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] \\ &= 2 + \frac{4}{3} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) \end{aligned}$$

$$\int_0^2 (x^2 + 1) dx = \lim_{n \rightarrow \infty} \left[2 + \frac{4}{3} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) \right] = \frac{14}{3}$$

$$13. \Delta x = \frac{3}{n}, \bar{x}_i = -2 + \frac{3i}{n}$$

$$f(\bar{x}_i) = 2 \left(-2 + \frac{3i}{n} \right) + \pi = \pi - 4 + \frac{6i}{n}$$

$$\sum_{i=1}^n f(\bar{x}_i) \Delta x = \sum_{i=1}^n \left[\pi - 4 + \frac{6i}{n} \right] \frac{3}{n}$$

$$16. \Delta x = \frac{20}{n}, \bar{x}_i = -10 + \frac{20i}{n}$$

$$f(\bar{x}_i) = \left(-10 + \frac{20i}{n} \right)^2 + \left(-10 + \frac{20i}{n} \right) = 90 - \frac{380i}{n} + \frac{400i^2}{n^2}$$

$$\sum_{i=1}^n f(\bar{x}_i) \Delta x = \sum_{i=1}^n \left[90 - i \left(\frac{380}{n} \right) + i^2 \left(\frac{400}{n^2} \right) \right] \frac{20}{n} = \frac{20}{n} \sum_{i=1}^n 90 - \frac{7600}{n^2} \sum_{i=1}^n i + \frac{8000}{n^3} \sum_{i=1}^n i^2$$

$$\begin{aligned} &= \frac{3}{n} \sum_{i=1}^n (\pi - 4) + \frac{18}{n^2} \sum_{i=1}^n i = 3(\pi - 4) + \frac{18}{n^2} \left[\frac{n(n+1)}{2} \right] \\ &= 3\pi - 12 + 9 \left(1 + \frac{1}{n} \right) \end{aligned}$$

$$\begin{aligned} \int_{-2}^1 (2x + \pi) dx &= \lim_{n \rightarrow \infty} \left[3\pi - 12 + 9 \left(1 + \frac{1}{n} \right) \right] \\ &= 3\pi - 3 \end{aligned}$$

$$14. \Delta x = \frac{3}{n}, \bar{x}_i = -2 + \frac{3i}{n}$$

$$f(\bar{x}_i) = 3 \left(-2 + \frac{3i}{n} \right)^2 + 2 = 14 - \frac{36i}{n} + \frac{27i^2}{n^2}$$

$$\begin{aligned} \sum_{i=1}^n f(\bar{x}_i) \Delta x &= \sum_{i=1}^n \left[14 - \left(\frac{36}{n} \right) i + \left(\frac{27}{n^2} \right) i^2 \right] \frac{3}{n} \\ &= \frac{3}{n} \sum_{i=1}^n 14 - \frac{108}{n^2} \sum_{i=1}^n i + \frac{81}{n^3} \sum_{i=1}^n i^2 \\ &= 42 - \frac{108}{n^2} \left[\frac{n(n+1)}{2} \right] + \frac{81}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] \\ &= 42 - 54 \left(1 + \frac{1}{n} \right) + \frac{27}{2} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) \end{aligned}$$

$$\begin{aligned} \int_{-2}^1 (3x^2 + 2) dx \\ &= \lim_{n \rightarrow \infty} \left[42 - 54 \left(1 + \frac{1}{n} \right) + \frac{27}{2} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) \right] = 15 \end{aligned}$$

$$15. \Delta x = \frac{5}{n}, \bar{x}_i = \frac{5i}{n}$$

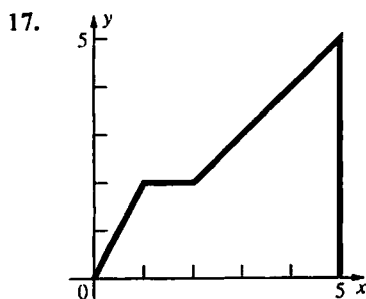
$$f(\bar{x}_i) = 1 + \frac{5i}{n}$$

$$\begin{aligned} \sum_{i=1}^n f(\bar{x}_i) \Delta x &= \sum_{i=1}^n \left[1 + i \left(\frac{5}{n} \right) \right] \frac{5}{n} \\ &= \frac{5}{n} \sum_{i=1}^n 1 + \frac{25}{n^2} \sum_{i=1}^n i = 5 + \frac{25}{n^2} \left[\frac{n(n+1)}{2} \right] \\ &= 5 + \frac{25}{2} \left(1 + \frac{1}{n} \right) \end{aligned}$$

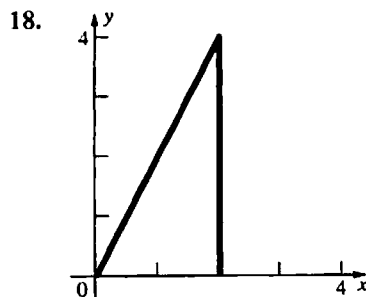
$$\int_0^5 (x+1) dx = \lim_{n \rightarrow \infty} \left[5 + \frac{25}{2} \left(1 + \frac{1}{n} \right) \right] = \frac{35}{2}$$

$$= 1800 - \frac{7600}{n^2} \left[\frac{n(n+1)}{2} \right] + \frac{8000}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] = 1800 - 3800 \left(1 + \frac{1}{n} \right) + \frac{4000}{3} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right)$$

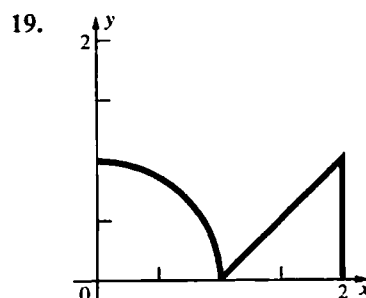
$$\int_{-10}^{10} (x^2 + x) dx = \lim_{n \rightarrow \infty} \left[1800 - 3800 \left(1 + \frac{1}{n} \right) + \frac{4000}{3} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) \right] = \frac{2000}{3}$$



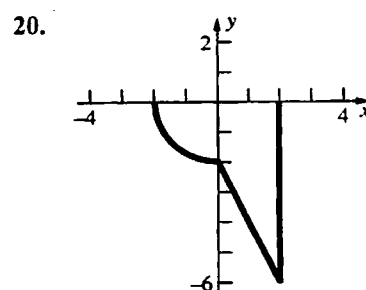
$$\int_0^5 f(x) dx = \frac{1}{2}(1)(2) + 1(2) + 3(2) + \frac{1}{2}(3)(3) = \frac{27}{2}$$



$$\int_0^2 f(x) dx = \frac{1}{2}(2)(4) = 4$$



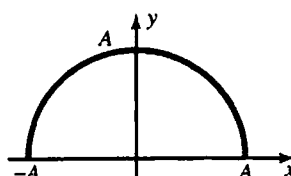
$$\int_0^2 f(x) dx = \frac{1}{4}(\pi \cdot 1^2) + \frac{1}{2}(1)(1) = \frac{1}{2} + \frac{\pi}{4}$$



$$\int_{-2}^2 f(x) dx = -\frac{1}{4}(\pi \cdot 2^2) - (2)(2) - \frac{1}{2}(2)(4)$$

$$= -\pi - 8$$

21. The area under the curve is equal to the area of a semi-circle: $\int_{-A}^A \sqrt{A^2 - x^2} dx = \frac{1}{2} \pi A^2$.



22. Partition $[0, 1]$ into n regular intervals, so

$$|P| = \frac{1}{n}.$$

$$\text{If } \bar{x}_i = \frac{i}{n} + \frac{1}{2n}, f(\bar{x}_i) = 1.$$

$$\lim_{|P| \rightarrow 0} \sum_{i=1}^n f(\bar{x}_i) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} = 1$$

$$\text{If } \bar{x}_i = \frac{i}{n} + \frac{1}{\pi n}, f(\bar{x}_i) = 0.$$

$$\lim_{|P| \rightarrow 0} \sum_{i=1}^n f(\bar{x}_i) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n 0 = 0$$

Thus f is not integrable on $[0, 1]$.

23. a. $\int_{-3}^3 [x] dx = (-3 - 2 - 1 + 0 + 1 + 2)(1) = -3$

b. $\int_{-3}^3 [x]^2 dx = [(-3)^2 + (-2)^2 + (-1)^2 + 0 + 1 + 4](1) = 19$

c. $\int_{-3}^3 (x - [x]) dx = 6 \left[\frac{1}{2}(1)(1) \right] = 3$

d. $\int_{-3}^3 (x - [x])^2 dx = \int_{-3}^3 (x^2 - 2x[x] + [x]^2) dx$
 $= \int_{-3}^3 x^2 dx - 2 \int_{-3}^3 x[x] dx + \int_{-3}^3 [x]^2 dx$
 $= 2 \frac{3^3}{3} - 2 \left[-3 \int_{-3}^{-2} x dx - 2 \int_{-2}^{-1} x dx - \int_{-1}^0 x dx \right]$
 $+ 0 \int_0^1 x dx + \int_1^2 x dx + 2 \int_2^3 x dx + 19$

$$\begin{aligned}
&= 18 - 2 \left[(-3) \left(1(-2) - \frac{1}{2}(1)(1) \right) \right. \\
&\quad \left. - 2 \left(1(-1) - \frac{1}{2}(1)(1) \right) - \left(-\frac{1}{2}(1)(1) \right) + 0 \right. \\
&\quad \left. + \left(1 \cdot 1 + \frac{1}{2}(1)(1) \right) + 2 \left(1 \cdot 2 + \frac{1}{2}(1)(1) \right) \right] + 19 \\
&= 37 - 2 \left[(-3) \left(-\frac{5}{2} \right) - 2 \left(-\frac{3}{2} \right) + \frac{1}{2} + \frac{3}{2} + 2 \left(\frac{5}{2} \right) \right] \\
&= 37 - 35 = 2
\end{aligned}$$

e. $\int_{-3}^3 |x| dx = \frac{1}{2}(3)(3) + \frac{1}{2}(3)(3) = 9$

f. $\int_{-3}^3 x|x| dx = \frac{(-3)^3}{3} + \frac{(3)^3}{3} = 0$

g. $\int_{-1}^2 |x| \llbracket x \rrbracket dx = -\int_{-1}^0 |x| dx + 0 \int_0^1 |x| dx + \int_1^2 |x| dx$
 $= -\frac{1}{2}(1)(1) + 1(1) + \frac{1}{2}(1)(1) = 1$

h. $\int_{-1}^2 x^2 \llbracket x \rrbracket dx = -\int_{-1}^0 x^2 dx + 0 \int_0^1 x^2 dx$
 $\quad + \int_1^2 x^2 dx$
 $= -\frac{1^3}{3} + \left(\frac{2^3}{3} - \frac{1^3}{3} \right) = 2$

24. a. $\int_{-1}^1 f(x) dx = -3 + 3 = 0$

b. $\int_{-1}^1 g(x) dx = 3 + 3 = 6$

c. $\int_{-1}^1 |f(x)| dx = 3 + 3 = 6$

d. $\int_{-1}^1 [-g(x)] dx = -3 + (-3) = -6$

e. $\int_{-1}^1 xg(x) dx = 0$ because $xg(x)$ is an odd function.

f. $\int_{-1}^1 f^3(x)g(x) dx = 0$ because $f^3(x)g(x)$ is an odd function.

25. $R_P = \frac{1}{2} \sum_{i=1}^n (x_i + x_{i-1})(x_i - x_{i-1})$
 $= \frac{1}{2} \sum_{i=1}^n (x_i^2 - x_{i-1}^2)$
 $= \frac{1}{2} [(x_1^2 - x_0^2) + (x_2^2 - x_1^2) + (x_3^2 - x_2^2)$
 $\quad + \cdots + (x_n^2 - x_{n-1}^2)]$
 $= \frac{1}{2} (x_n^2 - x_0^2)$
 $= \frac{1}{2} (b^2 - a^2)$
 $\lim_{n \rightarrow \infty} \frac{1}{2} (b^2 - a^2) = \frac{1}{2} (b^2 - a^2)$

26. Note that $\bar{x}_i = \left[\frac{1}{3} (x_{i-1}^2 + x_{i-1}x_i + x_i^2) \right]^{1/2}$
 $\geq \left[\frac{1}{3} (x_{i-1}^2 + x_{i-1}^2 + x_{i-1}^2) \right]^{1/2} = x_{i-1}$ and
 $\bar{x}_i = \left[\frac{1}{3} (x_{i-1}^2 + x_{i-1}x_i + x_i^2) \right]^{1/2}$
 $\leq \left[\frac{1}{3} (x_i^2 + x_i^2 + x_i^2) \right]^{1/2} = x_i$.
 $R_P = \sum_{i=1}^n \bar{x}_i^2 \Delta x_i$
 $= \sum_{i=1}^n \frac{1}{3} (x_i^2 + x_{i-1}x_i + x_{i-1}^2)(x_i - x_{i-1})$
 $= \frac{1}{3} \sum_{i=1}^n (x_i^3 - x_{i-1}^3)$
 $= \frac{1}{3} [(x_1^3 - x_0^3) + (x_2^3 - x_1^3) + (x_3^3 - x_2^3)$
 $\quad + \cdots + (x_n^3 - x_{n-1}^3)]$
 $= \frac{1}{3} (x_n^3 - x_0^3) = \frac{1}{3} (b^3 - a^3)$

27. Left: $\int_0^2 (x^3 + 1) dx \approx 5.24$

Right: $\int_0^2 (x^3 + 1) dx \approx 6.84$

Midpoint: $\int_0^2 (x^3 + 1) dx \approx 5.98$

28. Left: $\int_0^1 \tan x dx \approx 0.5398$

Right: $\int_0^1 \tan x dx \approx 0.6955$

Midpoint: $\int_0^1 \tan x dx \approx 0.6146$

29. Left: $\int_0^1 \cos x \, dx \approx 0.8638$

Right: $\int_0^1 \cos x \, dx \approx 0.8178$

Midpoint: $\int_0^1 \cos x \, dx \approx 0.8418$

30. Left: $\int_1^3 \left(\frac{1}{x}\right) dx \approx 1.1682$

Right: $\int_1^3 \left(\frac{1}{x}\right) dx \approx 1.0349$

Midpoint: $\int_1^3 \left(\frac{1}{x}\right) dx \approx 1.0971$

31. $\int_{-2}^4 (-1 + |x|) dx = 4$

32. $\int_0^6 \sin x \, dx \approx 0.0398$

33. $\int_{-1}^2 (x^4 - 3x^2 + 1) dx = 0.6$

34. $\int_{-2}^2 \left(\frac{x-1}{x^2+1}\right) dx \approx -2.2143$

35. $\int_0^1 \left(\frac{1}{x}\right) dx$ is undefined; $\frac{1}{x}$ is not bounded near $x = 0$.

36. $\int_0^2 \tan x \, dx$ is undefined; $\tan x$ is not bounded near $x = \frac{\pi}{2}$.

5.6 Concepts Review

1. $4(4-2) = 8$; $16(4-2) = 32$

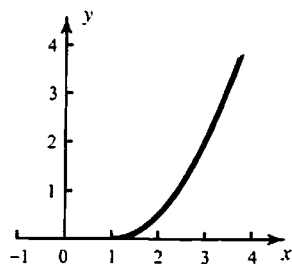
2. $\sin^3 x$

3. $\int_1^4 f(x) dx$; $\int_2^5 \sqrt{x} dx$

4. 5

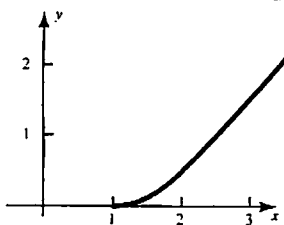
Problem Set 5.6

1. $A(x) = \frac{1}{2}(x-1)(-1+x)$, $x > 1$

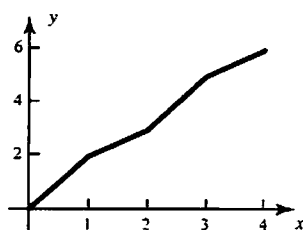


2. If $1 \leq x \leq 2$, then $A(x) = \frac{1}{2}(x-1)^2$.

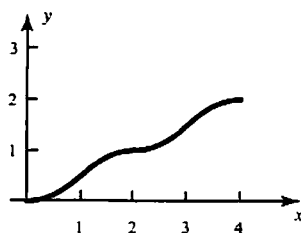
If $2 \leq x$, then $A(x) = x - \frac{3}{2}$



3.
$$A(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 2 + (x-1) & 1 < x \leq 2 \\ 3 + 2(x-2) & 2 < x \leq 3 \\ 5 + (x-3) & 3 < x \leq 4 \\ \text{etc.} \end{cases}$$



4.
$$A(x) = \begin{cases} \frac{1}{2}x^2 & 0 \leq x \leq 1 \\ \frac{1}{2} + \frac{1}{2}(3-x)(x-1) & 1 < x \leq 2 \\ 1 + \frac{1}{2}(x-2)^2 & 2 < x \leq 3 \\ \frac{3}{2} + \frac{1}{2}(5-x)(x-3) & 3 < x \leq 4 \\ 2 + \frac{1}{2}(x-4)^2 & 4 < x \leq 5 \\ \text{etc.} \end{cases}$$



$$5. \int_1^2 2f(x) dx = 2 \int_1^2 f(x) dx = 2(3) = 6$$

$$6. \int_0^2 2f(x) dx = 2 \int_0^2 f(x) dx \\ = 2 \left[\int_0^1 f(x) dx + \int_1^2 f(x) dx \right] = 2(2+3) = 10$$

$$7. \int_0^2 [2f(x) + g(x)] dx = 2 \int_0^2 f(x) dx + \int_0^2 g(x) dx \\ = 2 \left[\int_0^1 f(x) dx + \int_1^2 f(x) dx \right] + \int_0^2 g(x) dx \\ = 2(2+3) + 4 = 14$$

$$8. \int_0^1 [2f(s) + g(s)] ds = 2 \int_0^1 f(s) ds + \int_0^1 g(s) ds \\ = 2(2) + (-1) = 3$$

$$9. \int_2^1 [2f(s) + 5g(s)] ds = -2 \int_1^2 f(s) ds - 5 \int_1^2 g(s) ds \\ = -2(3) - 5 \left[\int_0^2 g(s) ds - \int_0^1 g(s) ds \right] \\ = -6 - 5[4+1] = -31$$

$$10. \int_1^1 [3f(x) + 2g(x)] dx = 0$$

$$11. \int_0^2 [3f(t) + 2g(t)] dt \\ = 3 \left[\int_0^1 f(t) dt + \int_1^2 f(t) dt \right] + 2 \int_0^2 g(t) dt \\ = 3(2+3) + 2(4) = 23$$

$$12. \int_0^2 [\sqrt{3}f(t) + \sqrt{2}g(t) + \pi] dt \\ = \sqrt{3} \left[\int_0^1 f(t) dt + \int_1^2 f(t) dt \right] + \sqrt{2} \int_0^2 g(t) dt \\ + \pi \int_0^2 dt \\ = \sqrt{3}(2+3) + \sqrt{2}(4) + 2\pi = 5\sqrt{3} + 4\sqrt{2} + 2\pi$$

$$13. G'(x) = D_x \left[\int_1^x 2t dt \right] = 2x$$

$$14. G'(x) = D_x \left[\int_x^1 2t dt \right] = D_x \left[- \int_1^x 2t dt \right] = -2x$$

$$15. G'(x) = D_x \left[\int_0^x (2t^2 + \sqrt{t}) dt \right] = 2x^2 + \sqrt{x}$$

$$16. G'(x) = D_x \left[\int_1^x \cos^3(2t) \tan(t) dt \right] \\ = \cos^3(2x) \tan(x)$$

$$17. G'(x) = D_x \left[\int_x^{\pi/2} (s-2) \cot(2s) ds \right] \\ = D_x \left[- \int_{\pi/2}^x (s-2) \cot(2s) ds \right] = -(x-2) \cot(2x)$$

$$18. G'(x) = D_x \left[\int_1^x xt dt \right] = D_x \left[x \int_1^x t dt \right] \\ = D_x \left[x \left[\frac{t^2}{2} \right]_1^x \right] = D_x \left[x \left(\frac{x^2-1}{2} \right) \right] \\ = D_x \left(\frac{x^3}{2} - \frac{x}{2} \right) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$19. G'(x) = D_x \left[\int_1^{x^2} \sin t dt \right] = 2x \sin(x^2)$$

$$20. G'(x) = D_x \left[\int_1^{x^2+x} \sqrt{2z + \sin z} dz \right] \\ = (2x+1) \sqrt{2(x^2+x) + \sin(x^2+x)} \\ = (2x+1) \sqrt{2x^2 + 2x + \sin(x^2+x)}$$

$$21. G(x) = \int_{-x^2}^x \frac{t^2}{1+t^2} dt = \int_{-x^2}^0 \frac{t^2}{1+t^2} dt + \int_0^x \frac{t^2}{1+t^2} dt \\ = - \int_0^{-x^2} \frac{t^2}{1+t^2} dt + \int_0^x \frac{t^2}{1+t^2} dt \\ G'(x) = - \frac{(-x^2)^2}{1+(-x^2)^2} (-2x) + \frac{x^2}{1+x^2} \\ = \frac{2x^5}{1+x^4} + \frac{x^2}{1+x^2}$$

$$22. G(x) = D_x \left[\int_{\cos x}^{\sin x} t^5 dt \right] \\ = D_x \left[\int_0^{\sin x} t^5 dt + \int_{\cos x}^0 t^5 dt \right] \\ = D_x \left[\int_0^{\sin x} t^5 dt - \int_0^{\cos x} t^5 dt \right] \\ = \sin^5 x \cos x + \cos^5 x \sin x$$

23. $f'(x) = \frac{x}{\sqrt{a^2 + x^2}}$

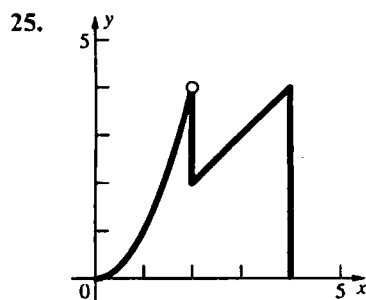
$$f''(x) = \frac{1}{\sqrt{a^2 + x^2}} - \frac{x^2}{\sqrt{(a^2 + x^2)^3}}$$

$$= \frac{a^2}{\sqrt{(a^2 + x^2)^3}}, \text{ which is always positive.}$$

24. The graph is concave up when $f''(x)$ is positive:

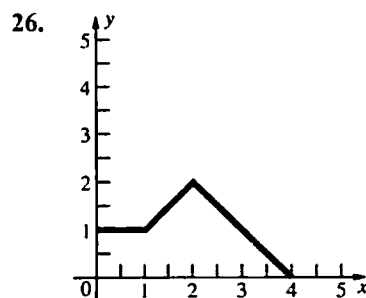
$$f'(x) = \frac{1+x}{1+x^2}, \quad f''(x) = \frac{1-2x-x^2}{(1+x^2)^2}, \quad f''(x) > 0$$

when $x^2 + 2x - 1 < 0$. By the quadratic formula the left side is zero for $x = -1 \pm \sqrt{2}$, and testing the point $x = 0$ indicates that $f''(x) > 0$ on $(-1 - \sqrt{2}, -1 + \sqrt{2})$.



$$\int_0^4 f(x) dx = \int_0^2 x^2 dx + \int_2^4 x dx$$

$$= \left[\frac{x^3}{3} \right]_0^2 + \left[\frac{x^2}{2} \right]_2^4 = \left(\frac{8}{3} - 0 \right) + (8 - 2) = \frac{26}{3}$$

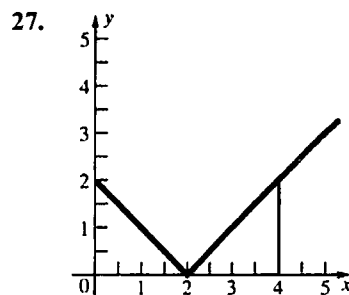


$$\int_0^4 f(x) dx = \int_0^1 dx + \int_1^2 x dx + \int_2^4 (4-x) dx$$

$$= \int_0^1 dx + \int_1^2 x dx + 4 \int_2^4 dx - \int_2^4 x dx$$

$$= [x]_0^1 + \left[\frac{x^2}{2} \right]_1^2 + 4[x]_2^4 - \left[\frac{x^2}{2} \right]_2^4$$

$$= (1-0) + \left(2 - \frac{1}{2} \right) + 4(4-2) - (8-2) = \frac{9}{2}$$

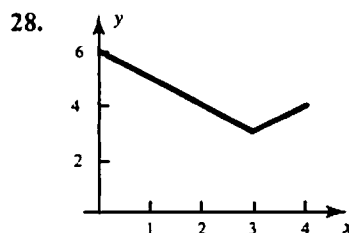


$$\int_0^4 f(x) dx = \int_0^2 (2-x) dx + \int_2^4 (x-2) dx$$

$$= 2 \int_0^2 dx - \int_0^2 x dx + \int_2^4 x dx - 2 \int_2^4 dx$$

$$= 2[x]_0^2 - \left[\frac{x^2}{2} \right]_0^2 + \left[\frac{x^2}{2} \right]_2^4 - 2[x]_2^4$$

$$= 2(2-0) - (2-0) + (8-2) - 2(4-2) = 4$$



$$\int_0^4 (3+|x-3|) dx$$

$$= \int_0^3 (3+|x-3|) dx + \int_3^4 (3+|x-3|) dx$$

$$= 3 \cdot \frac{9}{2} + 1 \cdot \frac{7}{2} = 17$$

29. a. $F(0) = \int_0^0 (t^4 + 1) dt = 0$

b. $y = F(x)$

$$\frac{dy}{dx} = F'(x) = x^4 + 1$$

$$dy = (x^4 + 1) dx$$

$$y = \frac{1}{5} x^5 + x + C$$

c. Now apply the initial condition $y(0) = 0$:

$$0 = \frac{1}{5} 0^5 + 0 + C$$

$$C = 0$$

Thus $y = F(x) = \frac{1}{5} x^5 + x$

d. $\int_0^1 (x^4 + 1) dx = F(1) = \frac{1}{5} 1^5 + 1 = \frac{6}{5}$

30. a.

$$G(x) = \int_0^x \sin t \, dt$$

$$G(0) = \int_0^0 \sin t \, dt = 0$$

$$G(2\pi) = \int_0^{2\pi} \sin t \, dt = 0$$

b. Let $y = G(x)$. Then

$$\frac{dy}{dx} = G'(x) = \sin x.$$

c. $dy = \sin x \, dx$

$$y = -\cos x + C$$

Apply the initial condition

$$0 = y(0) = -\cos 0 + C. \text{ Thus, } C = 1,$$

and hence $y = G(x) = 1 - \cos x$.

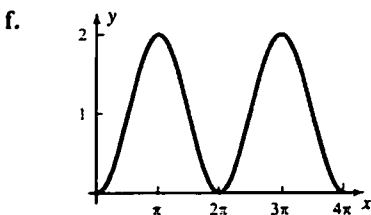
d. $\int_0^\pi \sin x \, dx = G(\pi) = 1 - \cos \pi = 2$

e. G attains the maximum of 2 when $x = 0, 2\pi, 4\pi$.

G attains the minimum of 0 when $x = \pi, 3\pi$

Inflection points of G occur at

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$



31. $1 \leq \sqrt{1+x^4} \leq 1+x^4$, so

$$\int_0^1 dx \leq \int_0^1 \sqrt{1+x^4} \, dx \leq \int_0^1 (1+x^4) \, dx$$

$$\int_0^1 dx \leq \int_0^1 \sqrt{1+x^4} \, dx \leq \int_0^1 dx + \int_0^1 x^4 \, dx$$

$$[x]_0^1 \leq \int_0^1 \sqrt{1+x^4} \, dx \leq [x]_0^1 + \left[\frac{x^5}{5} \right]_0^1$$

$$(1-0) \leq \int_0^1 \sqrt{1+x^4} \, dx \leq (1-0) + \left(\frac{1}{5} - 0 \right)$$

$$1 \leq \int_0^1 \sqrt{1+x^4} \, dx \leq \frac{6}{5}$$

32. On the interval $[0,1]$, $2 \leq \sqrt{4+x^4} \leq 4+x^4$. Thus

$$\int_0^1 2 \, dx \leq \int_0^1 \sqrt{4+x^4} \, dx \leq \int_0^1 (4+x^4) \, dx$$

$$2 \leq \int_0^1 \sqrt{4+x^4} \, dx \leq \frac{21}{5}$$

Here, we have used the result from problem 29:

$$\int_0^1 (4+x^4) \, dx = \int_0^1 (3+1+x^4) \, dx$$

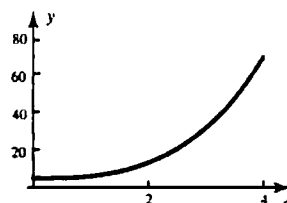
$$= \int_0^1 3 \, dx + \int_0^1 (1+x^4) \, dx$$

$$= 3 + \frac{6}{5} = \frac{21}{5}$$

33. $5 \leq f(x) \leq 69$ so

$$4 \cdot 5 \leq \int_0^4 (5+x^3) \, dx \leq 4 \cdot 69$$

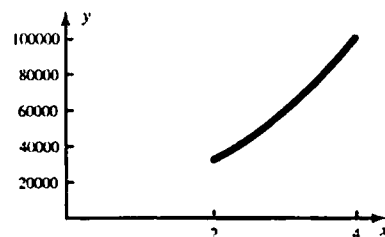
$$20 \leq \int_0^4 (5+x^3) \, dx \leq 276$$



34. On $[2,4]$, $8^5 \leq (x+6)^5 \leq 10^5$. Thus,

$$2 \cdot 8^5 \leq \int_2^4 (x+6)^5 \, dx \leq 2 \cdot 10^5$$

$$65,536 \leq \int_2^4 (x+6)^5 \, dx \leq 200,000$$

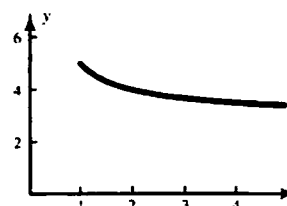


35. On $[1,5]$,

$$3 + \frac{2}{5} \leq 3 + \frac{2}{x} \leq 3 + \frac{2}{1}$$

$$4\left(\frac{17}{5}\right) \leq \int_1^5 \left(3 + \frac{2}{x}\right) \, dx \leq 4 \cdot 5$$

$$\frac{68}{5} \leq \int_1^5 \left(3 + \frac{2}{x}\right) \, dx \leq 20$$



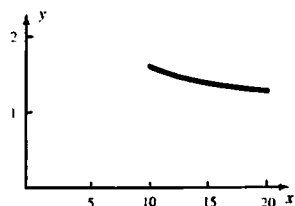
36. On $[10,20]$,

$$\left(1 + \frac{1}{20}\right)^5 \leq \left(1 + \frac{1}{x}\right)^5 \leq \left(1 + \frac{1}{10}\right)^5$$

$$10\left(\frac{21}{20}\right)^5 \leq \int_{10}^{20} \left(1 + \frac{1}{x}\right)^5 dx \leq 10\left(\frac{11}{10}\right)^5$$

$$\frac{4,084,101}{320,000} \leq \int_{10}^{20} \left(1 + \frac{1}{x}\right)^5 dx \leq \frac{161,051}{10,000}$$

$$12.7628 \leq \int_{10}^{20} \left(1 + \frac{1}{x}\right)^5 dx \leq 16.1051$$

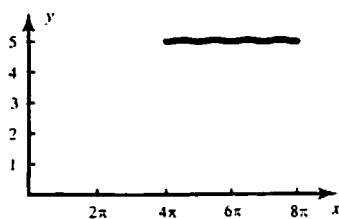


37. On $[4\pi, 8\pi]$

$$5 \leq 5 + \frac{1}{20} \sin^2 x \leq 5 + \frac{1}{20} x dx$$

$$(4\pi)(5) \leq \int_{4\pi}^{8\pi} \left(5 + \frac{1}{20} \sin^2 x\right) dx \leq (4\pi)\left(5 + \frac{1}{20}\right)$$

$$20\pi \leq \int_{4\pi}^{8\pi} \left(5 + \frac{1}{20} \sin^2 x\right) dx \leq \frac{101}{5}\pi$$



38. On $[0.2, 0.4]$,

$$0.002 + 0.0001 \cos^2 0.4 \leq 0.002 + 0.0001 \cos^2 x$$

$$\leq 0.002 + 0.0001 \cos^2 0.2$$

$$0.2(0.002 + 0.0001 \cos^2 0.4)$$

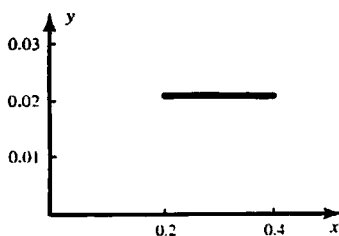
$$\leq \int_{0.2}^{0.4} (0.002 + 0.0001 \cos^2 x) dx$$

$$\leq 0.2(0.002 + 0.0001 \cos^2 0.2)$$

Thus,

$$0.000417 \leq \int_{0.2}^{0.4} (0.002 + 0.0001 \cos^2 x) dx$$

$$\leq 0.000419$$



39. Let $F(x) = \int_0^x \frac{1+t}{2+t} dt$. Then

$$\lim_{x \rightarrow 0} \frac{1}{x} \int_0^x \frac{1+t}{2+t} dt = \lim_{x \rightarrow 0} \frac{F(x) - F(0)}{x - 0}$$

$$= F'(0) = \frac{1+0}{2+0} = \frac{1}{2}$$

- 40.

$$\lim_{x \rightarrow 1} \frac{1}{x-1} \int_0^x \frac{1+t}{2+t} dt$$

$$= \lim_{x \rightarrow 1} \frac{1}{x-1} \left[\int_0^x \frac{1+t}{2+t} dt - \int_0^1 \frac{1+t}{2+t} dt \right]$$

$$= \lim_{x \rightarrow 1} \frac{F(x) - F(1)}{x-1}$$

$$= F'(1) = \frac{1+1}{2+1} = \frac{2}{3}$$

41. $\int_1^x f(t) dt = 2x - 2$

Differentiate both sides with respect to x :

$$\frac{d}{dx} \int_1^x f(t) dt = \frac{d}{dx} (2x - 2)$$

$$f(x) = 2$$

If such a function exists, it must satisfy $f(x) = 2$, but both sides of the first equality may differ by a constant yet still have equal derivatives. When

$x = 1$ the left side is $\int_1^1 f(t) dt = 0$ and the right side is $2 \cdot 1 - 2 = 0$. Thus the function $f(x) = 2$ satisfies $\int_1^x f(t) dt = 2x - 2$.

42. $\int_0^x f(t) dt = x^2$

Differentiate both sides with respect to x :

$$\frac{d}{dx} \int_0^x f(t) dt = \frac{d}{dx} x^2$$

$$f(x) = 2x$$

If such a function exists, it must satisfy $f(x) = 2x$, but both sides of the first equality may differ by a constant yet still have equal derivatives. When $x = 0$ the left side is

$$\int_0^0 f(t) dt = 0 \text{ and the right side is } 0^2 = 0.$$

Thus the function $f(x) = 2x$ satisfies

$$\int_0^x f(t) dt = x^2.$$

43. $\int_0^x f(t) dt = \frac{1}{3} x^3$

Differentiate both sides with respect to x :

$$\frac{d}{dx} \int_0^{x^2} f(t) dt = \frac{d}{dx} \left(\frac{1}{3} x^3 \right)$$

$$f(x^2)(2x) = x^2$$

$$f(x^2) = \frac{x}{2}$$

$$f(x) = \frac{\sqrt{x}}{2}$$

If such a function exists, it must satisfy

$$f(x) = \frac{\sqrt{x}}{2}, \text{ but both sides of the first equality}$$

may differ by a constant yet still have equal derivatives. When $x = 0$ the left side is

$$\int_0^0 f(t) dt = 0 \text{ and the right side is } \frac{\sqrt{0}}{2} = 0.$$

Thus the function $f(x) = \frac{\sqrt{x}}{2}$ satisfies

$$\int_0^{x^2} f(t) dt = \frac{1}{3} x^3.$$

44. No such function exists. When $x = 0$ the left side is 0, whereas the right side is 1
45. True. $f(x)$ will have a minimum, m on $[a, b]$, $m \geq 0$, so $\int_a^b f(x) dx \geq \int_a^b m dx = m(b-a) \geq 0$.
46. False. $a = -1, b = 2, f(x) = x$ is a counterexample.
47. False. $a = -1, b = 1, f(x) = x$ is a counterexample.
48. True. The integral can be zero only when the area above the x -axis equals the area below, but since $f(x) \geq 0$, the area below is zero. Hence $f(x)$ never rises above 0.

5.7 Concepts Review

- antiderivative; $F(b) - F(a)$
- $F(b) - F(a)$
- $\frac{(3)^3}{3} - \frac{(1)^3}{3} = \frac{26}{3}$
- $-\frac{1}{x}; 1 - \frac{1}{2} = \frac{1}{2}$

49. True. $\int_a^b f(x) dx - \int_a^b g(x) dx$
 $= \int_a^b [f(x) - g(x)] dx$

50. False. $a = 0, b = 1, f(x) = 0, g(x) = -1$ is a counterexample.

51. $-|f(x)| \leq f(x) \leq |f(x)|$, so

$$\int_a^b -|f(x)| dx \leq \int_a^b f(x) dx \Rightarrow$$

$$\int_a^b |f(x)| dx \geq -\int_a^b f(x) dx$$

and combining this with

$$\int_a^b |f(x)| dx \geq \int_a^b f(x) dx,$$

we can conclude that

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

52. If $x > a$, $\int_a^x |f'(x)| dx \leq M(x-a)$ by the Boundedness Property. If $x < a$,
- $$\int_x^a |f'(x)| dx = -\int_a^x |f'(x)| dx \geq -M(x-a)$$
- by the Boundedness Property. Thus
- $$\int_a^x |f'(x)| dx \leq M|x-a|.$$
- From Problem 51, $\int_a^x |f'(x)| dx \geq \left| \int_a^x f'(x) dx \right|$.
- $$\left| \int_a^x f'(x) dx \right| = |f(x) - f(a)| \geq |f(x)| - |f(a)|$$
- Therefore, $|f(x)| - |f(a)| \leq M|x-a|$ or
- $$|f(x)| \leq |f(a)| + M|x-a|.$$

Problem Set 5.7

- $\int_0^2 x^3 dx = \left[\frac{x^4}{4} \right]_0^2 = 4 - 0 = 4$
- $\int_{-1}^2 x^4 dx = \left[\frac{x^5}{5} \right]_{-1}^2 = \frac{32}{5} - \frac{1}{5} = \frac{31}{5}$
- $\int_{-1}^2 (3x^2 - 2x + 3) dx = \left[x^3 - x^2 + 3x \right]_{-1}^2$
 $= (8 - 4 + 6) - (-1 - 1 - 3) = 15$

$$4. \int_1^2 (4x^3 + 7) dx = \left[x^4 + 7x \right]_1^2 \\ = (16 + 14) - (1 + 7) = 22$$

$$5. \int_1^4 \frac{1}{w^2} dw = \left[-\frac{1}{w} \right]_1^4 = \left(-\frac{1}{4} \right) - (-1) = \frac{3}{4}$$

$$6. \int_1^3 \frac{2}{t^3} dt = \left[-\frac{1}{t^2} \right]_1^3 = \left(-\frac{1}{9} \right) - (-1) = \frac{8}{9}$$

$$7. \int_0^4 \sqrt{t} dt = \left[\frac{2}{3} t^{3/2} \right]_0^4 = \left(\frac{2}{3} \cdot 8 \right) - 0 = \frac{16}{3}$$

$$8. \int_1^8 \sqrt[3]{w} dw = \left[\frac{3}{4} w^{4/3} \right]_1^8 = \left(\frac{3}{4} \cdot 16 \right) - \left(\frac{3}{4} \cdot 1 \right) = \frac{45}{4}$$

$$9. \int_{-4}^{-2} \left(y^2 + \frac{1}{y^3} \right) dy = \left[\frac{y^3}{3} - \frac{1}{2y^2} \right]_{-4}^{-2} \\ = \left(-\frac{8}{3} - \frac{1}{8} \right) - \left(-\frac{64}{3} - \frac{1}{32} \right) = \frac{1783}{96}$$

$$10. \int_1^4 \frac{s^4 - 8}{s^2} ds = \int_1^4 (s^2 - 8s^{-2}) ds = \left[\frac{s^3}{3} + \frac{8}{s} \right]_1^4 \\ = \left(\frac{64}{3} + 2 \right) - \left(\frac{1}{3} + 8 \right) = 15$$

$$11. \int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = 1 - 0 = 1$$

$$12. \int_{\pi/6}^{\pi/2} 2 \sin t dt = [-2 \cos t]_{\pi/6}^{\pi/2} = 0 + \sqrt{3} = \sqrt{3}$$

$$13. \int_0^1 (2x^4 - 3x^2 + 5) dx = \left[\frac{2}{5} x^5 - x^3 + 5x \right]_0^1 \\ = \left(\frac{2}{5} - 1 + 5 \right) - 0 = \frac{22}{5}$$

$$14. \int_0^1 (x^{4/3} - 2x^{1/3}) dx = \left[\frac{3}{7} x^{7/3} - \frac{3}{2} x^{4/3} \right]_0^1 \\ = \left(\frac{3}{7} - \frac{3}{2} \right) - 0 = -\frac{15}{14}$$

$$15. u = x^2 + 1, du = 2x dx \\ \int (x^2 + 1)^{10} (2x) dx = \int u^{10} du = \frac{u^{11}}{11} + C \\ = \frac{1}{11} (x^2 + 1)^{11} + C$$

$$\int_0^1 (x^2 + 1)^{10} (2x) dx = \left[\frac{1}{11} (x^2 + 1)^{11} \right]_0^1 \\ = \left[\frac{1}{11} (2)^{11} \right] - \left[\frac{1}{11} (1)^{11} \right] = \frac{2047}{11}$$

$$16. u = x^3 + 1, du = 3x^2 dx \\ \int \sqrt{x^3 + 1} (3x^2) dx = \int \sqrt{u} du = \frac{2}{3} u^{3/2} + C \\ = \frac{2}{3} (x^3 + 1)^{3/2} + C \\ \int_{-1}^0 \sqrt{x^3 + 1} (3x^2) dx = \left[\frac{2}{3} (x^3 + 1)^{3/2} \right]_{-1}^0 \\ = \left(\frac{2}{3} \cdot 1^{3/2} \right) - \left(\frac{2}{3} \cdot 0 \right) = \frac{2}{3}$$

$$17. u = t + 2, du = dt \\ \int \frac{1}{(t+2)^2} dt = \int u^{-2} du = -\frac{1}{u} + C = -\frac{1}{t+2} + C \\ \int_{-1}^3 \frac{1}{(t+2)^2} dt = \left[-\frac{1}{t+2} \right]_{-1}^3 = \left[-\frac{1}{5} \right] - \left[-1 \right] = \frac{4}{5}$$

$$18. u = y - 1, du = dy \\ \int \sqrt{y-1} dy = \int \sqrt{u} du = \frac{2}{3} u^{3/2} + C \\ = \frac{2}{3} (y-1)^{3/2} + C \\ \int_2^{10} \sqrt{y-1} dy = \left[\frac{2}{3} (y-1)^{3/2} \right]_2^{10} \\ = \left[\frac{2}{3} (27) \right] - \left[\frac{2}{3} (1) \right] = \frac{52}{3}$$

$$19. u = 3x + 1, du = 3 dx \\ \int \sqrt{3x+1} dx = \frac{1}{3} \int \sqrt{3x+1} 3 dx = \frac{1}{3} \int \sqrt{u} du \\ = \frac{2}{9} u^{3/2} + C = \frac{2}{9} (3x+1)^{3/2} + C \\ \int_5^8 \sqrt{3x+1} dx = \left[\frac{2}{9} (3x+1)^{3/2} \right]_5^8 \\ = \left[\frac{2}{9} (125) \right] - \left[\frac{2}{9} (64) \right] = \frac{122}{9}$$

$$20. u = 2x + 2, du = 2 dx \\ \int \frac{1}{\sqrt{2x+2}} dx = \frac{1}{2} \int \frac{2}{\sqrt{2x+2}} dx = \frac{1}{2} \int u^{-1/2} du \\ = \sqrt{u} + C = \sqrt{2x+2} + C \\ \int_1^7 \frac{1}{\sqrt{2x+2}} dx = \left[\sqrt{2x+2} \right]_1^7 = 4 - 2 = 2$$

$$21. u = 7 + 2t^2, du = 4t dt$$

$$\int \sqrt{7+2t^2} (8t) dt = 2 \int \sqrt{7+2t^2} (4t) dt$$

$$= 2 \int \sqrt{2} du = \frac{4}{3} u^{3/2} + C = \frac{4}{3} (7+2t^2)^{3/2} + C$$

$$\int_{-3}^3 \sqrt{7+2t^2} (8t) dt = \left[\frac{4}{3} (7+2t^2)^{3/2} \right]_{-3}^3$$

$$= \left[\frac{4}{3} (125) \right] - \left[\frac{4}{3} (125) \right] = 0$$

$$22. u = x^3 + 3x, du = (3x^2 + 3) dx$$

$$\int \frac{x^2+1}{\sqrt{x^3+3x}} dx = \frac{1}{3} \int \frac{3x^2+3}{\sqrt{x^3+3x}} dx$$

$$= \frac{1}{3} \int u^{-1/2} du = \frac{2}{3} u^{1/2} + C = \frac{2}{3} \sqrt{x^3+3x} + C$$

$$\int_1^3 \frac{x^2+1}{\sqrt{x^3+3x}} dx = \left[\frac{2}{3} \sqrt{x^3+3x} \right]_1^3$$

$$= \left(\frac{2}{3} \cdot 6 \right) - \left(\frac{2}{3} \cdot 2 \right) = \frac{8}{3}$$

$$23. u = \cos x, du = -\sin x dx$$

$$\int \cos^2 x \sin x dx = -\int \cos^2 x (-\sin x) dx$$

$$= -\int u^2 du = -\frac{u^3}{3} + C = -\frac{1}{3} \cos^3 x + C$$

$$\int_0^{\pi/2} \cos^2 x \sin x dx = \left[-\frac{1}{3} \cos^3 x \right]_0^{\pi/2}$$

$$= 0 - \left(-\frac{1}{3} \right) = \frac{1}{3}$$

$$24. u = \sin 3x, du = 3 \cos 3x dx$$

$$\int \sin^2 3x \cos 3x dx = \frac{1}{3} \int \sin^2 3x (3 \cos 3x) dx$$

$$= \frac{1}{3} \int u^2 du = \frac{1}{9} u^3 + C = \frac{1}{9} \sin^3 3x + C$$

$$\int_0^{\pi/2} \sin^2 3x \cos 3x dx = \left[\frac{1}{9} \sin^3 3x \right]_0^{\pi/2}$$

$$= \left(-\frac{1}{9} \right) - 0 = -\frac{1}{9}$$

$$25. \int_0^{\pi/2} (2x + \sin x) dx$$

$$= 2 \int_0^{\pi/2} x dx + \int_0^{\pi/2} \sin x dx$$

$$= 2 \left[\frac{x^2}{2} \right]_0^{\pi/2} + [-\cos x]_0^{\pi/2}$$

$$= 2 \left(\frac{\pi^2}{8} - 0 \right) + (0 + 1) = \frac{\pi^2}{4} + 1$$

$$26. \int_0^{\pi/2} [4x + \cos x] dx$$

$$= 4 \int_0^{\pi/2} x dx + \int_0^{\pi/2} \cos x dx$$

$$= 4 \left[\frac{x^2}{2} \right]_0^{\pi/2} + [\sin x]_0^{\pi/2}$$

$$= 4 \left(\frac{\pi^2}{8} - 0 \right) + (1 - 0)$$

$$= \frac{\pi^2}{2} + 1 = \frac{\pi^2 + 2}{2}$$

$$27. \int_0^4 [\sqrt{x} + \sqrt{2x+1}] dx = \int_0^4 \sqrt{x} dx + \int_0^4 \sqrt{2x+1} dx$$

$$= \left[\frac{2}{3} x^{3/2} \right]_0^4 + \frac{1}{2} \left[\frac{2}{3} (2x+1)^{3/2} \right]_0^4$$

$$= \left(\frac{16}{3} - 0 \right) + \frac{1}{2} \left(18 - \frac{2}{3} \right) = 14$$

$$28. \int_{-4}^1 \frac{1-s^4}{2s^2} ds = \int_{-4}^1 \frac{1}{2s^2} ds - \int_{-4}^1 \frac{s^2}{2} ds$$

$$= \frac{1}{2} \left[-\frac{1}{s} \right]_{-4}^1 - \frac{1}{2} \left[\frac{s^3}{3} \right]_{-4}^1$$

$$= \frac{1}{2} \left(1 - \frac{1}{4} \right) - \frac{1}{2} \left(-\frac{1}{3} + \frac{64}{3} \right) = -\frac{81}{8}$$

$$29. \int_0^1 (x^2 + 2x)^2 dx = \int_0^1 (x^4 + 4x^3 + 4x^2) dx$$

$$= \int_0^1 x^4 dx + 4 \int_0^1 x^3 dx + 4 \int_0^1 x^2 dx$$

$$= \left[\frac{x^5}{5} \right]_0^1 + 4 \left[\frac{x^4}{4} \right]_0^1 + 4 \left[\frac{x^3}{3} \right]_0^1$$

$$= \left(\frac{1}{5} - 0 \right) + 4 \left(\frac{1}{4} - 0 \right) + 4 \left(\frac{1}{3} - 0 \right) = \frac{38}{15}$$

$$30. \int_a^{8a} (a^{1/3} - x^{1/3})^3 dx$$

$$= \int_a^{8a} (a - 3a^{2/3}x^{1/3} + 3a^{1/3}x^{2/3} - x) dx$$

$$\begin{aligned}
&= a \int_a^{8a} dx - 3a^{2/3} \int_a^{8a} x^{1/3} dx \\
&\quad + 3a^{1/3} \int_a^{8a} x^{2/3} dx - \int_a^{8a} x dx \\
&= a[x]_a^{8a} - 3a^{2/3} \left[\frac{3}{4} x^{4/3} \right]_a^{8a} \\
&\quad + 3a^{1/3} \left[\frac{3}{5} x^{5/3} \right]_a^{8a} - \left[\frac{x^2}{2} \right]_a^{8a} \\
&= a(8a - a) - 3a^{2/3} \left(12a^{4/3} - \frac{3}{4}a^{4/3} \right) \\
&\quad + 3a^{1/3} \left(\frac{96}{5}a^{5/3} - \frac{3}{5}a^{5/3} \right) - \left(32a^2 - \frac{a^2}{2} \right) \\
&= 7a^2 - \frac{135}{4}a^2 + \frac{279}{5}a^2 - \frac{63}{2}a^2 \\
&= -\frac{49}{20}a^2
\end{aligned}$$

$$31. \int_1^x t^3 dt = \left[\frac{t^4}{4} \right]_1^x = \frac{x^4}{4} - \frac{1}{4}$$

$$32. \int_{-1}^x (t + |t|) dt = \begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

$$33. \frac{1}{3-1} \int_1^3 4x^3 dx = \frac{1}{2} \left[x^4 \right]_1^3 = 40$$

$$34. \frac{1}{3-0} \int_0^3 \frac{x}{\sqrt{x^2+16}} dx = \frac{1}{3} \left[\sqrt{x^2+16} \right]_0^3 = \frac{1}{3}$$

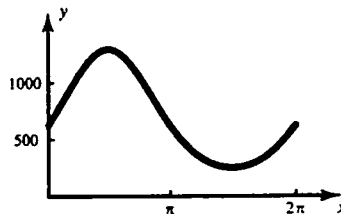
$$\begin{aligned}
35. \quad &\frac{1}{1+2} \int_{-2}^1 (2 + |x|) dx \\
&= \frac{1}{3} \left[\int_{-2}^0 (2 - x) dx + \int_0^1 (2 + x) dx \right] \\
&= \frac{1}{3} \left\{ \left[2x - \frac{1}{2}x^2 \right]_{-2}^0 + \left[2x + \frac{1}{2}x^2 \right]_0^1 \right\} \\
&= \frac{1}{3} \left(-2(-2) + \frac{1}{2}(-2)^2 + 2 + \frac{1}{2} \right) = \frac{17}{6}
\end{aligned}$$

$$\begin{aligned}
36. \quad &\frac{1}{2+3} \int_{-3}^2 (x + |x|) dx \\
&= \frac{1}{5} \left(\int_{-3}^0 (-x + x) dx + \int_0^2 2x dx \right) \\
&= \frac{1}{5} \left[x^2 \right]_0^2 = \frac{4}{5}
\end{aligned}$$

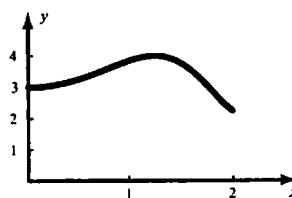
$$\begin{aligned}
37. \quad &\frac{1}{\pi} \int_0^\pi \cos x dx = \frac{1}{\pi} [\sin x]_0^\pi \\
&= \frac{1}{\pi} [\sin \pi - \sin 0] = 0
\end{aligned}$$

$$\begin{aligned}
38. \quad &\frac{1}{\pi/2 - 0} \int_0^{\pi/2} \sin^2 x \cos x dx \\
&= \frac{2}{\pi} \left[\frac{1}{3} \sin^3 x \right]_0^{\pi/2} = \frac{2}{3\pi}
\end{aligned}$$

$$39. \text{ Using } c = \pi \text{ yields } 2\pi(5)^4 = 1250\pi \approx 3927$$

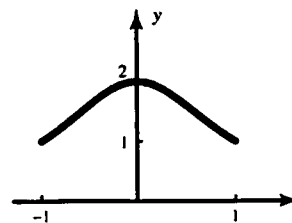


$$40. \text{ Using } c = 0.8 \text{ yields } 2(3 + \sin 0.8^2) \approx 7.19$$

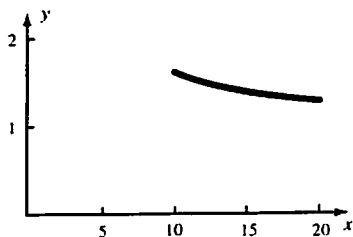


$$41. \text{ Using } c = 0.5 \text{ yields}$$

$$2 \frac{2}{1 + 0.5^2} = 3.2$$



$$42. \text{ Using } c = 15 \text{ yields } \left(\frac{16}{15} \right)^5 (20 - 10) \approx 13.8$$



$$\begin{aligned}
 43. \quad f(c)(3-0) &= \int_0^3 \sqrt{x+1} \, dx \\
 3\sqrt{c+1} &= \frac{2}{3} \left[(x+1)^{3/2} \right]_0^3 \\
 \sqrt{c+1} &= \frac{2}{9} (8-1) \\
 c &= \left(\frac{14}{9} \right)^2 - 1 = \frac{115}{81}
 \end{aligned}$$

$$\begin{aligned}
 44. \quad f(c)(1-(-1)) &= \int_{-1}^1 x^2 \, dx \\
 2f(c) &= \left[\frac{x^3}{3} \right]_{-1}^1 \\
 2c^2 &= \frac{2}{3} \\
 c &= \pm \frac{\sqrt{3}}{3}
 \end{aligned}$$

Both $\pm\sqrt{3}/3$ are in the interval $(-1, 1)$.

$$\begin{aligned}
 45. \quad 2f(c) &= \int_0^2 |x| \, dx \\
 2|c| &= \int_0^2 x \, dx \\
 2|c| &= \left[\frac{x^2}{2} \right]_0^2 \\
 2|c| &= 2 \\
 c &= \pm 1
 \end{aligned}$$

Only $c = 1$ is in the interval $[0, 2]$.

$$\begin{aligned}
 46. \quad 4f(c) &= \int_{-2}^2 |x| \, dx \\
 4|c| &= 2 \int_0^2 x \, dx = 2 \left[\frac{x^2}{2} \right]_0^2 \\
 4|c| &= 4 \\
 c &= \pm 1
 \end{aligned}$$

Both ± 1 are in the interval $[-2, 2]$.

$$47. \quad \int_0^3 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^3 = 9 - 0 = 9$$

$$48. \quad \int_0^2 x^3 \, dx = \left[\frac{x^4}{4} \right]_0^2 = 4 - 0 = 4$$

$$49. \quad \int_0^\pi \sin x \, dx = [-\cos x]_0^\pi = 1 + 1 = 2$$

$$\begin{aligned}
 50. \quad \int_0^2 (1+x+x^2) \, dx &= \int_0^2 1 \, dx + \int_0^2 x \, dx + \int_0^2 x^2 \, dx \\
 &= [x]_0^2 + \left[\frac{x^2}{2} \right]_0^2 + \left[\frac{x^3}{3} \right]_0^2 \\
 &= (2-0) + (2-0) + \left(\frac{8}{3} - 0 \right) = \frac{20}{3}
 \end{aligned}$$

$$\begin{aligned}
 51. \quad \text{The right-endpoint Riemann sum is} \\
 \sum_{i=1}^n \left(0 + \frac{1-0}{n} i \right) \left(\frac{1}{n} \right) &= \frac{1}{n^3} \sum_{i=1}^n i^2, \text{ which for} \\
 n = 10 \text{ equals } \frac{77}{200} &= 0.385.
 \end{aligned}$$

$$\int_0^1 x^2 \, dx = \left[\frac{1}{3} x^3 \right]_0^1 = \frac{1}{3} = 0.333$$

$$\begin{aligned}
 52. \quad \int_{-2}^4 (2\lfloor x \rfloor - 3|x|) \, dx &= 2 \int_{-2}^4 \lfloor x \rfloor \, dx - 3 \int_{-2}^4 |x| \, dx \\
 &= 2[(-2-1+0+1+2+3)(1)] \\
 &\quad - 3 \left[\frac{1}{2}(2)(2) + \frac{1}{2}(4)(4) \right] \\
 &= -24
 \end{aligned}$$

$$\begin{aligned}
 53. \quad \frac{d}{dx} \left(\frac{1}{2} x |x| \right) &= \frac{1}{2} x \left(\frac{|x|}{x} \right) + \frac{|x|}{2} = |x| \\
 \int_a^b |x| \, dx &= \left[\frac{1}{2} x |x| \right]_a^b = \frac{1}{2} (b|b| - a|a|)
 \end{aligned}$$

$$\begin{aligned}
 54. \quad \text{For } b > 0, \text{ if } b \text{ is an integer,} \\
 \int_0^b \lfloor x \rfloor \, dx &= 0 + 1 + 2 + \cdots + (b-1) \\
 &= \sum_{i=1}^{b-1} i = \frac{(b-1)b}{2}.
 \end{aligned}$$

If b is not an integer, let $n = \lfloor b \rfloor$. Then

$$\begin{aligned}\int_0^b \lfloor x \rfloor dx &= 0 + 1 + 2 + \cdots + (n-1) + n(b-n) \\ &= \frac{(n-1)n}{2} + n(b-n) \\ &= \frac{(\lfloor b \rfloor - 1)\lfloor b \rfloor}{2} + \lfloor b \rfloor(b - \lfloor b \rfloor).\end{aligned}$$

55. a. $\int_a^b x^n dx = B_n$; $\int_a^b \sqrt[n]{y} dy = A_n$

Using Figure 3 of the text,

$$(a)(a^n) + A_n + B_n = (b)(b^n) \text{ or}$$

$$B_n + A_n = b^{n+1} - a^{n+1}. \text{ Thus}$$

$$\int_a^b x^n dx + \int_a^b \sqrt[n]{y} dy = b^{n+1} - a^{n+1}$$

b. $\int_a^b x^n dx + \int_a^b \sqrt[n]{y} dy$

$$\begin{aligned}&= \left[\frac{x^{n+1}}{n+1} \right]_a^b + \left[\frac{n}{n+1} y^{(n+1)/n} \right]_a^b \\ &= \left(\frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} \right) + \left(\frac{n}{n+1} b^{n+1} - \frac{n}{n+1} a^{n+1} \right) \\ &= \frac{(n+1)b^{n+1} - (n+1)a^{n+1}}{n+1} = b^{n+1} - a^{n+1}\end{aligned}$$

c. $B_n = \int_a^b x^n dx = \frac{1}{n+1} \left[x^{n+1} \right]_a^b$

$$\begin{aligned}&= \frac{1}{n+1} (b^{n+1} - a^{n+1}) \\ A_n &= \int_a^b \sqrt[n]{y} dy = \left[\frac{n}{n+1} y^{(n+1)/n} \right]_a^b \\ &= \frac{n}{n+1} (b^{n+1} - a^{n+1})\end{aligned}$$

$$nB_n = \frac{n}{n+1} (b^{n+1} - a^{n+1}) = A_n$$

56. Since f is continuous on a closed interval $[a, b]$ there exist (by the Min-Max Existence Theorem) an m and M in $[a, b]$ such that $f(m) \leq f(x) \leq f(M)$ for all x in $[a, b]$. Thus

$$\begin{aligned}\int_a^b f(m) dx &\leq \int_a^b f(x) dx \leq \int_a^b f(M) dx \\ (b-a)f(m) &\leq \int_a^b f(x) dx \leq (b-a)f(M) \\ f(m) &\leq \frac{1}{b-a} \int_a^b f(x) dx \leq f(M)\end{aligned}$$

Since f is continuous, we can apply the Intermediate Value Theorem and say that f takes on every value between $f(m)$ and

$$f(M). \text{ Since } \frac{1}{b-a} \int_a^b f(x) dx \text{ is between } f(m) \text{ and } f(M), \text{ there exists a } c \text{ in } [a, b] \text{ such that } f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

57. a. Let c be in $[a, b]$. Then $G'(c) = f(c)$ by the First Fundamental Theorem of Calculus. Since G is differentiable at c , G is continuous there. Now suppose $c = a$. Then

$\lim_{x \rightarrow c} G(x) = \lim_{x \rightarrow a} \int_a^x f(t) dt$. Since f is continuous on $[a, b]$, there exist (by the Min-Max Existence Theorem) m and M such that $f(m) \leq f(x) \leq f(M)$ for all x in $[a, b]$. Then

$$\begin{aligned}\int_a^x f(m) dt &\leq \int_a^x f(t) dt \leq \int_a^x f(M) dt \\ (x-a)f(m) &\leq G(x) \leq (x-a)f(M)\end{aligned}$$

By the Squeeze Theorem

$$\begin{aligned}\lim_{x \rightarrow a^+} (x-a)f(m) &\leq \lim_{x \rightarrow a^+} G(x) \\ &\leq \lim_{x \rightarrow a^+} (x-a)f(M)\end{aligned}$$

Thus

$$\lim_{x \rightarrow a^+} G(x) = 0 = \int_a^a f(t) dt = G(a)$$

Therefore G is right-continuous at $x = a$.

Now, suppose $c = b$. Then

$$\lim_{x \rightarrow b^-} G(x) = \lim_{x \rightarrow b^-} \int_x^b f(t) dt$$

As before,

$$(b-x)f(m) \leq G(x) \leq (b-x)f(M)$$

so we can apply the Squeeze Theorem again to obtain

$$\begin{aligned}\lim_{x \rightarrow b^-} (b-x)f(m) &\leq \lim_{x \rightarrow b^-} G(x) \\ &\leq \lim_{x \rightarrow b^-} (b-x)f(M)\end{aligned}$$

Thus

$$\lim_{x \rightarrow b^-} G(x) = 0 = \int_b^b f(t) dt = G(b)$$

Therefore, G is left-continuous at $x = b$.

- b. Let F be any antiderivative of f . Note that G is also an antiderivative of f . Thus, $F(x) = G(x) + C$. We know from part (a) that $G(x)$ is continuous on $[a, b]$. Thus $F(x)$, being equal to $G(x)$ plus a constant, is also continuous on $[a, b]$.

58. Let

$$f(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

and $F(x) = \int_1^x f(t) dt$. If $x < 0$, then

$F(x) = 0$. If $x \geq 0$, then

$$\begin{aligned} F(x) &= \int_1^x f(t) dt \\ &= \int_1^0 0 dt + \int_0^x 1 dt \\ &= 0 + x = x \end{aligned}$$

Thus,

$$F(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

which is continuous everywhere even though $f(x)$ is not continuous everywhere.

59. a. Between 0 and 3, $f(x) > 0$. Thus,

$$\int_0^3 f(x) dx > 0.$$

- b. Since f is an antiderivative of f' ,

$$\begin{aligned} \int_0^3 f'(x) dx &= f(3) - f(0) \\ &= 0 - 2 = -2 < 0 \end{aligned}$$

- c. $\int_0^3 f''(x) dx = f'(3) - f'(0)$

$$= -1 - 0 = -1 < 0$$

- d. $\int_0^3 f'''(x) dx = f''(3) - f''(0)$

$$= 0 - (\text{negative number}) > 0$$

Since f is concave down at 0, $f''(0) < 0$.

60. a. On $[0, 4]$, $f(x) > 0$. Thus,

$$\int_0^4 f(x) dx > 0.$$

- b. Since f is an antiderivative of f' ,

$$\int_0^4 f'(x) dx = f(4) - f(0)$$

$$= 1 - 2 = -1 < 0$$

- c. $\int_0^4 f''(x) dx = f'(4) - f'(0)$

$$= \frac{1}{4} - (-2) = \frac{9}{4} > 0$$

- d. $\int_0^4 f'''(x) dx = f''(4) - f''(0)$

$$= (\text{positive}) - (\text{negative}) > 0$$

61. Let $y = G(x) = \int_a^x f(t) dt$. Then

$$\frac{dy}{dx} = G'(x) = f(x)$$

$$dy = f(x) dx$$

Let F be any antiderivative of f . Then

$y = F(x) + C$. When $x = a$, we must have

$y = 0$. Thus, $C = -F(a)$ and $y = F(x) - F(a)$.

Now choose $x = b$ to obtain

$$\int_a^b f(t) dt = y = F(b) - F(a)$$

62. a. A rectangle with height 30 and width 24 has approximately the same area as that under the curve. Thus,

$$\frac{1}{24-0} \int_0^{24} T(t) dt \approx 30$$

- b. Yes. The Mean Value Theorem for Integrals guarantees the existence of a c such that

$$\frac{1}{24-0} \int_0^{24} T(t) dt = T(c)$$

The figure indicates that there are actually two such values of c , roughly, $c = 11$ and $c = 16$.

63. A rectangle with height 25 and width 7 has approximately the same area as that under the curve. Thus

$$\frac{1}{7} \int_0^7 H(t) dt \approx 25$$

5.8 Concepts Review

- $\frac{1}{3} \int_1^2 u^4 du$
- $0; 2 \int_0^2 f(x) dx$
- $\int_0^2 (2+x^2) dx; 0$
- $f(x+p) = f(x); \text{period}$

Problem Set 5.8

- $u = 3x + 2, du = 3 dx$
 $\int \sqrt{u} \cdot \frac{1}{3} du = \frac{2}{9} u^{3/2} + C = \frac{2}{9} (3x + 2)^{3/2} + C$
- $u = 2x - 4, du = 2 dx$
 $\int u^{1/3} \cdot \frac{1}{2} du = \frac{3}{8} u^{4/3} + C = \frac{3}{8} (2x - 4)^{4/3} + C$
- $u = 6x - 7, du = 6 dx$
 $\int u^{1/8} \cdot \frac{1}{6} du = \frac{4}{27} u^{9/8} + C = \frac{4}{27} (6x - 7)^{9/8} + C$
- $x = 5u - \pi, dx = 5 du$
 $\int x^{21/8} \cdot \frac{1}{5} dx = \frac{8}{145} x^{29/8} + C$
 $= \frac{8}{145} (5u - \pi)^{29/8} + C$
- $u = 3x + 2, du = 3 dx$
 $\int \cos(u) \cdot \frac{1}{3} du = \frac{1}{3} \sin u + C = \frac{1}{3} \sin(3x + 2) + C$
- $u = 2x - 4, du = 2 dx$
 $\int \sin u \cdot \frac{1}{2} du = -\frac{1}{2} \cos u + C$
 $= -\frac{1}{2} \cos(2x - 4) + C$
- $u = 6x - 7, du = 6 dx$
 $\int \sin u \cdot \frac{1}{6} du = -\frac{1}{6} \cos u + C$
 $= -\frac{1}{6} \cos(6x - 7) + C$
- $u = \pi v - \sqrt{7}, du = \pi dv$
 $\int \cos u \cdot \frac{1}{\pi} du = \frac{1}{\pi} \sin u + C = \frac{1}{\pi} \sin(\pi v - \sqrt{7}) + C$
- $u = x^2 + 4, du = 2x dx$
 $\int \sqrt{u} \cdot \frac{1}{2} du = \frac{1}{3} u^{3/2} + C = \frac{1}{3} (x^2 + 4)^{3/2} + C$
- $u = x^3 + 5, du = 3x^2 dx$
 $\int u^9 \cdot \frac{1}{3} du = \frac{1}{30} u^{10} + C = \frac{1}{30} (x^3 + 5)^{10} + C$
- $u = x^2 + 3, du = 2x dx$
 $\int u^{-12/7} \cdot \frac{1}{2} du = -\frac{7}{10} u^{-5/7} + C$
 $= -\frac{7}{10} (x^2 + 3)^{-5/7} + C$
- $u = \sqrt{3} v^2 + \pi, du = 2\sqrt{3} v dv$
 $\int u^{7/8} \cdot \frac{1}{2\sqrt{3}} du = \frac{4}{15\sqrt{3}} u^{15/8} + C$
 $= \frac{4}{15\sqrt{3}} (\sqrt{3} v^2 + \pi)^{15/8} + C$
- $u = x^2 + 4, du = 2x dx$
 $\int \sin(u) \cdot \frac{1}{2} du = -\frac{1}{2} \cos u + C$
 $= -\frac{1}{2} \cos(x^2 + 4) + C$
- $u = x^3 + 5, du = 3x^2 dx$
 $\int \cos u \cdot \frac{1}{3} du = \frac{1}{3} \sin u + C = \frac{1}{3} \sin(x^3 + 5) + C$
- $y = 6x^3 - 7, du = 18x^2 dx$
 $\int \sin u \cdot \frac{1}{18} du = -\frac{1}{18} \cos u + C$
 $= -\frac{1}{18} \cos(6x^3 - 7) + C$
- $u = \pi v^5 - \sqrt{7}, du = 5\pi v^4 dv$
 $\int \cos u \cdot \frac{1}{5\pi} du = \frac{1}{5\pi} \sin u + C$
 $= \frac{1}{5\pi} \sin(\pi v^5 - \sqrt{7}) + C$

$$17. \quad u = \sqrt{x^2 + 4}, \quad du = \frac{x}{\sqrt{x^2 + 4}} dx$$

$$\int \sin u \, du = -\cos u + C = -\cos \sqrt{x^2 + 4} + C$$

$$18. \quad u = \sqrt[3]{z^2 + 3}, \quad du = \frac{2z}{3(\sqrt[3]{z^2 + 3})^2} dz$$

$$\int \cos u \cdot \frac{3}{2} du = \frac{3}{2} \sin u + C = \frac{3}{2} \sin \sqrt[3]{z^2 + 3} + C$$

$$19. \quad u = (x^3 + 5)^9,$$

$$du = 9(x^3 + 5)^8 (3x^2) dx = 27x^2 (x^3 + 5)^8 dx$$

$$\int \cos u \cdot \frac{1}{27} du = \frac{1}{27} \sin u + C$$

$$= \frac{1}{27} \sin[(x^3 + 5)^9] + C$$

$$20. \quad u = (7x^7 + \pi)^9, \quad du = 441x^6 (7x^7 + \pi)^8 dx$$

$$\int \sin u \cdot \frac{1}{441} du = -\frac{1}{441} \cos u + C$$

$$= -\frac{1}{441} \cos(7x^7 + \pi)^9 + C$$

$$21. \quad u = \sin(x^2 + 4), \quad du = 2x \cos(x^2 + 4) dx$$

$$\int \sqrt{u} \cdot \frac{1}{2} du = \frac{1}{3} u^{3/2} + C$$

$$= \frac{1}{3} [\sin(x^2 + 4)]^{3/2} + C$$

$$22. \quad u = \cos(3x^7 + 9)$$

$$du = -21x^6 \sin(3x^7 + 9) dx$$

$$\int \sqrt[3]{u} \cdot \left(-\frac{1}{21}\right) du = -\frac{1}{28} u^{4/3} + C$$

$$= -\frac{1}{28} [\cos(3x^7 + 9)]^{4/3} + C$$

$$23. \quad u = \cos(x^3 + 5), \quad du = -3x^2 \sin(x^3 + 5) dx$$

$$\int u^9 \cdot \left(-\frac{1}{3}\right) du = -\frac{1}{30} u^{10} + C$$

$$= -\frac{1}{30} \cos^{10}(x^3 + 5) + C$$

$$24. \quad u = \tan(x^{-3} + 1)$$

$$du = -3x^{-4} \sec^2(x^{-3} + 1) dx$$

$$\int \sqrt[5]{u} \cdot \left(-\frac{1}{3}\right) du = -\frac{5}{18} u^{6/5} + C$$

$$= -\frac{5}{18} [\tan(x^{-3} + 1)]^{6/5} + C$$

$$25. \quad \int (x+1) \sec^{1/2}(x^2 + 2x) \tan(x^2 + 2x) dx$$

$$= \int (x+1) \frac{\sin(x^2 + 2x)}{\cos^{3/2}(x^2 + 2x)} dx$$

$$u = \cos(x^2 + 2x), \quad du = -(2x+2) \sin(x^2 + 2x) dx$$

$$\int u^{-3/2} \cdot -\frac{1}{2} du = u^{-1/2} + C$$

$$= \frac{1}{\cos^{1/2}(x^2 + 2x)} + C = \sec^{1/2}(x^2 + 2x) + C$$

$$26. \quad u = \sqrt{x} + 4, \quad du = \frac{1}{2\sqrt{x}} dx$$

$$\int u^2 \cdot 2 du = \frac{2}{3} u^3 + C = \frac{2}{3} (\sqrt{x} + 4)^3 + C$$

$$27. \quad u = 3x + 1, \quad du = 3 dx$$

$$\frac{1}{3} \int_1^4 u^3 du = \left[\frac{1}{12} u^4 \right]_1^4 = \left(\frac{64}{3} - \frac{1}{12} \right) = \frac{85}{4}$$

$$28. \quad u = 2t + 1, \quad du = 2 dt$$

$$\frac{1}{2} \int_1^9 \sqrt{u} du = \frac{1}{3} \left[u^{3/2} \right]_1^9 = \frac{1}{3} (27 - 1) = \frac{26}{3}$$

$$29. \quad u = t^2 + 9, \quad du = 2t dt$$

$$\frac{1}{2} \int_9^{13} \frac{1}{u^2} du = -\frac{1}{2} \left[\frac{1}{u} \right]_9^{13} = -\frac{1}{2} \left(\frac{1}{13} - \frac{1}{9} \right) = \frac{2}{117}$$

$$30. \quad u = 9 - x^2, \quad du = -2x dx$$

$$-\frac{1}{2} \int_9^{27} \sqrt{u} du = -\frac{1}{3} \left[u^{3/2} \right]_9^{27} = -\frac{1}{3} (81 - 27) = \frac{19}{3}$$

$$31. \quad u = x^2 + 4x + 1, \quad du = (2x + 4) dx$$

$$\frac{1}{2} \int_1^6 u^{-2} du = -\frac{1}{2} \left[\frac{1}{u} \right]_1^6 = -\frac{1}{2} \left(\frac{1}{6} - 1 \right) = \frac{5}{12}$$

$$32. \quad u = 9 - x^3, \quad du = -3x^2 dx$$

$$-\frac{1}{3} \int_9^1 u^{-3/2} du = \frac{2}{3} \left[u^{-1/2} \right]_9^1 = \frac{2}{3} \left(1 - \frac{1}{3} \right) = \frac{4}{9}$$

$$33. \quad u = \sin \theta, \quad du = \cos \theta d\theta$$

$$\int_0^{1/2} u^3 du = \left[\frac{u^4}{4} \right]_0^{1/2} = \frac{1}{64} - 0 = \frac{1}{64}$$

$$34. u = \cos \theta, du = -\sin \theta d\theta$$

$$-\int_1^{\sqrt{3}/2} u^{-3} du = \frac{1}{2} [u^{-2}]_1^{\sqrt{3}/2} = \frac{1}{2} \left(\frac{4}{3} - 1 \right) = \frac{1}{6}$$

$$35. u = 3x - 3, du = 3dx$$

$$\frac{1}{3} \int_3^0 \cos u du = \frac{1}{3} [\sin u]_{-3}^0 = \frac{1}{3} (0 - \sin(-3)) \\ = \frac{\sin 3}{3}$$

$$36. u = 2\pi x, du = 2\pi dx$$

$$\frac{1}{2\pi} \int_0^\pi \sin u du = -\frac{1}{2\pi} [\cos u]_0^\pi = -\frac{1}{2\pi} (-1 - 1) \\ = \frac{1}{\pi}$$

$$37. u = \pi x^2, du = 2\pi x dx$$

$$\frac{1}{2\pi} \int_0^\pi \sin u du = -\frac{1}{2\pi} [\cos u]_0^\pi = -\frac{1}{2\pi} (-1 - 1) \\ = \frac{1}{\pi}$$

$$38. u = 2x^5, du = 10x^4 dx$$

$$\frac{1}{10} \int_0^{2\pi^5} \cos u du = \frac{1}{10} [\sin u]_0^{2\pi^5} \\ = \frac{1}{10} (\sin(2\pi^5) - 0) = \frac{1}{10} \sin(2\pi^5)$$

$$39. u = 2x, du = 2dx$$

$$\frac{1}{2} \int_0^{\pi/2} \cos u du + \frac{1}{2} \int_0^{\pi/2} \sin u du \\ = \frac{1}{2} [\sin u]_0^{\pi/2} - \frac{1}{2} [\cos u]_0^{\pi/2} \\ = \frac{1}{2} (1 - 0) - \frac{1}{2} (0 - 1) = 1$$

$$40. u = 3x, du = 3dx; v = 5x, dv = 5dx$$

$$\frac{1}{3} \int_{-3\pi/2}^{3\pi/2} \cos u du + \frac{1}{5} \int_{-5\pi/2}^{5\pi/2} \sin v dv \\ = \frac{1}{3} [\sin u]_{-3\pi/2}^{3\pi/2} - \frac{1}{5} [\cos v]_{-5\pi/2}^{5\pi/2} \\ = \frac{1}{3} [(-1) - 1] - \frac{1}{5} [0 - 0] = -\frac{2}{3}$$

$$41. u = \cos x, du = -\sin x dx$$

$$-\int_1^0 \sin u du = [\cos u]_1^0 = 1 - \cos 1$$

$$42. u = \pi \sin \theta, du = \pi \cos \theta d\theta$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos u du = \frac{1}{\pi} [\sin u]_{-\pi}^{\pi} = 0$$

$$43. u = \cos(x^2), du = -2x \sin(x^2) dx$$

$$-\frac{1}{2} \int_1^{\cos 1} u^3 du = -\frac{1}{2} \left[\frac{u^4}{4} \right]_1^{\cos 1} = -\frac{\cos^4 1}{8} + \frac{1}{8} \\ = \frac{1 - \cos^4 1}{8}$$

$$44. u = \sin(x^3), du = 3x^2 \cos(x^3) dx$$

$$\frac{1}{3} \int_{\sin(\pi^3/8)}^{\sin(\pi^3/8)} u^2 du = \frac{1}{9} [u^3]_{\sin(\pi^3/8)}^{\sin(\pi^3/8)} = \frac{2 \sin^3(\frac{\pi^3}{8})}{9}$$

$$45. u = \sqrt{t} + 1, du = \frac{1}{2\sqrt{t}} dt$$

$$2 \int_2^3 u^{-3} du = [-u^{-2}]_2^3 = \frac{5}{36}$$

$$46. u = 1 + \frac{1}{t}, du = -\frac{1}{t^2} dt$$

$$-\int_2^{3/2} u^2 du = -\frac{1}{3} [u^3]_2^{3/2} = \frac{37}{24}$$

$$47. \int_{-\pi}^{\pi} (\sin x + \cos x) dx = \int_{-\pi}^{\pi} \sin x dx + 2 \int_0^{\pi} \cos x dx \\ = 0 + 2 [\sin x]_0^{\pi} = 0$$

$$48. \int_{-1}^1 \frac{x^3}{(1+x^2)^4} dx = 0, \text{ since} \\ \frac{(-x)^3}{(1+(-x)^2)^4} = -\frac{x^3}{(1+x^2)^4}$$

$$49. \int_{-\pi/2}^{\pi/2} \frac{\sin x}{1 + \cos x} dx = 0, \text{ since} \\ \frac{\sin(-x)}{1 + \cos(-x)} = -\frac{\sin x}{1 + \cos x}$$

$$50. \int_{\sqrt{3}\pi}^{\sqrt{3}\pi} x^2 \cos(x^3) dx = 2 \int_0^{\sqrt{3}\pi} x^2 \cos(x^3) dx \\ = \frac{2}{3} [\sin(x^3)]_0^{\sqrt{3}\pi} = \frac{2}{3} \sin(3\sqrt{3}\pi^3)$$

$$\begin{aligned}
 51. \quad & \int_{-\pi}^{\pi} (\sin x + \cos x)^2 dx \\
 &= \int_{-\pi}^{\pi} (\sin^2 x + 2 \sin x \cos x + \cos^2 x) dx \\
 &= \int_{-\pi}^{\pi} (1 + 2 \sin x \cos x) dx = \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} \sin 2x dx \\
 &= 2 \int_0^{\pi} dx + 0 = 2[x]_0^{\pi} = 2\pi
 \end{aligned}$$

$$\begin{aligned}
 52. \quad & \int_{-\pi/2}^{\pi/2} z \sin^2(z^3) \cos(z^3) dz = 0, \text{ since} \\
 & (-z) \sin^2[(-z)^3] \cos[(-z)^3] \\
 &= -z \sin^2(-z^3) \cos(-z^3) \\
 &= -z[-\sin(z^3)]^2 \cos(z^3) \\
 &= -z \sin^2(z^3) \cos(z^3)
 \end{aligned}$$

$$\begin{aligned}
 53. \quad & \int_{-1}^1 (1 + x + x^2 + x^3) dx \\
 &= \int_{-1}^1 dx + \int_{-1}^1 x dx + \int_{-1}^1 x^2 dx + \int_{-1}^1 x^3 dx \\
 &= 2[x]_{-1}^1 + 0 + 2\left[\frac{x^3}{3}\right]_{-1}^1 + 0 = \frac{8}{3}
 \end{aligned}$$

$$\begin{aligned}
 54. \quad & \int_{-100}^{100} (v + \sin v + v \cos v + \sin^3 v)^5 dv = 0 \\
 & \text{since } (-v + \sin(-v) - v \cos(-v) + \sin^3(-v))^5 \\
 &= (-v - \sin v - v \cos v - \sin^3 v)^5 \\
 &= -(v + \sin v + v \cos v + \sin^3 v)^5
 \end{aligned}$$

$$\begin{aligned}
 55. \quad & \int_{-1}^1 (|x^3| + x^3) dx = 2 \int_0^1 |x^3| dx + \int_{-1}^1 x^3 dx \\
 &= 2 \left[\frac{x^4}{4} \right]_0^1 + 0 = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 56. \quad & \int_{-\pi/4}^{\pi/4} (|x| \sin^5 x + |x|^2 \tan x) dx = 0 \\
 & \text{since } |-x| \sin^5(-x) + |-x|^2 \tan(-x) \\
 &= -|x| \sin^5 x - |x|^2 \tan x
 \end{aligned}$$

$$\begin{aligned}
 57. \quad & \int_b^a f(x) dx = \int_a^b f(x) dx \text{ when } f \text{ is even.} \\
 & \int_b^a f(x) dx = -\int_a^b f(x) dx \text{ when } f \text{ is odd.}
 \end{aligned}$$

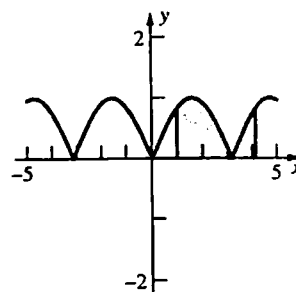
$$\begin{aligned}
 58. \quad & u = -x, du = -dx \\
 & -\int_a^b f(u) du = \int_b^a f(u) du = \int_b^a f(x) dx \\
 & \text{since the variable used in the integration is not important.}
 \end{aligned}$$

$$\begin{aligned}
 59. \quad & \int_0^{4\pi} |\cos x| dx = \int_0^{\pi} |\cos x| dx + \int_{\pi}^{2\pi} |\cos x| dx \\
 & \quad + \int_{2\pi}^{3\pi} |\cos x| dx + \int_{3\pi}^{4\pi} |\cos x| dx \\
 &= \int_0^{\pi} |\cos x| dx + \int_0^{\pi} |\cos x| dx + \int_0^{\pi} |\cos x| dx \\
 & \quad + \int_0^{\pi} |\cos x| dx \\
 &= 4 \int_0^{\pi} |\cos x| dx = 8 \int_0^{\pi/2} |\cos x| dx \\
 &= 8[\sin x]_0^{\pi/2} = 8
 \end{aligned}$$

60. Since $\sin x$ is periodic with period 2π , $\sin 2x$ is periodic with period π .

$$\begin{aligned}
 & \int_0^{4\pi} |\sin 2x| dx \\
 &= \int_0^{\pi} |\sin 2x| dx + \int_{\pi}^{2\pi} |\sin 2x| dx + \int_{2\pi}^{3\pi} |\sin 2x| dx \\
 & \quad + \int_{3\pi}^{4\pi} |\sin 2x| dx \\
 &= 4 \int_0^{\pi} |\sin 2x| dx = 8 \int_0^{\pi/2} \sin 2x dx \\
 &= 8 \left[-\frac{1}{2} \cos 2x \right]_0^{\pi/2} = -4(-1 - 1) = 8
 \end{aligned}$$

$$\begin{aligned}
 61. \quad & \int_1^{1+\pi} |\sin x| dx = \int_0^{\pi} |\sin x| dx = \int_0^{\pi} \sin x dx \\
 &= [-\cos x]_0^{\pi} = 2
 \end{aligned}$$



$$\begin{aligned}
 62. \quad & \int_2^{2+\pi/2} |\sin 2x| dx = \int_0^{\pi/2} |\sin 2x| dx \\
 &= \frac{1}{2} [-\cos 2x]_0^{\pi/2} = 1
 \end{aligned}$$

$$\begin{aligned}
 63. \quad & \frac{1}{12} \int_6^{18} T(t) dt = \left[70t - \frac{96}{\pi} \cos \left[\frac{\pi}{12} (t-9) \right] \right]_6^{18} \\
 &= 70(18-6) - \frac{96}{\pi} \left(\cos \frac{3\pi}{4} - \cos \frac{\pi}{4} \right) \\
 &= 840 + \frac{96\sqrt{2}}{\pi} \approx 883.2
 \end{aligned}$$

64. a. Written response.

$$\begin{aligned}
 \text{b. } A &= \int_0^a g(x) dx = \int_0^a \frac{a}{c} f\left(\frac{c}{a}x\right) dx \\
 &= \int_0^c \frac{a}{c} f(x) \frac{a}{c} dx = \frac{a^2}{c^2} \int_0^c f(x) dx \\
 B &= \int_0^b h(x) dx = \int_0^b \frac{b}{c} f\left(\frac{c}{b}x\right) dx \\
 &= \int_0^c \frac{b}{c} f(x) \frac{b}{c} dx = \frac{b^2}{c^2} \int_0^c f(x) dx \\
 \text{Thus, } \int_0^a g(x) dx + \int_0^b h(x) dx &= \frac{a^2}{c^2} \int_0^c f(x) dx + \frac{b^2}{c^2} \int_0^c f(x) dx \\
 &= \frac{a^2 + b^2}{c^2} \int_0^c f(x) dx = \int_0^c f(x) dx \text{ since } \\
 a^2 + b^2 &= c^2 \text{ from the triangle.}
 \end{aligned}$$

65. $y' = -k^2 \sin kx$

At $x = \frac{\pi}{2k}$, $y' = -k^2$. The equation of the tangent

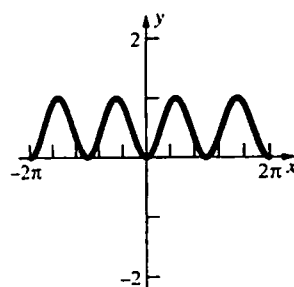
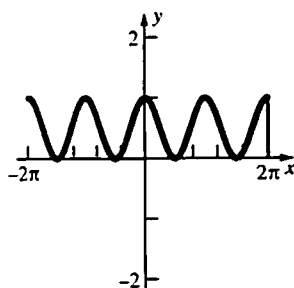
line at $x = \frac{\pi}{2k}$ is $y = -k^2 \left(x - \frac{\pi}{2k}\right)$.

When $x = 0$, $y = \frac{k\pi}{2}$.

$$\begin{aligned}
 A &= \frac{1}{2} \left(\frac{\pi}{k}\right) \left(\frac{k\pi}{2}\right) - \int_{-\pi/2k}^{\pi/2k} k \cos kx dx \\
 &= \frac{\pi^2}{4} - [\sin kx]_{-\pi/2k}^{\pi/2k} = \frac{\pi^2}{4} - 2
 \end{aligned}$$

66. a. $\int_0^{2\pi} (\sin^2 x + \cos^2 x) dx = \int_0^{2\pi} dx = [x]_0^{2\pi} = 2\pi$

b.



$$\begin{aligned}
 \text{c. } 2\pi &= \int_0^{2\pi} \cos^2 x dx + \int_0^{2\pi} \sin^2 x dx \\
 &= 2 \int_0^{2\pi} \cos^2 x dx, \text{ thus } \int_0^{2\pi} \cos^2 x dx \\
 &= \int_0^{2\pi} \sin^2 x dx = \pi
 \end{aligned}$$

67. a. Even

b. 2π

c. Interval	Value of Integral
$\left[0, \frac{\pi}{2}\right]$	0.46
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	0.92
$\left[0, \frac{3\pi}{2}\right]$	-0.46
$\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$	-0.92
$[0, 2\pi]$	0
$\left[\frac{\pi}{6}, \frac{13\pi}{6}\right]$	0
$\left[\frac{\pi}{6}, \frac{4\pi}{3}\right]$	-0.44
$\left[\frac{13\pi}{6}, \frac{10\pi}{3}\right]$	-0.44

68. a. Odd

b. 2π

c. Interval	Value of Integral
$\left[0, \frac{\pi}{2}\right]$	0.69
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	0
$\left[0, \frac{3\pi}{2}\right]$	0.69
$\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$	0
$[0, 2\pi]$	0
$\left[\frac{\pi}{6}, \frac{13\pi}{6}\right]$	0
$\left[\frac{\pi}{6}, \frac{4\pi}{3}\right]$	1.06
$\left[\frac{13\pi}{6}, \frac{10\pi}{3}\right]$	1.06

69. Using a right-Riemann sum,

$$\text{Distance} = \int_0^{24} v(t) dt$$

$$\approx \sum_{i=1}^8 v(t_i) \Delta t$$

$$= (31 + 54 + 53 + 52 + 35 + 31 + 28) \frac{3}{60}$$

$$= \frac{852}{60} = 14.2 \text{ miles}$$

70. Using a right-Riemann sum,

$$\text{Water Usage} = \int_0^{120} F(t) dt$$

$$\approx \sum_{i=1}^{10} F(t_i) \Delta t$$

$$= (71 + 68 + 78 + 105 + 111 + 108 \\ + 144 + 160 + 152 + 148) 12 \\ = 1145 \cdot 12 = 13,740 \text{ gallons}$$

71. If f is odd, then $f(-x) = -f(x)$ and we can write

$$\int_{-a}^0 f(x) dx = \int_{-a}^0 [-f(-x)] dx$$

$$= \int_{-a}^0 f(u) du$$

$$= - \int_0^a f(u) du$$

$$= - \int_0^a f(x) dx$$

On the second line, we have made the substitution $u = -x$.

5.9 Chapter Review

Concepts Test

- True: Theorem 5.1.C
- True: Obtained by integrating both sides of the Product Rule
- True: $(-\sin x)^2 = \sin^2 x = 1 - \cos^2 x$
- True: Theorem 5.6A
- True: If $F(x) = \int f(x) dx$, $f(x)$ is a derivative of $F(x)$.
- False: $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 + 7x - 5$ are a counterexample.
- False: The two sides will in general differ by a constant term.
- True: At any given height, speed on the downward trip is the negative of speed on the upward.
- True: $a_1 + a_0 + a_2 + a_1 + a_3 + a_2 \\ + \cdots + a_{n-1} + a_{n-2} + a_n + a_{n-1} \\ = a_0 + 2a_1 + 2a_2 + \cdots + 2a_{n-1} + a_n$
- True: $\sum_{i=1}^{100} (2i-1) = 2 \sum_{i=1}^{100} i - \sum_{i=1}^{100} 1 \\ = \frac{2(100)(100+1)}{2} - 100 = 10,000$
- True: $\sum_{i=1}^{10} (a_i + 1)^2 = \sum_{i=1}^{10} a_i^2 + 2 \sum_{i=1}^{10} a_i + \sum_{i=1}^{10} 1 \\ = 100 + 2(20) + 10 = 150$
- False: f must also be continuous except at a finite number of points on $[a, b]$.
- True: The area of a vertical line segment is 0.
- False: $\int_{-1}^1 x dx$ is a counterexample.
- True: $[f(x)]^2 \geq 0$, and if $[f(x)]^2$ is greater than 0 on $[a, b]$, the integral will be also.
- False: $D_x \left[\int_a^x f(z) dz \right] = f(x)$
- True: $\sin x + \cos x$ has period 2π , so $\int_x^{x+2\pi} (\sin x + \cos x) dx$ is independent of x .
- True: $\lim_{x \rightarrow a} kf(x) = k \lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} [f(x) + g(x)] \\ = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$ when all the limits exist.
- True: $\sin^{13} x$ is an odd function.
- True: Theorem 5.7.A
- False: The statement is not true if $c > d$.

22. False: $D_x \left[\int_0^{x^2} \frac{1}{1+t^2} dt \right] = \frac{2x}{1+x^2}$
23. True: Both sides equal 4.
24. True: Both sides equal 4.
25. True: The derivatives of even functions are odd.
26. False: $f(x) = x^2$ is a counterexample.
27. False: $f(x) = x^2$ is a counterexample.
28. False: $f(x) = x^2$ is a counterexample.
29. False: $f(x) = x^2$, $v(x) = 2x + 1$ is a counterexample.
30. False: $f(x) = x^3$ is a counterexample.
31. False: $f(x) = \sqrt{x}$ is a counterexample.
32. True: All rectangles have height 4, regardless of $\bar{x}_i$.
33. True: $F(b) - F(a) = \int_a^b F'(x) dx$
 $= \int_a^b G'(x) dx = G(b) - G(a)$
34. False: $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ because f is even.
35. False: $z(t) = t^2$ is a counterexample.
36. False: $\int_0^b f(x) dx = F(b) - F(0)$
37. True: Odd-exponent terms cancel themselves out over the interval, since they are odd.
38. False: $a = 0$, $b = 1$, $f(x) = -1$, $g(x) = 0$ is a counterexample.
39. False: $a = 0$, $b = 1$, $f(x) = -1$, $g(x) = 0$ is a counterexample.
40. True: $|a_1 + a_2 + a_3 + \cdots + a_n|$
 $\leq |a_1| + |a_2| + |a_3| + \cdots + |a_n|$ because any negative values of a_i make the left side smaller than the right side.

41. True: See Problem 28 of Section 5.7.
42. True: Definition of Definite Integral
43. True: Definition of Definite Integral

Sample Test Problems

- $\left[\frac{1}{4}x^4 - x^3 + 2x^{3/2} \right]_0^1 = \frac{5}{4}$
- $\frac{2}{3}x^3 - 3x - \frac{1}{x} + C$
- $\frac{1}{3}y^3 + 9 \cos y - \frac{26}{y} + C$
- $\frac{1}{3}(y^2 - 4)^{3/2} + C$
- $\frac{1}{4} \cdot \frac{3}{4} (2z^2 - 3)^{4/3} + C$
 $= \frac{3}{16} (2z^2 - 3)^{4/3} + C$
- $\left[-\frac{1}{5} \cos^5 x \right]_0^{\pi/2} = \frac{1}{5}$
- $u = \tan(3x^2 + 6x)$, $du = (6x + 6) \sec^2(3x^2 + 6x)$
 $\frac{1}{6} \int u^2 du = \frac{1}{18} u^3 + C$
 $\frac{1}{18} \left[\tan^3(3x^2 + 6x) \right]_0^{\pi} = \frac{1}{18} \tan^3(3\pi^2)$
- $u = t^4 + 9$, $du = 4t^3 dt$
 $\frac{1}{4} \int_9^{25} u^{-1/2} du = \frac{1}{2} \left[u^{1/2} \right]_9^{25} = 1$
- $\frac{1}{5} \left[\frac{3}{5} (t^5 + 5)^{5/3} \right]_1^2 \approx 46.9$
- $-\frac{1}{3(y^3 - 3y)} + C$
- $u = 2y^3 + 3y^2 + 6y$, $du = (6y^2 + 6y + 6) dy$
 $\frac{1}{6} \int u^{-1/5} du = \frac{5}{24} (2y^3 + 3y^2 + 6y)^{4/5} + C$
- $\int dy = \int \sin x dx$
 $y = -\cos x + C$
 $y = -\cos x + 3$

$$13. \int dy = \int \frac{1}{\sqrt{x+1}} dx$$

$$y = 2\sqrt{x+1} + C$$

$$y = 2\sqrt{x+1} + 14$$

$$14. \int \sin y dy = \int dx$$

$$-\cos y = x + C$$

$$x = -1 - \cos y$$

$$15. \int dy = \int \sqrt{2t-1} dt$$

$$y = \frac{1}{3}(2t-1)^{3/2} + C$$

$$y = \frac{1}{3}(2t-1)^{3/2} - 1$$

$$16. \int y^{-4} dy = \int t^2 dt$$

$$-\frac{1}{3y^3} = \frac{t^3}{3} + C$$

$$-\frac{1}{3y^3} = \frac{t^3}{3} - \frac{2}{3}$$

$$y = \sqrt[3]{\frac{1}{2-t^3}}$$

$$17. \int 2y dy = \int (6x - x^3) dx$$

$$y^2 = 3x^2 - \frac{1}{4}x^4 + C$$

$$y^2 = 3x^2 - \frac{1}{4}x^4 + 9$$

$$y = \sqrt{3x^2 - \frac{1}{4}x^4 + 9}$$

$$18. \int \cos y dy = \int x dx$$

$$\sin y = \frac{x^2}{2} + C$$

$$y = \sin^{-1}\left(\frac{x^2}{2}\right)$$

$$19. \text{ On } xy = 2, \frac{dy}{dx} = -\frac{2}{x^2}, \text{ so the curve has slope}$$

$$\frac{dy}{dx} = -\frac{2}{x^2}$$

$$\int dy = \int \frac{x^2}{2} dx$$

$$y = \frac{1}{6}x^3 + C$$

$$-\frac{1}{3} = -\frac{4}{3} + C$$

$$y = \frac{1}{6}x^3 + 1$$

$$20. a = 15\sqrt{t} + 8 = \frac{dv}{dt}$$

$$v = \int (15\sqrt{t} + 8) dt = 10t^{3/2} + 8t + v_0$$

$$v = 10t^{3/2} + 8t - 6 = \frac{dx}{dt}$$

$$x = \int (10t^{3/2} + 8t - 6) dt = 4t^{5/2} + 4t^2 - 6t + x_0$$

$$x = 4t^{5/2} + 4t^2 - 6t - 44$$

When $t = 4$ sec, $x = 124$ ft.

$$21. s = -16t^2 + 48t + 448; s = 0 \text{ at } t = 7;$$

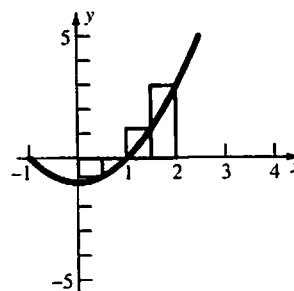
when $t = 7$, $v = -32(7) + 48 = -176$ ft/s

$$22. 45 \text{ miles per hour} = 66 \text{ feet per second};$$

$$60 \text{ miles per hour} = 88 \text{ feet per second.}$$

$$a = \frac{88-66}{10} = 2.2 \text{ ft/s}^2.$$

$$23. \sum_{i=1}^4 \left[\left(\frac{i}{2} \right)^2 - 1 \right] \left(\frac{1}{2} \right) = \frac{7}{4}$$



$$24. f'(x) = \frac{1}{x+3}, f'(7) = \frac{1}{10}$$

$$25. \int_0^3 (2 - \sqrt{x+1})^2 dx$$

$$= \int_0^3 (x + 5 - 4\sqrt{x+1}) dx$$

$$= \left[\frac{1}{2}x^2 + 5x - \frac{8}{3}(x+1)^{3/2} \right]_0^3 = \frac{5}{6}$$

$$26. \frac{1}{5-2} \int_2^5 3x^2 \sqrt{x^3-4} dx = \frac{1}{3} \left[\frac{2}{3}(x^3-4)^{3/2} \right]_2^5$$

$$= 294$$

$$27. \int_2^4 \left(5 - \frac{1}{x^2} \right) dx = \left[5x + \frac{1}{x} \right]_2^4 = \frac{39}{4}$$

$$\begin{aligned}
 28. \quad \sum_{i=1}^n (3^i - 3^{i-1}) \\
 &= (3-1) + (3^2-3) + (3^3-3^2) + \cdots + (3^n-3^{n-1}) \\
 &= 3^n - 1
 \end{aligned}$$

$$\begin{aligned}
 29. \quad \sum_{i=1}^{10} (6i^2 - 8i) &= 6 \sum_{i=1}^{10} i^2 - 8 \sum_{i=1}^{10} i \\
 &= 6 \left[\frac{10(11)(21)}{6} \right] - 8 \left[\frac{10(11)}{2} \right] = 1870
 \end{aligned}$$

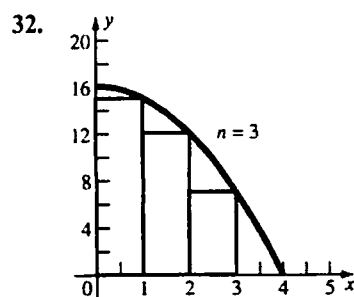
$$30. \quad \text{a.} \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

$$\text{b.} \quad 1 + 0 + (-1) + (-2) + (-3) + (-4) = -9$$

$$\text{c.} \quad 1 + \frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} - 1 = 0$$

$$31. \quad \text{a.} \quad \sum_{n=2}^{78} \frac{1}{n}$$

$$\text{b.} \quad \sum_{n=1}^{50} nx^{2n}$$



$$\begin{aligned}
 A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[16 - \left(\frac{3i}{n} \right)^2 \right] \left(\frac{3}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n \left[\frac{48}{n} - \frac{27}{n^3} i^2 \right] \right\} \\
 &= \lim_{n \rightarrow \infty} \left\{ \frac{48}{n} \sum_{i=1}^n 1 - \frac{27}{n^3} \sum_{i=1}^n i^2 \right\} \\
 &= \lim_{n \rightarrow \infty} \left\{ 48 - \frac{27}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] \right\} \\
 &= \lim_{n \rightarrow \infty} \left\{ 48 - \frac{9}{2} \left[2 + \frac{3}{n} + \frac{1}{n^2} \right] \right\} \\
 &= 48 - 9 = 39
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \text{a.} \quad \int_1^2 f(x) dx &= \int_1^0 f(x) dx + \int_0^2 f(x) dx \\
 &= -4 + 2 = -2
 \end{aligned}$$

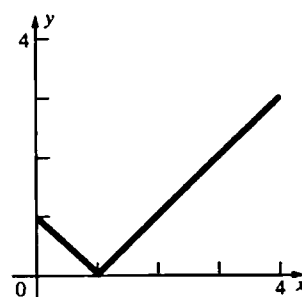
$$\text{b.} \quad \int_1^0 f(x) dx = - \int_0^1 f(x) dx = -4$$

$$\text{c.} \quad \int_0^2 3f(u) du = 3 \int_0^2 f(u) du = 3(2) = 6$$

$$\begin{aligned}
 \text{d.} \quad \int_0^2 [2g(x) - 3f(x)] dx \\
 &= 2 \int_0^2 g(x) dx - 3 \int_0^2 f(x) dx \\
 &= 2(-3) - 3(2) = -12
 \end{aligned}$$

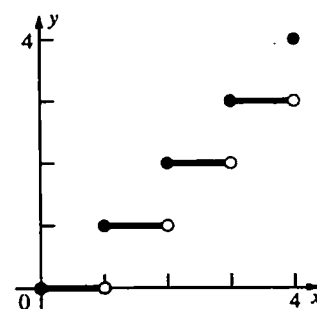
$$\text{e.} \quad \int_0^{-2} f(-x) dx = \int_0^2 f(x) dx = 2$$

34. a.



$$\int_0^4 |x-1| dx = \frac{1}{2}(1)(1) + \frac{1}{2}(3)(3) = 5$$

b.



$$\int_0^4 [x] dx = 1 + 2 + 3 = 6$$

$$\begin{aligned}
 \text{c.} \quad \int_0^4 (x - [x]) dx &= \int_0^4 x dx - \int_0^4 [x] dx \\
 &= \left[\frac{1}{2} x^2 \right]_0^4 - 6 = 8 - 6 = 2
 \end{aligned}$$

$$35. \quad \text{a.} \quad \int_{-2}^2 f(x) dx = 2 \int_0^2 f(x) dx = 2(-4) = -8$$

$$\text{b.} \quad \int_{-2}^2 |f(x)| dx = 2 \int_0^2 |f(x)| dx = 2|-4| = 8$$

$$\text{c.} \quad \int_{-2}^2 g(x) dx = 0$$

$$\begin{aligned} \text{d. } \int_{-2}^2 [f(x) + f(-x)] dx \\ &= 2 \int_0^2 f(x) dx + 2 \int_0^2 f(x) dx \\ &= 4(-4) = -16 \end{aligned}$$

$$\begin{aligned} \text{e. } \int_0^2 [2g(x) + 3f(x)] dx \\ &= 2 \int_0^2 g(x) dx + 3 \int_0^2 f(x) dx \\ &= 2(5) + 3(-4) = -2 \end{aligned}$$

$$\begin{aligned} \text{f. } \int_{-2}^0 g(x) dx &= \int_2^0 g(-x) \cdot -dx = \int_2^0 g(x) dx \\ &= - \int_0^2 g(x) dx = -5 \end{aligned}$$

$$36. \int_{-100}^{100} (x^3 + \sin^5 x) dx = 0$$

$$\begin{aligned} 37. \int_{-4}^1 3x^2 dx &= 3c^2(-1+4) \\ [x^3]_{-4}^1 &= 9c^2 \\ c^2 &= 7 \\ c &= -\sqrt{7} \approx -2.65 \end{aligned}$$

$$38. \text{ a. } G'(x) = \frac{1}{x^2 + 1}$$

$$\text{b. } G'(x) = \frac{2x}{x^4 + 1}$$

$$\text{c. } G'(x) = \frac{3x^2}{x^6 + 1} - \frac{1}{x^2 + 1}$$

$$39. \text{ a. } G'(x) = \sin^2 x$$

$$\text{b. } G'(x) = f(x+1) - f(x)$$

$$\text{c. } G'(x) = -\frac{1}{x^2} \int_0^x f(z) dz + \frac{1}{x} f(x)$$

$$\text{d. } G'(x) = \int_0^x f(t) dt$$

$$\begin{aligned} \text{e. } G(x) &= \int_0^{g(x)} \frac{dg(u)}{du} du = [g(u)]_0^{g(x)} \\ &= g(g(x)) - g(0) \\ G'(x) &= g'(g(x))g'(x) \end{aligned}$$

$$\begin{aligned} \text{f. } G(x) &= \int_0^x f(-t) dt = \int_0^x f(u)(-du) \\ &= - \int_0^x f(u) du \\ G'(x) &= -f(x) \end{aligned}$$

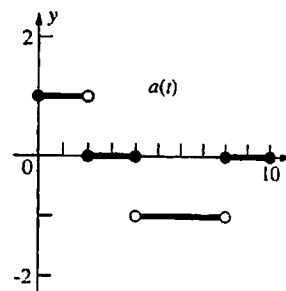
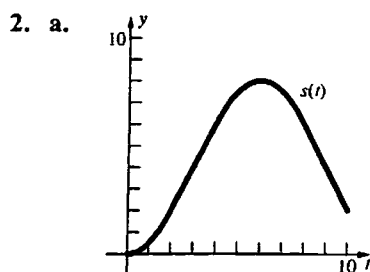
$$40. \text{ a. } \int_0^4 \sqrt{x} dx = \frac{2}{3} [x^{3/2}]_0^4 = \frac{16}{3}$$

$$\text{b. } \int_1^3 x^2 dx = \frac{1}{3} [x^3]_1^3 = \frac{26}{3}$$

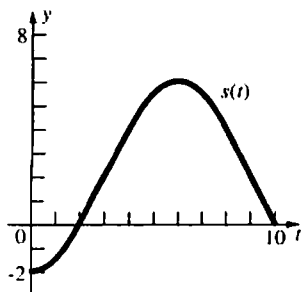
$$\begin{aligned} 41. f(x) &= \int_{2x}^{5x} \frac{1}{t} dt = \int_0^{5x} \frac{1}{t} dt - \int_0^{2x} \frac{1}{t} dt \\ f'(x) &= \frac{1}{5x} \cdot 5 - \frac{1}{2x} \cdot 2 = 0 \end{aligned}$$

5.10 Additional Problem Set

1. Recall that by implicitly differentiating the equation of the circle $x^2 + y^2 = r^2$, we get $\frac{dy}{dx} = -\frac{x}{y}$. Thus, the solution must be a graph of a circle. Since the point $(0, 1)$ must lie on the graph, the answer is Figure 1.

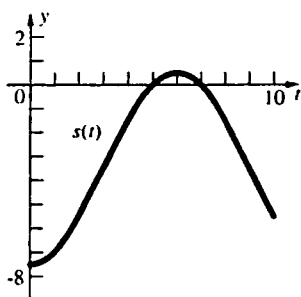


b.



The graph of $a(t)$ is the same as in part a.

c.



The graph of $a(t)$ is the same as in part a.

3. a. $30[88 + 80 + 66 + 51 + 37 + 26 + 14 + 5]$
 $= 11,010$ feet

b. $30[80 + 66 + 51 + 37 + 26 + 14 + 5 + 0]$
 $= 8370$ feet

c. $30\left[84 + 73 + \frac{117}{2} + 44 + \frac{63}{2} + 20 + \frac{19}{2} + \frac{5}{2}\right]$
 $= 9690$ feet

d. Part a. uses the largest value of each pair and b. uses the smallest. Part c. averages each pair, so the total value is the average of a. and b.

4. Recall that $\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx$.

$$\int_a^b \bar{f} dx = \bar{f} \int_a^b dx = \frac{1}{b-a} \int_a^b f(x) dx \cdot \int_a^b dx$$

$$= \frac{1}{b-a} \int_a^b f(x) dx \cdot (b-a) = \int_a^b f(x) dx$$

5. a. $\bar{u} + \bar{v} = \frac{1}{b-a} \int_a^b u dx + \frac{1}{b-a} \int_a^b v dx$
 $= \frac{1}{b-a} \int_a^b (u+v) dx = \overline{u+v}$

b. $k\bar{u} = \frac{k}{b-a} \int_a^b u dx = \frac{1}{b-a} \int_a^b ku dx = \overline{ku}$

c. $\bar{u} = \frac{1}{b-a} \int_a^b u dx \leq \frac{1}{b-a} \int_a^b v dx = \bar{v}$

6. a. $\int_{\phi/120\pi}^{\phi/120\pi+1} \hat{V} dt = \int_{\phi/120\pi}^{\phi/120\pi+1} \hat{V} \sin(120\pi t + \phi) dt = \left[-\frac{1}{120\pi} \hat{V} \cos(120\pi t + \phi) \right]_{\phi/120\pi}^{\phi/120\pi+1} = 0$

b. $\frac{1}{120} \int_{\phi/120\pi}^{\phi/120\pi+1/120} \hat{V} \sin(120\pi t + \phi) dt = 120 \left[-\frac{1}{120\pi} \hat{V} \cos(120\pi t + \phi) \right]_{\phi/120\pi}^{\phi/120\pi+1/120} = -\frac{\hat{V}}{\pi} (-1 - 1) = \frac{2\hat{V}}{\pi}$

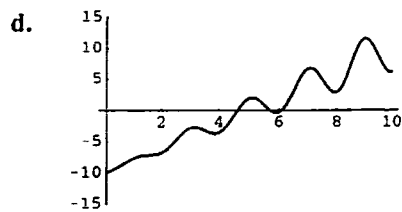
c. $V_{rms} = \sqrt{\int_{\phi}^{\phi+1} \hat{V}^2 \sin^2(120\pi t + \phi) dt} = \sqrt{\hat{V}^2 \left[-\frac{1}{240} \sin(120\pi t + \phi) \cos(120\pi t + \phi) + \frac{1}{2} t \right]_{\phi}^{\phi+1}}$
 $= \hat{V} \sqrt{-\frac{1}{240\pi} \sin(120\pi\phi + 120\pi + \phi) \cos(120\pi\phi + 120\pi + \phi) + \frac{1}{2} \phi + \frac{1}{2} - \left[-\frac{1}{240\pi} \sin(120\pi\phi + \phi) \cos(120\pi\phi + \phi) + \frac{1}{2} \phi \right]}$
 $= \hat{V} \sqrt{\frac{1}{2}} = \frac{\hat{V}\sqrt{2}}{2}$

d. $120 = \frac{\hat{V}\sqrt{2}}{2}$
 $\hat{V} = 120\sqrt{2} \approx 169.71$ Volts

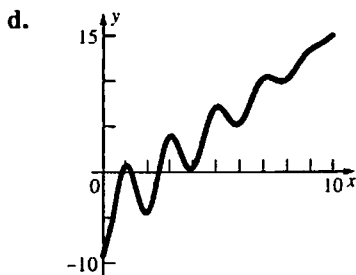
7. a. Local minima at 0, ≈ 3.8 , ≈ 5.8 , ≈ 7.9 , ≈ 9.9 , ≈ 10 ;
 local maxima at ≈ 3.1 , ≈ 5 , ≈ 7.1 , ≈ 9

b. Absolute minimum at 0, absolute maximum at ≈ 9

- c. $\approx (0.7, 1.5), (2.5, 3.5), (4.5, 5.5), (6.5, 7.5), (8.5, 9.5)$



8. a. Local minima at 0, ≈ 1.8 , ≈ 3.8 , ≈ 5.8 , ≈ 7.8 ;
local maxima at ≈ 1 , ≈ 3.1 , ≈ 5.2 , ≈ 7.3
- b. Absolute minimum at 0, absolute maximum at 10
- c. $(0.5, 1.5), (2.5, 3.5), (4.5, 5.5), (6.5, 7.5), (8.5, 9.5)$



9. a. $f(x) > 0$ so $\int_{-1}^3 f(x) dx > 0$
- b. $\int_{-1}^3 f'(x) dx = f(3) - f(-1) = 1 - 2 = -1 < 0$
- c. $\int_{-1}^3 f''(x) dx = f'(3) - f'(-1) = -1 - 1 = -2 < 0$
- d. $\int_{-1}^3 f'''(x) dx = f''(3) - f''(-1)$
 $= \text{positive} - \text{negative} > 0$

Applications of the
Integral

6.1 Concepts Review

- $\int_a^b f(x)dx; -\int_a^b f(x)dx$
- slice, approximate, integrate
- $g(x) - f(x); f(x) = g(x)$
- $\int_c^d [q(y) - p(y)]dy$

Problem Set 6.1

1. Slice vertically.

$$\Delta A \approx (x^2 + 1)\Delta x$$

$$A = \int_{-1}^2 (x^2 + 1)dx = \left[\frac{1}{3}x^3 + x \right]_{-1}^2$$

$$= \left(\frac{8}{3} + 2 \right) - \left(-\frac{1}{3} - 1 \right) = 6$$

2. Slice vertically.

$$\Delta A \approx (x^3 - x + 2)\Delta x$$

$$A = \int_{-1}^2 (x^3 - x + 2)dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 + 2x \right]_{-1}^2$$

$$= (4 - 2 + 4) - \left(\frac{1}{4} - \frac{1}{2} - 2 \right) = \frac{33}{4}$$

3. Slice vertically.

$$\Delta A \approx [(x^2 + 2) - (-x)]\Delta x = (x^2 + x + 2)\Delta x$$

$$A = \int_{-2}^2 (x^2 + x + 2)dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \right]_{-2}^2$$

$$= \left(\frac{8}{3} + 2 + 4 \right) - \left(-\frac{8}{3} + 2 - 4 \right) = \frac{40}{3}$$

4. Slice vertically.

$$\Delta A \approx -(x^2 + 2x - 3)\Delta x = (-x^2 - 2x + 3)\Delta x$$

$$A = \int_{-3}^1 (-x^2 - 2x + 3)dx = \left[-\frac{1}{3}x^3 - x^2 + 3x \right]_{-3}^1$$

$$= \left(-\frac{1}{3} - 1 + 3 \right) - (9 - 9 - 9) = \frac{32}{3}$$

5. To find the intersection points, solve
- $2 - x^2 = x$
- .

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2, 1$$

Slice vertically.

$$\Delta A \approx [(2 - x^2) - x]\Delta x = (-x^2 - x - 2)\Delta x$$

$$A = \int_{-2}^1 (-x^2 - x - 2)dx = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x \right]_{-2}^1$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) = \frac{9}{2}$$

6. To find the intersection points, solve

$$x + 4 = x^2 - 2$$

$$x^2 - x - 6 = 0$$

$$(x + 2)(x - 3) = 0$$

$$x = -2, 3$$

Slice vertically.

$$\Delta A \approx [(x + 4) - (x^2 - 2)]\Delta x = (-x^2 + x + 6)\Delta x$$

$$A = \int_{-2}^3 (-x^2 + x + 6)dx = \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x \right]_{-2}^3$$

$$\left(-9 + \frac{9}{2} + 18 \right) - \left(\frac{8}{3} + 2 - 12 \right) = \frac{125}{6}$$

7. Solve
- $x^3 - x^2 - 6x = 0$
- .

$$x(x^2 - x - 6) = 0$$

$$x(x + 2)(x - 3) = 0$$

$$x = -2, 0, 3$$

Slice vertically.

$$\Delta A_1 \approx (x^3 - x^2 - 6x)\Delta x$$

$$\Delta A_2 \approx -(x^3 - x^2 - 6x)\Delta x = (-x^3 + x^2 + 6x)\Delta x$$

$$A = A_1 + A_2$$

$$= \int_{-2}^0 (x^3 - x^2 - 6x)dx + \int_0^3 (-x^3 + x^2 + 6x)dx$$

$$\begin{aligned}
 &= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 \right]_{-2}^0 \\
 &+ \left[-\frac{1}{4}x^4 + \frac{1}{3}x^3 + 3x^2 \right]_0^3 \\
 &= \left[0 - \left(4 + \frac{8}{3} - 12 \right) \right] + \left[-\frac{81}{4} + 9 + 27 - 0 \right] \\
 &= \frac{16}{3} + \frac{63}{4} = \frac{253}{12}
 \end{aligned}$$

8. To find the intersection points, solve

$$-x + 2 = x^2.$$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2, 1$$

Slice vertically.

$$\Delta A \approx [(-x + 2) - x^2] \Delta x = (-x^2 - x + 2) \Delta x$$

$$\begin{aligned}
 A &= \int_{-2}^1 (-x^2 - x + 2) dx = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x \right]_{-2}^1 \\
 &= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) = \frac{9}{2}
 \end{aligned}$$

9. To find the intersection points, solve

$$y + 1 = 3 - y^2.$$

$$y^2 + y - 2 = 0$$

$$(y + 2)(y - 1) = 0$$

$$y = -2, 1$$

Slice horizontally.

$$\Delta A \approx [(3 - y^2) - (y + 1)] \Delta y = (-y^2 - y + 2) \Delta y$$

$$\begin{aligned}
 A &= \int_{-2}^1 (-y^2 - y + 2) dy = \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y \right]_{-2}^1 \\
 &= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) = \frac{9}{2}
 \end{aligned}$$

10. To find the intersection point, solve $y^2 = 6 - y$.

$$y^2 + y - 6 = 0$$

$$(y + 3)(y - 2) = 0$$

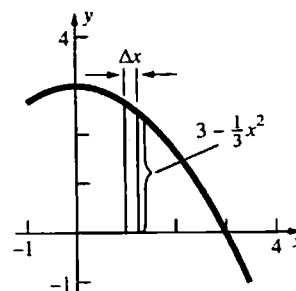
$$y = -3, 2$$

Slice horizontally.

$$\Delta A \approx [(6 - y) - y^2] \Delta y = (-y^2 - y + 6) \Delta y$$

$$\begin{aligned}
 A &= \int_0^2 (-y^2 - y + 6) dy = \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 6y \right]_0^2 \\
 &= -\frac{8}{3} - 2 + 12 = \frac{22}{3}
 \end{aligned}$$

11.

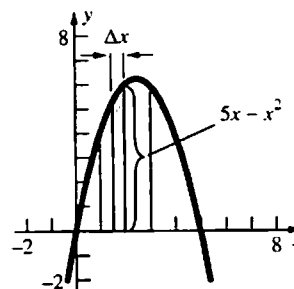


$$\Delta A \approx \left(3 - \frac{1}{3}x^2 \right) \Delta x$$

$$A = \int_0^3 \left(3 - \frac{1}{3}x^2 \right) dx = \left[3x - \frac{1}{9}x^3 \right]_0^3 = 9 - 3 = 6$$

Estimate the area to be $(3)(2) = 6$.

12.

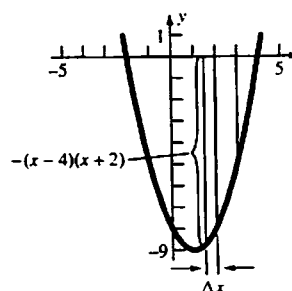


$$\Delta A \approx (5x - x^2) \Delta x$$

$$\begin{aligned}
 A &= \int_1^3 (5x - x^2) dx = \left[\frac{5}{2}x^2 - \frac{1}{3}x^3 \right]_1^3 \\
 &= \left(\frac{45}{2} - 9 \right) - \left(\frac{5}{2} - \frac{1}{3} \right) = \frac{34}{3} \approx 11.33
 \end{aligned}$$

Estimate the area to be $(2) \left(5 \frac{1}{2} \right) = 11$.

13.

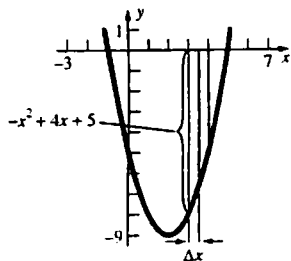


$$\Delta A \approx -(x - 4)(x + 2) \Delta x = (-x^2 + 2x + 8) \Delta x$$

$$\begin{aligned}
 A &= \int_{-2}^4 (-x^2 + 2x + 8) dx = \left[-\frac{1}{3}x^3 + x^2 + 8x \right]_{-2}^4 \\
 &= -9 + 9 + 24 = 24
 \end{aligned}$$

Estimate the area to be $(3)(8) = 24$.

14.



$$\Delta A \approx -(x^2 - 4x + 5)\Delta x = (-x^2 + 4x + 5)\Delta x$$

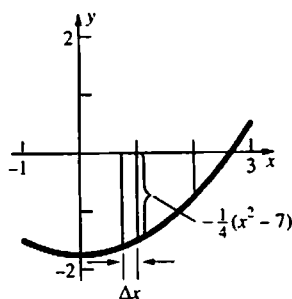
$$A = \int_{-1}^4 (-x^2 + 4x + 5) dx$$

$$= \left[-\frac{1}{3}x^3 + 2x^2 + 5x \right]_{-1}^4$$

$$= \left(-\frac{64}{3} + 32 + 20 \right) - \left(-\frac{1}{3} + 2 - 5 \right) = \frac{100}{3} \approx 33.33$$

Estimate the area to be $(5)\left(6\frac{1}{2}\right) = 32\frac{1}{2}$.

15.



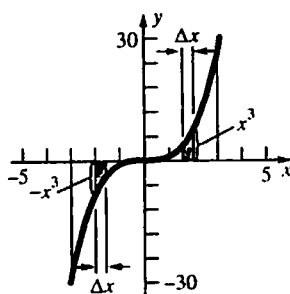
$$\Delta A \approx -\frac{1}{4}(x^2 - 7)\Delta x$$

$$A = \int_0^2 -\frac{1}{4}(x^2 - 7) dx = -\frac{1}{4} \left[\frac{1}{3}x^3 - 7x \right]_0^2$$

$$= -\frac{1}{4} \left(\frac{8}{3} - 14 \right) = \frac{17}{6} \approx 2.83$$

Estimate the area to be $(2)\left(1\frac{1}{2}\right) = 3$.

16.



$$\Delta A_1 \approx -x^3 \Delta x$$

$$\Delta A_2 \approx x^3 \Delta x$$

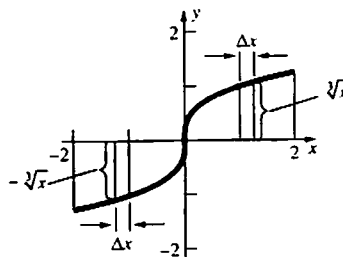
$$A = A_1 + A_2 = \int_{-3}^0 -x^3 dx + \int_0^3 x^3 dx$$

$$= \left[-\frac{1}{4}x^4 \right]_{-3}^0 + \left[\frac{1}{4}x^4 \right]_0^3 = \left(\frac{81}{4} \right) + \left(\frac{81}{4} \right) = \frac{81}{2}$$

$$= 40.5$$

Estimate the area to be $(3)(7) + (3)(7) = 42$.

17.



$$\Delta A_1 \approx -\sqrt[3]{x} \Delta x$$

$$\Delta A_2 \approx \sqrt[3]{x} \Delta x$$

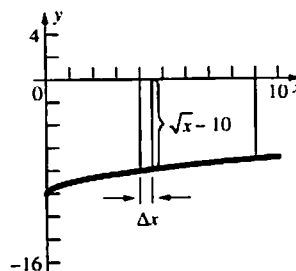
$$A = A_1 + A_2 = \int_{-2}^0 -\sqrt[3]{x} dx + \int_0^2 \sqrt[3]{x} dx$$

$$= \left[-\frac{3}{4}x^{4/3} \right]_{-2}^0 + \left[\frac{3}{4}x^{4/3} \right]_0^2 = \left(\frac{3\sqrt[3]{2}}{2} \right) + \left(\frac{3\sqrt[3]{2}}{2} \right)$$

$$= 3\sqrt[3]{2} \approx 3.78$$

Estimate the area to be $(2)(1) + (2)(1) = 4$.

18.



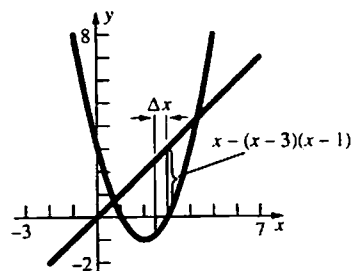
$$\Delta A \approx -(\sqrt{x} - 10)\Delta x = (10 - \sqrt{x})\Delta x$$

$$A = \int_0^9 (10 - \sqrt{x}) dx = \left[10x - \frac{2}{3}x^{3/2} \right]_0^9$$

$$= 90 - 18 = 72$$

Estimate the area to be $9 \cdot 8 = 72$.

19.



$$\Delta A \approx [x - (x-3)(x-1)] \Delta x$$

$$= [x - (x^2 - 4x + 3)] \Delta x = (-x^2 + 5x - 3) \Delta x$$

To find the intersection points, solve $x = (x-3)(x-1)$.

$$x^2 - 5x + 3 = 0$$

$$x = \frac{5 \pm \sqrt{25 - 12}}{2}$$

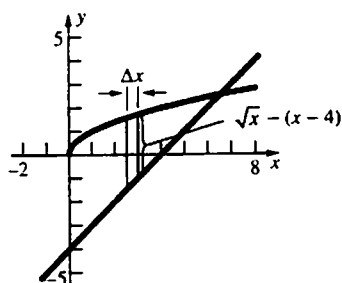
$$x = \frac{5 \pm \sqrt{13}}{2}$$

$$A = \int_{\frac{5-\sqrt{13}}{2}}^{\frac{5+\sqrt{13}}{2}} (-x^2 + 5x - 3) dx$$

$$= \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 3x \right]_{\frac{5-\sqrt{13}}{2}}^{\frac{5+\sqrt{13}}{2}} = \frac{13\sqrt{13}}{6} \approx 7.81$$

Estimate the area to be $\frac{1}{2}(4)(4) = 8$.

20.



$$\Delta A \approx [\sqrt{x} - (x - 4)] \Delta x = (\sqrt{x} - x + 4) \Delta x$$

To find the intersection point, solve

$$\sqrt{x} = (x - 4).$$

$$x = (x - 4)^2$$

$$x^2 - 9x + 16 = 0$$

$$x = \frac{9 \pm \sqrt{81 - 64}}{2}$$

$$x = \frac{9 \pm \sqrt{17}}{2}$$

$$\left(x = \frac{9 - \sqrt{17}}{2} \text{ is extraneous so } x = \frac{9 + \sqrt{17}}{2} \right)$$

$$A = \int_0^{\frac{9+\sqrt{17}}{2}} (\sqrt{x} - x + 4) dx$$

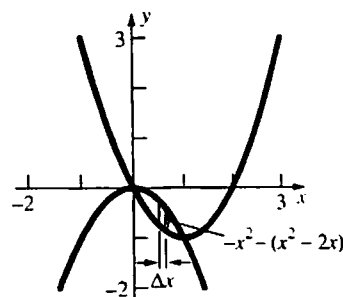
$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 4x \right]_0^{\frac{9+\sqrt{17}}{2}}$$

$$= \frac{2}{3} \left(\frac{9+\sqrt{17}}{2} \right)^{3/2} - \frac{1}{2} \left(\frac{9+\sqrt{17}}{2} \right)^2 + 4 \left(\frac{9+\sqrt{17}}{2} \right)$$

$$= \frac{2}{3} \left(\frac{9+\sqrt{17}}{2} \right)^{3/2} + \frac{23}{4} - \frac{\sqrt{17}}{4} \approx 15.92$$

Estimate the area to be $\frac{1}{2} \left(5\frac{1}{2} \right) \left(5\frac{1}{2} \right) = 15\frac{1}{8}$.

21.



$$\Delta A \approx [-x^2 - (x^2 - 2x)] \Delta x = (-2x^2 + 2x) \Delta x$$

To find the intersection points, solve

$$-x^2 = x^2 - 2x.$$

$$2x^2 - 2x = 0$$

$$2x(x - 1) = 0$$

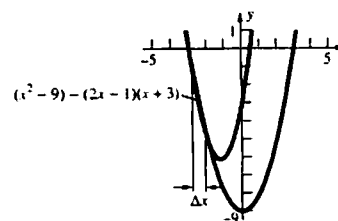
$$x = 0, x = 1$$

$$A = \int_0^1 (-2x^2 + 2x) dx = \left[-\frac{2}{3}x^3 + x^2 \right]_0^1$$

$$= -\frac{2}{3} + 1 = \frac{1}{3} \approx 0.33$$

Estimate the area to be $\left(\frac{1}{2} \right) \left(\frac{1}{2} \right) = \frac{1}{4}$.

22.



$$\Delta A \approx [(x^2 - 9) - (2x - 1)(x + 3)] \Delta x$$

$$= [(x^2 - 9) - (2x^2 + 5x - 3)] \Delta x$$

$$= (-x^2 - 5x - 6) \Delta x$$

To find the intersection points, solve

$$(2x - 1)(x + 3) = x^2 - 9.$$

$$x^2 + 5x + 6 = 0$$

$$(x + 3)(x + 2) = 0$$

$$x = -3, -2$$

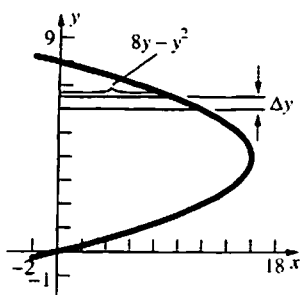
$$A = \int_{-3}^{-2} (-x^2 - 5x - 6) dx$$

$$= \left[-\frac{1}{3}x^3 - \frac{5}{2}x^2 - 6x \right]_{-3}^{-2}$$

$$= \left(\frac{8}{3} - 10 + 12 \right) - \left(9 - \frac{45}{2} + 18 \right) = \frac{1}{6} \approx 0.17$$

Estimate the area to be $\frac{1}{2} (1) \left(5 - 4\frac{2}{3} \right) = \frac{1}{6}$.

23.



$$\Delta A \approx (8y - y^2) \Delta y$$

To find the intersection points, solve

$$8y - y^2 = 0.$$

$$y(8 - y) = 0$$

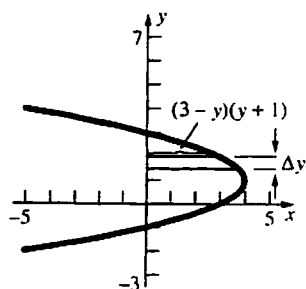
$$y = 0, 8$$

$$A = \int_0^8 (8y - y^2) dy = \left[4y^2 - \frac{1}{3}y^3 \right]_0^8$$

$$= 256 - \frac{512}{3} = \frac{256}{3} \approx 85.33$$

Estimate the area to be $(16)(5) = 80$.

24.



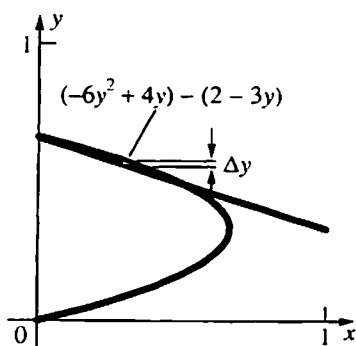
$$\Delta A \approx (3 - y)(y + 1) \Delta y = (-y^2 + 2y + 3) \Delta y$$

$$A = \int_{-1}^3 (-y^2 + 2y + 3) dy = \left[-\frac{1}{3}y^3 + y^2 + 3y \right]_{-1}^3$$

$$= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3 \right) = \frac{32}{3} \approx 10.67$$

Estimate the area to be $(4) \left(2\frac{1}{2} \right) = 10$.

25.



$$\Delta A \approx [(-6y^2 + 4y) - (2 - 3y)] \Delta y$$

$$= (-6y^2 + 7y - 2) \Delta y$$

To find the intersection points, solve

$$-6y^2 + 4y = 2 - 3y.$$

$$6y^2 - 7y + 2 = 0$$

$$(2y - 1)(3y - 2) = 0$$

$$y = \frac{1}{2}, \frac{2}{3}$$

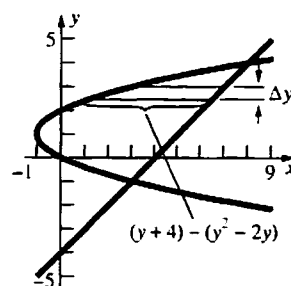
$$A = \int_{1/2}^{2/3} (-6y^2 + 7y - 2) dy = \left[-2y^3 + \frac{7}{2}y^2 - 2y \right]_{1/2}^{2/3}$$

$$= \left(-\frac{16}{27} + \frac{14}{9} - \frac{4}{3} \right) - \left(-\frac{1}{4} + \frac{7}{8} - 1 \right) = \frac{1}{216} \approx 0.0046$$

Estimate the area to be

$$\frac{1}{2} \left(\frac{1}{2} \right) \left(\frac{1}{5} \right) - \frac{1}{2} \left(\frac{1}{2} \right) \left(\frac{1}{6} \right) = \frac{1}{120}.$$

26.



$$\Delta A \approx [(y + 4) - (y^2 - 2y)] \Delta y = (-y^2 + 3y + 4) \Delta y$$

To find the intersection points, solve

$$y^2 - 2y = y + 4.$$

$$y^2 - 3y - 4 = 0$$

$$(y + 1)(y - 4) = 0$$

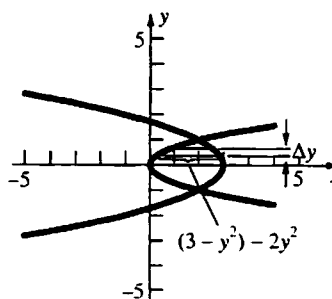
$$y = -1, 4$$

$$A = \int_{-1}^4 (-y^2 + 3y + 4) dy = \left[-\frac{1}{3}y^3 + \frac{3}{2}y^2 + 4y \right]_{-1}^4$$

$$= \left(-\frac{64}{3} + 24 + 16 \right) - \left(\frac{1}{3} + \frac{3}{2} - 4 \right) = \frac{125}{6} \approx 20.83$$

Estimate the area to be $(7)(3) = 21$.

27.



$$\Delta A \approx [(3 - y^2) - 2y^2] \Delta y = (-3y^2 + 3) \Delta y$$

To find the intersection points, solve

$$2y^2 = 3 - y^2.$$

$$3y^2 - 3 = 0$$

$$3(y + 1)(y - 1) = 0$$

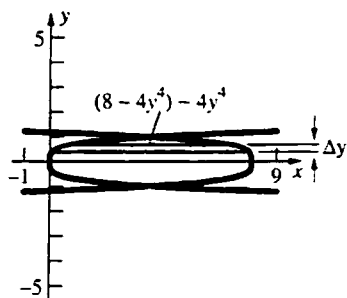
$$y = -1, 1$$

$$A = \int_{-1}^1 (-3y^2 + 3) dy = \left[-y^3 + 3y \right]_{-1}^1$$

$$= (-1 + 3) - (1 - 3) = 4$$

Estimate the value to be $(2)(2) = 4$.

28.



$$\Delta A \approx [(8 - 4y^4) - (4y^4)] \Delta y = (8 - 8y^4) \Delta y$$

To find the intersection points, solve

$$4y^4 = 8 - 4y^4.$$

$$8y^4 = 8$$

$$y^4 = 1$$

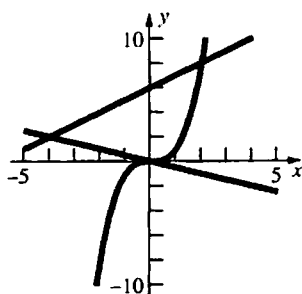
$$y = \pm 1$$

$$A = \int_{-1}^1 (8 - 8y^4) dy = \left[8y - \frac{8}{5}y^5 \right]_{-1}^1$$

$$= \left(8 - \frac{8}{5} \right) - \left(-8 + \frac{8}{5} \right) = \frac{64}{5} = 12.8$$

Estimate the area to be $(8) \left(1\frac{1}{2} \right) = 12$.

29.



Let R_1 be the region bounded by $2y + x = 0$,

$y = x + 6$, and $x = 0$.

$$A(R_1) = \int_{-4}^0 \left[(x+6) - \left(-\frac{1}{2}x \right) \right] dx$$

$$= \int_{-4}^0 \left(\frac{3}{2}x + 6 \right) dx$$

Let R_2 be the region bounded by $y = x + 6$,

$y = x^3$, and $x = 0$.

$$31. \int_{-1}^9 (3t^2 - 24t + 36) dt = \left[t^3 - 12t^2 + 36t \right]_{-1}^9 = (729 - 972 + 324) - (-1 - 12 - 36) = 130$$

The displacement is 130 ft. Solve $3t^2 - 24t + 36 = 0$.

$$3(t-2)(t-6) = 0$$

$$A(R_2) = \int_0^2 [(x+6) - x^3] dx = \int_0^2 (-x^3 + x + 6) dx$$

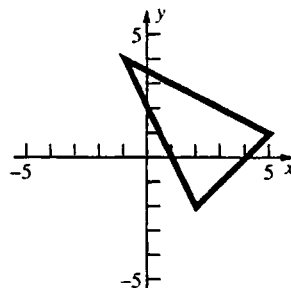
$$A(R) = A(R_1) + A(R_2)$$

$$= \int_{-4}^0 \left(\frac{3}{2}x + 6 \right) dx + \int_0^2 (-x^3 + x + 6) dx$$

$$= \left[\frac{3}{4}x^2 + 6x \right]_{-4}^0 + \left[-\frac{1}{4}x^4 + \frac{1}{2}x^2 + 6x \right]_0^2$$

$$= 12 + 10 = 22$$

30.



An equation of the line through $(-1, 4)$ and $(5, 1)$

is $y = -\frac{1}{2}x + \frac{7}{2}$. An equation of the line through

$(-1, 4)$ and $(2, -2)$ is $y = -2x + 2$. An equation of the line through $(2, -2)$ and $(5, 1)$ is $y = x - 4$.

Two integrals must be used. The left-hand part of the triangle has area

$$\int_{-1}^2 \left[-\frac{1}{2}x + \frac{7}{2} - (-2x + 2) \right] dx = \int_{-1}^2 \left(\frac{3}{2}x + \frac{3}{2} \right) dx.$$

The right-hand part of the triangle has area

$$\int_2^5 \left[-\frac{1}{2}x + \frac{7}{2} - (x - 4) \right] dx = \int_2^5 \left(-\frac{3}{2}x + \frac{15}{2} \right) dx.$$

The triangle has area

$$\int_{-1}^2 \left(\frac{3}{2}x + \frac{3}{2} \right) dx + \int_2^5 \left(-\frac{3}{2}x + \frac{15}{2} \right) dx$$

$$= \left[\frac{3}{4}x^2 + \frac{3}{2}x \right]_{-1}^2 + \left[-\frac{3}{4}x^2 + \frac{15}{2}x \right]_2^5$$

$$= \frac{27}{4} + \frac{27}{4} = \frac{27}{2} = 13.5$$

$$t = 2, 6$$

$$|v(t)| = \begin{cases} 3t^2 - 24t + 36 & t \leq 2, t \geq 6 \\ -3t^2 + 24t - 36 & 2 < t < 6 \end{cases}$$

$$\begin{aligned} \int_{-1}^9 |3t^2 - 24t + 36| dt &= \int_{-1}^2 (3t^2 - 24t + 36) dt + \int_2^6 (-3t^2 + 24t - 36) dt + \int_6^9 (3t^2 - 24t + 36) dt \\ &= \left[t^3 - 12t^2 + 36t \right]_{-1}^2 + \left[-t^3 + 12t^2 - 36t \right]_2^6 + \left[t^3 - 12t^2 + 36t \right]_6^9 = 81 + 32 + 81 = 194 \end{aligned}$$

The total distance traveled is 194 feet.

$$32. \quad \int_0^{3\pi/2} \left(\frac{1}{2} + \sin 2t \right) dt = \left[\frac{1}{2}t - \frac{1}{2}\cos 2t \right]_0^{3\pi/2} = \left(\frac{3\pi}{4} + \frac{1}{2} \right) - \left(0 - \frac{1}{2} \right) = \frac{3\pi}{4} + 1$$

The displacement is $\frac{3\pi}{4} + 1 \approx 3.36$ feet. Solve $\frac{1}{2} + \sin 2t = 0$ for $0 \leq t \leq \frac{3\pi}{2}$.

$$\sin 2t = -\frac{1}{2} \Rightarrow 2t = \frac{7\pi}{6}, \frac{11\pi}{6} \Rightarrow t = \frac{7\pi}{12}, \frac{11\pi}{12}$$

$$\left| \frac{1}{2} + \sin 2t \right| = \begin{cases} \frac{1}{2} + \sin 2t & 0 \leq t \leq \frac{7\pi}{12}, \frac{11\pi}{12} \leq t \leq \frac{3\pi}{2} \\ -\frac{1}{2} - \sin 2t & \frac{7\pi}{12} < t < \frac{11\pi}{12} \end{cases}$$

$$\begin{aligned} \int_0^{3\pi/2} \left| \frac{1}{2} + \sin 2t \right| dt &= \int_0^{7\pi/12} \left(\frac{1}{2} + \sin 2t \right) dt + \int_{7\pi/12}^{11\pi/12} \left(-\frac{1}{2} - \sin 2t \right) dt + \int_{11\pi/12}^{3\pi/2} \left(\frac{1}{2} + \sin 2t \right) dt \\ &= \left[\frac{1}{2}t - \frac{1}{2}\cos 2t \right]_0^{7\pi/12} + \left[-\frac{1}{2}t + \frac{1}{2}\cos 2t \right]_{7\pi/12}^{11\pi/12} + \left[\frac{1}{2}t - \frac{1}{2}\cos 2t \right]_{11\pi/12}^{3\pi/2} \\ &= \left(\frac{7\pi}{24} + \frac{\sqrt{3}}{4} + \frac{1}{2} \right) + \left(-\frac{\pi}{6} + \frac{\sqrt{3}}{2} \right) + \left(\frac{7\pi}{24} + \frac{\sqrt{3}}{4} + \frac{1}{2} \right) = \frac{5\pi}{12} + \sqrt{3} + 1 \end{aligned}$$

The total distance traveled is $\frac{5\pi}{12} + \sqrt{3} + 1 \approx 4.04$ feet.

$$33. \quad s(t) = \int v(t) dt = \int (2t - 4) dt = t^2 - 4t + C$$

Since $s(0) = 0$, $C = 0$ and $s(t) = t^2 - 4t$. $s = 12$ when $t = 6$, so it takes the object 6 seconds to get $s = 12$.

$$|2t - 4| = \begin{cases} 4 - 2t & 0 \leq t < 2 \\ 2t - 4 & 2 \leq t \end{cases}$$

$$\int_0^2 |2t - 4| dt = \left[-t^2 + 4t \right]_0^2 = 4, \text{ so the object}$$

travels a distance of 4 cm in the first two seconds.

$$\int_2^x |2t - 4| dt = \left[t^2 - 4t \right]_2^x = x^2 - 4x + 4$$

$x^2 - 4x + 4 = 8$ when $x = 2 + 2\sqrt{2}$, so the object takes $2 + 2\sqrt{2} \approx 4.83$ seconds to travel a total distance of 12 centimeters.

$$34. \text{ a. } A = \int_1^6 x^{-2} dx = \left[-\frac{1}{x} \right]_1^6 = -\frac{1}{6} + 1 = \frac{5}{6}$$

$$\text{b. Find } c \text{ so that } \int_1^c x^{-2} dx = \frac{5}{12}.$$

$$\int_1^c x^{-2} dx = \left[-\frac{1}{x} \right]_1^c = 1 - \frac{1}{c}$$

$$1 - \frac{1}{c} = \frac{5}{12}, c = \frac{12}{7}$$

The line $x = \frac{12}{7}$ bisects the area.

c. Slicing the region horizontally, the area is

$$\int_{1/36}^1 \frac{1}{\sqrt{y}} dy + \left(\frac{1}{36} \right) (5). \text{ Since } \frac{5}{36} < \frac{5}{12} \text{ the}$$

line that bisects the area is between $y = \frac{1}{36}$

and $y = 1$, so we find d such that

$$\int_d^1 \frac{1}{\sqrt{y}} dy = \frac{5}{12}; \int_d^1 \frac{1}{\sqrt{y}} dy = \left[2\sqrt{y} \right]_d^1$$

$$= 2 - 2\sqrt{d}; 2 - 2\sqrt{d} = \frac{5}{12};$$

$$d = \frac{361}{576} \approx 0.627.$$

The line $y = 0.627$ approximately bisects the area.

35. Equation of line through $(-2, 4)$ and $(3, 9)$:

$$y = x + 6$$

Equation of line through $(2, 4)$ and $(-3, 9)$:

$$y = -x + 6$$

$$A(A) = \int_{-3}^0 [9 - (-x + 6)] dx + \int_0^3 [9 - (x + 6)] dx$$

$$= \int_{-3}^0 (3 + x) dx + \int_0^3 (3 - x) dx$$

$$= \left[3x + \frac{1}{2}x^2 \right]_{-3}^0 + \left[3x - \frac{1}{2}x^2 \right]_0^3 = \frac{9}{2} + \frac{9}{2} = 9$$

$$A(B) = \int_{-3}^{-2} [(-x + 6) - x^2] dx$$

$$+ \int_{-2}^0 [(-x + 6) - (x + 6)] dx$$

$$= \int_{-3}^{-2} (-x^2 - x + 6) dx + \int_{-2}^0 (-2x) dx$$

$$= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x \right]_{-3}^{-2} + \left[-x^2 \right]_{-2}^0 = \frac{37}{6}$$

$$A(C) = A(B) = \frac{37}{6} \text{ (by symmetry)}$$

$$A(D) = \int_{-2}^0 [(x + 6) - x^2] dx + \int_0^2 [(-x + 6) - x^2] dx$$

$$= \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x \right]_{-2}^0 + \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x \right]_0^2$$

$$= \frac{44}{3}$$

$$A(A) + A(B) + A(C) + A(D) = 36$$

$$A(A + B + C + D) = \int_{-3}^3 (9 - x^2) dx = \left[9x - \frac{1}{3}x^3 \right]_{-3}^3$$

$$= 36$$

36. Let $f(x)$ be the width of region 1 at every x .

$$\Delta A_1 \approx f(x) \Delta x, \text{ so } A_1 = \int_a^b f(x) dx.$$

Let $g(x)$ be the width of region 2 at every x .

$$\Delta A_2 \approx g(x) \Delta x, \text{ so } A_2 = \int_a^b g(x) dx.$$

Since $f(x) = g(x)$ at every x in $[a, b]$,

$$A_1 = \int_a^b f(x) dx = \int_a^b g(x) dx = A_2.$$

37. The height of the triangular region is given by

for $0 \leq x \leq 1$. We need only show that the height of the second region is the same in order to apply Cavalieri's Principle. The height of the second region is

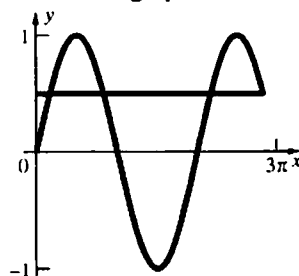
$$h_2 = (x^2 - 2x + 1) - (x^2 - 3x + 1)$$

$$= x^2 - 2x + 1 - x^2 + 3x - 1$$

$$= x \text{ for } 0 \leq x \leq 1.$$

Since $h_1 = h_2$ over the same closed interval, we can conclude that their areas are equal.

38. Sketch the graph.



Solve $\sin x = \frac{1}{2}$ for $0 \leq x \leq \frac{17\pi}{6}$.

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

The area of the trapped region is

$$\int_0^{\pi/6} \left(\frac{1}{2} - \sin x \right) dx + \int_{\pi/6}^{5\pi/6} \left(\sin x - \frac{1}{2} \right) dx$$

$$+ \int_{5\pi/6}^{13\pi/6} \left(\frac{1}{2} - \sin x \right) dx + \int_{13\pi/6}^{17\pi/6} \left(\sin x - \frac{1}{2} \right) dx$$

$$= \left[\frac{1}{2}x + \cos x \right]_0^{\pi/6} + \left[-\cos x - \frac{1}{2}x \right]_{\pi/6}^{5\pi/6}$$

$$+ \left[\frac{1}{2}x + \cos x \right]_{5\pi/6}^{13\pi/6} + \left[-\cos x - \frac{1}{2}x \right]_{13\pi/6}^{17\pi/6}$$

$$= \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \right) + \left(\sqrt{3} - \frac{\pi}{3} \right) + \left(\sqrt{3} + \frac{2\pi}{3} \right)$$

$$+ \left(\sqrt{3} - \frac{\pi}{3} \right) = \frac{\pi}{12} + \frac{7\sqrt{3}}{2} - 1 \approx 5.32$$

6.2 Concepts Review

1. $\pi r^2 h$

2. $\pi(R^2 - r^2)h$

3. $\pi x^4 \Delta x$

4. $\pi[(x^2 + 2)^2 - 4] \Delta x$

Problem Set 6.2

1. Slice vertically.

$$\Delta V \approx \pi(x^2 + 1)^2 \Delta x = \pi(x^4 + 2x^2 + 1)\Delta x$$

$$V = \pi \int_0^2 (x^4 + 2x^2 + 1) dx$$

$$= \pi \left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x \right]_0^2 = \pi \left(\frac{32}{5} + \frac{16}{3} + 2 \right) = \frac{206\pi}{15} \approx 43.14$$

2. Slice vertically.

$$\Delta V \approx \pi(-x^2 + 4x)^2 \Delta x = \pi(x^4 - 8x^3 + 16x^2)\Delta x$$

$$V = \pi \int_0^3 (x^4 - 8x^3 + 16x^2) dx$$

$$= \pi \left[\frac{1}{5}x^5 - 2x^4 + \frac{16}{3}x^3 \right]_0^3$$

$$= \pi \left(\frac{243}{5} - 162 + 144 \right)$$

$$= \frac{153\pi}{5} \approx 96.13$$

3. a. Slice vertically.

$$\Delta V \approx \pi(4 - x^2)^2 \Delta x = \pi(16 - 8x^2 + x^4)\Delta x$$

$$V = \pi \int_0^2 (16 - 8x^2 + x^4) dx$$

$$= \frac{256\pi}{15} \approx 53.62$$

- b. Slice horizontally.

$$x = \sqrt{4 - y}$$

Note that when $x = 0$, $y = 4$.

$$\Delta V \approx \pi(\sqrt{4 - y})^2 \Delta y = \pi(4 - y)\Delta y$$

$$V = \pi \int_0^4 (4 - y) dy = \pi \left[4y - \frac{1}{2}y^2 \right]_0^4$$

$$= \pi(16 - 8) = 8\pi \approx 25.13$$

4. a. Slice vertically.

$$\Delta V \approx \pi(4 - 2x)^2 \Delta x$$

$$0 \leq x \leq 2$$

$$V = \pi \int_0^2 (4 - 2x)^2 dx = \pi \left[-\frac{1}{6}(4 - 2x)^3 \right]_0^2$$

$$= \frac{32\pi}{3} \approx 33.51$$

- b. Slice vertically.

$$x = 2 - \frac{y}{2}$$

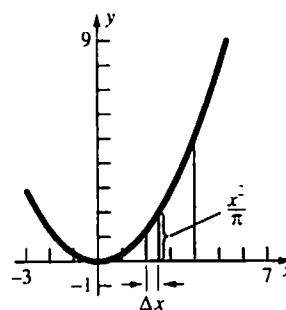
$$\Delta V \approx \pi \left(2 - \frac{y}{2} \right)^2 \Delta y$$

$$0 \leq y \leq 4$$

$$V = \pi \int_0^4 \left(2 - \frac{y}{2} \right)^2 dy = \pi \left[-\frac{2}{3} \left(2 - \frac{y}{2} \right)^3 \right]_0^4$$

$$= \frac{16\pi}{3} \approx 16.76$$

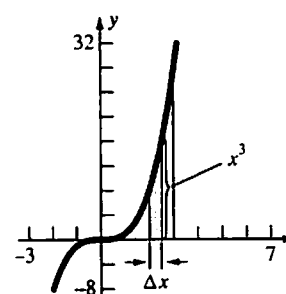
5.



$$\Delta V \approx \pi \left(\frac{x^2}{\pi} \right)^2 \Delta x = \frac{x^4}{\pi} \Delta x$$

$$V = \int_0^4 \frac{x^4}{\pi} dx = \frac{1}{\pi} \left[\frac{1}{5}x^5 \right]_0^4 = \frac{1024}{5\pi} \approx 65.19$$

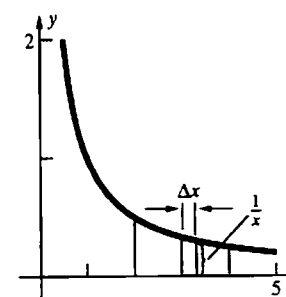
6.



$$\Delta V \approx \pi(x^3)^2 \Delta x = \pi x^6 \Delta x$$

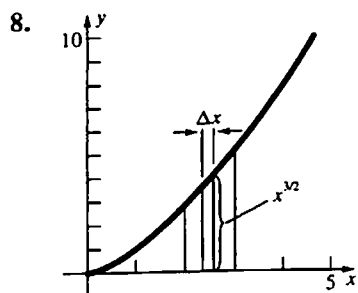
$$V = \pi \int_0^3 x^6 dx = \pi \left[\frac{1}{7}x^7 \right]_0^3 = \frac{2187\pi}{7} \approx 981.52$$

7.

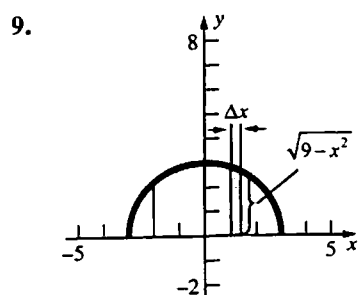


$$\Delta V \approx \pi \left(\frac{1}{x} \right)^2 \Delta x = \pi \left(\frac{1}{x^2} \right) \Delta x$$

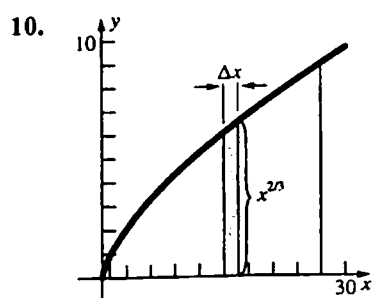
$$V = \pi \int_2^4 \frac{1}{x^2} dx = \pi \left[-\frac{1}{x} \right]_2^4 = \pi \left(-\frac{1}{4} + \frac{1}{2} \right) = \frac{\pi}{4} \approx 0.79$$



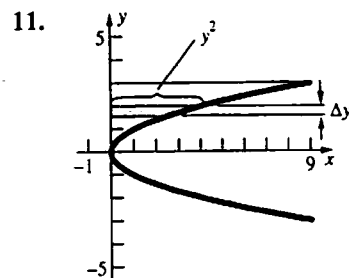
$$\begin{aligned}\Delta V &\approx \pi(x^{3/2})^2 \Delta x = \pi x^3 \Delta x \\ V &= \pi \int_2^3 x^3 dx = \pi \left[\frac{1}{4} x^4 \right]_2^3 = \pi \left(\frac{81}{4} - 4 \right) \\ &= \frac{65\pi}{4} \approx 51.05\end{aligned}$$



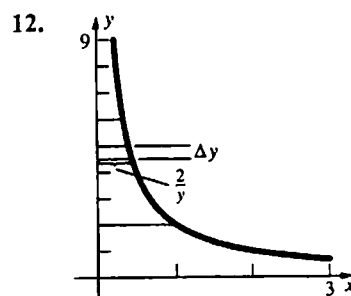
$$\begin{aligned}\Delta V &\approx \pi(\sqrt{9-x^2})^2 \Delta x = \pi(9-x^2) \Delta x \\ V &= \pi \int_{-2}^3 (9-x^2) dx = \pi \left[9x - \frac{1}{3} x^3 \right]_{-2}^3 \\ &= \pi \left[(27-9) - \left(-18 + \frac{8}{3} \right) \right] = \frac{100\pi}{3} \approx 104.72\end{aligned}$$



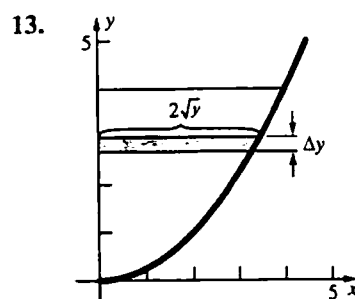
$$\begin{aligned}\Delta V &\approx \pi(x^{2/3})^2 \Delta x = \pi x^{4/3} \Delta x \\ V &= \pi \int_1^{27} x^{4/3} dx = \pi \left[\frac{3}{7} x^{7/3} \right]_1^{27} = \pi \left(\frac{6561}{7} - \frac{3}{7} \right) \\ &= \frac{6558\pi}{7} \approx 2943.22\end{aligned}$$



$$\begin{aligned}\Delta V &\approx \pi(y^2)^2 \Delta y = \pi y^4 \Delta y \\ V &= \pi \int_0^3 y^4 dy = \pi \left[\frac{1}{5} y^5 \right]_0^3 = \frac{243\pi}{5} \approx 152.68\end{aligned}$$

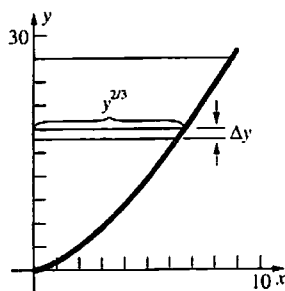


$$\begin{aligned}\Delta V &\approx \pi \left(\frac{2}{y} \right)^2 \Delta y = 4\pi \left(\frac{1}{y^2} \right) \Delta y \\ V &= 4\pi \int_2^6 \frac{1}{y^2} dy = 4\pi \left[-\frac{1}{y} \right]_2^6 = 4\pi \left(-\frac{1}{6} + \frac{1}{2} \right) \\ &= \frac{4\pi}{3} \approx 4.19\end{aligned}$$



$$\begin{aligned}\Delta V &\approx \pi(2\sqrt{y})^2 \Delta y = 4\pi y \Delta y \\ V &= 4\pi \int_0^4 y dy = 4\pi \left[\frac{1}{2} y^2 \right]_0^4 = 32\pi \approx 100.53\end{aligned}$$

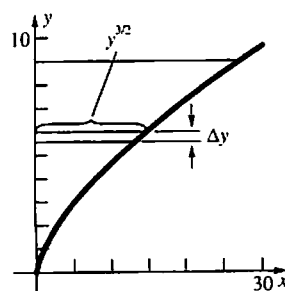
14.



$$\Delta V \approx \pi(y^{2/3})^2 \Delta y = \pi y^{4/3} \Delta y$$

$$V = \pi \int_0^{27} y^{4/3} dy = \pi \left[\frac{3}{7} y^{7/3} \right]_0^{27} = \frac{6561\pi}{7} \approx 2944.57$$

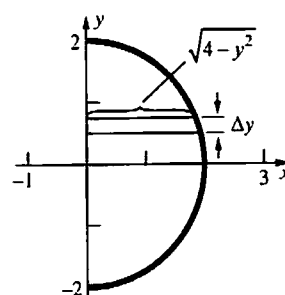
15.



$$\Delta V \approx \pi(y^{3/2})^2 \Delta y = \pi y^3 \Delta y$$

$$V = \pi \int_0^9 y^3 dy = \pi \left[\frac{1}{4} y^4 \right]_0^9 = \frac{6561\pi}{4} \approx 5153.00$$

16.



$$\Delta V \approx \pi(\sqrt{4-y^2})^2 \Delta y = \pi(4-y^2) \Delta y$$

$$V = \pi \int_{-2}^2 (4-y^2) dy = \pi \left[4y - \frac{1}{3} y^3 \right]_{-2}^2 = \pi \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] = \frac{32\pi}{3} \approx 33.51$$

17. The equation of the upper half of the ellipse is

$$y = b \sqrt{1 - \frac{x^2}{a^2}} \quad \text{or} \quad y = \frac{b}{a} \sqrt{a^2 - x^2}.$$

$$V = \pi \int_{-a}^a \frac{b^2}{a^2} (a^2 - x^2) dx$$

$$= \frac{b^2 \pi}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_{-a}^a = \frac{b^2 \pi}{a^2} \left[\left(a^3 - \frac{a^3}{3} \right) - \left(-a^3 + \frac{a^3}{3} \right) \right] = \frac{4}{3} ab^2 \pi$$

18. To find the intersection points, solve $6x = 6x^2$.

$$6(x^2 - x) = 0$$

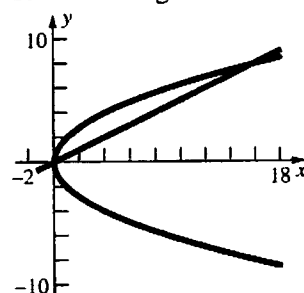
$$6x(x - 1) = 0$$

$$x = 0, 1$$

$$\Delta V \approx \pi[(6x)^2 - (6x^2)^2] \Delta x = 36\pi(x^2 - x^4) \Delta x$$

$$V = 36\pi \int_0^1 (x^2 - x^4) dx = 36\pi \left[\frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_0^1 = 36\pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{24\pi}{5} \approx 15.08$$

19. Sketch the region.

To find the intersection points, solve $\frac{x}{2} = 2\sqrt{x}$.

$$\frac{x^2}{4} = 4x$$

$$x^2 - 16x = 0$$

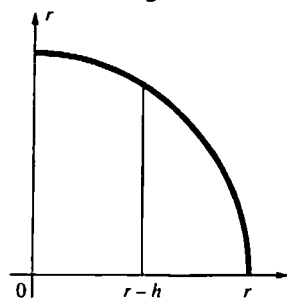
$$x(x - 16) = 0$$

$$x = 0, 16$$

$$\Delta V \approx \pi \left[(2\sqrt{x})^2 - \left(\frac{x}{2} \right)^2 \right] \Delta x = \pi \left(4x - \frac{x^2}{4} \right) \Delta x$$

$$V = \pi \int_0^{16} \left(4x - \frac{x^2}{4} \right) dx = \pi \left[2x^2 - \frac{x^3}{12} \right]_0^{16} = \pi \left(512 - \frac{1024}{3} \right) = \frac{512\pi}{3} \approx 536.17$$

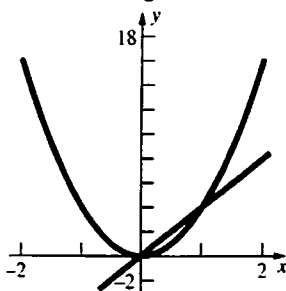
20. Sketch the region.



$$V = \pi \int_{r-h}^r (r^2 - x^2) dx = \pi \left[r^2 x - \frac{1}{3} x^3 \right]_{r-h}^r$$

$$= \frac{1}{3} \pi h^2 (3r - h)$$

21. Sketch the region.



To find the intersection points, solve $\frac{y}{4} = \frac{\sqrt{y}}{2}$.

$$\frac{y^2}{16} = \frac{y}{4}$$

$$y^2 - 4y = 0$$

$$y(y - 4) = 0$$

$$y = 0, 4$$

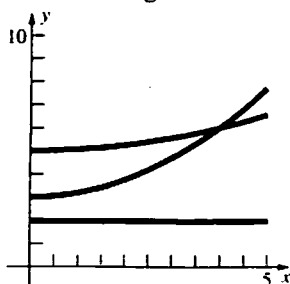
$$\Delta V \approx \pi \left[\left(\frac{\sqrt{y}}{2} \right)^2 - \left(\frac{y}{4} \right)^2 \right] \Delta y = \pi \left(\frac{y}{4} - \frac{y^2}{16} \right) \Delta y$$

$$V = \pi \int_0^4 \left(\frac{y}{4} - \frac{y^2}{16} \right) dy = \pi \left[\frac{y^2}{8} - \frac{y^3}{48} \right]_0^4$$

$$= \frac{2\pi}{3} \approx 2.0944$$

22. $y = \frac{3}{16}x^2 + 3$, $y = \frac{1}{16}x^2 + 5$

Sketch the region.



To find the intersection point, solve

$$\frac{3}{16}x^2 + 3 = \frac{1}{16}x^2 + 5$$

$$\frac{1}{8}x^2 - 2 = 0$$

$$x^2 - 16 = 0$$

$$(x + 4)(x - 4) = 0$$

$$x = -4, 4$$

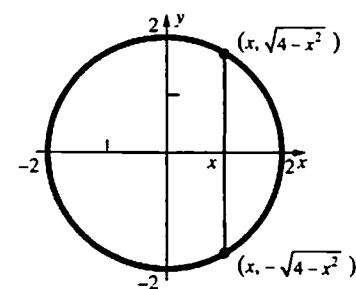
$$V = \pi \int_0^4 \left[\left(\frac{1}{16}x^2 + 5 - 2 \right) - \left(\frac{3}{16}x^2 + 3 - 2 \right) \right] dx$$

$$= \pi \int_0^4 \left[\left(\frac{1}{256}x^4 - \frac{3}{8}x^2 + 9 \right) - \left(\frac{9}{256}x^4 - \frac{3}{8}x^2 + 1 \right) \right] dx$$

$$= \pi \int_0^4 \left(8 - \frac{1}{32}x^4 \right) dx = \pi \left[8x - \frac{1}{160}x^5 \right]_0^4$$

$$= \pi \left(32 - \frac{32}{5} \right) = \frac{128\pi}{5} \approx 80.42$$

- 23.



The square at x has sides of length $2\sqrt{4-x^2}$, as shown.

$$V = \int_{-2}^2 \left(2\sqrt{4-x^2} \right)^2 dx = \int_{-2}^2 4(4-x^2) dx$$

$$= 4 \left[4x - \frac{x^3}{3} \right]_{-2}^2 = 4 \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] = \frac{128}{3} \approx 42.67$$

24. The area of each cross section perpendicular to the x -axis is $\frac{1}{2}(4)(2\sqrt{4-x^2}) = 4\sqrt{4-x^2}$.

The area of a semicircle with radius 2 is

$$\int_{-2}^2 \sqrt{4-x^2} dx = 2\pi$$

$$V = \int_{-2}^2 4\sqrt{4-x^2} dx = 4(2\pi) = 8\pi \approx 25.13$$

25. The square at x has sides of length $\sqrt{\cos x}$.

$$V = \int_{-\pi/2}^{\pi/2} \cos x dx = [\sin x]_{-\pi/2}^{\pi/2} = 2$$

26. The area of each cross section perpendicular to the x -axis is $[(1-x^2)-(1-x^4)]^2 = x^8 - 2x^6 + x^4$.

$$V = \int_{-1}^1 (x^8 - 2x^6 + x^4) dx$$

$$= \left[\frac{1}{9}x^9 - \frac{2}{7}x^7 + \frac{1}{5}x^5 \right]_{-1}^1 = \frac{16}{315} \approx 0.051$$

27. The square at x has sides of length $\sqrt{1-x^2}$.

$$V = \int_0^1 (1-x^2) dx = \left[x - \frac{x^3}{3} \right]_0^1 = \frac{2}{3} \approx 0.67$$

28. From Problem 27 we see that horizontal cross sections of one octant of the common region are squares. The length of a side at height y is

$\sqrt{r^2 - y^2}$ where r is the common radius of the cylinders. The volume of the "+" can be found by adding the volumes of each cylinder and subtracting off the volume of the common region (which is counted twice). The volume of one octant of the common region is

$$\int_0^r (r^2 - y^2) dy = r^2 y - \frac{1}{3} y^3 \Big|_0^r$$

$$= r^3 - \frac{1}{3} r^3 = \frac{2}{3} r^3$$

Thus, the volume of the "+" is

$V = \text{vol. of cylinders} - \text{vol. of common region}$

$$= 2(\pi r^2 l) - 8 \left(\frac{2}{3} r^3 \right)$$

$$= 2\pi(2^2)(12) - 8 \left(\frac{2}{3} (2)^3 \right) = 96\pi - \frac{128}{3}$$

$$\approx 258.93 \text{ in}^2$$

29. Using the result from Problem 28, the volume of one octant of the common region in the "+" is

$$\int_0^r (r^2 - y^2) dy = r^2 y - \frac{1}{3} y^3 \Big|_0^r$$

$$= r^3 - \frac{1}{3} r^3 = \frac{2}{3} r^3$$

Thus, the volume inside the "+" for two cylinders of radius r and length L is

$V = \text{vol. of cylinders} - \text{vol. of common region}$

$$= 2(\pi r^2 L) - 8 \left(\frac{2}{3} r^3 \right)$$

$$= 2\pi r^2 L - \frac{16}{3} r^3$$

30. From Problem 28, the volume of one octant of the common region is $\frac{2}{3} r^3$. We can find the volume of the "T" similarly. Since the "T" has one-half the common region of the "+" in

Problem 28, the volume of the "T" is given by
 $V = \text{vol. of cylinders} - \text{vol. of common region}$

$$= (\pi r^2)(L_1 + L_2) - 4 \left(\frac{2}{3} r^3 \right)$$

With $r = 2$, $L_1 = 12$, and $L_2 = 8$ (inches), the volume of the "T" is

$V = \text{vol. of cylinders} - \text{vol. of common region}$

$$= (\pi r^2)(L_1 + L_2) - 4 \left(\frac{2}{3} r^3 \right)$$

$$= (\pi 2^2)(12 + 8) - 4 \left(\frac{2}{3} 2^3 \right)$$

$$= 80\pi - \frac{64}{3} \text{ in}^3$$

$$\approx 229.99 \text{ in}^3$$

31. From Problem 30, the general form for the volume of a "T" formed by two cylinders with the same radius is

$V = \text{vol. of cylinders} - \text{vol. of common region}$

$$= (\pi r^2)(L_1 + L_2) - 4 \left(\frac{2}{3} r^3 \right)$$

$$= \pi r^2(L_1 + L_2) - \frac{8}{3} r^3$$

32. The area of each cross section perpendicular to

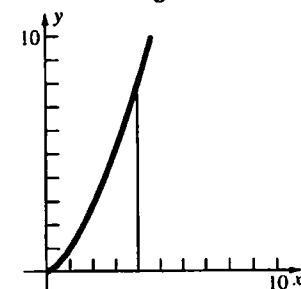
the x -axis is $\frac{1}{2} \pi \left[\frac{1}{2} (\sqrt{x} - x^2) \right]^2$

$$= \frac{\pi}{8} (x^4 - 2x^{5/2} + x)$$

$$V = \frac{\pi}{8} \int_0^1 (x^4 - 2x^{5/2} + x) dx$$

$$= \frac{\pi}{8} \left[\frac{1}{5} x^5 - \frac{4}{7} x^{7/2} + \frac{1}{2} x^2 \right]_0^1 = \frac{9\pi}{560} \approx 0.050$$

33. Sketch the region.



- a. Revolving about the line $x = 4$, the radius of the disk at y is $4 - \sqrt[3]{y^2} = 4 - y^{2/3}$.

$$V = \pi \int_0^8 (4 - y^{2/3})^2 dy$$

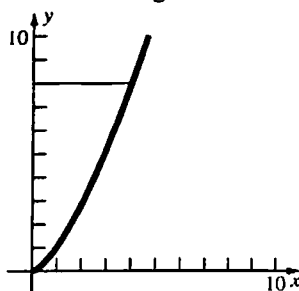
$$= \pi \int_0^8 (16 - 8y^{2/3} + y^{4/3}) dy$$

$$\begin{aligned}
&= \pi \left[16y - \frac{24}{5}y^{5/3} + \frac{3}{7}y^{7/3} \right]_0^8 \\
&= \pi \left(128 - \frac{768}{5} + \frac{384}{7} \right) \\
&= \frac{1024\pi}{35} \approx 91.91
\end{aligned}$$

- b. Revolving about the line $y = 8$, the inner radius of the disk at x is $8 - \sqrt{x^3} = 8 - x^{3/2}$.

$$\begin{aligned}
V &= \pi \int_0^4 \left[8^2 - (8 - x^{3/2})^2 \right] dx \\
&= \pi \int_0^4 (16x^{3/2} - x^3) dx \\
&= \pi \left[\frac{32}{5}x^{5/2} - \frac{1}{4}x^4 \right]_0^4 = \pi \left(\frac{1024}{5} - 64 \right) \\
&= \frac{704\pi}{5} \approx 442.34
\end{aligned}$$

34. Sketch the region.



- a. Revolving about the line $x = 4$, the inner radius of the disk at y is $4 - \sqrt[3]{y^2} = 4 - y^{2/3}$.

$$\begin{aligned}
V &= \pi \int_0^8 \left[4^2 - (4 - y^{2/3})^2 \right] dy \\
&= \pi \int_0^8 (8y^{2/3} - y^{4/3}) dy \\
&= \pi \left[\frac{24}{5}y^{5/3} - \frac{3}{7}y^{7/3} \right]_0^8 \\
&= \pi \left(\frac{768}{5} - \frac{384}{7} \right) = \frac{3456\pi}{35} \approx 310.21
\end{aligned}$$

- b. Revolving about the line $y = 8$, the radius of the disk at x is $8 - \sqrt{x^3} = 8 - x^{3/2}$.

$$\begin{aligned}
V &= \pi \int_0^4 (8 - x^{3/2})^2 dx \\
&= \pi \int_0^4 (64 - 16x^{3/2} + x^3) dx \\
&= \pi \left[64x - \frac{32}{5}x^{5/2} + \frac{1}{4}x^4 \right]_0^4 \\
&= \pi \left[256 - \frac{1024}{5} + 64 \right] = \frac{576\pi}{5} \approx 361.91
\end{aligned}$$

35. The area of a quarter circle with radius 2 is

$$\begin{aligned}
\int_0^2 \sqrt{4 - y^2} dy &= \pi. \\
\int_0^2 \left[2\sqrt{4 - y^2} + 4 - y^2 \right] dy \\
&= 2 \int_0^2 \sqrt{4 - y^2} dy + \int_0^2 (4 - y^2) dy \\
&= 2\pi + \left[4y - \frac{1}{3}y^3 \right]_0^2 = 2\pi + \left(8 - \frac{8}{3} \right) \\
&= 2\pi + \frac{16}{3} \approx 11.62
\end{aligned}$$

36. Let the x -axis lie along the diameter at the base perpendicular to the water level and slice perpendicular to the x -axis. Let $x = 0$ be at the center. The slice has base length $2\sqrt{r^2 - x^2}$ and height $\frac{hx}{r}$.

$$\begin{aligned}
V &= \frac{2h}{r} \int_0^r x\sqrt{r^2 - x^2} dx \\
&= \frac{2h}{r} \left[-\frac{1}{3}(r^2 - x^2)^{3/2} \right]_0^r = \frac{2h}{r} \left(\frac{1}{3}r^3 \right) = \frac{2}{3}r^2h
\end{aligned}$$

37. Let the x -axis lie on the base perpendicular to the diameter through the center of the base. The slice at x is a rectangle with base of length $2\sqrt{r^2 - x^2}$ and height $x \tan \theta$.

$$\begin{aligned}
V &= \int_0^r 2x \tan \theta \sqrt{r^2 - x^2} dx \\
&= \left[-\frac{2}{3} \tan \theta (r^2 - x^2)^{3/2} \right]_0^r \\
&= \frac{2}{3}r^3 \tan \theta
\end{aligned}$$

38. a. $x = \sqrt[4]{\frac{y}{k}}$

Slice horizontally.

$$\Delta V \approx \pi \left(\sqrt[4]{\frac{y}{k}} \right)^2 \Delta y = \pi \left(\sqrt{\frac{y}{k}} \right) \Delta y$$

If the depth of the tank is h , then

$$\begin{aligned}
V &= \pi \int_0^h \sqrt{\frac{y}{k}} dy = \frac{\pi}{\sqrt{k}} \left[\frac{2}{3}y^{3/2} \right]_0^h \\
&= \frac{2\pi}{3\sqrt{k}} h^{3/2}.
\end{aligned}$$

The volume as a function of the depth of the

$$\text{tank is } V(y) = \frac{2\pi}{3\sqrt{k}} y^{3/2}.$$

b. It is given that $\frac{dV}{dt} = -m\sqrt{y}$.

From part a, $\frac{dV}{dt} = \frac{\pi}{\sqrt{k}} y^{1/2} \frac{dy}{dt}$.

Thus, $\frac{\pi}{\sqrt{k}} \sqrt{y} \frac{dy}{dt} = -m\sqrt{y}$ and $\frac{dy}{dt} = \frac{-m\sqrt{k}}{\pi}$
which is constant.

39. Let A lie on the xy -plane. Suppose $\Delta A = f(x)\Delta x$ where $f(x)$ is the length at x , so $A = \int f(x)dx$. Slice the general cone at height z parallel to A . The slice of the resulting region is A_z and ΔA_z is a region related to $f(x)$ and Δx by similar triangles:

$$\Delta A_z = \left(1 - \frac{z}{h}\right) f(x) \cdot \left(1 - \frac{z}{h}\right) \Delta x$$

$$= \left(1 - \frac{z}{h}\right)^2 f(x) \Delta x$$

Therefore, $A_z = \left(1 - \frac{z}{h}\right)^2 \int f(x)dx = \left(1 - \frac{z}{h}\right)^2 A$.

$$\Delta V \approx A_z \Delta z = A \left(1 - \frac{z}{h}\right)^2 \Delta z \quad V = A \int_0^h \left(1 - \frac{z}{h}\right)^2 dz$$

$$= A \left[-\frac{h}{3} \left(1 - \frac{z}{h}\right)^3 \right]_0^h = \frac{1}{3} Ah.$$

a. $A = \pi r^2$

$$V = \frac{1}{3} Ah = \frac{1}{3} \pi r^2 h$$

- b. A face of a regular tetrahedron is an equilateral triangle. If the side of an equilateral triangle has length r , then the area is $A = \frac{1}{2} r \cdot \frac{\sqrt{3}}{2} r = \frac{\sqrt{3}}{4} r^2$.

The center of an equilateral triangle is $\frac{2}{3} \cdot \frac{\sqrt{3}}{2} r = \frac{1}{\sqrt{3}} r$ from a vertex. Then the height of a regular tetrahedron is

$$h = \sqrt{r^2 - \left(\frac{1}{\sqrt{3}} r\right)^2} = \sqrt{\frac{2}{3}} r^2 = \frac{\sqrt{2}}{\sqrt{3}} r.$$

$$V = \frac{1}{3} Ah = \frac{\sqrt{2}}{12} r^3$$

40. If two solids have the same cross sectional area at every x in $[a, b]$, then they have the same volume.

41. First we examine the cross-sectional areas of each shape.

Hemisphere: cross-sectional shape is a circle.

The radius of the circle at height y is $\sqrt{r^2 - y^2}$.

Therefore, the cross-sectional area for the hemisphere is

$$A_h = \pi(\sqrt{r^2 - y^2})^2 = \pi(r^2 - y^2)$$

Cylinder w/o cone: cross-sectional shape is a washer. The outer radius is a constant, r . The inner radius at height y is equal to y . Therefore, the cross-sectional area is

$$A_2 = \pi r^2 - \pi y^2 = \pi(r^2 - y^2).$$

Since both cross-sectional areas are the same, we can apply Cavalieri's Principle. The volume of the hemisphere of radius r is

$V = \text{vol. of cylinder} - \text{vol. of cone}$

$$= \pi r^2 h - \frac{1}{3} \pi r^2 h$$

$$= \frac{2}{3} \pi r^2 h$$

With the height of the cylinder and cone equal to r , the volume of the hemisphere is

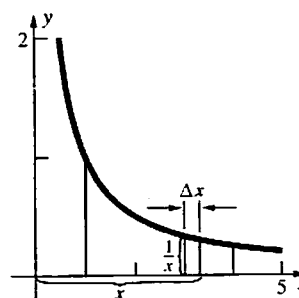
$$V = \frac{2}{3} \pi r^2 (r) = \frac{2}{3} \pi r^3$$

6.3 Concepts Review

- $2\pi x f(x) \Delta x$
- $2\pi \int_0^2 x^2 dx; \pi \int_0^2 (4 - y^2) dy$
- $2\pi \int_0^2 (1 + x)x dx$
- $2\pi \int_0^2 (1 + y)(2 - y) dy$

Problem Set 6.3

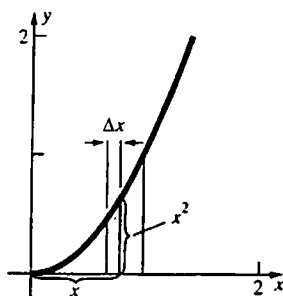
1. a, b.



c. $\Delta V \approx 2\pi x \left(\frac{1}{x}\right) \Delta x = 2\pi \Delta x$

d,e. $V = 2\pi \int_1^4 dx = 2\pi [x]_1^4 = 6\pi \approx 18.85$

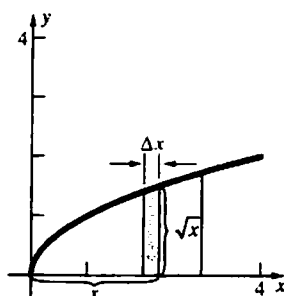
2. a,b.



c. $\Delta V \approx 2\pi x(x^2)\Delta x = 2\pi x^3 \Delta x$

d,e. $V = 2\pi \int_0^1 x^3 dx = 2\pi \left[\frac{1}{4}x^4 \right]_0^1 = \frac{\pi}{2} \approx 1.57$

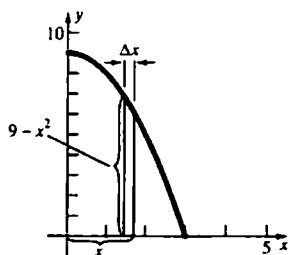
3. a,b.



c. $\Delta V \approx 2\pi x \sqrt{x} \Delta x = 2\pi x^{3/2} \Delta x$

d,e. $V = 2\pi \int_0^3 x^{3/2} dx = 2\pi \left[\frac{2}{5}x^{5/2} \right]_0^3$
 $= \frac{36\sqrt{3}}{5} \pi \approx 39.18$

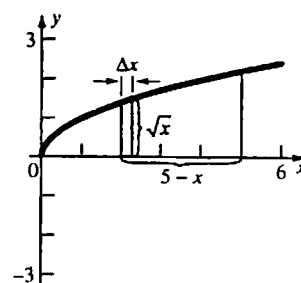
4. a,b.



c. $\Delta V \approx 2\pi x(9 - x^2)\Delta x = 2\pi(9x - x^3)\Delta x$

d,e. $V = 2\pi \int_0^3 (9x - x^3) dx = 2\pi \left[\frac{9}{2}x^2 - \frac{1}{4}x^4 \right]_0^3$
 $= 2\pi \left(\frac{81}{2} - \frac{81}{4} \right) = \frac{81\pi}{2} \approx 127.23$

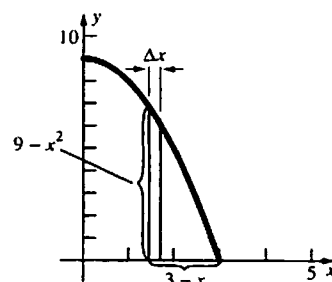
5. a,b.



c. $\Delta V \approx 2\pi(5-x)\sqrt{x} \Delta x$
 $= 2\pi(5x^{1/2} - x^{3/2})\Delta x$

d,e. $V = 2\pi \int_0^5 (5x^{1/2} - x^{3/2}) dx$
 $= 2\pi \left[\frac{10}{3}x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^5$
 $= 2\pi \left(\frac{50\sqrt{5}}{3} - 10\sqrt{5} \right) = \frac{40\sqrt{5}}{3} \pi \approx 93.66$

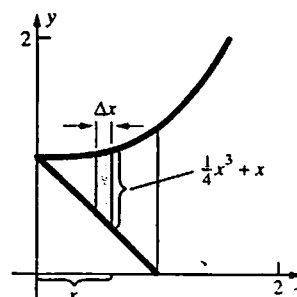
6. a,b.



c. $\Delta V \approx 2\pi(3-x)(9 - x^2)\Delta x$
 $= 2\pi(27 - 9x - 3x^2 + x^3)\Delta x$

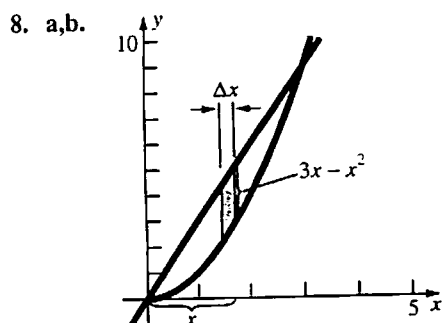
d,e. $V = 2\pi \int_0^3 (27 - 9x - 3x^2 + x^3) dx$
 $= 2\pi \left[27x - \frac{9}{2}x^2 - x^3 + \frac{1}{4}x^4 \right]_0^3$
 $= 2\pi \left(81 - \frac{81}{2} - 27 + \frac{81}{4} \right) = \frac{135\pi}{2} \approx 212.06$

7. a,b.



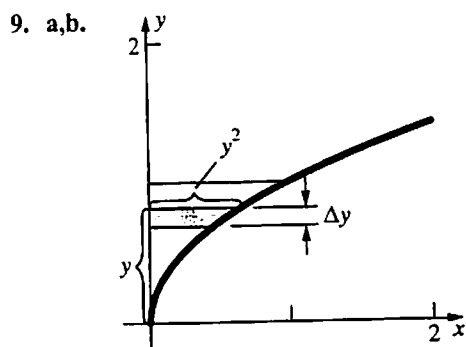
$$\begin{aligned} \text{c. } \Delta V &\approx 2\pi x \left[\left(\frac{1}{4}x^3 + 1 \right) - (1-x) \right] \Delta x \\ &= 2\pi \left(\frac{1}{4}x^4 + x^2 \right) \Delta x \end{aligned}$$

$$\begin{aligned} \text{d,e. } V &= 2\pi \int_0^1 \left(\frac{1}{4}x^4 + x^2 \right) dx \\ &= 2\pi \left[\frac{1}{20}x^5 + \frac{1}{3}x^3 \right]_0^1 = 2\pi \left(\frac{1}{20} + \frac{1}{3} \right) \\ &= \frac{23\pi}{30} \approx 2.41 \end{aligned}$$



$$\text{c. } \Delta V \approx 2\pi x(3x - x^2)\Delta x = 2\pi(3x^2 - x^3)\Delta x$$

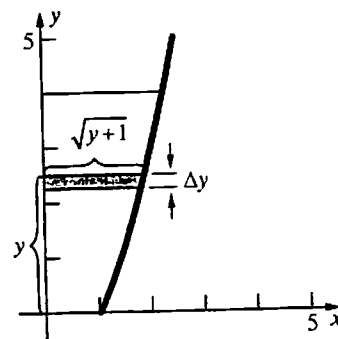
$$\begin{aligned} \text{d,e. } V &= 2\pi \int_0^3 (3x^2 - x^3) dx = 2\pi \left[x^3 - \frac{1}{4}x^4 \right]_0^3 \\ &= 2\pi \left(27 - \frac{81}{4} \right) = \frac{27\pi}{2} \approx 42.41 \end{aligned}$$



$$\text{c. } \Delta V \approx 2\pi y(y^2)\Delta y = 2\pi y^3 \Delta y$$

$$\text{d,e. } V = 2\pi \int_0^1 y^3 dy = 2\pi \left[\frac{1}{4}y^4 \right]_0^1 = \frac{\pi}{2} \approx 1.57$$

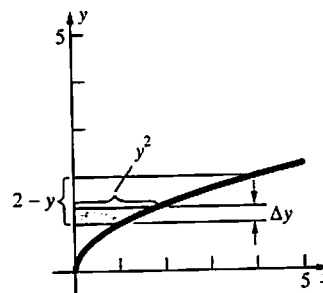
10. a,b.



$$\text{c. } \Delta V \approx 2\pi y(\sqrt{y+1})\Delta y = 2\pi(y^{3/2} + y)\Delta y$$

$$\begin{aligned} \text{d,e. } V &= 2\pi \int_0^4 (y^{3/2} + y) dy \\ &= 2\pi \left[\frac{2}{5}y^{5/2} + \frac{1}{2}y^2 \right]_0^4 = 2\pi \left(\frac{64}{5} + 8 \right) \\ &= \frac{208\pi}{5} \approx 130.69 \end{aligned}$$

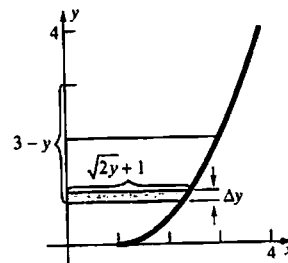
11. a,b.



$$\text{c. } \Delta V \approx 2\pi(2-y)y^2\Delta y = 2\pi(2y^2 - y^3)\Delta y$$

$$\begin{aligned} \text{d,e. } V &= 2\pi \int_0^2 (2y^2 - y^3) dy = 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^2 \\ &= 2\pi \left(\frac{16}{3} - 4 \right) = \frac{8\pi}{3} \approx 8.38 \end{aligned}$$

12. a,b.



$$\begin{aligned} \text{c. } \Delta V &\approx 2\pi(3-y)(\sqrt{2y+1})\Delta y \\ &= 2\pi(3 + 3\sqrt{2y+1} - y - \sqrt{2y+1})\Delta y \end{aligned}$$

$$\begin{aligned}
 \text{d.e. } V &= 2\pi \int_0^2 (3 + 3\sqrt{2}y^{1/2} - y - \sqrt{2}y^{3/2}) dy \\
 &= 2\pi \left[3y + 2\sqrt{2}y^{3/2} - \frac{1}{2}y^2 - \frac{2\sqrt{2}}{5}y^{5/2} \right]_0^2 \\
 &= 2\pi \left(6 + 8 - 2 - \frac{16}{5} \right) = \frac{88\pi}{5} \approx 55.29
 \end{aligned}$$

$$13. \text{ a. } \pi \int_a^b [f(x)^2 - g(x)^2] dx$$

$$\text{b. } 2\pi \int_a^b x[f(x) - g(x)] dx$$

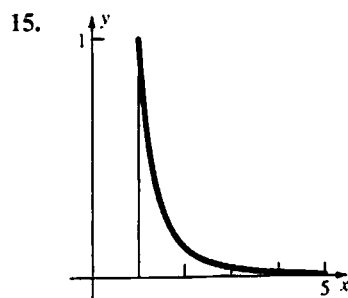
$$\text{c. } 2\pi \int_a^b (x-a)[f(x) - g(x)] dx$$

$$\text{d. } 2\pi \int_a^b (b-x)[f(x) - g(x)] dx$$

$$14. \text{ a. } \pi \int_c^d [f(y)^2 - g(y)^2] dy$$

$$\text{b. } 2\pi \int_c^d y[f(y) - g(y)] dy$$

$$\text{c. } 2\pi \int_c^d (3-y)[f(y) - g(y)] dy$$

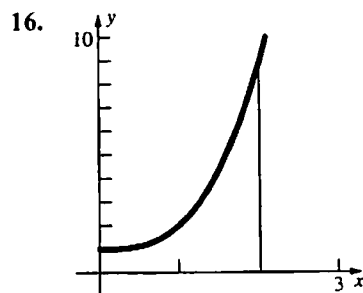


$$\text{a. } A = \int_1^3 \frac{1}{x^3} dx$$

$$\text{b. } V = 2\pi \int_1^3 x \left(\frac{1}{x^3} \right) dx = 2\pi \int_1^3 \frac{1}{x^2} dx$$

$$\begin{aligned}
 \text{c. } V &= \pi \int_1^3 \left[\left(\frac{1}{x^3} + 1 \right)^2 - (-1)^2 \right] dx \\
 &= \pi \int_1^3 \left(\frac{1}{x^6} + \frac{2}{x^3} \right) dx
 \end{aligned}$$

$$\begin{aligned}
 \text{d. } V &= 2\pi \int_1^3 (4-x) \left(\frac{1}{x^3} \right) dx \\
 &= 2\pi \int_1^3 \left(\frac{4}{x^3} - \frac{1}{x^2} \right) dx
 \end{aligned}$$



$$\text{a. } A = \int_0^2 (x^3 + 1) dx$$

$$\text{b. } V = 2\pi \int_0^2 x(x^3 + 1) dx = 2\pi \int_0^2 (x^4 + x) dx$$

$$\begin{aligned}
 \text{c. } V &= \pi \int_0^2 [(x^3 + 2)^2 - (-1)^2] \\
 &= \pi \int_0^2 (x^6 + 4x^3 + 3) dx
 \end{aligned}$$

$$\begin{aligned}
 \text{d. } V &= 2\pi \int_0^2 (4-x)(x^3 + 1) dx \\
 &= 2\pi \int_0^2 (-x^4 + 4x^3 - x + 4) dx
 \end{aligned}$$

17. To find the intersection point, solve $\sqrt{y} = \frac{y^3}{32}$.

$$y = \frac{y^6}{1024}$$

$$y^6 - 1024y = 0$$

$$y(y^5 - 1024) = 0$$

$$y = 0, 4$$

$$V = 2\pi \int_0^4 y \left(\sqrt{y} - \frac{y^3}{32} \right) dy$$

$$= 2\pi \int_0^4 \left(y^{3/2} - \frac{y^4}{32} \right) dy$$

$$\begin{aligned}
 &= 2\pi \left[\frac{2}{5} y^{5/2} - \frac{y^5}{160} \right]_0^4 = 2\pi \left(\frac{64}{5} - \frac{32}{5} \right) = \frac{64\pi}{5} \\
 &\approx 40.21
 \end{aligned}$$

$$\begin{aligned}
 18. \quad V &= 2\pi \int_0^4 (4-y) \left(\sqrt{y} - \frac{y^3}{32} \right) dy \\
 &= 2\pi \int_0^4 \left(4y^{1/2} - y^{3/2} - \frac{y^3}{8} + \frac{y^4}{32} \right) dy \\
 &= 2\pi \left[\frac{8}{3} y^{3/2} - \frac{2}{5} y^{5/2} - \frac{y^4}{32} + \frac{y^5}{160} \right]_0^4 \\
 &= 2\pi \left(\frac{64}{3} - \frac{64}{5} - 8 + \frac{32}{5} \right) = \frac{208\pi}{15} \approx 43.56
 \end{aligned}$$

19. Let R be the region bounded by $y = \sqrt{b^2 - x^2}$, $y = -\sqrt{b^2 - x^2}$, and $x = a$. When R is revolved about the y -axis, it produces the desired solid.

$$\begin{aligned}
 V &= 2\pi \int_a^b x \left(\sqrt{b^2 - x^2} + \sqrt{b^2 - x^2} \right) dx \\
 &= 4\pi \int_a^b x \sqrt{b^2 - x^2} dx = 4\pi \left[-\frac{1}{3} (b^2 - x^2)^{3/2} \right]_a^b \\
 &= 4\pi \left[\frac{1}{3} (b^2 - a^2)^{3/2} \right] = \frac{4\pi}{3} (b^2 - a^2)^{3/2}
 \end{aligned}$$

20. $y = \pm \sqrt{a^2 - x^2}$, $-a \leq x \leq a$

$$\begin{aligned}
 V &= 2\pi \int_{-a}^a (b-x) \left(2\sqrt{a^2 - x^2} \right) dx \\
 &= 4\pi b \int_{-a}^a \sqrt{a^2 - x^2} dx - 4\pi \int_{-a}^a x \sqrt{a^2 - x^2} dx \\
 &= 4\pi b \left(\frac{1}{2} \pi a^2 \right) - 4\pi \left[-\frac{1}{3} (a^2 - x^2)^{3/2} \right]_{-a}^a \\
 &= 2\pi^2 a^2 b
 \end{aligned}$$

(Note that the area of a semicircle with radius a is

$$\int_{-a}^a \sqrt{a^2 - x^2} dx = \frac{1}{2} \pi a^2 .)$$

21. To find the intersection point, solve

$$\sin(x^2) = \cos(x^2) .$$

$$\tan(x^2) = 1$$

$$x^2 = \frac{\pi}{4}$$

$$x = \frac{\sqrt{\pi}}{2}$$

$$\begin{aligned}
 V &= 2\pi \int_0^{\sqrt{\pi}/2} x \left[\cos(x^2) - \sin(x^2) \right] dx \\
 &= 2\pi \int_0^{\sqrt{\pi}/2} \left[x \cos(x^2) - x \sin(x^2) \right] dx
 \end{aligned}$$

$$\begin{aligned}
 &= 2\pi \left[\frac{1}{2} \sin(x^2) + \frac{1}{2} \cos(x^2) \right]_0^{\sqrt{\pi}/2} \\
 &= 2\pi \left[\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \right) - \frac{1}{2} \right] = \pi(\sqrt{2} - 1) \approx 1.30
 \end{aligned}$$

$$\begin{aligned}
 22. \quad V &= 2\pi \int_0^{2\pi} x(2 + \sin x) dx \\
 &= 2\pi \int_0^{2\pi} (2x + x \sin x) dx \\
 &= 2\pi \int_0^{2\pi} 2x dx + 2\pi \int_0^{2\pi} x \sin x dx \\
 &= 2\pi \left[x^2 \right]_0^{2\pi} + 2\pi \left[\sin x - x \cos x \right]_0^{2\pi} \\
 &= 2\pi(4\pi^2) + 2\pi(-2\pi) = 4\pi^2(2\pi - 1) \approx 208.57
 \end{aligned}$$

23. a. The curves intersect when $x = 0$ and $x = 1$.

$$\begin{aligned}
 V &= \pi \int_0^1 [x^2 - (x^2)^2] dx = \pi \int_0^1 (x^2 - x^4) dx \\
 &= \pi \left[\frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15} \approx 0.42
 \end{aligned}$$

$$\begin{aligned}
 b. \quad V &= 2\pi \int_0^1 x(x - x^2) dx = 2\pi \int_0^1 (x^2 - x^3) dx \\
 &= 2\pi \left[\frac{1}{3} x^3 - \frac{1}{4} x^4 \right]_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{\pi}{6} \\
 &\approx 0.52
 \end{aligned}$$

- c. Slice perpendicular to the line $y = x$. At (a, a) , the perpendicular line has equation $y = -(x - a) + a = -x + 2a$. Substitute $y = -x + 2a$ into $y = x^2$ and solve for $x \geq 0$.

$$x^2 + x - 2a = 0$$

$$x = \frac{-1 \pm \sqrt{1 + 8a}}{2}$$

$$x = \frac{-1 + \sqrt{1 + 8a}}{2}$$

Substitute into $y = -x + 2a$, so

$$y = \frac{1 + 4a - \sqrt{1 + 8a}}{2} . \text{ Find an expression for }$$

$$r^2, \text{ the square of the distance from } (a, a) \text{ to } \left(\frac{-1 + \sqrt{1 + 8a}}{2}, \frac{1 + 4a - \sqrt{1 + 8a}}{2} \right) .$$

$$\begin{aligned}
 r^2 &= \left[a - \frac{-1 + \sqrt{1 + 8a}}{2} \right]^2 \\
 &\quad + \left[a - \frac{1 + 4a - \sqrt{1 + 8a}}{2} \right]^2
 \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{2a+1-\sqrt{1+8a}}{2} \right]^2 \\
&\quad + \left[-\frac{2a+1-\sqrt{1+8a}}{2} \right]^2 \\
&= 2 \left[\frac{2a+1-\sqrt{1+8a}}{2} \right]^2 \\
&= 2a^2 + 6a + 1 - 2a\sqrt{1+8a} - \sqrt{1+8a} \\
\Delta V &\approx \pi r^2 \Delta a \\
V &= \pi \int_0^1 (2a^2 + 6a + 1 \\
&\quad - 2a\sqrt{1+8a} - \sqrt{1+8a}) da \\
&= \pi \left[\frac{2}{3}a^3 + 3a^2 + a - \frac{1}{12}(1+8a)^{3/2} \right]_0^1 \\
&\quad - \pi \int_0^1 2a\sqrt{1+8a} da \\
&= \pi \left[\left(\frac{2}{3} + 3 + 1 - \frac{9}{4} \right) - \left(-\frac{1}{12} \right) \right] \\
&\quad - \pi \int_0^1 2a\sqrt{1+8a} da \\
&= \frac{5\pi}{2} - \pi \int_0^1 2a\sqrt{1+8a} da
\end{aligned}$$

To integrate $\int_0^1 2a\sqrt{1+8a} da$, use the substitution $u = 1 + 8a$.

$$\begin{aligned}
\int_0^1 2a\sqrt{1+8a} da &= \int_1^9 \frac{1}{4}(u-1)\sqrt{u} \frac{1}{8} du \\
&= \frac{1}{32} \int_1^9 (u^{3/2} - u^{1/2}) du \\
&= \frac{1}{32} \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^9 \\
&= \frac{1}{32} \left[\left(\frac{486}{5} - 18 \right) - \left(\frac{2}{5} - \frac{2}{3} \right) \right] = \frac{149}{60} \\
V &= \frac{5\pi}{2} - \frac{149\pi}{60} = \frac{\pi}{60} \approx 0.052
\end{aligned}$$

24. $\Delta V \approx 4\pi x^2 \Delta x$

$$V = 4\pi \int_0^r x^2 dx = 4\pi \left[\frac{1}{3}x^3 \right]_0^r = \frac{4}{3}\pi r^3$$

25. $\Delta V \approx \frac{x^2}{r^2} S \Delta x$

$$V = \frac{S}{r^2} \int_0^r x^2 dx = \frac{S}{r^2} \left[\frac{1}{3}x^3 \right]_0^r = \frac{1}{3}rS$$

6.4 Concepts Review

1. Circle

$$x^2 + y^2 = 16 \cos^2 t + 16 \sin^2 t = 16$$

2. $x^2 + 1; x$

3. $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$

4. Mean Value Theorem (for derivatives)

Problem Set 6.4

1. $f(x) = x^2; a = -1; b = 3$

a. $n = 2; x = -1, 1, 3; y = 1, 1, 9$

$$\begin{aligned}
\sum_{i=1}^n \Delta w_i &= \sqrt{(1+1)^2 + (1-1)^2} \\
&\quad + \sqrt{(3-1)^2 + (9-1)^2} \\
&= \sqrt{4} + \sqrt{68} = 2 + 2\sqrt{17} \approx 10.2462
\end{aligned}$$

b. $n = 4; x = -1, 0, 1, 2, 3; y = 1, 0, 1, 4, 9$

$$\begin{aligned}
\sum_{i=1}^n \Delta w_i &= \sqrt{(0+1)^2 + (0-1)^2} \\
&\quad + \sqrt{(1-0)^2 + (1-0)^2} \\
&\quad + \sqrt{(2-1)^2 + (4-1)^2} \\
&\quad + \sqrt{(3-2)^2 + (9-4)^2} \\
&= \sqrt{2} + \sqrt{2} + \sqrt{10} + \sqrt{26} \\
&\approx 11.0897
\end{aligned}$$

2. $f(x) = \sqrt{x}; a = 0; b = 4$

a. $n = 2; x = 0, 2, 4; y = 0, \sqrt{2}, 2$

$$\begin{aligned}\sum_{i=1}^n \Delta w &= \sqrt{(2-0)^2 + (\sqrt{2}-0)^2} \\ &\quad + \sqrt{(4-2)^2 + (2-\sqrt{2})^2} \\ &\approx 4.5335\end{aligned}$$

b. $n = 4; x = 0, 1, 2, 3, 4; y = 0, 1, \sqrt{2}, \sqrt{3}, 2$

$$\begin{aligned}\sum_{i=1}^n \Delta w &= \sqrt{(1-0)^2 + (1-0)^2} \\ &\quad + \sqrt{(2-1)^2 + (\sqrt{2}-1)^2} \\ &\quad + \sqrt{(3-2)^2 + (\sqrt{3}-\sqrt{2})^2} \\ &\quad + \sqrt{(4-3)^2 + (2-\sqrt{3})^2} \\ &\approx 4.5812\end{aligned}$$

3. $f(x) = \sin x; a = 0; b = 2\pi$

a. $n = 2; x = 0, \pi, 2\pi; y = 0, 0, 0$

$$\begin{aligned}\sum_{i=1}^n \Delta w &= \sqrt{(\pi-0)^2 + (0-0)^2} \\ &\quad + \sqrt{(2\pi-\pi)^2 + (0-0)^2} \\ &= 2\pi \approx 6.2832\end{aligned}$$

b. $n = 4; x = 0, \pi/2, \pi, 3\pi/2, 2\pi;$
 $y = 0, 1, 0, -1, 0$

$$\begin{aligned}\sum_{i=1}^n \Delta w &= \sqrt{(\pi/2-0)^2 + (1-0)^2} \\ &\quad + \sqrt{(\pi-\pi/2)^2 + (0-1)^2} \\ &\quad + \sqrt{(3\pi/2-\pi)^2 + (-1-0)^2} \\ &\quad + \sqrt{(2\pi-3\pi/2)^2 + (0+1)^2} \\ &\approx 7.4484\end{aligned}$$

c. $n = 8; x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi;$
 $y = 0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, \frac{-\sqrt{2}}{2}, -1, \frac{-\sqrt{2}}{2}, 0$

$$\begin{aligned}\sum_{i=1}^n \Delta w &= \sqrt{(\pi/4-0)^2 + (\sqrt{2}/2-0)^2} \\ &\quad + \sqrt{(\pi/2-\pi/4)^2 + (1-\sqrt{2}/2)^2} \\ &\quad + \sqrt{(3\pi/4-\pi/2)^2 + (\sqrt{2}/2-1)^2} \\ &\quad + \sqrt{(\pi-3\pi/4)^2 + (0-\sqrt{2}/2)^2} \\ &\quad + \sqrt{(5\pi/4-\pi)^2 + (-\sqrt{2}/2-0)^2} \\ &\quad + \sqrt{(3\pi/2-5\pi/4)^2 + (-1+\sqrt{2}/2)^2} \\ &\quad + \sqrt{(7\pi/4-3\pi/2)^2 + (-\sqrt{2}/2+1)^2} \\ &\quad + \sqrt{(2\pi-7\pi/4)^2 + (0+\sqrt{2}/2)^2} \\ &\approx 7.5802\end{aligned}$$

4. $f(x) = \sin^2 x; a = 0; b = 2\pi$

a. $n = 2; x = 0, \pi, 2\pi; y = 0, 0, 0$

$$\begin{aligned}\sum_{i=1}^n \Delta w &= \sqrt{(\pi-0)^2 + (0-0)^2} \\ &\quad + \sqrt{(2\pi-\pi)^2 + (0-0)^2} \\ &= 2\pi \approx 6.2832\end{aligned}$$

b. $n = 4; x = 0, \pi/2, \pi, 3\pi/2, 2\pi;$
 $y = 0, 1, 0, 1, 0$

$$\begin{aligned}\sum_{i=1}^n \Delta w &= \sqrt{(\pi/2-0)^2 + (1-0)^2} \\ &\quad + \sqrt{(\pi-\pi/2)^2 + (0-1)^2} \\ &\quad + \sqrt{(3\pi/2-\pi)^2 + (1-0)^2} \\ &\quad + \sqrt{(2\pi-3\pi/2)^2 + (0-1)^2} \\ &\approx 7.4484\end{aligned}$$

c. $n = 8; x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi;$
 $y = 0, \frac{1}{2}, 1, \frac{1}{2}, 0, \frac{1}{2}, 1, \frac{1}{2}, 0$

$$\begin{aligned}\sum_{i=1}^n \Delta w &= \sqrt{(\pi/4-0)^2 + (1/2-0)^2} \\ &\quad + \sqrt{(\pi/2-\pi/4)^2 + (1-1/2)^2} \\ &\quad + \sqrt{(3\pi/4-\pi/2)^2 + (1/2-1)^2}\end{aligned}$$

$$\begin{aligned}
& +\sqrt{(\pi-3\pi/4)^2+(0-1/2)^2} \\
& +\sqrt{(5\pi/4-\pi)^2+(1/2-0)^2} \\
& +\sqrt{(3\pi/2-5\pi/4)^2+(1-1/2)^2} \\
& +\sqrt{(7\pi/4-3\pi/2)^2+(1/2-1)^2} \\
& +\sqrt{(2\pi-7\pi/4)^2+(0-1/2)^2} \\
& \approx 7.4484
\end{aligned}$$

5. $f(x) = 2x + 3$, $f'(x) = 2$

$$L = \int_1^3 \sqrt{1+(2)^2} dx = \sqrt{5} \int_1^3 dx = 2\sqrt{5}$$

At $x = 1$, $y = 2(1) + 3 = 5$.

At $x = 3$, $y = 2(3) + 3 = 9$.

$$d = \sqrt{(3-1)^2 + (9-5)^2} = \sqrt{20} = 2\sqrt{5}$$

6. $x = y + \frac{3}{2}$

$$g(y) = y + \frac{3}{2}, g'(y) = 1$$

$$L = \int_1^3 \sqrt{1+(1)^2} dy = \sqrt{2} \int_1^3 dy = 2\sqrt{2}$$

At $y = 1$, $x = 1 + \frac{3}{2} = \frac{5}{2}$.

At $y = 3$, $x = 3 + \frac{3}{2} = \frac{9}{2}$.

$$d = \sqrt{\left(\frac{9}{2} - \frac{5}{2}\right)^2 + (3-1)^2} = \sqrt{8} = 2\sqrt{2}$$

7. $f(x) = 4x^{3/2}$, $f'(x) = 6x^{1/2}$

$$L = \int_{1/3}^5 \sqrt{1+(6x^{1/2})^2} dx = \int_{1/3}^5 \sqrt{1+36x} dx$$

$$= \left[\frac{1}{36} \cdot \frac{2}{3} (1+36x)^{3/2} \right]_{1/3}^5$$

$$= \frac{1}{54} (181\sqrt{181} - 13\sqrt{13}) \approx 44.23$$

8. $f(x) = \frac{2}{3}(x^2+1)^{3/2}$, $f'(x) = 2x(x^2+1)^{1/2}$

$$L = \int_1^2 \sqrt{1+[2x(x^2+1)^{1/2}]^2} dx$$

$$= \int_1^2 \sqrt{4x^4+4x^2+1} dx = \int_1^2 (2x^2+1) dx$$

$$= \left[\frac{2}{3}x^3 + x \right]_1^2 = \left(\frac{16}{3} + 2 \right) - \left(\frac{2}{3} + 1 \right) = \frac{17}{3} \approx 5.67$$

9. $f(x) = (4-x^{2/3})^{3/2}$,

$$f'(x) = \frac{3}{2}(4-x^{2/3})^{1/2} \left(-\frac{2}{3}x^{-1/3} \right)$$

$$= -x^{-1/3}(4-x^{2/3})^{1/2}$$

$$L = \int_1^8 \sqrt{1+[-x^{-1/3}(4-x^{2/3})^{1/2}]^2} dx$$

$$= \int_1^8 \sqrt{4x^{-2/3}} dx = \int_1^8 2x^{-1/3} dx$$

$$= 2 \left[\frac{3}{2} x^{2/3} \right]_1^8 = 3(4-1) = 9$$

10. $f(x) = \frac{x^4+3}{6x} = \frac{x^3}{6} + \frac{1}{2x}$

$$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$L = \int_1^3 \sqrt{1+\left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} dx$$

$$= \int_1^3 \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} dx = \int_1^3 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} dx$$

$$= \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2x^2} \right) dx = \left[\frac{x^3}{6} - \frac{1}{2x} \right]_1^3$$

$$= \left(\frac{9}{2} - \frac{1}{6} \right) - \left(\frac{1}{6} - \frac{1}{2} \right) = \frac{14}{3} \approx 4.67$$

11. $g(y) = \frac{y^4}{16} + \frac{1}{2y^2}$, $g'(y) = \frac{y^3}{4} - \frac{1}{y^3}$

$$L = \int_{-3}^{-2} \sqrt{1+\left(\frac{y^3}{4} - \frac{1}{y^3}\right)^2} dy$$

$$= \int_{-3}^{-2} \sqrt{\frac{y^6}{16} + \frac{1}{2} + \frac{1}{y^6}} dy = \int_{-3}^{-2} \sqrt{\left(\frac{y^3}{4} + \frac{1}{y^3}\right)^2} dy$$

$$= \int_{-3}^{-2} \left(\frac{y^3}{4} + \frac{1}{y^3} \right) dy = \left[\frac{y^4}{16} - \frac{1}{2y^2} \right]_{-3}^{-2}$$

$$= -\left[\left(1 - \frac{1}{8} \right) - \left(\frac{81}{16} - \frac{1}{18} \right) \right] = \frac{595}{144} \approx 4.13$$

12. $x = \frac{y^5}{30} + \frac{1}{2y^3}$

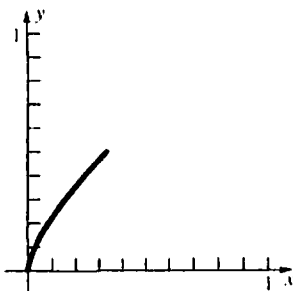
$$g(y) = \frac{y^5}{30} + \frac{1}{2y^3}, g'(y) = \frac{y^4}{6} - \frac{3}{2y^4}$$

$$L = \int_1^3 \sqrt{1+\left(\frac{y^4}{6} - \frac{3}{2y^4}\right)^2} dy$$

$$= \int_1^3 \sqrt{\frac{y^8}{36} + \frac{1}{2} + \frac{9}{4y^8}} dy$$

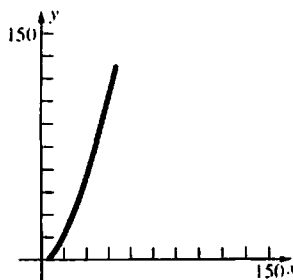
$$\begin{aligned}
 &= \int_1^3 \sqrt{\left(\frac{y^4}{6} + \frac{3}{2y^4}\right)^2} dy = \int_1^3 \left(\frac{y^4}{6} + \frac{3}{2y^4}\right) dy \\
 &= \left[\frac{y^5}{30} - \frac{1}{2y^3}\right]_1^3 = \left(\frac{81}{10} - \frac{1}{54}\right) - \left(\frac{1}{30} - \frac{1}{2}\right) = \frac{1154}{135} \\
 &\approx 8.55
 \end{aligned}$$

13.



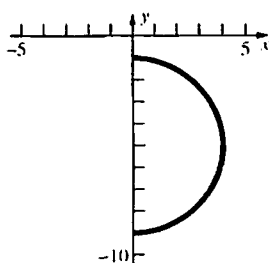
$$\begin{aligned}
 \frac{dx}{dt} &= t^2, \frac{dy}{dt} = t \\
 L &= \int_0^1 \sqrt{(t^2)^2 + (t)^2} dt = \int_0^1 \sqrt{t^4 + t^2} dt \\
 &= \int_0^1 t\sqrt{t^2 + 1} dt = \left[\frac{1}{3}(t^2 + 1)^{3/2}\right]_0^1 \\
 &= \frac{1}{3}(2\sqrt{2} - 1) \approx 0.61
 \end{aligned}$$

14.



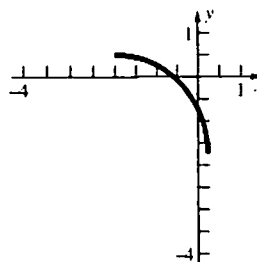
$$\begin{aligned}
 \frac{dx}{dt} &= 6t, \frac{dy}{dt} = 6t^2 \\
 L &= \int_1^4 \sqrt{(6t)^2 + (6t^2)^2} dt = \int_1^4 \sqrt{36t^2 + 36t^4} dt \\
 &= \int_1^4 6t\sqrt{1+t^2} dt = \left[2(1+t^2)^{3/2}\right]_1^4 \\
 &= 2(17\sqrt{17} - 2\sqrt{2}) \approx 134.53
 \end{aligned}$$

15.



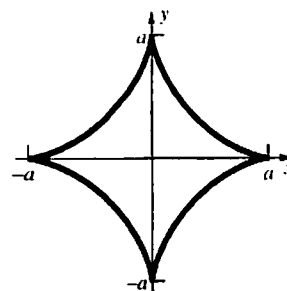
$$\begin{aligned}
 \frac{dx}{dt} &= 4 \cos t, \frac{dy}{dt} = -4 \sin t \\
 L &= \int_0^\pi \sqrt{(4 \cos t)^2 + (-4 \sin t)^2} dt \\
 &= \int_0^\pi \sqrt{16 \cos^2 t + 16 \sin^2 t} dt \\
 &= \int_0^\pi 4 dt = 4\pi \approx 12.57
 \end{aligned}$$

16.



$$\begin{aligned}
 \frac{dx}{dt} &= 2\sqrt{5} \cos 2t, \frac{dy}{dt} = -2\sqrt{5} \sin 2t \\
 L &= \int_0^{\pi/4} \sqrt{(2\sqrt{5} \cos 2t)^2 + (-2\sqrt{5} \sin 2t)^2} dt \\
 &= \int_0^{\pi/4} \sqrt{20 \cos^2 2t + 20 \sin^2 2t} dt = \int_0^{\pi/4} 2\sqrt{5} dt \\
 &= \frac{\sqrt{5} \pi}{2} \approx 3.51
 \end{aligned}$$

17.



$$\begin{aligned}
 \frac{dx}{dt} &= 3a \cos t \sin^2 t, \frac{dy}{dt} = -3a \sin t \cos^2 t \\
 \text{The first quadrant length is } L &= \int_0^{\pi/2} \sqrt{(3a \cos t \sin^2 t)^2 + (-3a \sin t \cos^2 t)^2} dt \\
 &= \int_0^{\pi/2} \sqrt{9a^2 \cos^2 t \sin^4 t + 9a^2 \sin^2 t \cos^4 t} dt \\
 &= \int_0^{\pi/2} \sqrt{9a^2 \cos^2 t \sin^2 t (\sin^2 t + \cos^2 t)} dt \\
 &= \int_0^{\pi/2} 3a \cos t \sin t dt = 3a \left[-\frac{1}{2} \cos^2 t\right]_0^{\pi/2} \\
 &= \frac{3a}{2}
 \end{aligned}$$

(The integral can also be evaluated as

$$3a \left[\frac{1}{2} \sin^2 t \right]_0^{\pi/2} \text{ with the same result.)}$$

The total length is $6a$.

18. a. $\overline{OT} = \text{length}(\widehat{PT}) = a\theta$

b. From Figure 18 of the text,

$$\sin \theta = \frac{\overline{PQ}}{\overline{PC}} = \frac{\overline{PQ}}{a} \text{ and } \cos \theta = \frac{\overline{QC}}{\overline{PC}} = \frac{\overline{QC}}{a}.$$

Therefore $\overline{PQ} = a \sin \theta$ and $\overline{QC} = a \cos \theta$.

c. $x = \overline{OT} - \overline{PQ} = a\theta - a \sin \theta = a(\theta - \sin \theta)$

$$y = \overline{CT} - \overline{CQ} = a - a \cos \theta = a(1 - \cos \theta)$$

19. From Problem 18,

$$x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$$

20. a. Using $\theta = \omega t$, the point P is at $x = a\omega t - a \sin(\omega t)$, $y = a - a \cos(\omega t)$ at time t .

$$\frac{dx}{dt} = a\omega - a\omega \cos(\omega t) = a\omega(1 - \cos(\omega t))$$

$$\frac{dy}{dt} = a\omega \sin(\omega t)$$

$$\frac{ds}{dt} = \sqrt{\left[\frac{dy}{dt} \right]^2 + \left[\frac{dx}{dt} \right]^2}$$

$$= \sqrt{a^2 \omega^2 \sin^2(\omega t) + a^2 \omega^2 - 2a^2 \omega^2 \cos(\omega t) + a^2 \omega^2 \cos^2(\omega t)} = \sqrt{2a^2 \omega^2 - 2a^2 \omega^2 \cos(\omega t)}$$

$$= 2a\omega \sqrt{\frac{1}{2}(1 - \cos(\omega t))}$$

$$= 2a\omega \sqrt{\sin^2 \frac{\omega t}{2}} = 2a\omega \left| \sin \frac{\omega t}{2} \right|$$

b. The speed is a maximum when $\left| \sin \frac{\omega t}{2} \right| = 1$, which occurs when $t = \frac{\pi}{\omega}(2k+1)$. The speed is a minimum when

$$\left| \sin \frac{\omega t}{2} \right| = 0, \text{ which occurs when } t = \frac{2k\pi}{\omega}.$$

c. From Problem 18a, the distance traveled by the wheel is $a\theta$, so at time t , the wheel has gone $a\theta = a\omega t$ miles. Since the car is going 60 miles per hour, the wheel has gone $60t$ miles at time t . Thus, $a\omega = 60$ and the maximum speed of the bug on the wheel is $2a\omega = 2(60) = 120$ miles per hour.

21. a. $\frac{dy}{dx} = \sqrt{x^3 - 1}$

$$L = \int_1^2 \sqrt{1 + x^3 - 1} dx = \int_1^2 x^{3/2} dx$$

$$= \left[\frac{2}{5} x^{5/2} \right]_1^2 = \frac{2}{5} (4\sqrt{2} - 1) \approx 1.86$$

$$\frac{dx}{d\theta} = a(1 - \cos \theta), \frac{dy}{d\theta} = a \sin \theta \text{ so}$$

$$\left(\frac{dx}{d\theta} \right)^2 + \left(\frac{dy}{d\theta} \right)^2 = [a(1 - \cos \theta)]^2 + [a \sin \theta]^2$$

$$= a^2 - 2a^2 \cos \theta + a^2 \cos^2 \theta + a^2 \sin^2 \theta$$

$$= 2a^2 - 2a^2 \cos \theta = 2a^2(1 - \cos \theta)$$

$$= 4a^2 \frac{1 - \cos \theta}{2} = 4a^2 \sin^2 \left(\frac{\theta}{2} \right).$$

The length of one arch of the cycloid is

$$\int_0^{2\pi} \sqrt{4a^2 \sin^2 \left(\frac{\theta}{2} \right)} d\theta = \int_0^{2\pi} 2a \sin \left(\frac{\theta}{2} \right) d\theta$$

$$= 2a \left[-2 \cos \frac{\theta}{2} \right]_0^{2\pi} = 2a(2 + 2) = 8a$$

b. $f'(t) = 1 - \cos t, g'(t) = \sin t$

$$L = \int_0^{4\pi} \sqrt{2 - 2 \cos t} dt = \int_0^{4\pi} 2 \left| \sin \left(\frac{t}{2} \right) \right| dt$$

$\sin \left(\frac{t}{2} \right)$ is positive for $0 < t < 2\pi$, and

by symmetry, we can double the integral

from 0 to 2π .

$$L = 4 \int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt = \left[-8\cos\frac{t}{2}\right]_0^{2\pi} \\ = 8 + 8 = 16$$

$$22. \text{ a. } \frac{dy}{dx} = \sqrt{64\sin^2 x \cos^4 x - 1} \\ L = \int_{\pi/6}^{\pi/3} \sqrt{1 + 64\sin^2 \cos^4 x - 1} dx \\ = \int_{\pi/6}^{\pi/3} 8\sin x \cos^2 x dx \\ = \left[-\frac{8}{3}\cos^3 x\right]_{\pi/6}^{\pi/3} = -\frac{1}{3} + \sqrt{3} \approx 1.40$$

$$\text{b. } \frac{dx}{dt} = -a \sin t + a \sin t + at \cos t = at \cos t \\ \frac{dy}{dt} = a \cos t - a \cos t + at \sin t = at \sin t \\ L = \int_{-1}^1 \sqrt{a^2 t^2 \cos^2 t + a^2 t^2 \sin^2 t} dt \\ = \int_{-1}^1 |at| dt = \int_0^1 at dt - \int_{-1}^0 at dt \\ = \left[\frac{a}{2}t^2\right]_0^1 - \left[\frac{a}{2}t^2\right]_{-1}^0 = \frac{a}{2} + \frac{a}{2} \\ = a$$

$$23. f(x) = 6x, f'(x) = 6 \\ A = 2\pi \int_0^1 6x\sqrt{1+36} dx = 12\sqrt{37}\pi \int_0^1 x dx \\ = 12\sqrt{37}\pi \left[\frac{1}{2}x^2\right]_0^1 = 6\sqrt{37}\pi \approx 114.66$$

$$24. f(x) = \sqrt{25-x^2}, f'(x) = -\frac{x}{\sqrt{25-x^2}} \\ A = 2\pi \int_{-2}^3 \sqrt{25-x^2} \sqrt{1+\frac{x^2}{25-x^2}} dx \\ = 2\pi \int_{-2}^3 \sqrt{25-x^2+x^2} dx \\ = 2\pi \int_{-2}^3 5 dx = 10\pi[x]_{-2}^3 = 50\pi \approx 157.08$$

$$25. f(x) = \frac{x^3}{3}, f'(x) = x^2 \\ A = 2\pi \int_1^{\sqrt{7}} \frac{x^3}{3} \sqrt{1+x^4} dx \\ = 2\pi \left[\frac{1}{18}(1+x^4)^{3/2}\right]_1^{\sqrt{7}} = \frac{\pi}{9}(250\sqrt{2}+2\sqrt{2}) \\ = 28\sqrt{2}\pi \approx 124.40$$

$$26. f(x) = \frac{x^6+2}{8x^2} = \frac{x^4}{8} + \frac{1}{4x^2}, f'(x) = \frac{x^3}{2} - \frac{1}{2x^3} \\ A = 2\pi \int_1^3 \left(\frac{x^4}{8} + \frac{1}{4x^2}\right) \sqrt{1+\left(\frac{x^3}{2} - \frac{1}{2x^3}\right)^2} dx \\ = 2\pi \int_1^3 \left(\frac{x^4}{8} + \frac{1}{4x^2}\right) \sqrt{\frac{x^6}{4} + \frac{1}{2} + \frac{1}{4x^6}} dx \\ = 2\pi \int_1^3 \left(\frac{x^4}{8} + \frac{1}{4x^2}\right) \left(\frac{x^3}{2} + \frac{1}{2x^3}\right) dx \\ = 2\pi \int_1^3 \left(\frac{x^7}{16} + \frac{3x}{16} + \frac{1}{8x^5}\right) dx \\ = 2\pi \left[\frac{x^8}{128} + \frac{3x^2}{32} - \frac{1}{32x^4}\right]_1^3 \\ = 2\pi \left[\left(\frac{6561}{128} + \frac{27}{32} - \frac{1}{2592}\right) - \left(\frac{1}{128} + \frac{3}{32} - \frac{1}{32}\right)\right] \\ = \frac{8429\pi}{81} \approx 326.92$$

$$27. \frac{dx}{dt} = 1, \frac{dy}{dt} = 3t^2 \\ A = 2\pi \int_0^1 t^3 \sqrt{1+9t^4} dt \\ = 2\pi \left[\frac{1}{54}(1+9t^4)^{3/2}\right]_0^1 = \frac{\pi}{27}(10\sqrt{10}-1) \\ \approx 3.56$$

$$28. \frac{dx}{dt} = -2t, \frac{dy}{dt} = 2 \\ A = 2\pi \int_0^1 2t\sqrt{4t^2+4} dt = 8\pi \int_0^1 t\sqrt{t^2+1} dt \\ = 8\pi \left[\frac{1}{3}(t^2+1)^{3/2}\right]_0^1 = \frac{8\pi}{3}(2\sqrt{2}-1) \\ \approx 15.32$$

$$29. y = f(x) = \sqrt{r^2-x^2} \\ f'(x) = -x(r^2-x^2)^{-1/2} \\ A = 2\pi \int_{-r}^r \sqrt{r^2-x^2} \sqrt{1+[-x(r^2-x^2)^{-1/2}]^2} dx \\ = 2\pi \int_{-r}^r \sqrt{r^2-x^2} \sqrt{1+x^2(r^2-x^2)^{-1}} dx \\ = 2\pi \int_{-r}^r \sqrt{(r^2-x^2)(1+x^2(r^2-x^2)^{-1})} dx \\ = 2\pi \int_{-r}^r \sqrt{r^2-x^2+x^2} dx \\ = 2\pi \int_{-r}^r \sqrt{r^2} dx = 2\pi \int_{-r}^r r dx \\ = 2\pi rx \Big|_{-r}^r = 4\pi r^2$$

30. $x = f(t) = r \cos t$

$y = g(t) = r \sin t$

$f'(t) = -r \sin t$

$g'(t) = r \cos t$

$$A = 2\pi \int_0^\pi r \sin t \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt$$

$$= 2\pi \int_0^\pi r \sin t \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} dt$$

$$= 2\pi \int_0^\pi r \sin t \sqrt{r^2} dt$$

$$= 2\pi \int_0^\pi r^2 \sin t dt = -2\pi r^2 \cos t \Big|_0^\pi$$

$$= -2r^2(-1 - 1) = 4\pi r^2$$

31. a. The base circumference is equal to the arc length of the sector, so $2\pi r = \theta l$. Therefore,

$$\theta = \frac{2\pi r}{l}.$$

- b. The area of the sector is equal to the lateral surface area. Therefore, the lateral surface area is

$$\frac{1}{2} l^2 \theta = \frac{1}{2} l^2 \left(\frac{2\pi r}{l} \right) = \pi r l.$$

- c. Assume $r_2 > r_1$. Let l_1 and l_2 be the slant heights for r_1 and r_2 , respectively. Then

$$A = \pi r_2 l_2 - \pi r_1 l_1 = \pi r_2 (l_1 + l) - \pi r_1 l_1.$$

$$\text{From part a, } \theta = \frac{2\pi r_2}{l_2} = \frac{2\pi r_2}{l_1 + l} = \frac{2\pi r_1}{l_1}.$$

$$\text{Solve for } l_1: l_1 r_2 = l_1 r_1 + l r_1$$

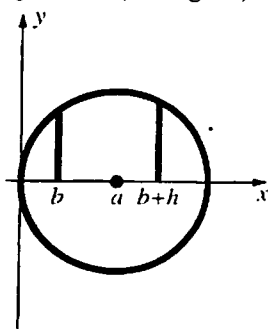
$$l_1 (r_2 - r_1) = l r_1$$

$$l_1 = \frac{l r_1}{r_2 - r_1}$$

$$A = \pi r_2 \left(\frac{l r_1}{r_2 - r_1} + l \right) - \pi r_1 \left(\frac{l r_1}{r_2 - r_1} \right)$$

$$= \pi (l r_1 + l r_2) = 2\pi \left[\frac{r_1 + r_2}{2} \right] l$$

32. Put the center of a circle of radius a at $(a, 0)$. Revolving the portion of the circle from $x = b$ to $x = b + h$ about the x -axis results in the surface in question. (See figure.)



The equation of the top half of the circle is

$$y = \sqrt{a^2 - (x - a)^2}.$$

$$\frac{dy}{dx} = \frac{-(x - a)}{\sqrt{a^2 - (x - a)^2}}$$

$$A = 2\pi \int_b^{b+h} \sqrt{a^2 - (x - a)^2} \sqrt{1 + \frac{(x - a)^2}{a^2 - (x - a)^2}} dx$$

$$= 2\pi \int_b^{b+h} \sqrt{a^2 - (x - a)^2} + (x - a)^2 dx$$

$$= 2\pi \int_b^{b+h} a dx = 2\pi a [x]_b^{b+h} = 2\pi ah$$

A right circular cylinder of radius a and height h has surface area $2\pi ah$.

33. a. $\frac{dx}{dt} = a(1 - \cos t), \frac{dy}{dt} = a \sin t$

$$A = 2\pi \int_0^{2\pi} a(1 - \cos t) \cdot$$

$$\sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} dt$$

$$= 2\pi a \int_0^{2\pi} (1 - \cos t) \sqrt{2a^2 - 2a^2 \cos t} dt$$

$$= 2\sqrt{2}\pi a^2 \int_0^{2\pi} (1 - \cos t)^{3/2} dt$$

b. $1 - \cos t = 2 \sin^2 \left(\frac{t}{2} \right)$, so

$$A = 2\sqrt{2}\pi a^2 \int_0^{2\pi} 2^{3/2} \sin^3 \left(\frac{t}{2} \right) dt$$

$$= 8\pi a^2 \int_0^{2\pi} \sin \left(\frac{t}{2} \right) \sin^2 \left(\frac{t}{2} \right) dt$$

$$= 8\pi a^2 \int_0^{2\pi} \sin \left(\frac{t}{2} \right) \left[1 - \cos^2 \left(\frac{t}{2} \right) \right] dt$$

$$= 8\pi a^2 \left[-2 \cos \left(\frac{t}{2} \right) + \frac{2}{3} \cos^3 \left(\frac{t}{2} \right) \right]_0^{2\pi}$$

$$= 8\pi a^2 \left[\left(2 - \frac{2}{3} \right) - \left(-2 + \frac{2}{3} \right) \right] = \frac{64}{3} \pi a^2$$

34. $\frac{dx}{dt} = -a \sin t, \frac{dy}{dt} = a \cos t$

Since the circle is being revolved about the line $x = b$, the surface area is

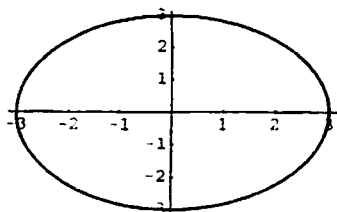
$$A = 2\pi \int_0^{2\pi} (b - a \cos t) \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} dt$$

$$= 2\pi a \int_0^{2\pi} (b - a \cos t) dt$$

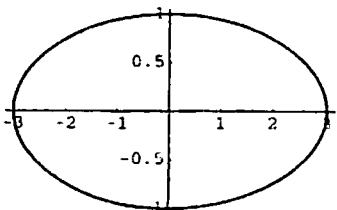
$$= 2\pi a [bt - a \sin t]_0^{2\pi}$$

$$= 4\pi^2 ab$$

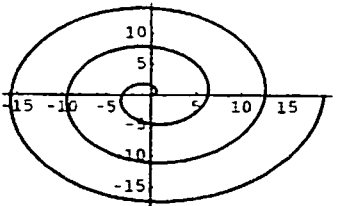
35. a.



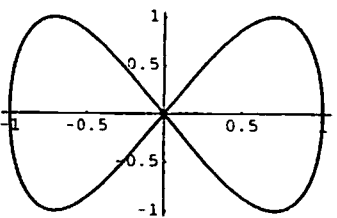
b.



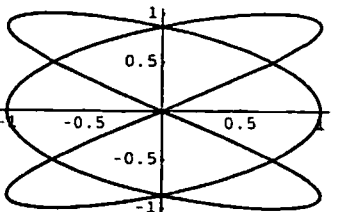
c.



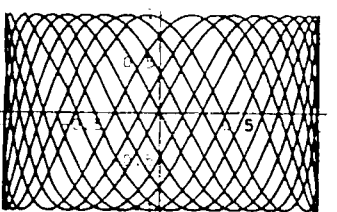
d.



e.



f.



36. a. $f'(t) = -3\sin t$, $g'(t) = 3\cos t$

$$L = \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t} dt$$

$$= \int_0^{2\pi} 3 dt = 3[t]_0^{2\pi} = 6\pi \approx 18.850$$

b. $f'(t) = -3\sin t$, $g'(t) = \cos t$

$$L = \int_0^{2\pi} \sqrt{9\sin^2 t + \cos^2 t} dt \approx 13.365$$

c. $f'(t) = \cos t - t \sin t$, $g'(t) = t \cos t + \sin t$

$$L = \int_0^{6\pi} \sqrt{(\cos t - t \sin t)^2 + (t \cos t + \sin t)^2} dt$$

$$= \int_0^{6\pi} \sqrt{1 + t^2} dt \approx 179.718$$

d. $f'(t) = -\sin t$, $g'(t) = 2\cos 2t$

$$L = \int_0^{2\pi} \sqrt{\sin^2 t + 4\cos^2 2t} dt \approx 9.429$$

e. $f'(t) = -3\sin 3t$, $g'(t) = 2\cos 2t$

$$L = \int_0^{2\pi} \sqrt{9\sin^2 3t + 4\cos^2 2t} dt$$

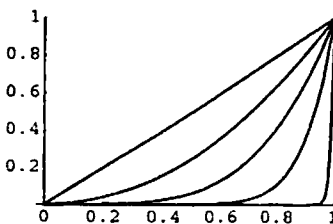
$$\approx 15.289$$

f. $f'(t) = -\sin t$, $g'(t) = \pi \cos \pi t$

$$L = \int_0^{10} \sqrt{\sin^2 t + \pi^2 \cos^2 \pi t} dt$$

$$\approx 86.58$$

37.



$y = x$, $y' = 1$,

$$L = \int_0^1 \sqrt{2} dx = [\sqrt{2}x]_0^1 = \sqrt{2} \approx 1.41421$$

$$y = x^2$$
, $y' = 2x$, $L = \int_0^1 \sqrt{1 + 4x^2} dx \approx 1.47894$

$$y = x^4$$
, $y' = 4x^3$, $L = \int_0^1 \sqrt{1 + 16x^6} dx \approx 1.60023$

$$y = x^{10}$$
, $y' = 10x^9$,

$$L = \int_0^1 \sqrt{1 + 100^{18}} dx \approx 1.75441$$

$$y = x^{100}$$
, $y' = 100x^{99}$,

$$L = \int_0^1 \sqrt{1 + 10,000x^{198}} dx \approx 1.95167$$

When $n = 10,000$ the length will be close to 2.

6.5 Concepts Review

1. $F \cdot (b-a); \int_a^b F(x) dx$
2. kx
3. $30 \cdot 10 = 300$
4. $\delta \pi 5^2 \Delta y (12-y) = 62.4(12-y)\pi(25)\Delta y;$
 $62.4\pi(25) \int_0^{12} (12-y) dy$

Problem Set 6.5

1. $F\left(\frac{1}{2}\right) = 6; k \cdot \frac{1}{2} = 6, k = 12$
 $F(x) = 12x$
 $W = \int_0^{1/2} 12x dx = \left[6x^2\right]_0^{1/2} = \frac{3}{2} = 1.5 \text{ ft-lb}$
2. From Problem 1, $F(x) = 12x$.
 $W = \int_0^2 12x dx = \left[6x^2\right]_0^2 = 24 \text{ ft-lb}$
3. $F(0.01) = 0.6; k = 60$
 $F(x) = 60x$
 $W = \int_0^{0.02} 60x dx = \left[30x^2\right]_0^{0.02} = 0.012 \text{ Joules}$
4. $F(x) = kx$ and let l be the natural length of the spring.
 $W = \int_{8-l}^{9-l} kx dx = \left[\frac{1}{2}kx^2\right]_{8-l}^{9-l}$
 $= \frac{1}{2}k[(81-18l+l^2) - (64-16l+l^2)]$
 $= \frac{1}{2}k(17-2l) = 0.05$
 Thus, $k = \frac{0.1}{17-2l}$.
 $W = \int_{9-l}^{10-l} kx dx = \left[\frac{1}{2}kx^2\right]_{9-l}^{10-l}$
 $= \frac{1}{2}k[(100-20l+l^2) - (81-18l+l^2)]$
 $= \frac{1}{2}k(19-2l) = 0.1$
 Thus, $k = \frac{0.2}{19-2l}$.
 Solving $\frac{0.1}{17-2l} = \frac{0.2}{19-2l}, l = \frac{15}{2}$.
 Thus $k = 0.05$, and the natural length is 7.5 cm.

$$5. W = \int_0^d kx dx = \left[\frac{1}{2}kx^2\right]_0^d$$

$$= \frac{1}{2}k(d^2 - 0) = \frac{1}{2}kd^2$$

$$6. F(8) = 2; k16 = 2, k = \frac{1}{8}$$

$$W = \int_0^{27} \frac{1}{8}s^{4/3} ds = \frac{1}{8} \left[\frac{3}{7}s^{7/3}\right]_0^{27} = \frac{6561}{56}$$

$$\approx 117.16 \text{ inch-pounds}$$

$$7. W = \int_0^2 9s ds = 9 \left[\frac{1}{2}s^2\right]_0^2 = 18 \text{ ft-lb}$$

8. One spring will move from 2 feet beyond its natural length to 3 feet beyond its natural length. The other will move from 2 feet beyond its natural length to 1 foot beyond its natural length.

$$W = \int_2^3 6s ds + \int_2^1 6s ds = \left[3s^2\right]_2^3 + \left[3s^2\right]_2^1$$

$$= 3(9-4) + 3(1-4) = 6 \text{ ft-lb}$$

9. A slab of thickness Δy at height y has width

$4 - \frac{4}{5}y$ and length 10. The slab will be lifted a distance $10 - y$.

$$\Delta W \approx \delta \cdot 10 \cdot \left(4 - \frac{4}{5}y\right) \Delta y (10 - y)$$

$$= 8\delta(y^2 - 15y + 50)\Delta y$$

$$W = \int_0^5 8\delta(y^2 - 15y + 50) dy$$

$$= 8(62.4) \left[\frac{1}{3}y^3 - \frac{15}{2}y^2 + 50y\right]_0^5$$

$$= 8(62.4) \left(\frac{125}{3} - \frac{375}{2} + 250\right)$$

$$= 52,000 \text{ ft-lb}$$

10. A slab of thickness Δy at height y has width

$4 - \frac{4}{3}y$ and length 10. The slab will be lifted a distance $8 - y$.

$$\Delta W \approx \delta \cdot 10 \cdot \left(4 - \frac{4}{3}y\right) \Delta y (8 - y)$$

$$= \frac{40}{3} \delta (24 - 11y + y^2) \Delta y$$

$$W = \int_0^3 \frac{40}{3} \delta (24 - 11y + y^2) dy$$

$$= \frac{40}{3} (62.4) \left[24y - \frac{11}{2}y^2 + \frac{1}{3}y^3\right]_0^3$$

$$= \frac{40}{3}(62.4)\left(72 - \frac{99}{2} + 9\right)$$

$$= 26,208 \text{ ft-lb}$$

11. A slab of thickness Δy at height y has width $\frac{3}{4}y + 3$ and length 10. The slab will be lifted a

distance $9 - y$. $\Delta W \approx \delta \cdot 10 \cdot \left(\frac{3}{4}y + 3\right) \Delta y (9 - y)$

$$= \frac{15}{2} \delta (36 + 5y - y^2) \Delta y$$

$$W = \int_0^4 \frac{15}{2} \delta (36 + 5y - y^2) dy$$

$$= \frac{15}{2} (62.4) \left[36y + \frac{5}{2}y^2 - \frac{1}{3}y^3 \right]_0^4$$

$$= \frac{15}{2} (62.4) \left(144 + 40 - \frac{64}{3} \right)$$

$$= 76,128 \text{ ft-lb}$$

12. A slab of thickness Δy at height y has width

$2\sqrt{6y - y^2}$ and length 10. The slab will be lifted

a distance $8 - y$.

$$\Delta W \approx \delta \cdot 10 \cdot 2\sqrt{6y - y^2} \Delta y (8 - y)$$

$$= 20\delta \sqrt{6y - y^2} (8 - y) \Delta y$$

$$W = \int_0^3 20\delta \sqrt{6y - y^2} (8 - y) dy$$

$$= 20\delta \int_0^3 \sqrt{6y - y^2} (3 - y) dy$$

$$+ 20\delta \int_0^3 \sqrt{6y - y^2} (5) dy$$

$$= 20\delta \left[\frac{1}{3} (6y - y^2)^{3/2} \right]_0^3$$

$$+ 100\delta \int_0^3 \sqrt{6y - y^2} dy$$

Notice that $\int_0^3 \sqrt{6y - y^2} dy$ is the area of a quarter of a circle with radius 3.

$$W = 20\delta(9) + 100\delta \left(\frac{1}{4} \pi 9 \right)$$

$$= (62.4)(180 + 225\pi) \approx 55,340 \text{ ft-lb}$$

13. The volume of a disk with thickness Δy is $16\pi\Delta y$. If it is at height y , it will be lifted a distance $10 - y$.

$$\Delta W \approx \delta 16\pi\Delta y(10 - y) = 16\pi\delta(10 - y)\Delta y$$

$$W = \int_0^{10} 16\pi\delta(10 - y)dy = 16\pi(50) \left[10y - \frac{1}{2}y^2 \right]_0^{10}$$

$$= 16\pi(50)(100 - 50) \approx 125,664 \text{ ft-lb}$$

14. The volume of a disk with thickness Δx at height x is $\pi(4 + x)^2 \Delta x$. It will be lifted a distance of $10 - x$.

$$\Delta W \approx \delta \pi(4 + x)^2 \Delta x (10 - x)$$

$$= \pi\delta(160 + 64x + 2x^2 - x^3) \Delta x$$

$$W = \int_0^{10} \pi\delta(160 + 64x + 2x^2 - x^3) dx$$

$$= \pi(50) \left[160x + 32x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^{10}$$

$$= \pi(50) \left(1600 + 3200 + \frac{2000}{3} - 2500 \right)$$

$$\approx 466,003 \text{ ft-lb}$$

15. The total force on the face of the piston is $A \cdot f(x)$ if the piston is x inches from the cylinder head. The work done by moving the piston from

$$x_1 \text{ to } x_2 \text{ is } W = \int_{x_1}^{x_2} A \cdot f(x) dx = A \int_{x_1}^{x_2} f(x) dx.$$

This is the work done by the gas in moving the piston. The work done by the piston to compress

the gas is the opposite of this or $A \int_{x_2}^{x_1} f(x) dx$.

16. $c = 40(16)^{1.4}$

$$A = 1; p(v) = cv^{-1.4}$$

$$f(x) = cx^{-1.4}$$

$$x_1 = \frac{16}{1} = 16, x_2 = \frac{2}{1} = 2$$

$$W = \int_2^{16} cx^{-1.4} dx = c \left[-2.5x^{-0.4} \right]_2^{16}$$

$$= 40(16)^{1.4} (-2.5)(16^{-0.4} - 2^{-0.4})$$

$$\approx 2075.83 \text{ in.-lb}$$

17. $c = 40(16)^{1.4}$

$$A = 2; p(v) = cv^{-1.4}$$

$$f(x) = c(2x)^{-1.4}$$

$$x_1 = \frac{16}{2} = 8, x_2 = \frac{2}{2} = 1$$

$$W = 2 \int_1^8 c(2x)^{-1.4} dx = 2c \left[-1.25(2x)^{-0.4} \right]_1^8$$

$$= 80(16)^{1.4} (-1.25)(16^{-0.4} - 2^{-0.4})$$

$$\approx 2075.83 \text{ in.-lb}$$

18. $80 \text{ lb/in.}^2 = 11,520 \text{ lb/ft}^2$

$$c = 11,520(1)^{1.4} = 11,520$$

$$\Delta W \approx p(v)\Delta v = 11,520v^{-1.4}\Delta v$$

$$W = \int_1^4 11,520v^{-1.4} dv = \left[-28,800v^{-0.4} \right]_1^4$$

$$= -28,800(4^{-0.4} - 1^{-0.4}) \approx 12,259 \text{ ft-lb}$$

19. The total work is equal to the work W_1 to haul the load by itself and the work W_2 to haul the rope by itself.

$$W_1 = 200 \cdot 500 = 100,000 \text{ ft-lb}$$

Let $y = 0$ be the bottom of the shaft. When the rope is at y , $\Delta W_2 \approx 2\Delta y(500 - y)$.

$$W_2 = \int_0^{500} 2(500 - y)dy = 2 \left[500y - \frac{1}{2}y^2 \right]_0^{500}$$

$$= 2(250,000 - 125,000) = 250,000 \text{ ft-lb}$$

$$W = W_1 + W_2 = 100,000 + 250,000$$

$$= 350,000 \text{ ft-lb}$$

20. The total work is equal to the work W_1 to lift the monkey plus the work W_2 to lift the chain.

$$W_1 = 10 \cdot 20 = 200 \text{ ft-lb}$$

Let $y = 20$ represent the top. As the monkey climbs the chain, the piece of chain at height y ($0 \leq y \leq 10$) will be lifted $20 - 2y$ ft.

$$\Delta W_2 \approx \frac{1}{2} \Delta y(20 - 2y) = (10 - y)\Delta y$$

$$W_2 = \int_0^{10} (10 - y)dy = \left[10y - \frac{1}{2}y^2 \right]_0^{10}$$

$$= 100 - 50 = 50 \text{ ft-lb}$$

$$W = W_1 + W_2 = 250 \text{ ft-lb}$$

21. $f(x) = \frac{k}{x^2}$; $f(4000) = 5000$

$$\frac{k}{4000^2} = 5000, k = 80,000,000,000$$

$$W = \int_{4000}^{4200} \frac{80,000,000,000}{x^2} dx$$

$$= 80,000,000,000 \left[-\frac{1}{x} \right]_{4000}^{4200}$$

$$= \frac{20,000,000}{21} \approx 952.381 \text{ mi-lb}$$

22. $F(x) = \frac{k}{x^2}$ where x is the distance between the

charges. $F(2) = 10$; $\frac{k}{4} = 10$, $k = 40$

$$W = \int_1^5 \frac{40}{x^2} dx = \left[-\frac{40}{x} \right]_1^5 = 32 \text{ ergs}$$

23. The relationship between the height of the bucket and time is $y = 2t$, so $t = \frac{1}{2}y$. When the bucket is a height y , the sand has been leaking out of the bucket for $\frac{1}{2}y$ seconds. The weight of the bucket

$$\text{and sand is } 100 + 500 - 3\left(\frac{1}{2}y\right) = 600 - \frac{3}{2}y.$$

$$\Delta W \approx \left(600 - \frac{3}{2}y \right) \Delta y$$

$$W = \int_0^{80} \left(600 - \frac{3}{2}y \right) dy = \left[600y - \frac{3}{4}y^2 \right]_0^{80}$$

$$= 48,000 - 4800 = 43,200 \text{ ft-lb}$$

24. The total work is equal to the work W_1 needed to fill the pipe plus the work W_2 needed to fill the tank.

$$\Delta W_1 = \delta \pi \left(\frac{1}{2} \right)^2 \Delta y(y) = \frac{\delta \pi y}{4} \Delta y$$

$$W_1 = \int_0^{30} \frac{\delta \pi y}{4} dy = \frac{(62.4)\pi}{4} \left[\frac{1}{2}y^2 \right]_0^{30}$$

$$\approx 22,054 \text{ ft-lb}$$

The cross sectional area at height y feet

($30 \leq y \leq 50$) is πr^2 where

$$r = \sqrt{10^2 - (40 - y)^2} = \sqrt{-y^2 + 80y - 1500}.$$

$$\Delta W_2 = \delta \pi r^2 \Delta y y = \delta \pi (-y^3 + 80y^2 - 1500y) \Delta y$$

$$W_2 = \int_{30}^{50} \delta \pi (-y^3 + 80y^2 - 1500y) dy$$

$$= (62.4)\pi \left[-\frac{1}{4}y^4 + \frac{80}{3}y^3 - 750y^2 \right]_{30}^{50}$$

$$= (62.4)\pi \left[\left(-1,562,500 + \frac{10,000,000}{3} - 1,875,000 \right) \right.$$

$$\left. - \left(-202,500 + 720,000 - 675,000 \right) \right]$$

$$\approx 10,455,220 \text{ ft-lb}$$

$$W = W_1 + W_2 \approx 10,477,274 \text{ ft-lb}$$

25. Let W_1 be the work to lift V to the surface and W_2 be the work to lift V from the surface to 15 feet above the surface. The volume displaced by the buoy y feet above its original position is

$$\frac{1}{3} \pi \left(a - \frac{a}{h}y \right)^2 (h - y) = \frac{1}{3} \pi a^2 h \left(1 - \frac{y}{h} \right)^3.$$

$$\text{The weight displaced is } \frac{\delta}{3} \pi a^2 h \left(1 - \frac{y}{h} \right)^3.$$

Note by Archimede's Principle $m = \frac{\delta}{3} \pi a^2 h$ or

$$a^2 h = \frac{3m}{\delta \pi}, \text{ so the displaced weight is}$$

$$m \left(1 - \frac{y}{h} \right)^3.$$

$$\Delta W_1 \approx \left(m - m \left(1 - \frac{y}{h} \right)^3 \right) \Delta y = m \left(1 - \left(1 - \frac{y}{h} \right)^3 \right) \Delta y$$

$$W_1 = m \int_0^h \left(1 - \left(1 - \frac{y}{h} \right)^3 \right) dy$$

$$= m \left[y + \frac{h}{4} \left(1 - \frac{y}{h} \right)^4 \right]_0^h = \frac{3mh}{4}$$

$$W_2 = m \cdot 15 = 15m$$

$$W = W_1 + W_2 = \frac{3mh}{4} + 15m$$

26. Since $\delta \left(\frac{1}{3} \pi a^2 \right) (8) = 300$, $a = \sqrt{\frac{225}{2\pi\delta}}$.

When the buoy is at z feet ($0 \leq z \leq 2$) below floating position, the radius r at the water level is

$$r = \left(\frac{8+z}{8} \right) a = \sqrt{\frac{225}{2\pi\delta}} \left(\frac{8+z}{8} \right).$$

$$F = \delta \left(\frac{1}{3} \pi r^2 \right) (8+z) - 300$$

$$= \frac{75}{128} (8+z)^3 - 300$$

$$W = \int_0^2 \left[\frac{75}{128} (8+z)^3 - 300 \right] dz$$

$$= \left[\frac{75}{512} (8+z)^4 - 300z \right]_0^2$$

$$= \left(\frac{46,875}{32} - 600 \right) - (600 - 0)$$

$$= \frac{8475}{32} \approx 264.84 \text{ ft-lb}$$

27. First calculate the work W_1 needed to lift the contents of the bottom tank to 10 feet.

$$\Delta W_1 \approx \delta 40 \Delta y (10 - y)$$

$$W_1 = \int_0^4 \delta 40 (10 - y) dy$$

$$= (62.4)(40) \left[-\frac{1}{2} (10 - y)^2 \right]_0^4$$

$$= (62.4)(40)(-18 + 50)$$

$$= 79,872 \text{ ft-lb}$$

Next calculate the work W_2 needed to fill the top tank. Let y be the distance from the bottom of the top tank.

$$\Delta W_2 \approx \delta (36\pi) \Delta y y$$

Solve for the height of the top tank:

$$36\pi h = 160; h = \frac{160}{36\pi} = \frac{40}{9\pi}$$

$$W_2 = \int_0^{40/9\pi} \delta 36\pi y dy$$

$$= (62.4)(36\pi) \left[\frac{1}{2} y^2 \right]_0^{40/9\pi}$$

$$= (62.4)(36\pi) \left(\frac{800}{81\pi^2} \right)$$

$$\approx 7062 \text{ ft-lbs}$$

$$W = W_1 + W_2 \approx 86,934 \text{ ft-lbs}$$

6.6 Concepts Review

1. right; $\frac{4 \cdot 1 + 6 \cdot 3}{4 + 6} = 2.2$

2. 2.5; right; $x(1+x)$; $1+x$

3. 1; 3

4. $\frac{24}{16}$; $\frac{40}{16}$

The second lamina balances at $\bar{x} = 3$, $\bar{y} = 1$.

The first lamina has area 12 and the second lamina has area 4.

$$\bar{x} = \frac{12 \cdot 1 + 4 \cdot 3}{12 + 4} = \frac{24}{16}, \bar{y} = \frac{12 \cdot 3 + 4 \cdot 1}{12 + 4} = \frac{40}{16}$$

Problem Set 6.6

1. $\bar{x} = \frac{2 \cdot 5 + (-2) \cdot 7 + 1 \cdot 9}{5 + 7 + 9} = \frac{5}{21}$

2. Let x measure the distance from the end where John sits.

$$\frac{180 \cdot 0 + 80 \cdot x + 110 \cdot 12}{180 + 80 + 110} = 6$$

$$80x + 1320 = 6 \cdot 370$$

$$80x = 900$$

$$x = 11.25$$

Tom should be 11.25 feet from John, or, equivalently, 0.75 feet from Mary.

$$3. \quad \bar{x} = \frac{\int_0^7 x\sqrt{x} dx}{\int_0^7 \sqrt{x} dx} = \frac{\left[\frac{2}{5}x^{5/2}\right]_0^7}{\left[\frac{2}{3}x^{3/2}\right]_0^7} = \frac{\frac{2}{5}(49\sqrt{7})}{\frac{2}{3}(7\sqrt{7})} = \frac{21}{5}$$

$$4. \quad \bar{x} = \frac{\int_0^7 x(1+x^3) dx}{\int_0^7 (1+x^3) dx} = \frac{\left[\frac{1}{2}x^2 + \frac{1}{5}x^5\right]_0^7}{\left[x + \frac{1}{4}x^4\right]_0^7} \\ = \frac{\left(\frac{49}{2} + \frac{16,807}{5}\right)}{\left(7 + \frac{2401}{4}\right)} = \frac{\frac{33,859}{10}}{\frac{2429}{4}} = \frac{9674}{1735} \approx 5.58$$

$$5. \quad M_y = 1 \cdot 2 + 7 \cdot 3 + (-2) \cdot 4 + (-1) \cdot 6 + 4 \cdot 2 = 17$$

$$M_x = 1 \cdot 2 + 1 \cdot 3 + (-5) \cdot 4 + 0 \cdot 6 + 6 \cdot 2 = -3$$

$$m = 2 + 3 + 4 + 6 + 2 = 17$$

$$\bar{x} = \frac{M_y}{m} = 1, \bar{y} = \frac{M_x}{m} = -\frac{3}{17}$$

$$6. \quad M_y = (-3) \cdot 5 + (-2) \cdot 6 + 3 \cdot 2 + 4 \cdot 7 + 7 \cdot 1 = 14$$

$$M_x = 2 \cdot 5 + (-2) \cdot 6 + 5 \cdot 2 + 3 \cdot 7 + (-1) \cdot 1 = 28$$

$$m = 5 + 6 + 2 + 7 + 1 = 21$$

$$\bar{x} = \frac{M_y}{m} = \frac{2}{3}, \bar{y} = \frac{M_x}{m} = \frac{4}{3}$$

7. Consider two regions R_1 and R_2 such that R_1 is bounded by $f(x)$ and the x -axis, and R_2 is bounded by $g(x)$ and the x -axis. Let R_3 be the region formed by $R_1 - R_2$. Make a regular partition of the homogeneous region R_3 such that each sub-region is of width Δx and let x be the distance from the y -axis to the center of mass of a sub-region. The heights of R_1 and R_2 at x are approximately $f(x)$ and $g(x)$ respectively. The mass of R_3 is approximately

$$\Delta m = \Delta m_1 - \Delta m_2$$

$$\approx \delta f(x) \Delta x - \delta g(x) \Delta x$$

$$= \delta [f(x) - g(x)] \Delta x$$

where δ is the density. The moments for R_3 are approximately

$$M_x = M_x(R_1) - M_x(R_2)$$

$$\approx \frac{\delta}{2} [f(x)]^2 \Delta x - \frac{\delta}{2} [g(x)]^2 \Delta x$$

$$= \frac{\delta}{2} [(f(x))^2 - (g(x))^2] \Delta x$$

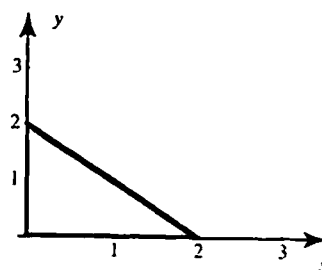
$$M_y = M_y(R_1) - M_y(R_2)$$

$$\approx x \delta f(x) \Delta x - x \delta g(x) \Delta x$$

$$= x \delta [f(x) - g(x)] \Delta x$$

Taking the limit of the regular partition as $\Delta x \rightarrow 0$ yields the resulting integrals in Figure 10.

8.



$$f(x) = 2 - x; g(x) = 0$$

$$\bar{x} = \frac{\int_0^2 x[(2-x)-0] dx}{\int_0^2 [(2-x)-0] dx}$$

$$= \frac{\int_0^2 [2x - x^2] dx}{\int_0^2 [2 - x] dx}$$

$$= \frac{(x^2 - \frac{1}{3}x^3)_0^2}{(2x - \frac{1}{2}x^2)_0^2} = \frac{4 - \frac{8}{3}}{4 - 2}$$

$$= \frac{2}{3}$$

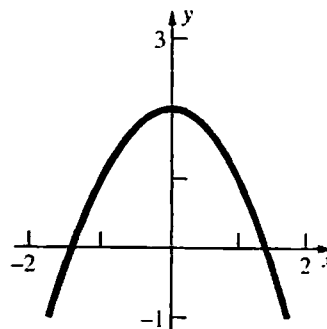
$$\bar{y} = \frac{\frac{1}{2} \int_0^2 [(2-x)^2 - 0^2] dx}{\int_0^2 [(2-x)-0] dx}$$

$$= \frac{\int_0^2 [4 - 4x + x^2] dx}{4}$$

$$= \frac{(4x - 2x^2 + \frac{1}{3}x^3)_0^2}{4} = \frac{8 - 8 + \frac{8}{3}}{4}$$

$$= \frac{2}{3}$$

9.

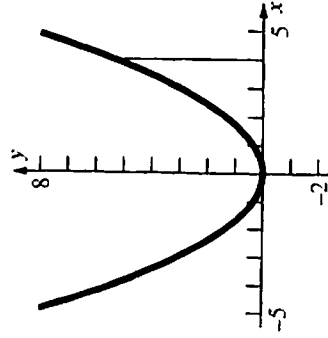


$$\bar{x} = 0 \text{ (by symmetry)}$$

$$\bar{y} = \frac{\frac{1}{2} \int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)^2 dx}{\int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2) dx}$$

$$\begin{aligned}
 &= \frac{\frac{1}{2} \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 4x^2 + x^4) dx}{\left[2x - \frac{1}{3}x^3 \right]_{-\sqrt{2}}^{\sqrt{2}}} \\
 &= \frac{\frac{1}{2} \left[4x - \frac{4}{3}x^3 + \frac{1}{5}x^5 \right]_{-\sqrt{2}}^{\sqrt{2}}}{\frac{8\sqrt{2}}{3}} = \frac{\frac{32\sqrt{2}}{15}}{\frac{8\sqrt{2}}{3}} = \frac{4}{5}
 \end{aligned}$$

10.

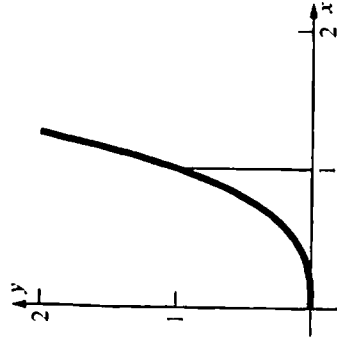


$$\bar{x} = \frac{\int_{-\sqrt{2}}^{\sqrt{2}} x \left(\frac{1}{3}x^2 \right) dx}{\int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{3}x^2 dx} = \frac{\frac{1}{3} \int_{-\sqrt{2}}^{\sqrt{2}} x^3 dx}{\frac{1}{3} \int_{-\sqrt{2}}^{\sqrt{2}} x^2 dx}$$

$$\begin{aligned}
 &= \frac{\frac{1}{3} \left[\frac{1}{4}x^4 \right]_0^{\sqrt{2}}}{\frac{1}{3} \left[\frac{1}{3}x^3 \right]_0^{\sqrt{2}}} = \frac{\frac{64}{3}}{\frac{64}{9}} = 3
 \end{aligned}$$

$$\begin{aligned}
 \bar{y} &= \frac{\frac{1}{2} \int_0^1 \left(\frac{1}{3}x^2 \right)^2 dx}{\int_0^1 \frac{1}{3}x^2 dx} = \frac{\frac{1}{18} \int_0^1 x^4 dx}{\frac{64}{9}} = \frac{\frac{1}{18} \left[\frac{1}{5}x^5 \right]_0^1}{\frac{64}{9}} \\
 &= \frac{\frac{512}{45}}{\frac{64}{9}} = \frac{8}{5}
 \end{aligned}$$

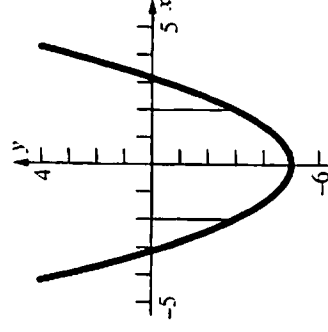
11.



$$\bar{x} = \frac{\int_0^1 x(x^3) dx}{\int_0^1 x^3 dx} = \frac{\int_0^1 x^4 dx}{\left[\frac{1}{4}x^4 \right]_0^1} = \frac{\left[\frac{1}{5}x^5 \right]_0^1}{\frac{1}{4}} = \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{4}{5}$$

$$\bar{y} = \frac{\frac{1}{2} \int_0^1 (x^3)^2 dx}{\int_0^1 x^3 dx} = \frac{\frac{1}{2} \int_0^1 x^6 dx}{\frac{1}{4}} = \frac{\left[\frac{1}{14}x^7 \right]_0^1}{\frac{1}{4}} = \frac{\frac{1}{14}}{\frac{1}{4}} = \frac{2}{7}$$

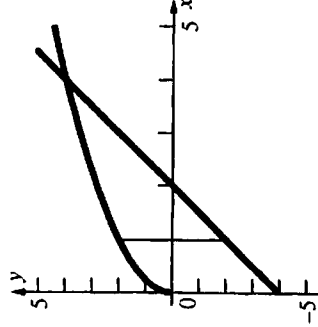
12.



$\bar{x} = 0$ (by symmetry)

$$\begin{aligned}
 \bar{y} &= \frac{\frac{1}{2} \int_{-2}^2 \left[-\left(\frac{1}{2}(x^2 - 10) \right)^2 \right] dx}{\int_{-2}^2 \left[-\frac{1}{2}(x^2 - 10) \right] dx} \\
 &= \frac{-\frac{1}{8} \int_{-2}^2 (x^4 - 20x^2 + 100) dx}{-\frac{1}{2} \int_{-2}^2 (x^2 - 10) dx} \\
 &= \frac{-\frac{1}{8} \left[\frac{1}{5}x^5 - \frac{20}{3}x^3 + 100x \right]_{-2}^2}{-\frac{1}{2} \left[\frac{1}{3}x^3 - 10x \right]_{-2}^2} = \frac{-\frac{574}{15}}{\frac{32}{3}} = -\frac{287}{130}
 \end{aligned}$$

13.



To find the intersection point, solve

$$2x - 4 = 2\sqrt{x}.$$

$$x - 2 = \sqrt{x}$$

$$x^2 - 4x + 4 = x$$

$$x^2 - 5x + 4 = 0$$

$$(x - 4)(x - 1) = 0$$

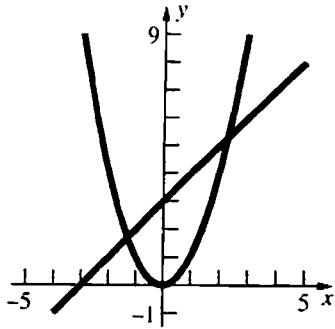
$$x = 4 \quad (x = 1 \text{ is extraneous.})$$

$$\begin{aligned}
 \bar{x} &= \frac{\int_1^4 x[2\sqrt{x} - (2x - 4)] dx}{\int_1^4 [2\sqrt{x} - (2x - 4)] dx}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2 \int_1^4 (x^{3/2} - x^2 + 2x) dx}{2 \int_1^4 (x^{1/2} - x + 2) dx} \\
 &= \frac{2 \left[\frac{2}{5}x^{5/2} - \frac{1}{3}x^3 + x^2 \right]_1^4}{2 \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x \right]_1^4} = \frac{\frac{64}{5}}{\frac{19}{3}} = \frac{192}{95}
 \end{aligned}$$

$$\begin{aligned}\bar{y} &= \frac{\frac{1}{2} \int_1^4 \left[(2\sqrt{x})^2 - (2x-4)^2 \right] dx}{\int_1^4 [2\sqrt{x} - (2x-4)] dx} \\ &= \frac{2 \int_1^4 (-x^2 + 5x - 4) dx}{\frac{19}{3}} \\ &= \frac{2 \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x \right]_1^4}{\frac{19}{3}} = \frac{9}{\frac{19}{3}} = \frac{27}{19}\end{aligned}$$

14.



To find the intersection points, $x^2 = x + 3$.

$$x^2 - x - 3 = 0$$

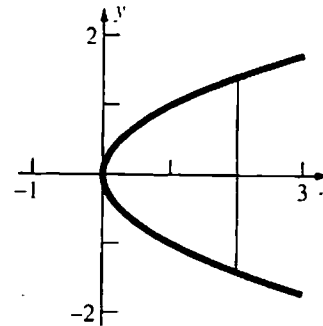
$$x = \frac{1 \pm \sqrt{13}}{2}$$

$$\begin{aligned}\bar{x} &= \frac{\frac{(1+\sqrt{13})}{2} \int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}} x(x+3-x^2) dx}{\frac{(1+\sqrt{13})}{2} \int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}} (x+3-x^2) dx} \\ &= \frac{\frac{(1+\sqrt{13})}{2} \left[\frac{1}{2}x^2 + 3x - \frac{1}{3}x^3 \right]_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}}{\frac{(1+\sqrt{13})}{2} \left[\frac{1}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{4}x^4 \right]_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}} = \frac{\frac{13\sqrt{13}}{12}}{\frac{13\sqrt{13}}{6}} = \frac{1}{2}\end{aligned}$$

$$\bar{y} = \frac{\frac{1}{2} \int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}} \left[(x+3)^2 - (x^2)^2 \right] dx}{\frac{(1+\sqrt{13})}{2} \int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}} (x+3-x^2) dx}$$

$$\begin{aligned}& \frac{(1+\sqrt{13})}{2} \int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}} (x^2 + 6x + 9 - x^4) dx \\ &= \frac{\frac{13\sqrt{13}}{6}}{\frac{(1+\sqrt{13})}{2} \int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}} \left[\frac{1}{3}x^3 + 3x^2 + 9x - \frac{1}{5}x^5 \right]_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}} dx} \\ &= \frac{\frac{143\sqrt{13}}{30}}{\frac{13\sqrt{13}}{6}} = \frac{11}{5}\end{aligned}$$

15.



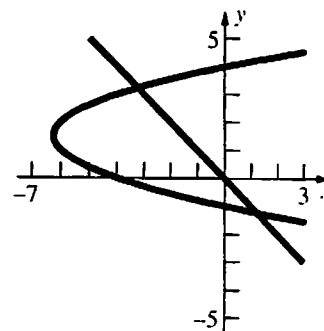
To find the intersection points, solve $y^2 = 2$.

$$y = \pm\sqrt{2}$$

$$\begin{aligned}\bar{x} &= \frac{\frac{1}{2} \int_{-\sqrt{2}}^{\sqrt{2}} [2^2 - (y^2)^2] dy}{\int_{-\sqrt{2}}^{\sqrt{2}} (2 - y^2) dy} = \frac{\frac{1}{2} \int_{-\sqrt{2}}^{\sqrt{2}} (4 - y^4) dy}{\left[2y - \frac{1}{3}y^3 \right]_{-\sqrt{2}}^{\sqrt{2}}} \\ &= \frac{\frac{1}{2} \left[4y - \frac{1}{5}y^5 \right]_{-\sqrt{2}}^{\sqrt{2}}}{\frac{8\sqrt{2}}{3}} = \frac{\frac{16\sqrt{2}}{5}}{\frac{8\sqrt{2}}{3}} = \frac{6}{5}\end{aligned}$$

$$\bar{y} = 0 \text{ (by symmetry)}$$

16.



To find the intersection points, solve

$$y^2 - 3y - 4 = -y.$$

$$y^2 - 2y - 4 = 0$$

$$y = \frac{2 \pm \sqrt{20}}{2}$$

$$\begin{aligned}
 y &= 1 \pm \sqrt{5} \\
 \bar{x} &= \frac{\frac{1}{2} \int_{1-\sqrt{5}}^{1+\sqrt{5}} [(-y)^2 - (y^2 - 3y - 4)^2] dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}} [(-y) - (y^2 - 3y - 4)] dy} \\
 &= \frac{\frac{1}{2} \int_{1-\sqrt{5}}^{1+\sqrt{5}} (-y^4 + 6y^3 - 24y - 16) dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}} (-y^2 + 2y + 4) dy} \\
 &= \frac{\frac{1}{2} \left[-\frac{1}{5}y^5 + \frac{3}{2}y^4 - 12y^2 - 16y \right]_{1-\sqrt{5}}^{1+\sqrt{5}}}{\left[-\frac{1}{3}y^3 + y^2 + 4y \right]_{1-\sqrt{5}}^{1+\sqrt{5}}} \\
 &= \frac{-20\sqrt{5}}{\frac{20\sqrt{5}}{3}} = -3 \\
 \bar{y} &= \frac{\int_{1-\sqrt{5}}^{1+\sqrt{5}} y [(-y) - (y^2 - 3y - 4)] dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}} [(-y) - (y^2 - 3y - 4)] dy} \\
 &= \frac{\int_{1-\sqrt{5}}^{1+\sqrt{5}} (-y^3 + 2y^2 + 4y) dy}{\frac{20\sqrt{5}}{3}} \\
 &= \frac{\left[-\frac{1}{4}y^4 + \frac{2}{3}y^3 + 2y^2 \right]_{1-\sqrt{5}}^{1+\sqrt{5}}}{\frac{20\sqrt{5}}{3}} = \frac{\frac{20\sqrt{5}}{3}}{\frac{20\sqrt{5}}{3}} = 1
 \end{aligned}$$

17. We let δ be the density of the regions and A_i be the area of region i .

Region R_1 :

$$m(R_1) = \delta A_1 = \delta(1/2)(1)(1) = \frac{1}{2}\delta$$

$$\bar{x}_1 = \frac{\int_0^1 x(x) dx}{\int_0^1 x dx} = \frac{\frac{1}{3}x^3 \Big|_0^1}{\frac{1}{2}x^2 \Big|_0^1} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Since R_1 is symmetric about the line $y = 1 - x$, the centroid must lie on this line. Therefore,

$$\bar{y}_1 = 1 - \bar{x}_1 = 1 - \frac{2}{3} = \frac{1}{3}; \text{ and we have}$$

$$M_y(R_1) = \bar{x}_2 \cdot m(R_1) = \frac{1}{3}\delta$$

$$M_x(R_1) = \bar{y}_2 \cdot m(R_1) = \frac{1}{6}\delta$$

Region R_2 :

$$m(R_2) = \delta A_2 = \delta(2)(1) = 2\delta$$

By symmetry we get

$$\bar{x}_2 = 2 \quad \text{and} \quad \bar{y}_2 = \frac{1}{2}.$$

Thus,

$$M_y(R_2) = \bar{x}_2 \cdot m(R_2) = 4\delta$$

$$M_x(R_2) = \bar{y}_2 \cdot m(R_2) = \delta$$

18. We can obtain the mass and moments for the whole region by adding the individual regions. Using the results from Problem 17 we get that

$$m = m(R_1) + m(R_2) = \frac{1}{2}\delta + 2\delta = \frac{5}{2}\delta$$

$$M_y = M_y(R_1) + M_y(R_2) = \frac{1}{3}\delta + 4\delta = \frac{13}{3}\delta$$

$$M_x = M_x(R_1) + M_x(R_2) = \frac{1}{6}\delta + \delta = \frac{7}{6}\delta$$

Therefore, the centroid is given by

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{13}{3}\delta}{\frac{5}{2}\delta} = \frac{26}{15}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{7}{6}\delta}{\frac{5}{2}\delta} = \frac{7}{15}$$

$$19. \quad m(R_1) = \delta \int_a^b (g(x) - f(x)) dx$$

$$m(R_2) = \delta \int_b^c (g(x) - f(x)) dx$$

$$M_x(R_1) = \frac{\delta}{2} \int_a^b ((g(x))^2 - (f(x))^2) dx$$

$$M_x(R_2) = \frac{\delta}{2} \int_b^c ((g(x))^2 - (f(x))^2) dx$$

$$M_y(R_1) = \delta \int_a^b x(g(x) - f(x)) dx$$

$$M_y(R_2) = \delta \int_b^c x(g(x) - f(x)) dx$$

Now,

$$\begin{aligned}
 m(R_3) &= \delta \int_a^c (g(x) - f(x)) dx \\
 &= \delta \int_a^b (g(x) - f(x)) dx + \delta \int_b^c (g(x) - f(x)) dx \\
 &= m(R_1) + m(R_2)
 \end{aligned}$$

$$\begin{aligned}
 M_x(R_3) &= \frac{\delta}{2} \int_a^c ((g(x))^2 - (f(x))^2) dx \\
 &= \frac{\delta}{2} \int_a^b ((g(x))^2 - (f(x))^2) dx \\
 &\quad + \frac{\delta}{2} \int_b^c ((g(x))^2 - (f(x))^2) dx \\
 &= M_x(R_1) + M_x(R_2)
 \end{aligned}$$

$$\begin{aligned}
 M_y(R_3) &= \delta \int_a^c x(g(x) - f(x))dx \\
 &= \delta \int_a^b x(g(x) - f(x))dx \\
 &\quad + \delta \int_b^c x(g(x) - f(x))dx \\
 &= M_y(R_1) + M_y(R_2)
 \end{aligned}$$

$$20. \quad m(R_1) = \delta \int_a^b (h(x) - g(x))dx$$

$$m(R_2) = \delta \int_a^b (g(x) - f(x))dx$$

$$M_x(R_1) = \frac{\delta}{2} \int_a^b ((h(x))^2 - (g(x))^2)dx$$

$$M_x(R_2) = \frac{\delta}{2} \int_a^b ((g(x))^2 - (f(x))^2)dx$$

$$M_y(R_1) = \delta \int_a^b x(h(x) - g(x))dx$$

$$M_y(R_2) = \delta \int_a^b x(g(x) - f(x))dx$$

Now,

$$\begin{aligned}
 m(R_3) &= \delta \int_a^b (h(x) - f(x))dx \\
 &= \delta \int_a^b (h(x) - g(x) + g(x) - f(x))dx \\
 &= \delta \int_a^b (h(x) - g(x))dx + \delta \int_a^b (g(x) - f(x))dx \\
 &= m(R_1) + m(R_2)
 \end{aligned}$$

$$\begin{aligned}
 M_x(R_3) &= \frac{\delta}{2} \int_a^b ((h(x))^2 - (f(x))^2)dx \\
 &= \frac{\delta}{2} \int_a^b ((h(x))^2 - (g(x))^2 + (g(x))^2 - (f(x))^2)dx \\
 &= \frac{\delta}{2} \int_a^b ((h(x))^2 - (g(x))^2)dx \\
 &\quad + \frac{\delta}{2} \int_a^b ((g(x))^2 - (f(x))^2)dx \\
 &= M_x(R_1) + M_x(R_2)
 \end{aligned}$$

$$\begin{aligned}
 M_y(R_3) &= \delta \int_a^b x(h(x) - f(x))dx \\
 &= \delta \int_a^b x(h(x) - g(x) + g(x) - f(x))dx \\
 &= \delta \int_a^b x(h(x) - g(x))dx + \delta \int_a^b x(g(x) - f(x))dx \\
 &= M_y(R_1) + M_y(R_2)
 \end{aligned}$$

21. Let region 1 be the region bounded by $x = -2$, $x = 2$, $y = 0$, and $y = 1$, so $m_1 = 4 \cdot 1 = 4$.

By symmetry, $\bar{x}_1 = 0$ and $\bar{y}_1 = \frac{1}{2}$. Therefore

$$M_{1y} = \bar{x}_1 m_1 = 0 \text{ and } M_{1x} = \bar{y}_1 m_1 = 2.$$

Let region 2 be the region bounded by $x = -2$, $x = 1$, $y = -1$, and $y = 0$, so $m_2 = 3 \cdot 1 = 3$.

By symmetry, $\bar{x}_2 = -\frac{1}{2}$ and $\bar{y}_2 = -\frac{1}{2}$. Therefore

$$M_{2y} = \bar{x}_2 m_2 = -\frac{3}{2} \text{ and } M_{2x} = \bar{y}_2 m_2 = -\frac{3}{2}.$$

$$\bar{x} = \frac{M_{1y} + M_{2y}}{m_1 + m_2} = \frac{-\frac{3}{2}}{7} = -\frac{3}{14}$$

$$\bar{y} = \frac{M_{1x} + M_{2x}}{m_1 + m_2} = \frac{\frac{1}{2}}{7} = \frac{1}{14}$$

22. Let region 1 be the region bounded by $x = -3$, $x = 1$, $y = -1$, and $y = 4$, so $m_1 = 20$. By

symmetry, $\bar{x} = -1$ and $\bar{y}_1 = \frac{3}{2}$. Therefore,

$$M_{1y} = \bar{x}_1 m_1 = -20 \text{ and } M_{1x} = \bar{y}_1 m_1 = 30.$$

Let region 2 be the region bounded by $x = -3$, $x = -2$, $y = -3$, and $y = -1$, so $m_2 = 2$. By symmetry,

$$\bar{x}_2 = -\frac{5}{2} \text{ and } \bar{y}_2 = -2.$$

$$M_{2y} = \bar{x}_2 m_2 = -5 \text{ and } M_{2x} = \bar{y}_2 m_2 = -4.$$

Let region 3 be the region bounded by $x = 0$, $x = 1$, $y = -2$, and $y = -1$, so $m_3 = 1$. By symmetry,

$$\bar{x}_3 = \frac{1}{2} \text{ and } \bar{y}_3 = -\frac{3}{2}.$$

$$M_{3y} = \bar{x}_3 m_3 = \frac{1}{2} \text{ and } M_{3x} = \bar{y}_3 m_3 = -\frac{3}{2}.$$

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{-\frac{49}{2}}{23} = -\frac{49}{46}$$

$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = \frac{\frac{49}{2}}{23} = \frac{49}{46}$$

23. Let region 1 be the region bounded by $x = -2$, $x = 2$, $y = 2$, and $y = 4$, so $m_1 = 4 \cdot 2 = 8$. By

symmetry, $\bar{x}_1 = 0$ and $\bar{y}_1 = 3$. Therefore,

$$M_{1y} = \bar{x}_1 m_1 = 0 \text{ and } M_{1x} = \bar{y}_1 m_1 = 24.$$

Let region 2 be the region bounded by $x = -1$, $x = 2$, $y = 0$, and $y = 2$, so $m_2 = 3 \cdot 2 = 6$. By

symmetry, $\bar{x}_2 = \frac{1}{2}$ and $\bar{y}_2 = 1$. Therefore,

$$M_{2y} = \bar{x}_2 m_2 = 3 \text{ and } M_{2x} = \bar{y}_2 m_2 = 6.$$

Let region 3 be the region bounded by $x = 2$, $x = 4$, $y = 0$, and $y = 1$, so $m_3 = 2 \cdot 1 = 2$. By symmetry,

$$\bar{x}_3 = 3 \text{ and } \bar{y}_2 = \frac{1}{2}.$$

$$\text{Therefore, } M_{3y} = \bar{x}_3 m_3 = 6$$

$$\text{and } M_{3x} = \bar{y}_3 m_3 = 1.$$

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{9}{16}$$

$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = \frac{31}{16}$$

24. Let region 1 be the region bounded by $x = -3$, $x = -1$, $y = -2$, and $y = 1$, so $m_1 = 6$. By

symmetry, $\bar{x}_1 = -2$ and $\bar{y}_1 = -\frac{1}{2}$. Therefore,

$M_{1y} = \bar{x}_1 m_1 = -12$ and $M_{1x} = \bar{y}_1 m_1 = -3$. Let region 2 be the region bounded by $x = -1$, $x = 0$, $y = -2$, and $y = 0$, so $m_2 = 2$. By symmetry,

$\bar{x}_2 = -\frac{1}{2}$ and $\bar{y}_2 = -1$. Therefore,

$M_{2y} = \bar{x}_2 m_2 = -1$ and $M_{2x} = \bar{y}_2 m_2 = -2$. Let

region 3 be the remaining region, so $m_3 = 22$.

By symmetry, $\bar{x}_3 = 2$ and $\bar{y}_3 = -\frac{1}{2}$. Therefore,

$M_{3y} = \bar{x}_3 m_3 = 44$ and $M_{3x} = \bar{y}_3 m_3 = -11$.

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{31}{30}$$

$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = -\frac{16}{30} = -\frac{8}{15}$$

25. $A = \int_0^1 x^3 dx = \left[\frac{1}{4} x^4 \right]_0^1 = \frac{1}{4}$

From Problem 11, $\bar{x} = \frac{4}{5}$.

$$V = A(2\pi\bar{x}) = \frac{1}{4} \left(2\pi \cdot \frac{4}{5} \right) = \frac{2\pi}{5}$$

Using cylindrical shells:

$$V = 2\pi \int_0^1 x \cdot x^3 dx = 2\pi \int_0^1 x^4 dx = 2\pi \left[\frac{1}{5} x^5 \right]_0^1 = \frac{2\pi}{5}$$

26. The area of the region is πa^2 . The centroid is the center $(0, 0)$ of the circle. It travels a distance of $2\pi(2a) = 4\pi a$. $V = 4\pi^2 a^3$

27. The volume of a sphere of radius a is $\frac{4}{3}\pi a^3$. If

the semicircle $y = \sqrt{a^2 - x^2}$ is revolved about the x -axis the result is a sphere of radius a . The centroid of the region travels a distance of $2\pi\bar{y}$.

The area of the region is $\frac{1}{2}\pi a^2$. Pappus's

Theorem says that

$$(2\pi\bar{y}) \left(\frac{1}{2}\pi a^2 \right) = \pi^2 a^2 \bar{y} = \frac{4}{3}\pi a^3.$$

$$\bar{y} = \frac{4a}{3\pi}, \quad \bar{x} = 0 \quad (\text{by symmetry})$$

28. Consider a slice at x rotated about the y -axis.

$$\Delta V = 2\pi x h(x) \Delta x, \text{ so } V = 2\pi \int_a^b x h(x) dx.$$

$$\Delta m \approx h(x) \Delta x, \text{ so } m = \int_a^b h(x) dx = A.$$

$$\Delta M_y \approx x h(x) \Delta x, \text{ so } M_y = \int_a^b x h(x) dx.$$

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x h(x) dx}{A}$$

The distance traveled by the centroid is $2\pi\bar{x}$.

$$(2\pi\bar{x})A = 2\pi \int_a^b x h(x) dx$$

Therefore, $V = 2\pi\bar{x}A$.

29. a. $\Delta V \approx 2\pi(K - y)w(y)\Delta y$

$$V = 2\pi \int_c^d (K - y)w(y)dy$$

b. $\Delta m \approx w(y)\Delta y$, so $m = \int_c^d w(y)dy = A$.

$$\Delta M_x \approx y w(y) \Delta y, \text{ so } M_x = \int_c^d y w(y) dy.$$

$$\bar{y} = \frac{\int_c^d y w(y) dy}{A}$$

The distance traveled by the centroid is $2\pi(K - \bar{y})$.

$$2\pi(K - \bar{y})A = 2\pi(KA - M_x)$$

$$= 2\pi \left(\int_c^d K w(y) dy - \int_c^d y w(y) dy \right)$$

$$= 2\pi \int_c^d (K - y)w(y) dy$$

Therefore, $V = 2\pi(K - \bar{y})A$.

30. a. $m = \frac{1}{2}bh$

The length of a segment at y is $b - \frac{b}{h}y$.

$$\Delta M_x \approx y \left(b - \frac{b}{h}y \right) \Delta y = \left(by - \frac{b}{h}y^2 \right) \Delta y$$

$$M_x = \int_0^h \left(by - \frac{b}{h}y^2 \right) dy$$

$$= \left[\frac{1}{2}by^2 - \frac{b}{3h}y^3 \right]_0^h = \frac{1}{6}bh^2$$

$$\bar{y} = \frac{M_x}{m} = \frac{h}{3}$$

- b. $A = \frac{1}{2}bh$; the distance traveled by the

centroid is $2\pi \left(k - \frac{h}{3} \right)$.

$$V = 2\pi \left(k - \frac{h}{3} \right) \left(\frac{1}{2}bh \right) = \frac{\pi bh}{3} (3k - h)$$

31. a. The area of a regular polygon P of $2n$ sides is $2r^2 n \sin \frac{\pi}{2n} \cos \frac{\pi}{2n}$. (To find this consider the isosceles triangles with one vertex at the center of the polygon and the other vertices on adjacent corners of the polygon. Each such triangle has base of length $2r \sin \frac{\pi}{2n}$ and height $r \cos \frac{\pi}{2n}$.) Since P is a regular polygon the centroid is at its center. The distance from the centroid to any side is $r \cos \frac{\pi}{2n}$, so the centroid travels a distance of $2\pi r \cos \frac{\pi}{2n}$.
- Thus, by Pappus's Theorem, the volume of the resulting solid is
- $$\left(2\pi r \cos \frac{\pi}{2n} \right) \left(2r^2 n \sin \frac{\pi}{2n} \cos \frac{\pi}{2n} \right)$$
- $$= 4\pi r^3 n \sin \frac{\pi}{2n} \cos^2 \frac{\pi}{2n}.$$

b. $\lim_{n \rightarrow \infty} 4\pi r^3 n \sin \frac{\pi}{2n} \cos^2 \frac{\pi}{2n}$

$$\lim_{n \rightarrow \infty} \frac{\sin \frac{\pi}{2n}}{\frac{\pi}{2n}} 2\pi^2 r^3 \cos^2 \frac{\pi}{2n} = 2\pi^2 r^3$$

As $n \rightarrow \infty$, the regular polygon approaches a circle. Using Pappus's Theorem on the circle of area πr^2 whose centroid (= center) travels a distance of $2\pi r$, the volume of the solid is $(\pi r^2)(2\pi r) = 2\pi^2 r^3$ which agrees with the results from the polygon.

32. a. The graph of $f(\sin x)$ on $[0, \pi]$ is symmetric about the line $x = \frac{\pi}{2}$ since

$$f(\sin x) = f(\sin(\pi - x)). \text{ Thus } \bar{x} = \frac{\pi}{2}.$$

$$\bar{x} = \frac{\int_0^\pi x f(\sin x) dx}{\int_0^\pi f(\sin x) dx} = \frac{\pi}{2}$$

Therefore

$$\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$$

- b. $\sin x \cos^4 x = \sin x(1 - \sin^2 x)^2$, so $f(x) = x(1 - x^2)^2$.

$$\int_0^\pi x \sin x \cos^4 x dx = \frac{\pi}{2} \int_0^\pi \sin x \cos^4 x dx$$

$$= \frac{\pi}{2} \left[-\frac{1}{5} \cos^5 x \right]_0^\pi = \frac{\pi}{5}$$

33. Consider the region $S - R$.

$$\bar{y}_{S-R} = \frac{\frac{1}{2} \int_0^1 [g^2(x) - f^2(x)] dx}{S - R} \geq \bar{y}_R$$

$$= \frac{\frac{1}{2} \int_0^1 f^2(x) dx}{R}$$

$$\frac{1}{2} R \int_0^1 [g^2(x) - f^2(x)] dx \geq \frac{1}{2} (S - R) \int_0^1 f^2(x) dx$$

$$\frac{1}{2} R \int_0^1 [g^2(x) - f^2(x)] dx + \frac{1}{2} R \int_0^1 f^2(x) dx$$

$$\geq \frac{1}{2} (S - R) \int_0^1 f^2(x) dx + \frac{1}{2} R \int_0^1 f^2(x) dx$$

$$\frac{1}{2} R \int_0^1 g^2(x) dx \geq \frac{1}{2} S \int_0^1 f^2(x) dx$$

$$\frac{\frac{1}{2} \int_0^1 g^2(x) dx}{S} \geq \frac{\frac{1}{2} \int_0^1 f^2(x) dx}{R}$$

$$\bar{y}_S \geq \bar{y}_R$$

34. To approximate the centroid, we can lay the figure on the x -axis (flat side down) and put the shortest side against the y -axis. Next we can use the eight regions between measurements to approximate the centroid. We will let h_i , the height of the i th region, be approximated by the height at the right end of the interval. Each interval is of width $\Delta x = 5$ cm. The centroid can be approximated as

$$\bar{x} \approx \frac{\sum_{i=1}^8 x_i h_i}{\sum_{i=1}^8 h_i} = \frac{(5)(6.5) + (10)(8) + \cdots + (35)(10) + (40)(8)}{6.5 + 8 + \cdots + 10 + 8}$$

$$= \frac{1695}{72.5} \approx 23.38$$

$$\bar{y} \approx \frac{\frac{1}{2} \sum_{i=1}^8 (h_i)^2}{\sum_{i=1}^8 h_i} = \frac{(1/2)(6.5^2 + 8^2 + \cdots + 10^2 + 8^2)}{(6.5 + 8 + \cdots + 10 + 8)}$$

$$= \frac{335.875}{72.5} \approx 4.63$$

35. First we place the lamina so that the origin is centered inside the hole. We then recompute the centroid of Problem 34 (in this position) as

$$\begin{aligned}\bar{x} &\approx \frac{\sum_{i=1}^8 x_i h_i}{\sum_{i=1}^8 h_i} = \frac{(-25)(6.5) + (-15)(8) + \cdots + (5)(10) + (10)(8)}{6.5 + 8 + \cdots + 10 + 8} \\ &= \frac{-480}{72.5} \approx -6.62 \\ \bar{y} &\approx \frac{\frac{1}{2} \sum_{i=1}^8 ((h_i - 4)^2 - (-4)^2)}{\sum_{i=1}^8 h_i} \\ &= \frac{(1/2)((2.5^2 - (-4)^2) + \cdots + (4^2 - (-4)^2))}{6.5 + 8 + \cdots + 10 + 8} \\ &= \frac{45.875}{72.5} \approx 0.633\end{aligned}$$

A quick computation will show that these values agree with those in Problem 34 (using a different reference point).

Now consider the whole lamina as R_3 , the circular hole as R_2 , and the remaining lamina as R_1 . We can find the centroid of R_1 by noting that

$$M_x(R_1) = M_x(R_3) - M_x(R_2)$$

and similarly for $M_y(R_1)$.

From symmetry, we know that the centroid of a circle is at the center. Therefore, both

$M_x(R_2)$ and $M_y(R_2)$ must be zero in our case. This

leads to the following equations

$$\begin{aligned}\bar{x} &= \frac{M_y(R_3) - M_y(R_2)}{m(R_3) - m(R_2)} \\ &= \frac{\delta \Delta x (-480)}{\delta \Delta x (72.5) - \delta \pi (2.5)^2} \\ &= \frac{-2400}{342.87} \approx -7\end{aligned}$$

$$\begin{aligned}\bar{y} &= \frac{M_x(R_3) - M_x(R_2)}{m(R_3) - m(R_2)} \\ &= \frac{\delta \Delta x (45.875)}{\delta \Delta x (72.5) - \delta \pi (2.5)^2} \\ &= \frac{229.375}{342.87} \approx 0.669\end{aligned}$$

36. This problem is much like Problem 34 except we don't have one side that is completely flat. In this problem, it will be necessary, in some regions, to find the value of $g(x)$ instead of just $f(x) - g(x)$. We will use the 19 regions in the figure to approximate the centroid. Again we choose the height of a region to be approximately the value at the right end of that region. Each region has a width of 20 miles. We will place the north-east corner of the state at the origin. The centroid is approximately

$$\begin{aligned}\bar{x} &\approx \frac{\sum_{i=1}^{19} x_i (f(x_i) - g(x_i))}{\sum_{i=1}^{19} (f(x_i) - g(x_i))} \\ &= \frac{(20)(145 - 13) + (40)(149 - 10) + \cdots + (380)(85 - 85)}{(145 - 13) + (149 - 19) + \cdots + (85 - 85)} \\ &= \frac{482,860}{2780} \approx 173.69 \\ \bar{y} &\approx \frac{\frac{1}{2} \sum_{i=1}^{19} [(f(x_i))^2 - (g(x_i))^2]}{\sum_{i=1}^{19} (f(x_i) - g(x_i))} \\ &= \frac{\frac{1}{2} [(145^2 - 13^2) + (149^2 - 10^2) + \cdots + (85^2 - 85^2)]}{(145 - 13) + (149 - 19) + \cdots + (85 - 85)} \\ &= \frac{230,805}{2780} \approx 83.02\end{aligned}$$

This would put the geographic center of Illinois just south-east of Lincoln, IL.

6.7 Chapter Review

Concepts Test

- False: $\int_0^{\pi} \cos x \, dx = 0$ because half of the area lies above the x -axis and half below the x -axis.

- True: The integral represents the area of the region in the first quadrant if the center of the circle is at the origin.
- False: The statement would be true if either $f(x) \geq g(x)$ or $g(x) \geq f(x)$ for $a \leq x \leq b$. Consider Problem 1 with $f(x) = \cos x$ and $g(x) = 0$.
- True: The area of a cross section of a cylinder will be the same in any plane parallel to the base.

5. True: Since the cross sections in all planes parallel to the bases have the same area, the integrals used to compute the volumes will be equal.
6. False: The volume of a right circular cone of radius r and height h is $\frac{1}{3}\pi r^2 h$. If the radius is doubled and the height halved the volume is $\frac{2}{3}\pi r^2 h$.
7. False: Using the method of shells, $V = 2\pi \int_0^1 x(-x^2 + x)dx$. To use the method of washers we need to solve $y = -x^2 + x$ for x in terms of y .
8. True: The bounded region is symmetric about the line $x = \frac{1}{2}$. Thus the solids obtained by revolving about the lines $x = 0$ and $x = 1$ have the same volume.
9. False: Consider the curve given by $x = \frac{\cos t}{t}$, $y = \frac{\sin t}{t}$, $2 \leq t < \infty$.
10. False: The work required to stretch a spring 2 inches beyond its natural length is $\int_0^2 kx dx = 2k$, while the work required to stretch it 1 inch beyond its natural length is $\int_0^1 kx dx = \frac{1}{2}k$.
11. False: If the cone-shaped tank is placed with the point downward, then the amount of water that needs to be pumped from near the bottom of the tank is much less than the amount that needs to be pumped from near the bottom of the cylindrical tank.
12. True: $(100)(10) + (100)(15) + (200)(-12.5) = 0$
13. True: This is the definition of the center of mass.
14. True: The region is symmetric about the point $(\pi, 0)$.
15. True: By symmetry, the centroid is on the line $x = \frac{\pi}{2}$, so the centroid travels a distance of $2\pi \left(\frac{\pi}{2}\right) = \pi^2$.

16. True: At slice y , $\Delta A \approx (9 - y^2)\Delta y$.
17. True: Since the density is proportional to the square of the distance from the midpoint, equal masses are on either side of the midpoint.
18. True: See Problem 30 in Section 6.6.

Sample Test Problems

- $A = \int_0^1 (x - x^2)dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{6}$
- $V = \pi \int_0^1 (x - x^2)^2 dx$
 $= \pi \int_0^1 (x^2 - 2x^3 + x^4) dx$
 $= \pi \left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_0^1 = \frac{\pi}{30}$
- $V = 2\pi \int_0^1 x(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx$
 $= 2\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = \frac{\pi}{6}$
- $V = \pi \int_0^1 [(x - x^2 + 2)^2 - (2)^2] dx$
 $= \pi \int_0^1 (x^4 - 2x^3 - 3x^2 + 4x) dx$
 $= \pi \left[\frac{1}{5}x^5 - \frac{1}{2}x^4 - x^3 + 2x^2 \right]_0^1 = \frac{7\pi}{10}$
- $V = 2\pi \int_0^1 (3 - x)(x - x^2)dx$
 $= 2\pi \int_0^1 (x^3 - 4x^2 + 3x) dx$
 $= 2\pi \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2 \right]_0^1 = \frac{5\pi}{6}$
- $\bar{x} = \frac{\int_0^1 x(x - x^2)dx}{\int_0^1 (x - x^2)dx} = \frac{\left[\frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1}{\left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1} = \frac{1}{2}$
 $\bar{y} = \frac{\frac{1}{2} \int_0^1 (x - x^2)^2 dx}{\int_0^1 (x - x^2)dx} = \frac{\frac{1}{2} \left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_0^1}{\left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1}$
 $= \frac{1}{10}$

7. From Problem 1, $A = \frac{1}{6}$.

From Problem 6, $\bar{x} = \frac{1}{2}$ and $\bar{y} = \frac{1}{10}$.

$$V(S_1) = 2\pi \left(\frac{1}{10} \right) \left(\frac{1}{6} \right) = \frac{\pi}{30}$$

$$V(S_2) = 2\pi \left(\frac{1}{2} \right) \left(\frac{1}{6} \right) = \frac{\pi}{6}$$

$$V(S_3) = 2\pi \left(\frac{1}{10} + 2 \right) \left(\frac{1}{6} \right) = \frac{7\pi}{10}$$

$$V(S_4) = 2\pi \left(3 - \frac{1}{2} \right) \left(\frac{1}{6} \right) = \frac{5\pi}{6}$$

8. $8 = F(8) = 8k$, $k = 1$

a. $W = \int_2^8 x \, dx = \left[\frac{1}{2} x^2 \right]_2^8 = \frac{1}{2} (64 - 4)$
 $= 30 \text{ in.-lb}$

b. $W = \int_0^{12} x \, dx = \left[\frac{1}{2} x^2 \right]_0^{12} = 72 \text{ in.-lb}$

9. $W = \int_0^6 (62.4)(5^2)\pi(10 - y) \, dy$
 $= 1560\pi \int_0^6 (10 - y) \, dy$
 $= 1560\pi \left[10y - \frac{1}{2} y^2 \right]_0^6 = 65,520\pi \approx 205,837 \text{ ft.-lb}$

10. The total work is equal to the work W_1 to pull up the object to the top without the cable and the work W_2 to pull up the cable.

$$W_1 = 200 \cdot 100 = 20,000 \text{ ft.-lb}$$

The cable weighs $\frac{120}{100} = \frac{6}{5} \text{ lb/ft.}$

$$\Delta W_2 = \frac{6}{5} \Delta y \cdot y = \frac{6}{5} y \Delta y$$

$$W_2 = \int_0^{100} \frac{6}{5} y \, dy = \frac{6}{5} \left[\frac{1}{2} y^2 \right]_0^{100}$$

$$= 6000 \text{ ft.-lb}$$

$$W = W_1 + W_2 = 26,000 \text{ ft.-lb}$$

11. a. To find the intersection points, solve

$$4x = x^2$$

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x = 0, 4$$

$$A = \int_0^4 (4x - x^2) \, dx = \left[2x^2 - \frac{1}{3} x^3 \right]_0^4$$

$$= \left(32 - \frac{64}{3} \right) = \frac{32}{3}$$

b. To find the intersection points, solve

$$\frac{y}{4} = \sqrt{y}$$

$$\frac{y^2}{16} = y$$

$$y^2 - 16y = 0$$

$$y(y - 16) = 0$$

$$y = 0, 16$$

$$A = \int_0^{16} \left(\sqrt{y} - \frac{y}{4} \right) \, dy = \left[\frac{2}{3} y^{3/2} - \frac{1}{8} y^2 \right]_0^{16}$$

$$= \left(\frac{128}{3} - 32 \right) = \frac{32}{3}$$

12. $\bar{x} = \frac{\int_0^4 x(4x - x^2) \, dx}{\int_0^4 (4x - x^2) \, dx} = \frac{\int_0^4 (4x^2 - x^3) \, dx}{\frac{32}{3}}$

$$= \frac{\left[\frac{4}{3} x^3 - \frac{1}{4} x^4 \right]_0^4}{\frac{32}{3}} = \frac{\frac{64}{3}}{\frac{32}{3}} = 2$$

$$\bar{y} = \frac{\frac{1}{2} \int_0^4 [(4x)^2 - (x^2)^2] \, dx}{\int_0^4 (4x - x^2) \, dx}$$

$$= \frac{\frac{1}{2} \int_0^4 (16x^2 - x^4) \, dx}{\frac{32}{3}}$$

$$= \frac{\frac{1}{2} \left[\frac{16}{3} x^3 - \frac{1}{5} x^5 \right]_0^4}{\frac{32}{3}} = \frac{\frac{1024}{15}}{\frac{32}{3}} = \frac{32}{5}$$

13. $V = \pi \int_0^4 [(4x)^2 - (x^2)^2] \, dx$

$$= \pi \int_0^4 (16x^2 - x^4) \, dx$$

$$= \pi \left[\frac{16}{3} x^3 - \frac{1}{5} x^5 \right]_0^4 = \frac{2048\pi}{15}$$

Using Pappus's Theorem:

From Problem 11, $A = \frac{32}{3}$.

From Problem 12, $\bar{y} = \frac{32}{5}$.

$$V = 2\pi \bar{y} \cdot A = 2\pi \left(\frac{32}{5} \right) \left(\frac{32}{3} \right) = \frac{2048\pi}{15}$$

$$\begin{aligned}
 14. \quad V &= 2\pi \int_0^4 x(4x - x^2) dx \\
 &= 2\pi \int_0^4 (4x^2 - x^3) dx \\
 &= 2\pi \left[\frac{4}{3}x^3 - \frac{1}{4}x^4 \right]_0^4 = \frac{128\pi}{3}
 \end{aligned}$$

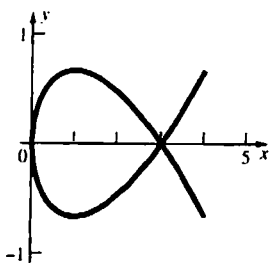
Using Pappus's Theorem:

$$\text{From Problem 11, } A = \frac{32}{3}.$$

From Problem 12, $\bar{x} = 2$.

$$V = 2\pi\bar{x} \cdot A = 2\pi(2) \left(\frac{32}{3} \right) = \frac{128\pi}{3}$$

16.



The loop is $-\sqrt{3} \leq t \leq \sqrt{3}$. By symmetry, we can double the length of the loop from $t = 0$ to

$$t = \sqrt{3}, \quad \frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = t^2 - 1$$

$$L = 2 \int_0^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1} dt = 2 \int_0^{\sqrt{3}} (t^2 + 1) dt$$

$$= 2 \left[\frac{1}{3}t^3 + t \right]_0^{\sqrt{3}} = 4\sqrt{3}k$$

$$\begin{aligned}
 17. \quad V &= \int_{-3}^3 \left(\sqrt{9-x^2} \right)^2 dx = \int_{-3}^3 (9-x^2) dx \\
 &= \left[9x - \frac{1}{3}x^3 \right]_{-3}^3 = (27-9) - (-27+9) = 36
 \end{aligned}$$

$$18. \quad A = \int_a^b [f(x) - g(x)] dx$$

$$15. \quad \frac{dy}{dx} = x^2 - \frac{1}{4x^2}$$

$$\begin{aligned}
 L &= \int_1^3 \sqrt{1 + \left(x^2 - \frac{1}{4x^2} \right)^2} dx \\
 &= \int_1^3 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}} dx = \int_1^3 \left(x^2 + \frac{1}{4x^2} \right) dx \\
 &= \left[\frac{1}{3}x^3 - \frac{1}{4x} \right]_1^3 = \left(9 - \frac{1}{12} \right) - \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{53}{6}
 \end{aligned}$$

$$19. \quad V = \pi \int_a^b [f^2(x) - g^2(x)] dx$$

$$20. \quad V = 2\pi \int_a^b (x-a)[f(x) - g(x)] dx$$

$$21. \quad M_y = \delta \int_a^b x[f(x) - g(x)] dx$$

$$M_x = \frac{\delta}{2} \int_a^b [f^2(x) - g^2(x)] dx$$

$$22. \quad L_1 = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$L_2 = \int_a^b \sqrt{1 + [g'(x)]^2} dx$$

$$L_3 = f(a) - g(a)$$

$$L_4 = f(b) - g(b)$$

$$\text{Total length} = L_1 + L_2 + L_3 + L_4$$

$$23. \quad A_1 = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx$$

$$A_2 = 2\pi \int_a^b g(x) \sqrt{1 + [g'(x)]^2} dx$$

$$A_3 = \pi [f^2(a) - g^2(a)]$$

$$A_4 = \pi [f^2(b) - g^2(b)]$$

$$\text{Total surface area} = A_1 + A_2 + A_3 + A_4.$$

6.8 Additional Problem Set

1. a. feet/second

$F(T)$ is the change in velocity from 6 seconds to T seconds.

b. foot-pounds

$F(s)$ is the change in work from 3 feet to s feet.

c. inches

$F(r)$ is the center of mass of an object r inches long whose mass at x is $f(x)$.

2. This follows from the Fundamental Theorem of Calculus.

$$3. F(t) = \int_0^t f(u) du$$

$$a. \lim_{t \rightarrow \infty} F(t) = \int_0^3 A du = [Au]_0^3 = 3A = 1$$

$$A = \frac{1}{3}$$

$$b. \lim_{t \rightarrow \infty} F(t) = \int_0^7 A(u^2 - 7u) du$$

$$= A \left[\frac{1}{3} u^3 - \frac{7}{2} u^2 \right]_0^7 = \frac{343}{6} A = 1$$

$$A = \frac{6}{343}$$

$$c. \lim_{t \rightarrow \infty} F(t) = \int_0^3 Au^2 du + \int_3^9 A(9-u) du$$

$$= A \left[\frac{1}{3} u^3 \right]_0^3 + A \left[9u - \frac{1}{2} u^2 \right]_3^9$$

$$= 9A + 18A = 27A = 1$$

$$A = \frac{1}{27}$$

$$4. a. \lim_{t \rightarrow \infty} F(t) = \int_0^4 15Au^2(4-u)^2 du$$

$$= 15A \int_0^4 (u^4 - 8u^3 + 16u^2) du$$

$$= 15A \left[\frac{1}{5} u^5 - 2u^4 + \frac{16}{3} u^3 \right]_0^4$$

$$= 15A \left(\frac{1024}{5} - 512 + \frac{1024}{3} \right) = 512A = 1$$

$$A = \frac{1}{512}$$

$$b. F(3) = \int_0^3 \frac{15}{512} u^2(4-u)^2 du$$

$$= \frac{15}{512} \left[\frac{1}{5} u^5 - 2u^4 + \frac{16}{3} u^3 \right]_0^3$$

$$= \frac{15}{512} \left(\frac{153}{5} \right) = \frac{459}{512}$$

$$\text{Pr(fail after 3 years)} = 1 - F(3)$$

$$= 1 - \left(\frac{459}{512} \right) \approx 0.1035$$

$$c. F(t) = 15A \int_0^t [x^2(16-8x+x^2)] dx + F(0)$$

$$= 15A \int_0^t (16x^2 - 8x^3 + x^4) dx + F(0)$$

$$= 15A \left[\frac{16}{3} t^3 - 2t^4 + \frac{1}{5} t^5 \right] + F(0)$$

From the initial conditions, we know that

$F(0) = 0$ and that

$$F(4) = 1 = 15A \left[\frac{1024}{3} - 512 + \frac{1024}{5} \right]$$

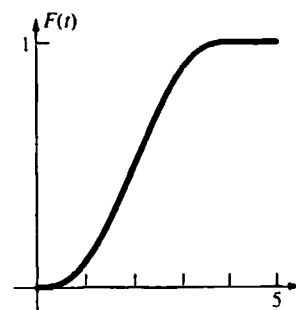
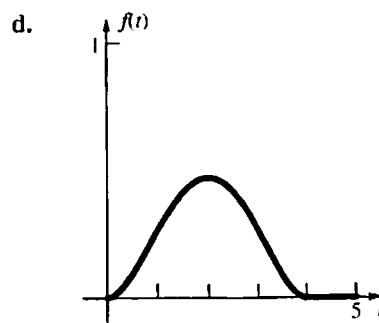
$$= A[512] = 1$$

Therefore,

$$A = \frac{1}{512}$$

and we get:

$$F(t) = \begin{cases} \frac{15}{512} \left(\frac{1}{5} t^5 - 2t^4 + \frac{16}{3} t^3 \right) & 0 \leq t \leq 4 \\ 1 & t > 4 \end{cases}$$



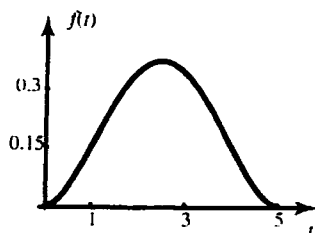
5. a. We can consider the expression

$$\int_0^B t f(t) dt$$

as a moment if we consider that the total mass of the region is one unit. The total area under the curve must equal one as well, and so we can consider the region to have a density of $\delta = 1$. A moment is defined to be the product of a region's mass times its directed distance from the rotation point. $f(t)$ represents the height of small region and dt represents a very small change in t . $f(t)dt$ can be considered as the area of a very small sliver of the region. Since the density is equal to 1, the quantity $f(t)dt$ also represents the mass of the sliver. Multiplying by t , the direct distance from the y-axis, we obtain the moment for that sliver about the y-axis. The integral represents an infinite sum over all the slivers that make up the region, and represents the moment of the region about the y-axis.

$$\begin{aligned}
 \text{b. } E &= \int_0^4 \frac{15}{512} t [t^2(4-t)^2] dt \\
 &= \frac{15}{512} \int_0^4 (16t^3 - 8t^4 + t^5) dt \\
 &= \frac{15}{512} \left[\frac{16}{4} t^4 - \frac{8}{5} t^5 + \frac{1}{6} t^6 \right]_0^4 \\
 &= \frac{15}{512} \left[1024 - \frac{8192}{5} + \frac{4096}{6} \right] = 2
 \end{aligned}$$

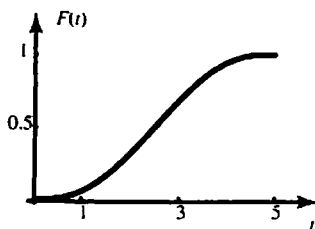
6. a.



$$\text{b. } F(t) = \int_0^t f(u) du$$

For $0 \leq t \leq 5$:

$$\begin{aligned}
 F(t) &= \frac{6}{625} \int_0^t u^2(5-u)^2 du \\
 &= \frac{6}{625} \left(\frac{1}{5} t^5 - \frac{5}{2} t^4 + \frac{25}{3} t^3 \right) \\
 &= \frac{6}{3125} t^5 - \frac{3}{125} t^4 + \frac{2}{25} t^3 \\
 F(t) &= \begin{cases} \frac{6}{3125} t^5 - \frac{3}{125} t^4 + \frac{2}{25} t^3 & \text{if } 0 \leq t \leq 5 \\ 1 & t > 5 \end{cases}
 \end{aligned}$$



$$\begin{aligned}
 \text{c. } F(t \geq 3) &= 1 - F(3) \\
 &= 1 - \frac{6}{3125} 3^5 + \frac{3}{125} 3^4 - \frac{2}{25} 3^3 = 0.31744
 \end{aligned}$$

d. We would expect
 $100,000 \cdot 0.31744 = 31,744$
 batteries to be working.

$$\begin{aligned}
 7. \quad F &= (62.4)(10)(0.0004) = 0.2496 \text{ lb} \\
 p &= (62.4)(10) = 624 \text{ lb/ft}^2
 \end{aligned}$$

8. a. The total force is the sum of the forces at each depth. For example, the force will be approximately equal to the sum of the forces at every $\frac{1}{100}$ foot. We can approximate the

force on the wall by using $n = 100$,

$\Delta h = \frac{1}{100}$, and depth $= 10 - h_i$, where

$h_i = \frac{i}{10}$ is the i th height. At a particular

depth, the force on the wall will be approximately $F \approx 62.4 \cdot 75 \cdot (10 - h)(\Delta h)$

b. Given the coordinate system, the depth at h is $(10 - h)$. On the interval $[0, 10]$,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} 62.4 \cdot 75 \sum_{i=1}^n (10 - h_i) \Delta h_i \\
 = 62.4 \cdot 75 \int_0^{10} (10 - h) dh.
 \end{aligned}$$

9. a. At height h , a strip Δh has
 $F \approx 62.4(10 - h)\Delta h \cdot W(h)$.

$$\text{Thus } F = 62.4 \int_0^{10} (10 - h)W(h) dh.$$

b. $W(h) = \frac{2}{\sqrt{3}}h$ follows from the geometry of an equilateral triangle. (Similar triangles, 30-60-90 triangle, etc.)

$$\begin{aligned}
 \text{c. } F &= 62.4 \int_0^{10} (10 - h)(2h/\sqrt{3}) dh \\
 &= 62.4 \int_0^{10} \left(\frac{20}{\sqrt{3}}h - \frac{2}{\sqrt{3}}h^2 \right) dh \\
 &= 62.4 \left[\frac{10}{\sqrt{3}}h^2 - \frac{2}{3\sqrt{3}}h^3 \right]_0^{10} \approx 12009
 \end{aligned}$$

10. Let h measure the distance from the top of the plate, where $0 \leq h \leq 3\sqrt{3}$. This also gives the depth at that distance. The width of the plate at h is given by $W(h) = -\frac{2}{\sqrt{3}}(h - 3\sqrt{3})$. At distance h

from the top, a strip Δh has

$$F \approx 62.4h\Delta h \left[-\frac{2}{\sqrt{3}}(h - 3\sqrt{3}) \right].$$

$$F = -\frac{124.8}{\sqrt{3}} \int_0^{3\sqrt{3}} h(h - 3\sqrt{3}) dh$$

$$= -\frac{124.8}{\sqrt{3}} \left[\frac{1}{3}h^3 - \frac{3\sqrt{3}}{2}h^2 \right]_0^{3\sqrt{3}}$$

$$= -\frac{124.8}{\sqrt{3}} \left(-\frac{27\sqrt{3}}{2} \right) = 1684.8 \text{ lb}$$

11. Let h measure the distance from the top of the plate, where $0 \leq h \leq 3\sqrt{3}$. The depth at that distance is $h + 3$. The width of the plate at h is given by $W(h) = -\frac{2}{\sqrt{3}}(h - 3\sqrt{3})$. At distance h from the top a strip Δh has

$$\begin{aligned} F &\approx 62.4(h+3)\Delta h \left[-\frac{2}{\sqrt{3}}(h-3\sqrt{3}) \right] \\ F &= -\frac{124.8}{\sqrt{3}} \int_0^{3\sqrt{3}} (h+3)(h-3\sqrt{3}) dh \\ &= -\frac{124.8}{\sqrt{3}} \int_0^{3\sqrt{3}} [h^2 + 3(1-\sqrt{3})h - 9\sqrt{3}h] dh \\ &= -\frac{124.8}{\sqrt{3}} \left[\frac{1}{3}h^3 + \frac{3(1-\sqrt{3})}{2}h^2 - 9\sqrt{3}h \right]_0^{3\sqrt{3}} \\ &= -\frac{124.8}{\sqrt{3}} \left[-\frac{27(3+\sqrt{3})}{2} \right] \\ &= 1684.8(1+\sqrt{3}) \approx 4602.96 \text{ lb} \end{aligned}$$

12. Let y measure the distance from the bottom of the patch, where $0 \leq y \leq 2$. Let l be the length of a side of the cube. The depth at y is $l - y$. The width of the patch at y is $2\sqrt{y}$. At y , a strip Δy has

$$\begin{aligned} F &\approx 62.4(l-y)\Delta y \cdot 2\sqrt{y} \\ F &= 124.8 \int_0^2 \sqrt{y}(l-y) dy \\ &= 124.8 \int_0^2 (l\sqrt{y} - y^{3/2}) dy \\ &= 124.8 \left[\frac{2l}{3}y^{3/2} - \frac{2}{5}y^{5/2} \right]_0^2 \\ &= 33.28\sqrt{2}(5l-6) \\ 33.28\sqrt{2}(5l-6) &= 200, \text{ so } l \approx 2.05 \text{ ft.} \end{aligned}$$

13. Let y measure the distance from the bottom of the window, where $0 \leq y \leq 2$. Let h be the height of the tank. The depth at y is $h - y$. The width of the window at y is y . At y , the strip Δy has

$$\begin{aligned} F &\approx 62.4(h-y)\Delta y \cdot y \\ F &= 62.4 \int_0^2 y(h-y) dy \\ &= 62.4 \int_0^2 (hy - y^2) dy \\ &= 62.4 \left[\frac{h}{2}y^2 - \frac{1}{3}y^3 \right]_0^2 \\ &= 62.4 \left(2h - \frac{8}{3} \right) \\ 62.4 \left(2h - \frac{8}{3} \right) &= 400, \text{ so } h \approx 4.54 \text{ ft.} \end{aligned}$$

14. a. The mass of the car is

$$\frac{3200 \text{ lb}}{32 \text{ ft/s}^2} = 100 \text{ lb-s}^2/\text{ft}.$$

Since acceleration is uniform,

$$a = \frac{66-0}{6} = 11 \text{ ft/s}^2. \text{ Thus,}$$

$$F = ma = (100 \text{ lb-s}^2/\text{ft})(11 \text{ ft/s}^2) = 1100 \text{ lb}.$$

$v(t) = 11t$, so

$$P(t) = F \cdot v(t) = (1100)(11t) = 12,100t. \text{ The}$$

most power needed is at 6 seconds, so

$$P(6) = 12,100 \cdot 6 = 72,600 \text{ ft-lb/s}.$$

$$\frac{72,600 \text{ ft-lb/s}}{550 \text{ (ft-lb/s)/hp}} = 132 \text{ hp is needed.}$$

- b. Since the amount of horsepower needed depends on the acceleration, we cannot necessarily determine the horsepower needed, but we can determine the minimum horsepower needed. To calculate the minimum horsepower needed, assume that power is constant. Then

$$P = 100a(t)v(t). \text{ Since } a(t) = \frac{dv(t)}{dt},$$

$P dt = 100v(t) dv(t)$. Integrate both sides to get $Pt = 50v(t)^2 + C$. Since $v(0) = 0$, $C = 0$,

$$\text{so } P = \frac{50v(t)^2}{t}. \text{ At } t = 6,$$

$$P = \frac{50(66)^2}{6} = 36,300 \text{ ft-lb/s}.$$

$$\frac{36,300 \text{ ft-lb/s}}{550 \text{ (ft-lb/s)/hp}} = 66 \text{ hp is the minimum horsepower needed.}$$

15. a. The additional force on the car is the component of the force due to gravity along the incline.

$$F = 100 \cdot 32 \sin(\tan^{-1} 0.05) \approx 159.8 \text{ lb}.$$

$$P \approx 159.8 \cdot 66 = 10,546.8 \text{ ft-lb/s}$$

$$\frac{10,546.8 \text{ ft-lb/s}}{550 \text{ (ft-lb/s)/hp}} \approx 19 \text{ hp additional is required.}$$

- b. Since acceleration is not assumed to be uniform, we will calculate the minimum horsepower required, so we will assume that the additional power is constant while accelerating.

$$P = 100a(t)v(t). \text{ Since } a(t) = \frac{dv(t)}{dt},$$

$$P dt = 100v(t) dv(t).$$

Integrate both sides to get $Pt = 50v(t)^2 + C$.

$$60 \text{ mi/h} = 88 \text{ ft/s}$$

$$\text{At } t = 0, v = 66 \text{ and at } t = 4, v = 88.$$

$$\text{Therefore, } C = -217,800 \text{ and}$$

$$P = \frac{50(88)^2 - 217,800}{4} = 42,350 \text{ ft-lb/s.}$$

$$\frac{42,350 \text{ ft-lb/s}}{550 \text{ (ft-lb/s)/hp}} = 77 \text{ hp additional is}$$

required. Note that since the car is already on the incline, we do not need to consider the power needed to overcome the force due to gravity.

If we assume uniform acceleration, then

$$a = \frac{88 - 66}{4} = 5.5 \text{ ft/s}^2, \text{ so the additional}$$

force is $F = 100 \cdot 5.5 = 550 \text{ lb}$. The most

power is needed when $v = 88 \text{ ft/s}$, so

$$P = 550 \cdot 88 = 48,400 \text{ ft-lb/s.}$$

$$\frac{48,400 \text{ ft-lb/s}}{550 \text{ (ft-lb/s)/hp}} = 88 \text{ hp additional is}$$

required.

- c. The force due to gravity along the incline is

$$F = 100 \cdot 32 \sin(\tan^{-1} 0.05) \approx 159.8 \text{ lb. The}$$

distance traveled is

$$(66 \text{ ft/s})(180 \text{ s}) = 11,880 \text{ ft.}$$

Thus the work is

$$W \approx 159.8 \cdot 11,800 = 1,898,424 \text{ ft-lb}$$

7.1 Concepts Review

1. $\int_1^x \frac{1}{t} dt; (0, \infty); (-\infty, \infty)$

2. $\frac{1}{x}$

3. $\frac{1}{x}; \ln|x| + C$

4. $\ln x + \ln y; \ln x - \ln y; r \ln x$

Problem Set 7.1

1. a. $\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3$
 $= 0.693 + 1.099 = 1.792$

b. $\ln 1.5 = \ln\left(\frac{3}{2}\right) = \ln 3 - \ln 2 = 1.099 - 0.693$
 $= 0.406$

c. $\ln 81 = \ln 3^4 = 4 \ln 3 = 4(1.099) = 4.396$

d. $\ln \sqrt{2} = \ln 2^{1/2} = \frac{1}{2} \ln 2 = \frac{1}{2}(0.693) = 0.3465$

e. $\ln\left(\frac{1}{36}\right) = -\ln 36 = -\ln(2^2 \cdot 3^2)$
 $= -2 \ln 2 - 2 \ln 3 = -2(0.693) - 2(1.099)$
 $= -3.584$

f. $\ln 48 = \ln(2^4 \cdot 3) = 4 \ln 2 + \ln 3$
 $= 4(0.693) + 1.099 = 3.871$

2. a. 1.792 b. 0.405

c. 4.394 d. 0.3466

e. -3.584 f. 3.871

3. $D_x \ln(x^2 + 3x + \pi)$

$$= \frac{1}{x^2 + 3x + \pi} \cdot D_x(x^2 + 3x + \pi)$$

$$= \frac{2x + 3}{x^2 + 3x + \pi}$$

4. $D_x \ln(3x^3 + 2x) = \frac{1}{3x^3 + 2x} D_x(3x^3 + 2x)$

$$= \frac{9x^2 + 2}{3x^3 + 2x}$$

5. $D_x \ln(x-4)^3 = D_x 3 \ln(x-4)$

$$= 3 \cdot \frac{1}{x-4} D_x(x-4) = \frac{3}{x-4}$$

6. $D_x \ln \sqrt{3x-2} = D_x \frac{1}{2} \ln(3x-2)$

$$= \frac{1}{2} \cdot \frac{1}{3x-2} D_x(3x-2) = \frac{3}{2(3x-2)}$$

7. $\frac{dy}{dx} = 3 \cdot \frac{1}{x} = \frac{3}{x}$

8. $\frac{dy}{dx} = x^2 \cdot \frac{1}{x} + 2x \cdot \ln x = x(1 + 2 \ln x)$

9. $z = x^2 \ln x^2 + (\ln x)^3 = x^2 \cdot 2 \ln x + (\ln x)^3$

$$\frac{dz}{dx} = x^2 \cdot \frac{2}{x} + 2x \cdot 2 \ln x + 3(\ln x)^2 \cdot \frac{1}{x}$$

$$= 2x + 4x \ln x + \frac{3}{x} (\ln x)^2$$

10. $r = \frac{\ln x}{x^2 \ln x^2} + \left(\ln \frac{1}{x}\right)^3 = \frac{\ln x}{x^2 \cdot 2 \ln x} + (-\ln x)^3$

$$= \frac{1}{2} x^{-2} - (\ln x)^3$$

$$\frac{dr}{dx} = \frac{-2}{2} x^{-3} - 3(\ln x)^2 \cdot \frac{1}{x} = -\frac{1}{x^3} - \frac{3(\ln x)^2}{x}$$

$$11. \quad g'(x) = \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x \right] \\ = \frac{1}{\sqrt{x^2 + 1}}$$

$$12. \quad h'(x) = \frac{1}{x + \sqrt{x^2 - 1}} \left[1 + \frac{1}{2}(x^2 - 1)^{-1/2} \cdot 2x \right] \\ = \frac{1}{\sqrt{x^2 - 1}}$$

$$13. \quad f(x) = \ln \sqrt[3]{x} = \frac{1}{3} \ln x$$

$$f'(x) = \frac{1}{3} \cdot \frac{1}{x} = \frac{1}{3x}$$

$$f'(81) = \frac{1}{3 \cdot 81} = \frac{1}{243}$$

$$14. \quad f'(x) = \frac{1}{\cos x} (-\sin x) = -\tan x$$

$$f'\left(\frac{\pi}{2}\right) = -\tan\left(\frac{\pi}{2}\right), \text{ which is undefined.}$$

$$15. \quad \text{Let } u = 2x + 1 \text{ so } du = 2 \, dx.$$

$$\int \frac{1}{2x+1} dx = \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |2x + 1| + C$$

$$16. \quad \text{Let } u = 1 - 2x \text{ so } du = -2 \, dx.$$

$$\int \frac{1}{1-2x} dx = -\frac{1}{2} \int \frac{1}{u} du$$

$$= -\frac{1}{2} \ln |u| + C = -\frac{1}{2} \ln |1 - 2x| + C$$

$$17. \quad \text{Let } u = 3v^2 + 9v \text{ so } du = 6v + 9.$$

$$\int \frac{6v+9}{3v^2+9v} dv = \int \frac{1}{u} du = \ln |u| + C$$

$$= \ln |3v^2 + 9v| + C$$

$$18. \quad \text{Let } u = 2z^2 + 8 \text{ so } du = 4z \, dz.$$

$$\int \frac{z}{2z^2+8} dz = \frac{1}{4} \int \frac{1}{u} du$$

$$= \frac{1}{4} \ln |u| + C = \frac{1}{4} \ln |2z^2 + 8| + C$$

$$19. \quad \text{Let } u = \ln x \text{ so } du = \frac{1}{x} dx$$

$$\int \frac{2 \ln x}{x} dx = 2 \int u du$$

$$= u^2 + C = (\ln x)^2 + C$$

$$20. \quad \text{Let } u = \ln x, \text{ so } du = \frac{1}{x} dx.$$

$$\int \frac{-1}{x(\ln x)^2} dx = -\int u^{-2} du$$

$$= \frac{1}{u} + C = \frac{1}{\ln x} + C$$

$$21. \quad \text{Let } u = 2x^5 + \pi \text{ so } du = 10x^4 dx.$$

$$\int \frac{x^4}{2x^5 + \pi} dx = \frac{1}{10} \int \frac{1}{u} du$$

$$= \frac{1}{10} \ln |u| + C = \frac{1}{10} \ln |2x^5 + \pi| + C$$

$$\int_0^3 \frac{x^4}{2x^5 + \pi} dx = \left[\frac{1}{10} \ln |2x^5 + \pi| \right]_0^3$$

$$= \frac{1}{10} [\ln(486 + \pi) - \ln \pi] = \ln 10 \sqrt[10]{\frac{486 + \pi}{\pi}}$$

$$22. \quad \text{Let } u = 2t^2 + 4t + 3 \text{ so } du = (4t + 4) dt.$$

$$\int \frac{t+1}{2t^2+4t+3} dt = \frac{1}{4} \int \frac{1}{u} du$$

$$= \frac{1}{4} \ln |u| + C = \frac{1}{4} \ln |2t^2 + 4t + 3| + C$$

$$\int_0^1 \frac{t+1}{2t^2+4t+3} dt = \left[\frac{1}{4} \ln |2t^2 + 4t + 3| \right]_0^1$$

$$= \frac{1}{4} \ln 9 - \frac{1}{4} \ln 3 = \ln \sqrt[4]{\frac{9}{3}} = \ln \sqrt[4]{3} = \frac{1}{4} \ln 3$$

$$23. \quad 2 \ln(x+1) - \ln x = \ln(x+1)^2 - \ln x$$

$$= \ln \frac{(x+1)^2}{x}$$

$$24. \quad \frac{1}{2} \ln(x-9) + \frac{1}{2} \ln x = \ln \sqrt{x-9} - \ln \sqrt{x}$$

$$= \ln \frac{\sqrt{x-9}}{\sqrt{x}} = \ln \sqrt{\frac{x-9}{x}}$$

$$25. \quad \ln(x-2) - \ln(x+2) + 2 \ln x$$

$$= \ln(x-2) - \ln(x+2) + \ln x^2$$

$$= \ln \frac{x^2(x-2)}{x+2}$$

$$\begin{aligned}
 26. \quad & \ln(x^2 - 9) - 2\ln(x - 3) - \ln(x + 3) \\
 &= \ln(x^2 - 9) - \ln(x - 3)^2 - \ln(x + 3) \\
 &= \ln \frac{x^2 - 9}{(x - 3)^2(x + 3)} = \ln \frac{1}{x - 3}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy}{dx} &= y \cdot \left[\frac{1}{x+11} - \frac{3x^2}{2(x^3-4)} \right] \\
 &= \frac{x+11}{\sqrt{x^3-4}} \left[\frac{1}{x+11} - \frac{3x^2}{2(x^3-4)} \right] \\
 &= -\frac{x^3+33x^2+8}{2(x^3-4)^{3/2}}
 \end{aligned}$$

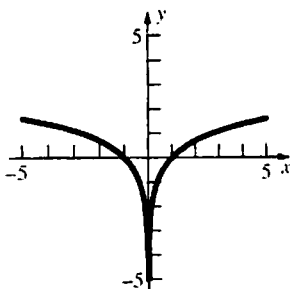
$$\begin{aligned}
 27. \quad & \ln y = \ln(x+11) - \frac{1}{2}\ln(x^3-4) \\
 \frac{1}{y} \frac{dy}{dx} &= \frac{1}{x+11} \cdot 1 - \frac{1}{2} \cdot \frac{1}{x^3-4} \cdot 3x^2 \\
 &= \frac{1}{x+11} - \frac{3x^2}{2(x^3-4)}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad & \ln y = \ln(x^2+3x) + \ln(x-2) + \ln(x^2+1) \\
 \frac{1}{y} \frac{dy}{dx} &= \frac{2x+3}{x^2+3x} + \frac{1}{x-2} + \frac{2x}{x^2+1} \\
 \frac{dy}{dx} &= (x^2+3x)(x-2)(x^2+1) \left(\frac{2x+3}{x^2+3x} + \frac{1}{x-2} + \frac{2x}{x^2+1} \right) = 5x^4 + 4x^3 - 15x^2 + 2x - 6
 \end{aligned}$$

$$\begin{aligned}
 29. \quad & \ln y = \frac{1}{2}\ln(x+13) - \ln(x-4) - \frac{1}{3}\ln(2x+1) \\
 \frac{1}{y} \frac{dy}{dx} &= \frac{1}{2(x+13)} - \frac{1}{x-4} - \frac{2}{3(2x+1)} \\
 \frac{dy}{dx} &= \frac{\sqrt{x+13}}{(x-4)\sqrt[3]{2x+1}} \left[\frac{1}{2(x+13)} - \frac{1}{x-4} - \frac{2}{3(2x+1)} \right] = -\frac{10x^2+219x-118}{6(x-4)^2(x+13)^{1/2}(2x+1)^{4/3}}
 \end{aligned}$$

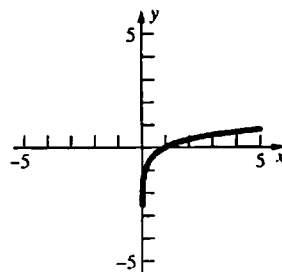
$$\begin{aligned}
 30. \quad & \ln y = \frac{2}{3}\ln(x^2+3) + 2\ln(3x+2) - \frac{1}{2}\ln(x+1) \\
 \frac{1}{y} \frac{dy}{dx} &= \frac{2}{3} \cdot \frac{2x}{x^2+3} + \frac{2 \cdot 3}{3x+2} - \frac{1}{2(x+1)} \\
 \frac{dy}{dx} &= \frac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}} \left[\frac{4x}{3(x^2+3)} + \frac{6}{3x+2} - \frac{1}{2(x+1)} \right] = \frac{(3x+2)(51x^3+70x^2+97x+90)}{6(x^2+3)^{1/3}(x+1)^{3/2}}
 \end{aligned}$$

31.



$y = \ln x$ is reflected across the y -axis.

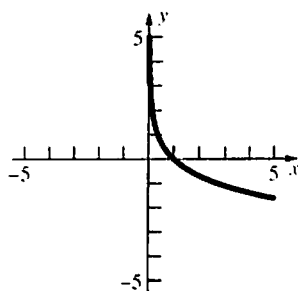
32.



The y -values of $y = \ln x$ are multiplied by $\frac{1}{2}$,

since $\ln \sqrt{x} = \frac{1}{2} \ln x$.

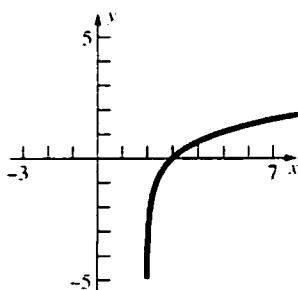
33.



$y = \ln x$ is reflected across the x -axis since

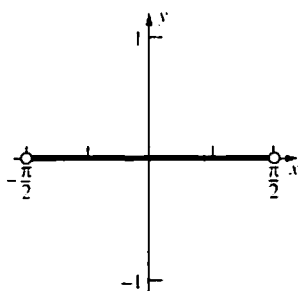
$$\ln\left(\frac{1}{x}\right) = -\ln x.$$

34.



$y = \ln x$ is shifted two units to the right.

35.



$$y = \ln \cos x + \ln \sec x$$

$$= \ln \cos x + \ln \frac{1}{\cos x}$$

$$= \ln \cos x - \ln \cos x = 0 \text{ on } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

36. Since $\ln$ is continuous,

$$\lim_{x \rightarrow 0} \ln \frac{\sin x}{x} = \ln \lim_{x \rightarrow 0} \frac{\sin x}{x} = \ln 1 = 0$$

$$37. f'(x) = 4x \ln x + 2x^2 \left(\frac{1}{x}\right) - 2x = 4x \ln x$$

so $f(1) = -1$ is a minimum.

38. Let $r(x)$ = rate of transmission

$$= kv^2 \ln \frac{1}{x} = -kv^2 \ln x.$$

$$r'(x) = -2kv^2 \ln x - kv^2 \left(\frac{1}{x}\right) = -kv^2(2 \ln x + 1)$$

$$r'(x) = 0 \text{ if } \ln x = -\frac{1}{2}, \text{ or } -\ln x = \frac{1}{2}, \text{ so}$$

$$\ln \frac{1}{x} = \frac{1}{2}.$$

$$\ln 1.65 \approx \frac{1}{2}, \text{ so } x \approx \frac{1}{1.65} \approx 0.606.$$

$$r''(x) = -k(2 \ln x + 1) - kv \left(2 \cdot \frac{1}{x}\right) = -k(2 \ln x + 3)$$

$$r''(0.606) \approx -2k < 0 \text{ since } k > 0, \text{ so}$$

$x \approx 0.606$ gives the maximum rate of transmission.

39. $\ln 4 > 1$

$$\text{so } \ln 4^m = m \ln 4 > m \cdot 1 = m$$

$$\text{Thus } x > 4^m \Rightarrow \ln x > m$$

$$\text{so } \lim_{x \rightarrow \infty} \ln x = \infty$$

40. Let $z = \frac{1}{x}$ so $z \rightarrow \infty$ as $x \rightarrow 0^+$

$$\begin{aligned} \text{Then } \lim_{x \rightarrow 0^+} \ln x &= \lim_{z \rightarrow \infty} \ln \left(\frac{1}{z}\right) = \lim_{z \rightarrow \infty} (-\ln z) \\ &= -\lim_{z \rightarrow \infty} \ln z = -\infty \end{aligned}$$

$$41. \int_{1/3}^x \frac{1}{t} dt = 2 \int_1^x \frac{1}{t} dt$$

$$\int_{1/3}^1 \frac{1}{t} dt + \int_1^x \frac{1}{t} dt = 2 \int_1^x \frac{1}{t} dt$$

$$\int_{1/3}^1 \frac{1}{t} dt = \int_1^x \frac{1}{t} dt$$

$$-\int_1^{1/3} \frac{1}{t} dt = \int_1^x \frac{1}{t} dt$$

$$-\ln \frac{1}{3} = \ln x$$

$$\ln 3 = \ln x$$

$$x = 3$$

42. a. $\frac{1}{t} < \frac{1}{\sqrt{t}}$ for $t > 1$.

$$\text{so } \ln x = \int_1^x \frac{1}{t} dt < \int_1^x \frac{1}{\sqrt{t}} dt = \int_1^x t^{-1/2} dt$$

$$= \left[2\sqrt{t} \right]_1^x = 2(\sqrt{x} - 1)$$

$$\text{so } \ln x < 2(\sqrt{x} - 1)$$

b. If $x > 1$, $0 < \ln x < 2(\sqrt{x} - 1)$,

$$\text{so } 0 < \frac{\ln x}{x} < \frac{2(\sqrt{x} - 1)}{x}.$$

$$\text{Hence } 0 \leq \lim_{x \rightarrow \infty} \frac{\ln x}{x} \leq \lim_{x \rightarrow \infty} \frac{2(\sqrt{x}+1)}{x} = 0$$

$$\text{and } \lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0.$$

$$43. \lim_{n \rightarrow \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}} \right] \cdot \frac{1}{n}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{1}{1+\frac{i}{n}} \right) \cdot \frac{1}{n} = \int_1^2 \frac{1}{x} dx = \ln 2 \approx 0.693$$

$$44. \frac{1,000,000}{\ln 1,000,000} \approx 72,382$$

$$45. \text{ a. } f(x) = \ln \left(\frac{ax-b}{ax+b} \right)^c = c \ln \left(\frac{ax-b}{ax+b} \right)$$

$$= \frac{a^2-b^2}{2ab} [\ln(ax-b) - \ln(ax+b)]$$

$$f'(x) = \frac{a^2-b^2}{2ab} \left[\frac{a}{ax-b} - \frac{a}{ax+b} \right]$$

$$= \frac{a^2-b^2}{2ab} \left[\frac{2ab}{(ax-b)(ax+b)} \right] = \frac{a^2-b^2}{a^2x^2-b^2}$$

$$f'(1) = \frac{a^2-b^2}{a^2-b^2} = 1$$

$$\text{b. } f'(x) = \cos^2 u \cdot \frac{du}{dx}$$

$$= \cos^2 [\ln(x^2+x-1)] \cdot \frac{2x+1}{x^2+x-1}$$

$$f'(1) = \cos^2 [\ln(1^2+1-1)] \cdot \frac{2 \cdot 1 + 1}{1^2+1-1}$$

$$= 3 \cos^2(0) = 3$$

$$46. \int_0^{\pi/3} \tan x dx = \int_0^{\pi/3} \frac{\sin x}{\cos x} dx$$

Let $u = \cos x$ so $du = -\sin x dx$.

$$\int \frac{\sin x}{\cos x} dx = - \int \frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$$

$$\int_0^{\pi/3} \tan x dx = [-\ln|\cos x|]_0^{\pi/3}$$

$$= -\ln \left| \cos \frac{\pi}{3} \right| + \ln|\cos 0| = -\ln \frac{1}{2} + \ln 1 = \ln 2 \approx 0.693$$

$$47. V = 2\pi \int_1^4 xf(x) dx = \int_1^4 \frac{2\pi x}{x^2+4} dx$$

Let $u = x^2 + 4$ so $du = 2x dx$.

$$\int \frac{2\pi x}{x^2+4} dx = \pi \int \frac{1}{u} du = \pi \ln|u| + C$$

$$= \pi \ln|x^2+4| + C$$

$$\int_1^4 \frac{2\pi x}{x^2+4} dx = \left[\pi \ln|x^2+4| \right]_1^4$$

$$= \pi \ln 20 - \pi \ln 5 = \pi \ln 4 \approx 4.355$$

$$48. y = \frac{x^2}{4} - \ln \sqrt{x} = \frac{x^2}{4} - \frac{1}{2} \ln x$$

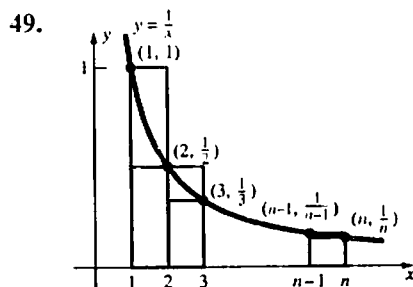
$$\frac{dy}{dx} = \frac{2x}{4} - \frac{1}{2} \cdot \frac{1}{x} = \frac{x}{2} - \frac{1}{2x}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx = \int_1^2 \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x} \right)^2} dx$$

$$= \int_1^2 \sqrt{\left(\frac{x}{2} + \frac{1}{2x} \right)^2} dx = \int_1^2 \left(\frac{x}{2} + \frac{1}{2x} \right) dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} + \ln|x| \right]_1^2 = \frac{1}{2} \left[2 + \ln 2 - \left(\frac{1}{2} + \ln 1 \right) \right]$$

$$= \frac{3}{4} + \frac{1}{2} \ln 2 \approx 1.097$$



$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \text{the lower approximate area}$$

$$1 + \frac{1}{2} + \cdots + \frac{1}{n-1} = \text{the upper approximate area}$$

$\ln n = \text{the exact area under the curve}$

Thus,

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}.$$

$$50. \frac{\ln y - \ln x}{y - x} = \frac{\int_x^y \frac{1}{t} dt - \int_x^x \frac{1}{t} dt}{y - x} = \frac{\int_x^y \frac{1}{t} dt}{y - x}$$

= the average value of $\frac{1}{t}$ on $[x, y]$.

Since $\frac{1}{t}$ is decreasing on the interval $[x, y]$, the average value is between the minimum value of $\frac{1}{y}$ and the maximum value of $\frac{1}{x}$.

51. a. $f'(x) = \frac{1}{1.5 + \sin x} \cdot \cos x = \frac{\cos x}{1.5 + \sin x}$
 $f'(x) = 0$ when $\cos x = 0$.

Critical points: $0, \frac{\pi}{2}, \frac{3\pi}{2}, 3\pi$

$f(0) \approx 0.405$,

$f\left(\frac{\pi}{2}\right) \approx 0.916, f\left(\frac{3\pi}{2}\right) \approx -0.693$,

$f\left(\frac{5\pi}{2}\right) \approx 0.916, f(3\pi) \approx 0.405$.

On $[0, 3\pi]$, the maximum value points are

$\left(\frac{\pi}{2}, 0.916\right), \left(\frac{5\pi}{2}, 0.916\right)$ and the minimum

value point is $\left(\frac{3\pi}{2}, -0.693\right)$.

b. $f''(x) = -\frac{1 + 1.5 \sin x}{(1.5 + \sin x)^2}$

On $[0, 3\pi]$, $f''(x) = 0$ when $x \approx 3.871, 5.553$.

Inflection points are $(3.871, -0.182), (5.553, -0.183)$.

c. $\int_0^{3\pi} \ln(1.5 + \sin x) dx \approx 4.042$

52. a. $f'(x) = -\frac{\sin(\ln x)}{x}$

On $[0.1, 20]$, $f'(x) = 0$ when $x = 1$.

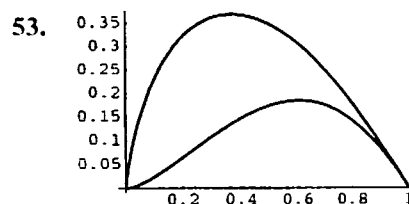
Critical points: $0.1, 1, 20$

$f(0.1) \approx -0.668, f(1) = 1, f(20) \approx -0.989$

On $[0.1, 20]$, the maximum value point is $(1, 1)$ and minimum value point is $(20, -0.989)$.

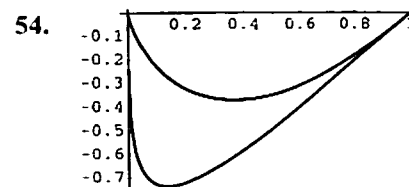
b. On $[0.01, 0.1]$, $f'(x) = 0$ when $x \approx 0.043$.
 $f(0.01) \approx -0.107, f(0.043) \approx -1$
On $[0.01, 20]$, the maximum value point is $(1, 1)$ and the minimum value point is $(0.043, -1)$.

c. $\int_{0.1}^{20} \cos(\ln x) dx = -8.37$



a. $\int_0^1 \left[x \ln\left(\frac{1}{x}\right) - x^2 \ln\left(\frac{1}{x}\right) \right] dx \approx 0.139$

b. Maximum of ≈ 0.260 at $x \approx 0.236$



a. $\int_0^1 [x \ln x - \sqrt{x} \ln x] dx \approx 0.194$

b. Maximum of ≈ 0.521 at $x \approx 0.0555$

7.2 Concepts Review

1. $f(x_1) \neq f(x_2)$

2. $x: f^{-1}(y)$

3. monotonic: increasing; decreasing

4. $(f^{-1})'(y) = \frac{1}{f'(x)}$

Problem Set 7.2

1. $f(x)$ is one-to-one, so it has an inverse.

Since $f(4) = 2$, $f^{-1}(2) = 4$.

2. $f(x)$ is one-to-one, so it has an inverse.

Since $f(1) = 2$, $f^{-1}(2) = 1$.

3. $f(x)$ is not one-to-one, so it does not have an inverse.

4. $f(x)$ is not one-to-one, so it does not have an inverse.

5. $f(x)$ is one-to-one, so it has an inverse.

Since $f(-1) = 2$, $f^{-1}(2) = -1$.

6. $f(x)$ is one-to-one, so it has an inverse. Since

$f\left(\frac{1}{2}\right) = 2$, $f^{-1}(2) = \frac{1}{2}$.

7. $f'(x) = -5x^4 - 3x^2 = -(5x^4 + 3x^2) < 0$ for all $x \neq 0$. $f(x)$ is strictly decreasing at $x = 0$ because $f(x) > 0$ for $x < 0$ and $f(x) < 0$ for $x > 0$. Therefore $f(x)$ is strictly decreasing for x and so it has an inverse.

8. $f'(x) = 7x^6 + 15x^2 > 0$ for all $x \neq 0$. $f(x)$ is strictly increasing at $x = 0$ because $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$. Therefore $f(x)$ is strictly increasing for all x and so it has an inverse.

9. $f'(\theta) = -\sin \theta < 0$ for $0 < \theta < \pi$. $f(\theta)$ is decreasing at $\theta = 0$ because $f(0) = 1$ and $f(\theta) < 1$ for $0 < \theta < \pi$. $f(\theta)$ is decreasing at $\theta = \pi$ because $f(\pi) = -1$ and $f(\theta) > -1$ for $0 < \theta < \pi$. Therefore $f(\theta)$ is strictly decreasing on $0 \leq \theta \leq \pi$ and so it has an inverse.

10. $f'(x) = -\csc^2 x < 0$ for $0 < x < \frac{\pi}{2}$. $f(x)$ is decreasing on $0 < x < \frac{\pi}{2}$ and so it has an inverse.

11. $f'(z) = 2(z-1) > 0$ for $z > 1$. $f(z)$ is increasing at $z = 1$ because $f(1) = 0$ and $f(z) > 0$ for $z > 1$. Therefore, $f(z)$ is strictly increasing on $z \geq 1$ and so it has an inverse.

12. $f'(x) = 2x + 1 > 0$ for $x \geq 2$. $f(x)$ is strictly increasing on $x \geq 2$ and so it has an inverse.

13. $f'(x) = \sqrt{x^4 + x^2} + 10 > 0$ for all real x . $f(x)$ is strictly increasing and so it has an inverse.

14. $f(r) = \int_r^1 \cos^4 t \, dt = -\int_1^r \cos^4 t \, dt$
 $f'(r) = -\cos^4 r < 0$ for all $r \neq k\pi + \frac{\pi}{2}$, k any integer.
 $f(r)$ is decreasing at $r = k\pi + \frac{\pi}{2}$ since $f'(r) < 0$ on the deleted neighborhood $\left(k\pi + \frac{\pi}{2} - \varepsilon, k\pi + \frac{\pi}{2} + \varepsilon\right)$. Therefore, $f(r)$ is strictly decreasing for all r and so it has an inverse.

15. Step 1:
 $y = x + 1$
 $x = y - 1$
 Step 2: $f^{-1}(y) = y - 1$
 Step 3: $f^{-1}(x) = x - 1$

Check:

$$f^{-1}(f(x)) = (x+1) - 1 = x$$

$$f(f^{-1}(x)) = (x-1) + 1 = x$$

16. Step 1:

$$y = -\frac{x}{3} + 1$$

$$-\frac{x}{3} = y - 1$$

$$x = -3(y - 1) = 3 - 3y$$

$$\text{Step 2: } f^{-1}(y) = 3 - 3y$$

$$\text{Step 3: } f^{-1}(x) = 3 - 3x$$

Check:

$$f^{-1}(f(x)) = 3 - 3\left(-\frac{x}{3} + 1\right)$$

$$= 3 + (x - 3) = x$$

$$f(f^{-1}(x)) = \frac{-(3 - 3x)}{3} + 1$$

$$= (-1 + x) + 1 = x$$

17. Step 1:

$$y = \sqrt{x+1} \quad (\text{note that } y \geq 0)$$

$$x + 1 = y^2$$

$$x = y^2 - 1, y \geq 0$$

$$\text{Step 2: } f^{-1}(y) = y^2 - 1, y \geq 0$$

$$\text{Step 3: } f^{-1}(x) = x^2 - 1, x \geq 0$$

Check:

$$f^{-1}(f(x)) = (\sqrt{x+1})^2 - 1 = (x+1) - 1 = x$$

$$f(f^{-1}(x)) = \sqrt{(x^2 - 1) + 1} = \sqrt{x^2} = |x| = x$$

18. Step 1:

$$y = -\sqrt{1-x} \quad (\text{note that } y \leq 0)$$

$$\sqrt{1-x} = -y$$

$$1 - x = (-y)^2 = y^2$$

$$x = 1 - y^2, y \leq 0$$

$$\text{Step 2: } f^{-1}(y) = 1 - y^2, y \leq 0$$

$$\text{Step 3: } f^{-1}(x) = 1 - x^2, x \leq 0$$

Check:

$$f^{-1}(f(x)) = 1 - (-\sqrt{1-x})^2 = 1 - (1-x) = x$$

$$f(f^{-1}(x)) = -\sqrt{1 - (1 - x^2)} = -\sqrt{x^2} = -|x|$$

$$= -(-x) = x$$

19. Step 1:

$$y = -\frac{1}{x-3}$$

$$x - 3 = -\frac{1}{y}$$

$$x = 3 - \frac{1}{y}$$

$$\text{Step 2: } f^{-1}(y) = 3 - \frac{1}{y}$$

$$\text{Step 3: } f^{-1}(x) = 3 - \frac{1}{x}$$

Check:

$$f^{-1}(f(x)) = 3 - \frac{1}{-\frac{1}{x-3}} = 3 + (x-3) = x$$

$$f(f^{-1}(x)) = -\frac{1}{\left(3 - \frac{1}{x}\right) - 3} = -\frac{1}{-\frac{1}{x}} = x$$

20. Step 1:

$$y = \sqrt{\frac{1}{x-2}} \quad (\text{note that } y > 0)$$

$$y^2 = \frac{1}{x-2}$$

$$x - 2 = \frac{1}{y^2}$$

$$x = 2 + \frac{1}{y^2}, y > 0$$

$$\text{Step 2: } f^{-1}(y) = 2 + \frac{1}{y^2}, y > 0$$

$$\text{Step 3: } f^{-1}(x) = 2 + \frac{1}{x^2}, x > 0$$

Check:

$$f^{-1}(f(x)) = 2 + \frac{1}{\left(\sqrt{\frac{1}{x-2}}\right)^2} = 2 + \frac{1}{\left(\frac{1}{x-2}\right)}$$

$$= 2 + (x-2) = x$$

$$f(f^{-1}(x)) = \sqrt{\frac{1}{\left(2 + \frac{1}{x^2}\right) - 2}} = \sqrt{\frac{1}{\left(\frac{1}{x^2}\right)}} = \sqrt{x^2}$$

$$= |x| = x$$

21. Step 1:

$$y = 4x^2, x \leq 0 \quad (\text{note that } y \geq 0)$$

$$x^2 = \frac{y}{4}$$

$$x = -\sqrt{\frac{y}{4}} = -\frac{\sqrt{y}}{2}, \text{ negative since } x \leq 0$$

$$\text{Step 2: } f^{-1}(y) = -\frac{\sqrt{y}}{2}$$

$$\text{Step 3: } f^{-1}(x) = -\frac{\sqrt{x}}{2}$$

Check:

$$f^{-1}(f(x)) = -\frac{\sqrt{4x^2}}{2} = -\sqrt{x^2} = -|x| = -(-x) = x$$

$$f(f^{-1}(x)) = 4\left(-\frac{\sqrt{x}}{2}\right)^2 = 4 \cdot \frac{x}{4} = x$$

22. Step 1:

$$y = (x-3)^2, x \geq 3 \quad (\text{note that } y \geq 0)$$

$$x - 3 = \sqrt{y}$$

$$x = 3 + \sqrt{y}$$

$$\text{Step 2: } f^{-1}(y) = 3 + \sqrt{y}$$

$$\text{Step 3: } f^{-1}(x) = 3 + \sqrt{x}$$

Check:

$$f^{-1}(f(x)) = 3 + \sqrt{(x-3)^2} = 3 + |x-3|$$

$$= 3 + (x-3) = x$$

$$f(f^{-1}(x)) = [(3 + \sqrt{x}) - 3]^2 = (\sqrt{x})^2 = x$$

23. Step 1:

$$y = (x-1)^3$$

$$x - 1 = \sqrt[3]{y}$$

$$x = 1 + \sqrt[3]{y}$$

$$\text{Step 2: } f^{-1}(y) = 1 + \sqrt[3]{y}$$

$$\text{Step 3: } f^{-1}(x) = 1 + \sqrt[3]{x}$$

$$\text{Check: } f^{-1}(f(x)) = 1 + \sqrt[3]{(x-1)^3} = 1 + (x-1) = x$$

$$f(f^{-1}(x)) = [(1 + \sqrt[3]{x}) - 1]^3 = (\sqrt[3]{x})^3 = x$$

24. Step 1:

$$y = x^{5/2}, x \geq 0$$

$$x = y^{2/5}$$

$$\text{Step 2: } f^{-1}(y) = y^{2/5}$$

$$\text{Step 3: } f^{-1}(x) = x^{2/5}$$

Check:

$$f^{-1}(f(x)) = (x^{5/2})^{2/5} = x$$

$$f(f^{-1}(x)) = (x^{2/5})^{5/2} = x$$

25. Step 1:

$$y = \frac{x-1}{x+1}$$

$$xy + y = x - 1$$

$$x - xy = 1 + y$$

$$x = \frac{1+y}{1-y}$$

$$\text{Step 2: } f^{-1}(y) = \frac{1+y}{1-y}$$

$$\text{Step 3: } f^{-1}(x) = \frac{1+x}{1-x}$$

Check:

$$f^{-1}(f(x)) = \frac{1 + \frac{x-1}{x+1}}{1 - \frac{x-1}{x+1}} = \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

$$f(f^{-1}(x)) = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{1+x-1+x}{1+x+1-x} = \frac{2x}{2} = x$$

26. Step 1:

$$y = \left(\frac{x-1}{x+1} \right)^3$$

$$y^{1/3} = \frac{x-1}{x+1}$$

$$xy^{1/3} + y^{1/3} = x-1$$

$$x - xy^{1/3} = 1 + y^{1/3}$$

$$x = \frac{1 + y^{1/3}}{1 - y^{1/3}}$$

$$\text{Step 2: } f^{-1}(y) = \frac{1 + y^{1/3}}{1 - y^{1/3}}$$

$$\text{Step 3: } f^{-1}(x) = \frac{1 + x^{1/3}}{1 - x^{1/3}}$$

Check:

$$f^{-1}(f(x)) = \frac{1 + \left[\left(\frac{x-1}{x+1} \right)^3 \right]^{1/3}}{1 - \left[\left(\frac{x-1}{x+1} \right)^3 \right]^{1/3}} = \frac{1 + \frac{x-1}{x+1}}{1 - \frac{x-1}{x+1}}$$

$$= \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

$$f(f^{-1}(x)) = \left(\frac{\frac{1+x^{1/3}}{1-x^{1/3}} - 1}{\frac{1+x^{1/3}}{1-x^{1/3}} + 1} \right)^3 = \left(\frac{1+x^{1/3}-1+x^{1/3}}{1+x^{1/3}+1-x^{1/3}} \right)^3$$

$$= \left(\frac{2x^{1/3}}{2} \right)^3 = (x^{1/3})^3 = x$$

27. Step 1:

$$y = \frac{x^3 + 2}{x^3 + 1}$$

$$x^3 y + y = x^3 + 2$$

$$x^3 y - x^3 = 2 - y$$

$$x^3 = \frac{2-y}{y-1}$$

$$x = \left(\frac{2-y}{y-1} \right)^{1/3}$$

$$\text{Step 2: } f^{-1}(y) = \left(\frac{2-y}{y-1} \right)^{1/3}$$

$$\text{Step 3: } f^{-1}(x) = \left(\frac{2-x}{x-1} \right)^{1/3}$$

Check:

$$f^{-1}(f(x)) = \left(\frac{2 - \frac{x^3+2}{x^3+1}}{\frac{x^3+2}{x^3+1} - 1} \right)^{1/3} = \left(\frac{2x^3 + 2 - x^3 - 2}{x^3 + 2 - x^3 - 1} \right)^{1/3}$$

$$= \left(\frac{x^3}{1} \right)^{1/3} = x$$

$$f(f^{-1}(x)) = \frac{\left[\left(\frac{2-x}{x-1} \right)^{1/3} \right]^3 + 2}{\left[\left(\frac{2-x}{x-1} \right)^{1/3} \right]^3 + 1} = \frac{\frac{2-x}{x-1} + 2}{\frac{2-x}{x-1} + 1}$$

$$= \frac{2-x+2x-2}{2-x+x-1} = \frac{x}{1} = x$$

28. Step 1:

$$y = \left(\frac{x^3 + 2}{x^3 + 1} \right)^5$$

$$y^{1/5} = \frac{x^3 + 2}{x^3 + 1}$$

$$x^3 y^{1/5} + y^{1/5} = x^3 + 2$$

$$x^3 y^{1/5} - x^3 = 2 - y^{1/5}$$

$$x^3 = \frac{2 - y^{1/5}}{y^{1/5} - 1}$$

$$x = \left(\frac{2 - y^{1/5}}{y^{1/5} - 1} \right)^{1/3}$$

$$\text{Step 2: } f^{-1}(y) = \left(\frac{2 - y^{1/5}}{y^{1/5} - 1} \right)^{1/3}$$

$$\text{Step 3: } f^{-1}(x) = \left(\frac{2 - x^{1/5}}{x^{1/5} - 1} \right)^{1/3}$$

Check:

$$f^{-1}(f(x)) = \left(\frac{2 - \left[\left(\frac{x^3+2}{x^3+1} \right)^5 \right]^{1/5}}{\left[\left(\frac{x^3+2}{x^3+1} \right)^5 \right]^{1/5} - 1} \right)^{1/3}$$

$$= \left(\frac{2 - \frac{x^3+2}{x^3+1}}{\frac{x^3+2}{x^3+1} - 1} \right)^{1/3} = \left(\frac{2x^3 + 2 - x^3 - 2}{x^3 + 2 - x^3 - 1} \right)^{1/3}$$

$$= \left(\frac{x^3}{1} \right)^{1/3} = x$$

$$f(f^{-1}(x)) = \frac{\left[\left(\frac{2-x^{1/5}}{x^{1/5}-1} \right)^{1/3} + 2 \right]^5}{\left[\left(\frac{2-x^{1/5}}{x^{1/5}-1} \right)^{1/3} + 1 \right]^5}$$

$$= \left(\frac{2-x^{1/5}}{x^{1/5}-1} + 2 \right)^5 = \left(\frac{2-x^{1/5} + 2x^{1/5} - 2}{2-x^{1/5} + x^{1/5} - 1} \right)^5$$

$$= \left(\frac{x^{1/5}}{1} \right)^5 = x$$

29. By similar triangles $\frac{r}{h} = \frac{4}{6}$. Thus, $r = \frac{2h}{3}$

This gives

$$V = \frac{\pi r^2 h}{3} = \frac{\pi (4h^2/9) h}{3} = \frac{4\pi h^3}{27}$$

$$h^3 = \frac{27V}{4\pi}$$

$$h = 3\sqrt[3]{\frac{V}{4\pi}}$$

30. $v = v_0 - 32t$

$v = 0$ when $v_0 = 32t$, that is, when

$t = \frac{v_0}{32}$. The position function is

$s(t) = v_0 t - 16t^2$. The ball then reaches a height of

$$H = s(v_0/32) = v_0 \frac{v_0}{32} - 16 \frac{v_0^2}{32^2} = \frac{v_0^2}{64}$$

$$v_0^2 = 64H$$

$$v_0 = 8\sqrt{H}$$

31. $f'(x) = 4x + 1$; $f'(x) > 0$ when $x > -\frac{1}{4}$ and

$$f'(x) < 0 \text{ when } x < -\frac{1}{4}.$$

The function is decreasing on $\left(-\infty, -\frac{1}{4}\right]$ and

increasing on $\left[-\frac{1}{4}, \infty\right)$. Restrict the domain to

$\left(-\infty, -\frac{1}{4}\right]$ or restrict it to $\left[-\frac{1}{4}, \infty\right)$.

Then $f^{-1}(x) = \frac{1}{4}(-1 - \sqrt{8x+33})$ or

$$f^{-1}(x) = \frac{1}{4}(-1 + \sqrt{8x+33}).$$

32. $f'(x) = 2x - 3$; $f'(x) > 0$ when $x > \frac{3}{2}$

and $f'(x) < 0$ when $x < \frac{3}{2}$.

The function is decreasing on $\left(-\infty, \frac{3}{2}\right]$ and

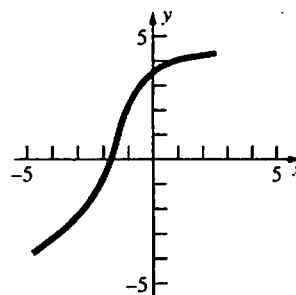
increasing on $\left[\frac{3}{2}, \infty\right)$. Restrict the domain to

$\left(-\infty, \frac{3}{2}\right]$ or restrict it to $\left[\frac{3}{2}, \infty\right)$. Then

$$f^{-1}(x) = \frac{1}{2}(3 - \sqrt{4x+5}) \text{ or}$$

$$f^{-1}(x) = \frac{1}{2}(3 + \sqrt{4x+5}).$$

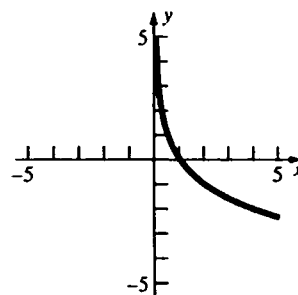
- 33.



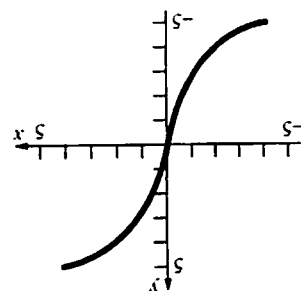
$$(f^{-1})'(3) \approx \frac{1}{10}$$

- 34.

$$(f^{-1})'(3) \approx -\frac{1}{2}$$



36. $\frac{1}{2} \approx (f^{-1})'(3)$



$$\frac{161}{1} = \frac{1+51}{1} = \frac{(1), f}{1} = (2), (1-f) \text{ os}$$

$$\frac{01}{1} = \frac{5+5}{1} = \frac{(1), f}{1} = (2), (1-f) \text{ os}$$

$$\text{so } (f^{-1})'(2) = \frac{f'\left(\frac{\pi}{4}\right)}{1} = \frac{2 \sec^2\left(\frac{\pi}{4}\right)}{1} = \frac{2}{1} \cos^2\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{4}.$$

$$\text{so } (f^{-1})'(2) = \frac{f'(3)}{1} = 2\sqrt{3+1} = 4.$$

$$(((x)(\downarrow_{-}f \circ \downarrow_{-}g))g)f = (((x)(\downarrow_{-}f \circ \downarrow_{-}g))h)$$

$$f \circ g = 4 \sin \theta$$

$$\frac{A}{1} = x \quad , \quad \frac{x}{1} = A$$

$$\frac{x}{1} = (x)_{1-f}$$

$$x = \frac{3}{y-2}$$

$$\frac{\xi}{x-2} = (x)_{1-\delta}$$

$$\frac{x}{1} = \left(\frac{x}{1} \right)_{1-\mathcal{G}} = ((x)_{1-f})_{1-\mathcal{G}} = (x)_{1-y}$$

$$x = \frac{\left(\frac{x}{1}\right)}{1} = \frac{2 + \left[2 - \left(\frac{x}{1}\right)\right]}{1} = \left(\frac{3}{2 - \left(\frac{x}{1}\right)}\right) y = ((x)_1 - y)y$$

$$0 < x_2 \cos \alpha + 1 = f(x),$$

$$I = \frac{\int_0^{\frac{\pi}{2}} \cos^2 z \sqrt{1 + \cos^2 z} \, dz}{\int_0^{\frac{\pi}{2}} \cos^2 z \, dz} = \frac{f(\frac{\pi}{2})}{f(0)} = (f^{-1})'(0) \quad \text{a.}$$

$$\frac{\mathcal{L}}{2} =$$

$$\frac{z^{\wedge}}{1} = \frac{(0)z^{\cos 2} + 1^{\wedge}}{1} = \frac{(0), f}{1} = (0), (1-f) \quad \cdot$$

$$\begin{aligned} 4p - q &= x(v - 12) \\ q + xv &= 4p + 12x \end{aligned}$$

$$x = \frac{b - dy}{cy - a} = -\frac{dy - b}{cy - a}$$

$$f^{-1}(y) = -\frac{dy - b}{cy - a}$$

$$f^{-1}(x) = -\frac{dx - b}{cx - a}$$

b. If $bc - ad = 0$, then $f(x)$ is either a constant function or undefined.

c. If $f = f^{-1}$, then for all x in the domain we have:

$$\frac{ax + b}{cx + d} + \frac{dx - b}{cx - a} = 0$$

$$(ax + b)(cx - a) + (dx - b)(cx + d) = 0$$

$$acx^2 + (bc - a^2)x - ab + dcx^2$$

$$+ (d^2 - bc)x - bd = 0$$

$$(ac + dc)x^2 + (d^2 - a^2)x + (-ab - bd) = 0$$

Setting the coefficients equal to 0 gives three requirements:

$$(1) a = -d \text{ or } c = 0$$

$$(2) a = \pm d$$

$$(3) a = -d \text{ or } b = 0$$

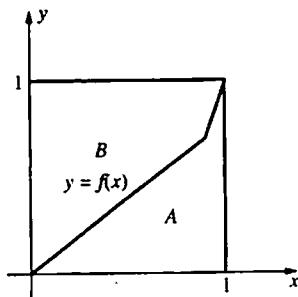
If $a = d$, then $f = f^{-1}$ requires $b = 0$ and

$c = 0$, so $f(x) = \frac{ax}{d} = x$. If $a = -d$, there are

no requirements on b and c (other than

$bc - ad \neq 0$). Therefore, $f = f^{-1}$ if $a = -d$ or if f is the identity function.

45.



$$\int_0^1 f^{-1}(y) dy = (\text{Area of region B})$$

$$= 1 - (\text{Area of region A})$$

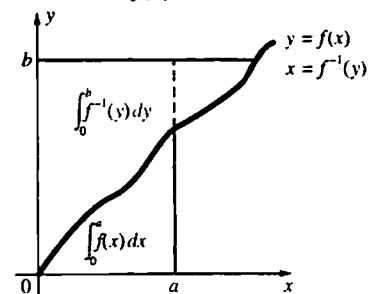
$$= 1 - \int_0^1 f(x) dx = 1 - \frac{2}{5} = \frac{3}{5}$$

46. $\int_0^a f(x) dx$ = the area bounded by $y = f(x)$, $y = 0$, and $x = a$ [the area under the curve].

$\int_0^b f^{-1}(y) dy$ = the area bounded by $x = f^{-1}(y)$, $x = 0$, and $y = b$.

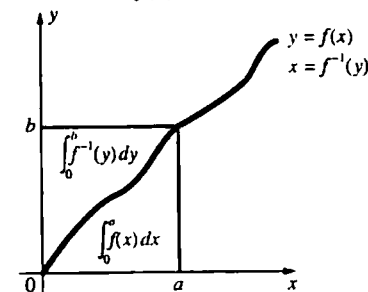
ab = the area of the rectangle bounded by $x = 0$, $x = a$, $y = 0$, and $y = b$.

Case 1: $b > f(a)$



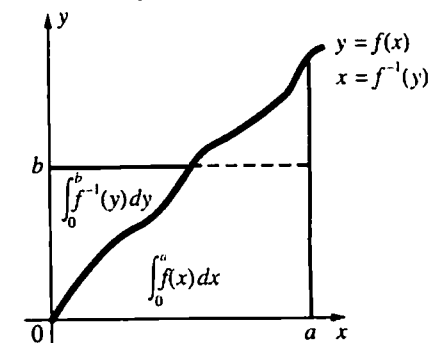
The area above the curve is greater than the area of the part of the rectangle above the curve, so the total area represented by the sum of the two integrals is greater than the area ab of the rectangle.

Case 2: $b = f(a)$



The area represented by the sum of the two integrals = the area ab of the rectangle.

Case 3: $b < f(a)$



The area below the curve is greater than the area of the part of the rectangle which is below the curve, so the total area represented by the sum of the two integrals is greater than the area ab of the rectangle.

$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy$ with equality holding when $b = f(a)$.

47. Given $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f(x) = x^{p-1}$,

solving $\frac{1}{p} + \frac{1}{q} = 1$ for p gives $p = \frac{q}{q-1}$, so

$$\frac{1}{p-1} = \frac{1}{\frac{q}{q-1} - 1} = \frac{1}{\left[\frac{q-(q-1)}{q-1}\right]} = \frac{q-1}{1} = q-1.$$

Thus, if $y = x^{p-1}$ then $x = y^{\frac{1}{p-1}} = y^{q-1}$, so $f^{-1}(y) = y^{q-1}$.

By Problem 44, since $f(x) = x^{p-1}$ is strictly

increasing for $p > 1$, $ab \leq \int_0^a x^{p-1} dx + \int_0^b y^{q-1} dy$

$$ab \leq \left[\frac{x^p}{p} \right]_0^a + \left[\frac{y^q}{q} \right]_0^b$$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

7.3 Concepts Review

1. increasing; exp
2. $\ln e = 1$; 2.72
3. x ; x
4. e^x ; $e^x + C$

Problem Set 7.3

1. a. 20.086
b. 8.1662
c. $e^{\sqrt{2}} \approx e^{1.41} = e^{1.4} \cdot e^{0.01} \approx 4.1$
d. $e^{\cos(\ln 4)} \approx e^{0.18} \approx 1.20$
2. a. $e^{3 \ln 2} = e^{\ln(2^3)} = e^{\ln 8} = 8$
b. $e^{\frac{\ln 64}{2}} = e^{\ln(64^{1/2})} = e^{\ln 8} = 8$
3. $e^{3 \ln x} = e^{\ln x^3} = x^3$
4. $e^{-2 \ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$
5. $\ln e^{\cos x} = \cos x$
6. $\ln e^{-2x-3} = -2x-3$
7. $\ln(x^3 e^{-3x}) = \ln x^3 + \ln e^{-3x} = 3 \ln x - 3x$
8. $e^{x-\ln x} = \frac{e^x}{e^{\ln x}} = \frac{e^x}{x}$
9. $e^{\ln 3 + 2 \ln x} = e^{\ln 3} \cdot e^{2 \ln x} = 3 \cdot e^{\ln x^2} = 3x^2$

$$10. e^{\ln x^2 - y \ln x} = \frac{e^{\ln x^2}}{e^{y \ln x}} = \frac{x^2}{e^{\ln x^y}} = \frac{x^2}{x^y} = x^{2-y}$$

$$11. D_x e^{x+2} = e^{x+2} D_x (x+2) = e^{x+2}$$

$$12. D_x e^{2x^2-x} = e^{2x^2-x} D_x (2x^2-x) \\ = e^{2x^2-x} (4x-1)$$

$$13. D_x e^{\sqrt{x+2}} = e^{\sqrt{x+2}} D_x \sqrt{x+2} = \frac{e^{\sqrt{x+2}}}{2\sqrt{x+2}}$$

$$14. D_x e^{-\frac{1}{x^2}} = e^{-\frac{1}{x^2}} D_x \left(-\frac{1}{x^2} \right) \\ = e^{-\frac{1}{x^2}} \cdot 2x^{-3} = \frac{2e^{-\frac{1}{x^2}}}{x^3}$$

$$15. D_x e^{2 \ln x} = D_x e^{\ln x^2} = D_x x^2 = 2x$$

$$16. D_x e^{\frac{x}{\ln x}} = e^{\frac{x}{\ln x}} D_x \frac{x}{\ln x} = e^{\frac{x}{\ln x}} \cdot \frac{(\ln x) \cdot 1 - x \cdot \frac{1}{x}}{(\ln x)^2} \\ = \frac{e^{\frac{x}{\ln x}} (\ln x - 1)}{(\ln x)^2}$$

$$17. D_x (x^3 e^x) = x^3 D_x e^x + e^x D_x (x^3) \\ = x^3 e^x + e^x \cdot 3x^2 = x^2 e^x (x+3)$$

$$18. D_x e^{x^3 \ln x} = e^{x^3 \ln x} D_x (x^3 \ln x) \\ = e^{x^3 \ln x} \left(x^3 \cdot \frac{1}{x} + \ln x \cdot 3x^2 \right) \\ = e^{x^3 \ln x} (x^2 + 3x^2 \ln x) \\ = x^2 e^{x^3 \ln x} (1 + 3 \ln x)$$

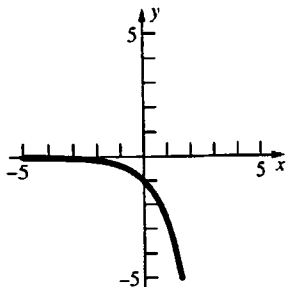
$$\begin{aligned}
 19. \quad D_x[\sqrt{e^{x^2}} + e^{\sqrt{x^2}}] &= D_x(e^{x^2})^{1/2} + D_x e^{\sqrt{x^2}} \\
 &= \frac{1}{2}(e^{x^2})^{-1/2} D_x e^{x^2} + e^{\sqrt{x^2}} D_x \sqrt{x^2} \\
 &= \frac{1}{2}(e^{x^2})^{-1/2} e^{x^2} D_x x^2 + e^{\sqrt{x^2}} \cdot \frac{x}{\sqrt{x^2}} \\
 &= \frac{1}{2}(e^{x^2})^{1/2} 2x + e^{\sqrt{x^2}} \cdot \frac{x}{|x|} \\
 &= x\sqrt{e^{x^2}} + \frac{xe^{\sqrt{x^2}}}{|x|}
 \end{aligned}$$

$$\begin{aligned}
 20. \quad D_x \left[e^{1/x^2} + \frac{1}{e^{x^2}} \right] &= D_x e^{x^{-2}} + D_x e^{-x^2} \\
 &= e^{x^{-2}} D_x x^{-2} + e^{-x^2} D_x [-x^2] \\
 &= e^{x^{-2}} \cdot (-2x^{-3}) + e^{-x^2} \cdot (-2x) \\
 &= -\frac{2e^{1/x^2}}{x^3} - \frac{2x}{e^{x^2}}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad D_x[e^{xy} + xy] &= D_x[2] \\
 e^{xy}(xD_x y + y) + (xD_x y + y) &= 0 \\
 xe^{xy} D_x y + ye^{xy} + xD_x y + y &= 0 \\
 xe^{xy} D_x y + xD_x y &= -ye^{xy} - y \\
 D_x y &= \frac{-ye^{xy} - y}{xe^{xy} + x} = -\frac{y(e^{xy} + 1)}{x(e^{xy} + 1)} = -\frac{y}{x}
 \end{aligned}$$

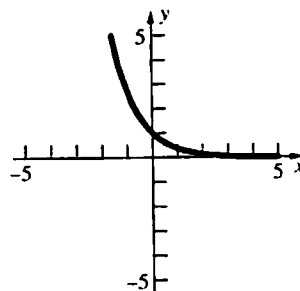
$$\begin{aligned}
 22. \quad D_x[e^{x+y}] &= D_x[x+y] \\
 e^{x+y}(1 + D_x y) &= 1 + D_x y \\
 e^{x+y} + e^{x+y} D_x y &= 1 + D_x y \\
 e^{x+y} D_x y - D_x y &= 1 - e^{x+y} \\
 D_x y &= \frac{1 - e^{x+y}}{e^{x+y} - 1} = -1
 \end{aligned}$$

23. a.



The graph of $y = e^x$ is reflected across the x -axis.

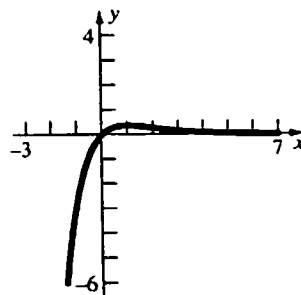
b.



The graph of $y = e^x$ is reflected across the y -axis.

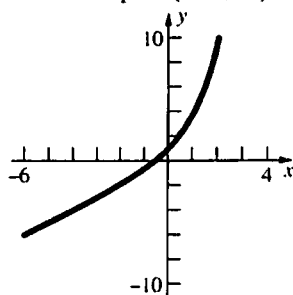
24. $a < b \Rightarrow -a > -b \Rightarrow e^{-a} > e^{-b}$, since e^x is an increasing function.

25. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x)$;
 $f'(x) = 0$ when $x = 1$
 $f'(x) > 0$ on $(-\infty, 1)$
 $f''(x) = xe^{-x} - 2e^{-x} = e^{-x}(x-2)$;
 $f''(x) = 0$ when $x = 2$
 $f''(x) > 0$ on $(2, \infty)$
 Increasing on $(-\infty, 1]$
 Decreasing on $[1, \infty)$
 Maximum at $\left(1, \frac{1}{e}\right) \approx (1, 0.4)$
 Concave up on $(2, \infty)$
 Concave down on $(-\infty, 2)$
 Inflection point at $\left(2, \frac{2}{e^2}\right) \approx (2, 0.3)$



26. $f'(x) = e^x + 1$
 $f'(x) > 0$ for all real x .
 $f''(x) = e^x$
 $f''(x) > 0$ for all real x .
 Increasing on $(-\infty, \infty)$

Concave up on $(-\infty, \infty)$



27. $f'(x) = -2(x-2)e^{-(x-2)^2}$;

$f'(x) = 0$ when $x = 2$

$f'(x) > 0$ on $(-\infty, 2)$

$f''(x) = 4(x-2)^2 e^{-(x-2)^2} - 2e^{-(x-2)^2}$
 $= 2e^{-(x-2)^2} [2(x-2)^2 - 1]$;

$f''(x) = 0$ when $x = 2 \pm \frac{1}{\sqrt{2}}$

$f''(x) > 0$ on $(-\infty, 2 - \frac{1}{\sqrt{2}})$ and

$(2 + \frac{1}{\sqrt{2}}, \infty)$

Increasing on $(-\infty, 2]$

Decreasing on $[2, \infty)$

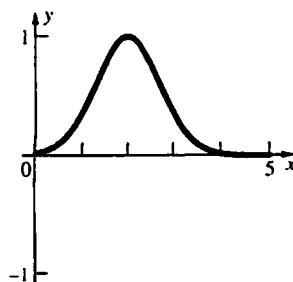
Maximum at $(2, 1)$

Concave up on $(-\infty, 2 - \frac{1}{\sqrt{2}}) \cup (2 + \frac{1}{\sqrt{2}}, \infty)$

Concave down on $(2 - \frac{1}{\sqrt{2}}, 2 + \frac{1}{\sqrt{2}})$

Inflection points at $(2 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}) \approx (1.3, 0.6)$

and $(2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}) \approx (2.7, 0.6)$



28. $f'(x) = e^x + e^{-x}$

$f'(x) > 0$ for all x

$f''(x) = e^x - e^{-x} = e^{-x}(e^{2x} - 1)$;

$f''(x) = 0$ when $e^{2x} = 1$ or $x = 0$.

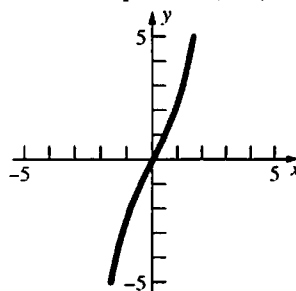
$f''(x) > 0$ when $x > 0$

Increasing for all x

Concave up on $(0, \infty)$

Concave down on $(-\infty, 0)$

Inflection point at $(0, 0)$



29. Let $u = 3x + 1$, so $du = 3dx$.

$\int e^{3x+1} dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C$
 $= \frac{1}{3} e^{3x+1} + C$

30. Let $u = x^2 - 3$, so $du = 2x dx$.

$\int x e^{x^2-3} dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C$
 $= \frac{1}{2} e^{x^2-3} + C$

31. Let $u = x^2 + 6x$, so $du = (2x + 6)dx$.

$\int (x+3)e^{x^2+6x} dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C$
 $= \frac{1}{2} e^{x^2+6x} + C$

32. Let $u = e^x - 1$, so $du = e^x dx$.

$\int \frac{e^x}{e^x - 1} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|e^x - 1| + C$

33. Let $u = -\frac{1}{x}$, so $du = \frac{1}{x^2} dx$.

$\int \frac{e^{-1/x}}{x^2} dx = \int e^u du = e^u + C = e^{-1/x} + C$

34. $\int e^{x+e^x} dx = \int e^x \cdot e^{e^x} dx$

Let $u = e^x$, so $du = e^x dx$.

$\int e^x \cdot e^{e^x} dx = \int e^u du = e^u + C = e^{e^x} + C$

35. Let $u = 2x + 3$, so $du = 2dx$

$\int e^{2x+3} dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{2x+3} + C$

$$\int_0^1 e^{2x+3} dx = \left[\frac{1}{2} e^{2x+3} \right]_0^1 = \frac{1}{2} e^5 - \frac{1}{2} e^3$$

$$= \frac{1}{2} e^3 (e^2 - 1) \approx 64.2$$

36. Let $u = \frac{3}{x}$, so $du = -\frac{3}{x^2} dx$.

$$\int \frac{e^{3/x}}{x^2} dx = -\frac{1}{3} \int e^u du = -\frac{1}{3} e^u + C$$

$$= -\frac{1}{3} e^{3/x} + C$$

$$\int_1^2 \frac{e^{3/x}}{x^2} dx = \left[-\frac{1}{3} e^{3/x} \right]_1^2 = -\frac{1}{3} e^{3/2} + \frac{1}{3} e^3 \approx 5.2$$

37. $V = \pi \int_0^{\ln 3} (e^x)^2 dx = \pi \int_0^{\ln 3} e^{2x} dx$

$$= \pi \left[\frac{1}{2} e^{2x} \right]_0^{\ln 3} = \pi \left(\frac{1}{2} e^{2 \ln 3} - \frac{1}{2} e^0 \right) \approx 12.57$$

38. $V = \int_0^1 2\pi x e^{-x^2} dx$.

Let $u = -x^2$, so $du = -2x dx$.

$$\int 2\pi x e^{-x^2} dx = -\pi \int e^{-x^2} (-2x) dx = -\pi \int e^u du$$

$$= -\pi e^u + C = -\pi e^{-x^2} + C$$

$$\int_0^1 2\pi x e^{-x^2} dx = -\pi \left[e^{-x^2} \right]_0^1 = -\pi(e^{-1} - e^0)$$

$$= \pi(1 - e^{-1}) \approx 1.99$$

39. The line through $(0, 1)$ and $\left(1, \frac{1}{e}\right)$ has slope

$$\frac{\frac{1}{e} - 1}{1 - 0} = \frac{1}{e} - 1 = \frac{1 - e}{e} \Rightarrow y - 1 = \frac{1 - e}{e} (x - 0);$$

$$y = \frac{1 - e}{e} x + 1$$

$$\int_0^1 \left[\left(\frac{1 - e}{e} x + 1 \right) - e^{-x} \right] dx = \left[\frac{1 - e}{2e} x^2 + x + e^{-x} \right]_0^1$$

$$= \frac{1 - e}{2e} + 1 + \frac{1}{e} - 1 = \frac{3 - e}{2e} \approx 0.052$$

40. $f'(x) = \frac{(e^x - 1)(1) - x(e^x)}{(e^x - 1)^2} - \frac{1}{1 - e^{-x}} (-e^{-x})(-1)$

$$= \frac{e^x - 1 - xe^x}{(e^x - 1)^2} - \frac{1}{1 - e^{-x}} \left(\frac{1}{e^x} \right)$$

$$= \frac{e^x - 1 - xe^x}{(e^x - 1)^2} - \frac{1}{e^x - 1} = \frac{e^x - 1 - xe^x - (e^x - 1)}{(e^x - 1)^2}$$

$$= -\frac{xe^x}{(e^x - 1)^2}$$

When $x > 0$, $f'(x) < 0$, so $f(x)$ is decreasing for $x > 0$.

41. a. Exact:

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 3,628,800$$

Approximate:

$$10! \approx \sqrt{20\pi} \left(\frac{10}{e} \right)^{10} \approx 3,598,696$$

b. $60! \approx \sqrt{120\pi} \left(\frac{60}{e} \right)^{60} \approx 8.31 \times 10^{81}$

42. $e^{0.3} \approx \left\{ \left[\left(\frac{0.3}{4} + 1 \right) \frac{0.3}{3} + 1 \right] \frac{0.3}{2} + 1 \right\} (0.3) + 1$

$$= 1.3498375$$

$$e^{0.3} \approx 1.3498588 \text{ by direct calculation}$$

43. $x = e^t \sin t$, so $dx = (e^t \sin t + e^t \cos t) dt$

$y = e^t \cos t$, so $dy = (e^t \cos t - e^t \sin t) dt$

$$ds = \sqrt{dx^2 + dy^2}$$

$$= e^t \sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2} dt$$

$$= e^t \sqrt{2 \sin^2 t + 2 \cos^2 t} dt = \sqrt{2} e^t dt$$

The length of the curve is

$$\int_0^\pi \sqrt{2} e^t dt = \sqrt{2} \left[e^t \right]_0^\pi = \sqrt{2} (e^\pi - 1) \approx 31.312$$

44. Use $x = 30$, $n = 8$, and $k = 0.25$.

$$P_n(x) = \frac{(kx)^n e^{-kx}}{n!} = \frac{(0.25 \cdot 30)^8 e^{-0.25 \cdot 30}}{8!} \approx 0.14$$

45. a. $\lim_{x \rightarrow 0^+} \frac{\ln x}{1 + (\ln x)^2}$ is of the form $\frac{\infty}{\infty}$.

$$= \lim_{x \rightarrow 0^+} \frac{D_x \ln x}{D_x [1 + (\ln x)^2]} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{2 \ln x \cdot \frac{1}{x}}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{2 \ln x} = 0$$

$$\lim_{x \rightarrow \infty} \frac{\ln x}{1 + (\ln x)^2} = \lim_{x \rightarrow \infty} \frac{1}{2 \ln x} = 0$$

b. $f'(x) = \frac{[1 + (\ln x)^2] \cdot \frac{1}{x} - \ln x \cdot 2 \ln x \cdot \frac{1}{x}}{[1 + (\ln x)^2]^2}$

$$= \frac{1 - (\ln x)^2}{x[1 + (\ln x)^2]^2}$$

$$f'(x) = 0 \text{ when } \ln x = \pm 1 \text{ so } x = e^1 = e$$

$$\text{or } x = e^{-1} = \frac{1}{e}$$

$$f(e) = \frac{\ln e}{1 + (\ln e)^2} = \frac{1}{1 + 1^2} = \frac{1}{2}$$

$$f\left(\frac{1}{e}\right) = \frac{\ln \frac{1}{e}}{1 + \left(\ln \frac{1}{e}\right)^2} = \frac{-1}{1 + (-1)^2} = -\frac{1}{2}$$

Maximum value of $\frac{1}{2}$ at $x = e$; minimum

value of $-\frac{1}{2}$ at $x = e^{-1}$.

$$c. \quad F(x) = \int_1^{x^2} \frac{\ln t}{1 + (\ln t)^2} dt$$

$$F'(x) = \frac{\ln x^2}{1 + (\ln x^2)^2} \cdot 2x$$

$$F'(\sqrt{e}) = \frac{\ln(\sqrt{e})^2}{1 + [\ln(\sqrt{e})^2]^2} \cdot 2\sqrt{e} = \frac{1}{1 + 1^2} \cdot 2\sqrt{e}$$

$$= \sqrt{e} \approx 1.65$$

46. Let (x_0, e^{x_0}) be the point of tangency. Then

$$\frac{e^{x_0} - 0}{x_0 - 0} = f'(x_0) = e^{x_0} \Rightarrow e^{x_0} = x_0 e^{x_0} \Rightarrow x_0 = 1$$

so the line is $y = e^{x_0}x$ or $y = ex$.

$$a. \quad A = \int_0^1 (e^x - ex) dx = \left[e^x - \frac{ex^2}{2} \right]_0^1$$

$$= e - \frac{e}{2} - (e^0 - 0) = \frac{e}{2} - 1 \approx 0.36$$

$$b. \quad V = \pi \int_0^1 [(e^x)^2 - (ex)^2] dx$$

$$= \pi \int_0^1 (e^{2x} - e^2 x^2) dx = \pi \left[\frac{1}{2} e^{2x} - \frac{e^2 x^3}{3} \right]_0^1$$

$$= \pi \left[\frac{1}{2} e^2 - \frac{e^2}{3} - \left(\frac{1}{2} e^0 - \frac{e^2}{3} \right) \right] = \frac{\pi}{6} (e^2 - 3) \approx 2.30$$

$$47. \quad \lim_{n \rightarrow \infty} (e^{1/n} + e^{2/n} + \dots + e^{n/n}) \left(\frac{1}{n} \right) = \int_0^1 e^x dx$$

$$= [e^x]_0^1 = e - 1 \approx 1.718$$

48. a. The reflection of the point $(x, f(x))$ through the line $x = \mu$ is $(2\mu - x, f(x))$.

Thus we want to show that $f(2\mu - x) = f(x)$

$$f(2\mu - x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{2\mu - x - \mu}{\sigma} \right)^2 \right] = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{\mu - x}{\sigma} \right)^2 \right] = f(x) \text{ since } (\mu - x)^2 = (x - \mu)^2$$

$$b. \quad f'(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right] \left[-\frac{1}{2} \cdot 2 \left(\frac{x - \mu}{\sigma} \right) \cdot \frac{1}{\sigma} \right] = -\frac{x - \mu}{\sigma^3 \sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right]$$

$$f'(x) = 0 \text{ when } x = \mu$$

$$f''(x) = -\frac{x - \mu}{\sigma^3 \sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right] \left[-\frac{1}{2} \cdot 2 \left(\frac{x - \mu}{\sigma} \right) \cdot \frac{1}{\sigma} \right] + \exp \left[-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right] \left(-\frac{1}{\sigma^3 \sqrt{2\pi}} \right)$$

$$= \frac{1}{\sigma^5 \sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right] [(x - \mu)^2 - \sigma^2]$$

$$f''(\mu) = -\frac{1}{\sigma^3 \sqrt{2\pi}} < 0 \text{ so } f(x) \text{ has a maximum at } x = \mu.$$

$$f''(x) = 0 \text{ when } (x - \mu)^2 = \sigma^2 \text{ so } f(x) \text{ has inflection points at } x = \mu \pm \sigma.$$

$$49. a. \quad \int_{-3}^3 \exp \left(-\frac{1}{x^2} \right) dx = 2 \int_0^3 \exp \left(-\frac{1}{x^2} \right) dx \approx 3.11$$

$$b. \quad \int_0^{8\pi} e^{-0.1x} \sin x dx \approx 0.910$$

50. a. $\lim_{x \rightarrow 0} (1+x)^{1/x} = e \approx 2.72$

b. $\lim_{x \rightarrow 0} (1+x)^{-1/x} = \frac{1}{e} \approx 0.368$

51. $f(x) = e^{-x^2}$

$$f'(x) = -2xe^{-x^2}$$

$$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 2e^{-x^2}(2x^2 - 1)$$

$y = f(x)$ and $y = f''(x)$ intersect when

$$e^{-x^2} = 2e^{-x^2}(2x^2 - 1); 1 = 4x^2 - 2;$$

$$4x^2 - 3 = 0, x = \pm \frac{\sqrt{3}}{2}$$

Both graphs are symmetric with respect to the y-axis so the area is

$$2 \left\{ \int_0^{\frac{\sqrt{3}}{2}} [e^{-x^2} - 2e^{-x^2}(2x^2 - 1)] dx + \int_{\frac{\sqrt{3}}{2}}^{\infty} [2e^{-x^2}(2x^2 - 1) - e^{-x^2}] dx \right\}$$

$$\approx 4.2614$$

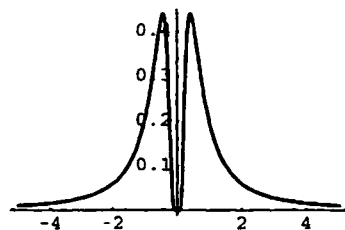
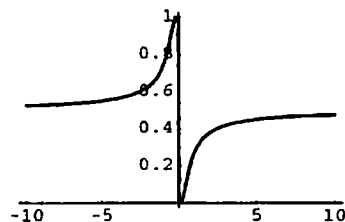
52. a. $\lim_{x \rightarrow \infty} x^p e^{-x} = 0$

b. $f'(x) = x^p e^{-x}(-1) + e^{-x} \cdot px^{p-1}$
 $= x^{p-1} e^{-x}(p - x)$
 $f'(x) = 0$ when $x = p$

53. $\lim_{x \rightarrow -\infty} \ln(x^2 + e^{-x}) = \infty$ (behaves like $-x$)

$\lim_{x \rightarrow \infty} \ln(x^2 + e^{-x}) = \infty$ (behaves like $2 \ln x$)

54. $f'(x) = -(1 + e^{x^{-1}})^{-2} \cdot e^{x^{-1}}(-x^{-2})$
 $= \frac{e^{1/x}}{x^2(1 + e^{1/x})^2}$



a. $\lim_{x \rightarrow 0^+} f(x) = 0$

b. $\lim_{x \rightarrow 0^-} f(x) = 1$

c. $\lim_{x \rightarrow \pm\infty} f(x) = \frac{1}{2}$

d. $\lim_{x \rightarrow 0} f'(x) = 0$

e. f has no minimum or maximum values.

7.4 Concepts Review

1. $e^{\sqrt{3} \ln \pi}; e^{x \ln a}$

2. e

3. $\frac{\ln x}{\ln a}$

4. $ax^{a-1}; a^x \ln a$

Problem Set 7.4

1. $2^x = 8 = 2^3; x = 3$

2. $x = 5^2 = 25$

3. $x = 4^{3/2} = 8$

4. $x^4 = 64$

$$x = \sqrt[4]{64} = 2\sqrt{2}$$

$$5. \log_9 \left(\frac{x}{3} \right) = \frac{1}{2}$$

$$\frac{x}{3} = 9^{1/2} = 3$$

$$x = 9$$

$$6. 4^3 = \frac{1}{2x}$$

$$x = \frac{1}{2 \cdot 4^3} = \frac{1}{128}$$

$$7. \log_2(x+3) - \log_2 x = 2$$

$$\log_2 \frac{x+3}{x} = 2$$

$$\frac{x+3}{x} = 2^2 = 4$$

$$x+3 = 4x$$

$$x = 1$$

$$8. \log_5(x+3) - \log_5 x = 1$$

$$\log_5 \frac{x+3}{x} = 1$$

$$\frac{x+3}{x} = 5^1 = 5$$

$$x+3 = 5x$$

$$x = \frac{3}{4}$$

$$9. \log_5 12 = \frac{\ln 12}{\ln 5} \approx 1.544$$

$$10. \log_7 0.11 = \frac{\ln 0.11}{\ln 7} \approx -1.1343$$

$$11. \log_{11}(8.12)^{1/5} = \frac{1}{5} \frac{\ln 8.12}{\ln 11} \approx 0.1747$$

$$12. \log_{10}(8.57)^7 = 7 \frac{\ln 8.57}{\ln 10} \approx 6.5309$$

$$13. x \ln 2 = \ln 17$$

$$x = \frac{\ln 17}{\ln 2} \approx 4.08746$$

$$14. x \ln 5 = \ln 13$$

$$x = \frac{\ln 13}{\ln 5} \approx 1.5937$$

$$15. (2s-3) \ln 5 = \ln 4$$

$$2s-3 = \frac{\ln 4}{\ln 5}$$

$$s = \frac{1}{2} \left(3 + \frac{\ln 4}{\ln 5} \right) \approx 1.9307$$

$$16. \frac{1}{\theta-1} \ln 12 = \ln 4$$

$$\frac{\ln 12}{\ln 4} = \theta - 1$$

$$\theta = 1 + \frac{\ln 12}{\ln 4} \approx 2.7925$$

$$17. D_x(6^{2x}) = 6^{2x} \ln 6 \cdot D_x(2x) = 2 \cdot 6^{2x} \ln 6$$

$$18. D_x(3^{2x^2-3x}) = 3^{2x^2-3x} \ln 3 \cdot D_x(2x^2-3x) \\ = (4x-3) \cdot 3^{2x^2-3x} \ln 3$$

$$19. D_x \log_3 e^x = \frac{1}{e^x \ln 3} \cdot D_x e^x$$

$$= \frac{e^x}{e^x \ln 3} = \frac{1}{\ln 3} \approx 0.9102$$

Alternate method:

$$D_x \log_3 e^x = D_x(x \log_3 e) = \log_3 e$$

$$= \frac{\ln e}{\ln 3} = \frac{1}{\ln 3} \approx 0.9102$$

$$20. D_x \log_{10}(x^3+9) = \frac{1}{(x^3+9) \ln 10} \cdot D_x(x^3+9)$$

$$= \frac{3x^2}{(x^3+9) \ln 10}$$

$$21. D_z[3^z \ln(z+5)]$$

$$= 3^z \cdot \frac{1}{z+5} (1) + \ln(z+5) \cdot 3^z \ln 3$$

$$= 3^z \left[\frac{1}{z+5} + \ln(z+5) \ln 3 \right]$$

$$22. D_\theta \sqrt{\log_{10}(3^{\theta^2-\theta})} = D_\theta \sqrt{(\theta^2-\theta) \log_{10} 3}$$

$$= D_\theta \sqrt{\frac{(\theta^2-\theta) \ln 3}{\ln 10}} = \sqrt{\frac{\ln 3}{\ln 10}} \cdot D_\theta \sqrt{\theta^2-\theta}$$

$$= \sqrt{\frac{\ln 3}{\ln 10}} \cdot \frac{1}{2} (\theta^2-\theta)^{-1/2} (2\theta-1)$$

$$= \frac{2\theta-1}{2\sqrt{\theta^2-\theta}} \sqrt{\frac{\ln 3}{\ln 10}}$$

23. Let $u = x^2$ so $du = 2x dx$.

$$\int x \cdot 2^{x^2} dx = \frac{1}{2} \int 2^u du = \frac{1}{2} \cdot \frac{2^u}{\ln 2} + C$$

$$= \frac{2^{x^2}}{2 \ln 2} + C = \frac{2^{x^2-1}}{\ln 2} + C$$

24. Let $u = 5x - 1$, so $du = 5 dx$.

$$\int 10^{5x-1} dx = \frac{1}{5} \int 10^u du = \frac{1}{5} \cdot \frac{10^u}{\ln 10} + C$$

$$= \frac{10^{5x-1}}{5 \ln 10} + C$$

25. Let $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{5^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int 5^u du = 2 \cdot \frac{5^u}{\ln 5} + C$$

$$= \frac{2 \cdot 5^{\sqrt{x}}}{\ln 5} + C$$

$$\int_1^4 \frac{5^{\sqrt{x}}}{\sqrt{x}} dx = 2 \left[\frac{5^{\sqrt{x}}}{\ln 5} \right]_1^4 = 2 \left(\frac{25}{\ln 5} - \frac{5}{\ln 5} \right)$$

$$= \frac{40}{\ln 5} \approx 24.85$$

26. $\int_0^1 (10^{3x} + 10^{-3x}) dx = \int_0^1 10^{3x} dx + \int_0^1 10^{-3x} dx$

Let $u = 3x$, so $du = 3 dx$.

$$\int 10^{3x} dx = \frac{1}{3} \int 10^u du = \frac{1}{3} \cdot \frac{10^u}{\ln 10} + C$$

$$= \frac{10^{3x}}{3 \ln 10} + C$$

Now let $u = -3x$, so $du = -3 dx$.

$$\int 10^{-3x} dx = -\frac{1}{3} \int 10^u du = -\frac{1}{3} \cdot \frac{10^u}{\ln 10} + C$$

$$= -\frac{10^{-3x}}{3 \ln 10} + C$$

Thus, $\int_0^1 (10^{3x} + 10^{-3x}) dx = \left[\frac{10^{3x} - 10^{-3x}}{3 \ln 10} \right]_0^1$

$$= \frac{1}{3 \ln 10} \left(1000 - \frac{1}{1000} \right) = \frac{999,999}{3000 \ln 10}$$

$$\approx 144.76$$

27. $\frac{d}{dx} 10^{(x^2)} = 10^{(x^2)} \ln 10 \frac{d}{dx} x^2 = 10^{(x^2)} 2x \ln 10$

$$\frac{d}{dx} (x^2)^{10} = \frac{d}{dx} x^{20} = 20x^{19}$$

$$\frac{dy}{dx} = \frac{d}{dx} [10^{(x^2)} + (x^2)^{10}]$$

$$= 10^{(x^2)} 2x \ln 10 + 20x^{19}$$

28. $\frac{d}{dx} \sin^2 x = 2 \sin x \frac{d}{dx} \sin x = 2 \sin x \cos x$

$$\frac{d}{dx} 2^{\sin x} = 2^{\sin x} \ln 2 \frac{d}{dx} \sin x = 2^{\sin x} \ln 2 \cos x$$

$$\frac{dy}{dx} = \frac{d}{dx} (\sin^2 x + 2^{\sin x})$$

$$= 2 \sin x \cos x + 2^{\sin x} \cos x \ln 2$$

29. $\frac{d}{dx} x^{\pi+1} = (\pi+1)x^{\pi}$

$$\frac{d}{dx} (\pi+1)^x = (\pi+1)^x \ln(\pi+1)$$

$$\frac{dy}{dx} = \frac{d}{dx} [x^{\pi+1} + (\pi+1)^x]$$

$$= (\pi+1)x^{\pi} + (\pi+1)^x \ln(\pi+1)$$

30. $\frac{d}{dx} 2^{(e^x)} = 2^{(e^x)} \ln 2 \frac{d}{dx} e^x = 2^{(e^x)} e^x \ln 2$

$$\frac{d}{dx} (2^e)^x = (2^e)^x \ln 2^e = (2^e)^x e \ln 2$$

$$\frac{dy}{dx} = \frac{d}{dx} [2^{(e^x)} + (2^e)^x]$$

$$= 2^{(e^x)} e^x \ln 2 + (2^e)^x e \ln 2$$

31. $y = (x^2 + 1)^{\ln x} = e^{(\ln x) \ln(x^2+1)}$

$$\frac{dy}{dx} = e^{(\ln x) \ln(x^2+1)} \frac{d}{dx} [(\ln x) \ln(x^2+1)]$$

$$= e^{(\ln x) \ln(x^2+1)} \left[\frac{1}{x} \ln(x^2+1) + \ln x \frac{2x}{x^2+1} \right]$$

$$= (x^2+1)^{\ln x} \left(\frac{\ln(x^2+1)}{x} + \frac{2x \ln x}{x^2+1} \right)$$

32. $y = (\ln x^2)^{2x+3} = e^{(2x+3) \ln(\ln x^2)}$

$$\frac{dy}{dx} = e^{(2x+3) \ln(\ln x^2)} \frac{d}{dx} [(2x+3) \ln(\ln x^2)]$$

$$= e^{(2x+3) \ln(\ln x^2)} \left[2 \ln(\ln x^2) + (2x+3) \frac{1}{\ln x^2} \frac{1}{x^2} (2x) \right]$$

$$= \underbrace{(2 \ln x)^{2x+3}}_{\ln x^2} \left[2 \ln \underbrace{(2 \ln x)}_{\ln x^2} + \frac{2x+3}{x \ln x} \right]$$

$$33. f(x) = x^{\sin x} = e^{\sin x \ln x}$$

$$f'(x) = e^{\sin x \ln x} \frac{d}{dx}(\sin x \ln x)$$

$$= e^{\sin x \ln x} \left[(\sin x) \left(\frac{1}{x} \right) + (\cos x)(\ln x) \right]$$

$$= x^{\sin x} \left(\frac{\sin x}{x} + \cos x \ln x \right)$$

$$f'(1) = 1^{\sin 1} \left(\frac{\sin 1}{1} + \cos 1 \ln 1 \right) = \sin 1 \approx 0.8415$$

$$34. f(e) = \pi^e \approx 22.46$$

$$g(e) = e^\pi \approx 23.14$$

$g(e)$ is larger than $f(e)$.

$$f'(x) = \frac{d}{dx} \pi^x = \pi^x \ln \pi$$

$$f'(e) = \pi^e \ln \pi \approx 25.71$$

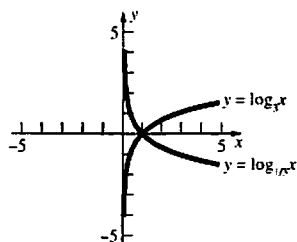
$$g'(x) = \frac{d}{dx} x^\pi = \pi x^{\pi-1}$$

$$g'(e) = \pi e^{\pi-1} \approx 26.74$$

$g'(e)$ is larger than $f'(e)$.

$$35. \log_{1/2} x = \frac{\ln x}{\ln \frac{1}{2}} = \frac{\ln x}{-\ln 2} = -\log_2 x$$

36.



$$37. M = 0.67 \log_{10}(0.37E) + 1.46$$

$$\log_{10}(0.37E) = \frac{M - 1.46}{0.67}$$

$$E = \frac{10^{\frac{M - 1.46}{0.67}}}{0.37}$$

Evaluating this expression for $M = 7$ and $M = 8$ gives $E \approx 5.017 \times 10^8$ kW-h and

$E \approx 1.560 \times 10^{10}$ kW-h, respectively.

$$38. 115 = 20 \log_{10}(121.3P)$$

$$\log_{10}(121.3P) = 5.75$$

$$P = \frac{10^{5.75}}{121.3} \approx 4636 \text{ lb/in.}^2$$

39. If r is the ratio between the frequencies of successive notes, then the frequency of $\bar{C} = r^{12}$

(the frequency of C). Since $\bar{C}$ has twice the

frequency of C , $r = 2^{1/12} \approx 1.0595$

Frequency of $\bar{C} = 440(2^{1/12})^3 = 440\sqrt[4]{2} \approx 523.25$

40. Assume $\log_2 3 = \frac{p}{q}$ where p and q are integers,

$q \neq 0$. Then $2^{p/q} = 3$ or $2^p = 3^q$. But

$2^p = 2 \cdot 2 \dots 2$ (p times) and has only powers of 2 as factors and $3^q = 3 \cdot 3 \dots 3$ (q times) and has only powers of 3 as factors.

$2^p = 3^q$ only for $p = q = 0$ which contradicts our assumption, so $\log_2 3$ cannot be rational.

41. If $y = A \cdot b^x$, then $\ln y = \ln A + x \ln b$, so the $\ln y$ vs. x plot will be linear.

If $y = C \cdot x^d$, then $\ln y = \ln C + d \ln x$, so the $\ln y$ vs. $\ln x$ plot will be linear.

42. WRONG 1:

$$y = f(x)^{g(x)}$$

$$y' = g(x)f'(x)^{g(x)-1} f'(x)$$

WRONG 2:

$$y = f(x)^{g(x)}$$

$$y' = f(x)^{g(x)} (\ln f(x)) \cdot g'(x) = f(x)^{g(x)} g'(x) \ln f(x)$$

RIGHT:

$$y = f(x)^{g(x)} = e^{g(x) \ln f(x)}$$

$$y' = e^{g(x) \ln f(x)} \frac{d}{dx} [g(x) \ln f(x)]$$

$$= f(x)^{g(x)} \left[g'(x) \ln f(x) + g(x) \frac{1}{f(x)} f'(x) \right]$$

$$= f(x)^{g(x)} g'(x) \ln f(x) + f(x)^{g(x)-1} g(x) f'(x)$$

Note that RIGHT = WRONG 2 + WRONG 1.

$$43. f(x) = (x^x)^x = x^{(x^2)} \neq x^{(x^x)} = g(x)$$

$$f(x) = x^{(x^2)} = e^{x^2 \ln x}$$

$$f'(x) = e^{x^2 \ln x} \frac{d}{dx} (x^2 \ln x) = e^{x^2 \ln x} \left(2x \ln x + x^2 \cdot \frac{1}{x} \right)$$

$$= x^{(x^2)} (2x \ln x + x)$$

$$g(x) = x^{(x^x)} = e^{x^x \ln x}$$

Using the result from Example 5

$$\left(\frac{d}{dx} x^x = x^x (1 + \ln x) \right):$$

$$g'(x) = e^{x^x \ln x} \frac{d}{dx} (x^x \ln x)$$

$$\begin{aligned}
&= e^{x^x \ln x} \left[x^x (1 + \ln x) \ln x + x^x \cdot \frac{1}{x} \right] \\
&= x^{(x^x)} x^x \left[(1 + \ln x) \ln x + \frac{1}{x} \right] \\
&= x^{x^x + x} \left[\ln x + (\ln x)^2 + \frac{1}{x} \right]
\end{aligned}$$

44. $f(x) = \frac{a^x - 1}{a^x + 1}$

$$f'(x) = \frac{(a^x + 1)a^x \ln a - (a^x - 1)a^x \ln a}{(a^x + 1)^2} = \frac{2a^x \ln a}{(a^x + 1)^2}$$

Since a is positive, a^x is always positive. $(a^x + 1)^2$ is also always positive, thus $f'(x) > 0$ if $\ln a > 0$ and $f'(x) < 0$ if $\ln a < 0$. $f(x)$ is either always increasing or always decreasing, depending on a , so $f(x)$ has an inverse.

$$y = \frac{a^x - 1}{a^x + 1}$$

$$y(a^x + 1) = a^x - 1$$

$$a^x(y - 1) = -1 - y$$

$$a^x = \frac{1 + y}{1 - y}$$

$$x \ln a = \ln \frac{1 + y}{1 - y}$$

$$x = \frac{\ln \frac{1 + y}{1 - y}}{\ln a} = \log_a \frac{1 + y}{1 - y}$$

$$f^{-1}(y) = \log_a \frac{1 + y}{1 - y}$$

$$f^{-1}(x) = \log_a \frac{1 + x}{1 - x}$$

45. a. Let $g(x) = \ln f(x) = \ln \left(\frac{x^a}{a^x} \right) = a \ln x - x \ln a$.

$$g'(x) = \left(\frac{a}{x} \right) - \ln a$$

$$g'(x) < 0 \text{ when } x > \frac{a}{\ln a}, \text{ so as } x \rightarrow \infty, g(x)$$

$$\text{is decreasing. } g''(x) = -\frac{a}{x^2}, \text{ so } g(x) \text{ is}$$

$$\text{concave down. Thus, } \lim_{x \rightarrow \infty} g(x) = -\infty, \text{ so}$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{g(x)} = 0.$$

- b. Again let $g(x) = \ln f(x) = a \ln x - x \ln a$. Since $y = \ln x$ is an increasing function, $f(x)$ is maximized when $g(x)$ is maximized.

$$g'(x) = \left(\frac{a}{x} \right) - \ln a, \text{ so } g'(x) > 0 \text{ on } \left(0, \frac{a}{\ln a} \right)$$

$$\text{and } g'(x) < 0 \text{ on } \left(\frac{a}{\ln a}, \infty \right).$$

Therefore, $g(x)$ (and hence $f(x)$) is

$$\text{maximized at } x_0 = \frac{a}{\ln a}.$$

- c. Note that $x^a = a^x$ is equivalent to $g(x) = 0$.

$$\text{By part b., } g(x) \text{ is maximized at } x_0 = \frac{a}{\ln a}.$$

If $a = e$, then

$$g(x_0) = g\left(\frac{e}{\ln e}\right) = g(e) = e \ln e - e \ln e = 0.$$

Since $g(x) < g(x_0) = 0$ for all $x \neq x_0$, the equation $g(x) = 0$ (and hence $x^a = a^x$) has just one positive solution. If $a \neq e$, then

$$\begin{aligned}
g(x_0) &= g\left(\frac{a}{\ln a}\right) = a \ln\left(\frac{a}{\ln a}\right) - \frac{a}{\ln a} (\ln a) \\
&= a \left[\ln\left(\frac{a}{\ln a}\right) - 1 \right].
\end{aligned}$$

Now $\frac{a}{\ln a} > e$ (justified below), so

$$g(x_0) = a \left[\ln \frac{a}{\ln a} - 1 \right] > a(\ln e - 1) = 0. \text{ Since}$$

$$g'(x) > 0 \text{ on } (0, x_0), g(x_0) > 0, \text{ and}$$

$$\lim_{x \rightarrow 0} g(x) = -\infty, g(x) = 0 \text{ has exactly one solution on } (0, x_0).$$

$$\text{Since } g'(x) < 0 \text{ on } (x_0, \infty),$$

$$g(x_0) > 0, \text{ and } \lim_{x \rightarrow \infty} g(x) = -\infty, g(x) = 0 \text{ has}$$

$$\text{exactly one solution on } (x_0, \infty). \text{ Therefore,}$$

$$\text{the equation } g(x) = 0 \text{ (and hence } x^a = a^x) \text{ has exactly two positive solutions.}$$

$$\text{To show that } \frac{a}{\ln a} > e \text{ when } a \neq e:$$

$$\text{Consider the function } h(x) = \frac{x}{\ln x}, \text{ for } x > 1.$$

$$h'(x) = \frac{\ln(x)(1) - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

$$\text{Note that } h'(x) < 0 \text{ on } (1, e) \text{ and } h'(x) > 0 \text{ on } (e, \infty), \text{ so } h(x) \text{ has its minimum at } (e, e).$$

$$\text{Therefore } \frac{x}{\ln x} > e \text{ for all } x \neq e, x > 1.$$

- d. For the case $a = e$, part c. shows that $g(x) = e \ln x - x \ln e < 0$ for $x \neq e$. Therefore, when $x \neq e$, $\ln x^e < \ln e^x$, which implies $x^e < e^x$. In particular, $\pi^e < e^\pi$.

46. a. $f_u(x) = x^u e^{-x}$
 $f'_u(x) = ux^{u-1}e^{-x} - x^u e^{-x} = (u-x)x^{u-1}e^{-x}$
 Since $f'_u(x) > 0$ on $(0, u)$ and $f'_u(x) < 0$ on (u, ∞) , $f_u(x)$ attains its maximum at $x_0 = u$.

b. $f_u(u) > f_u(u+1)$ means
 $u^u e^{-u} > (u+1)^u e^{-(u+1)}$.

Multiplying by $\frac{e^{u+1}}{u^u}$ gives $e > \left(\frac{u+1}{u}\right)^u$.

$f_{u+1}(u+1) > f_{u+1}(u)$ means
 $(u+1)^{u+1} e^{-(u+1)} > u^{u+1} e^{-u}$.

Multiplying by $\frac{e^{u+1}}{u^{u+1}}$ gives $\left(\frac{u+1}{u}\right)^{u+1} > e$.

Combining the two inequalities,

$$\left(\frac{u+1}{u}\right)^u < e < \left(\frac{u+1}{u}\right)^{u+1}.$$

c. From part b., $e < \left(\frac{u+1}{u}\right)^{u+1}$.

Multiplying by $\frac{u}{u+1}$ gives $\frac{u}{u+1}e < \left(\frac{u+1}{u}\right)^u$.

We showed $\left(\frac{u+1}{u}\right)^u < e$ in part b., so

$$\frac{u}{u+1}e < \left(\frac{u+1}{u}\right)^u < e.$$

Since $\lim_{u \rightarrow \infty} \frac{u}{u+1}e = e$, this implies that

$$\lim_{u \rightarrow \infty} \left(\frac{u+1}{u}\right)^u = e, \text{ i.e., } \lim_{u \rightarrow \infty} \left(1 + \frac{1}{u}\right)^u = e.$$

47. $f(x) = x^x = e^{x \ln x}$

Let $g(x) = x \ln x$.

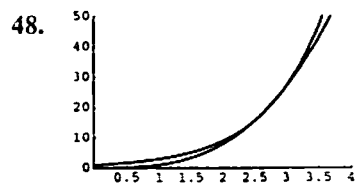
Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0^+} g(x) &= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = 0 \end{aligned}$$

Therefore, $\lim_{x \rightarrow 0^+} x^x = e^0 = 1$.

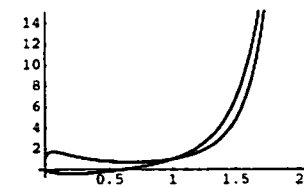
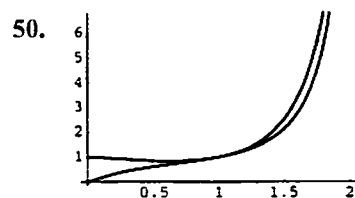
$g'(x) = \ln x + 1$

Since $g'(x) < 0$ on $(0, e)$ and $g'(x) > 0$ on (e, ∞) , $g(x)$ has its minimum at (e, e) . Therefore, $f(x)$ has its minimum at (e, e^e) .



$(2.4781, 15.2171), (3, 27)$

49. $\int_0^{4\pi} x^{\sin x} dx \approx 20.2259$



7.5 Concepts Review

1. ky ; $ky(L - y)$
2. $2^3 = 8$
3. half-life
4. $(1 + h)^{1/h}$

Problem Set 7.5

1. $k = -6$, $y_0 = 4$, so $y = 4e^{-6t}$

2. $k = 6$, $y_0 = 1$, so $y = e^{6t}$

3. $k = 0.005$, so $y = y_0 e^{0.005t}$
 $y(10) = y_0 e^{0.005(10)} = y_0 e^{0.05}$
 $y(10) = 2 \Rightarrow y_0 = \frac{2}{e^{0.05}}$

$$y = \frac{2}{e^{0.05}} e^{0.005t} = 2e^{0.005t - 0.05} = 2e^{0.005(t-10)}$$

4. $k = -0.003$, so $y = y_0 e^{-0.003t}$

$$y(-2) = y_0 e^{(-0.003)(-2)} = y_0 e^{0.006}$$

$$y(-2) = 3 \Rightarrow y_0 = \frac{3}{e^{0.006}}$$

$$y = \frac{3}{e^{0.006}} e^{-0.003t} = 3e^{-0.003t-0.006} = 3e^{-0.003(t+2)}$$

5. $y_0 = 10,000$, $y(10) = 20,000$

$$20,000 = 10,000 e^{k(10)}$$

$$2 = e^{10k}$$

$$\ln 2 = 10k$$

$$k = \frac{\ln 2}{10} \approx 0.069$$

$$y = 10,000 e^{0.069t}$$

After 25 days, $y = 10,000 e^{0.069 \cdot 25} \approx 56,125$.

6. Since the growth is exponential and it doubles in 10 days (from $t = 0$ to $t = 10$), it will always double in 10 days.

7. $3y_0 = y_0 e^{0.069t}$

$$3 = e^{0.069t}$$

$$\ln 3 = 0.069t$$

$$t = \frac{\ln 3}{0.069} \approx 15.9 \text{ days}$$

8. Let $P(t)$ = population (in millions) in year $1790 + t$.

In 1960, $t = 170$.

$$P(t) = P_0 e^{kt}$$

$$178 = 3.9 e^{170k}$$

$$45.64 = e^{170k}$$

$$k = \frac{\ln 45.64}{170} \approx 0.02248$$

In 2000, $t = 210$

$$P(210) \approx 3.9 e^{0.02248 \cdot 210} \approx 438$$

The model predicts that the population will be about 438 million. The actual number, 275 million, is quite a bit smaller because the rate of growth has declined in recent decades.

9. 1 year: (4.5 million) $(1.032) \approx 4.64$ million

2 years: (4.5 million) $(1.032)^2 \approx 4.79$ million

10 years: (4.5 million) $(1.032)^{10} \approx 6.17$ million

100 years: (4.5 million) $(1.032)^{100} \approx 105$ million

10. $y = y_0 e^{kt}$

$$1.032A = Ae^{k(1)}$$

$$k = \ln 1.032 \approx 0.03150$$

At $t = 100$, $y = 4.5 e^{(0.03150)(100)} \approx 105$.

After 100 years, the population will be about 105 million.

11. $\frac{1}{2} = e^{k(700)}$ and $y_0 = 10$

$$-\ln 2 = 700k$$

$$k = -\frac{\ln 2}{700} \approx -0.00099$$

$$y = 10 e^{-0.00099t}$$

At $t = 300$, $y = 10 e^{-0.00099 \cdot 300} \approx 7.43$.

After 10 years there will be about 7.43 g.

12. $0.85 = e^{k(2)}$

$$\ln 0.85 = 2k$$

$$k = \frac{\ln 0.85}{2} \approx -0.0813$$

$$\frac{1}{2} = e^{-0.0813t}$$

$$-\ln 2 = -0.0813t$$

$$t = \frac{\ln 2}{0.0813} \approx 8.53$$

The half-life is about 8.53 days.

13. $\frac{1}{2} = e^{5730k}$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -1.210 \times 10^{-4}$$

$$0.7y_0 = y_0 e^{(-1.210 \times 10^{-4})t}$$

$$t = \frac{\ln 0.7}{-1.210 \times 10^{-4}} \approx 2950$$

The fort burned down about 2950 years ago.

14. $\frac{1}{2} = e^{5730k}$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -1.210 \times 10^{-4}$$

$$0.51y_0 = y_0 e^{(-1.210 \times 10^{-4})t}$$

$$t = \frac{\ln 0.51}{-1.210 \times 10^{-4}} \approx 5565$$

The body was buried about 5565 years ago.

15. $\frac{dT}{dt} = k(T - 75)$

$$\int \frac{1}{T - 75} dT = \int k dt$$

$$\ln|T - 75| = kt + C$$

$$|T - 75| = e^{kt+C}$$

$$T = 75 + Ae^{kt}$$

$$T(0) = 75 + A$$

$$T(0) = 300 \Rightarrow A = 225$$

$$T(0.5) = 200 \Rightarrow 200 = 75 + 225e^{0.5k};$$

$$\frac{125}{225} = e^{0.5k}; k = 2 \ln \frac{5}{9} \approx -1.176$$

$$T = 75 + 225e^{-1.176t}$$

$$\text{At } t = 3, T = 75 + 225e^{-1.176 \cdot 3} \approx 81.6.$$

After 3 hours, the temperature will be about 81.6°F.

16. $\frac{dT}{dt} = k(T - 24)$

$$\int \frac{dT}{T - 24} = \int k dt$$

$$\ln|T - 24| = kt + C$$

$$|T - 24| = e^{kt+C}$$

$$T = 24 + Ae^{kt}$$

$$T(0) = -20 \Rightarrow A = -44.$$

$$T = 24 - 44e^{kt}$$

$$T(5) = 0 \Rightarrow 0 = 24 - 44e^{5k}.$$

$$k = \frac{\ln\left(\frac{24}{44}\right)}{5} \approx -0.1212$$

$$T = 24 - 44e^{-0.1212t}$$

$$20 = 24 - 44e^{-0.1212t}$$

$$e^{-0.1212t} = \frac{1}{11}$$

$$t = \frac{\ln\left(\frac{1}{11}\right)}{-0.1212} \approx 19.78$$

It will register 20°C after about 19.78 minutes.

17. a. $(\$375)(1.095)^2 \approx \449.63

b. $(\$375)\left(1 + \frac{0.095}{12}\right)^{24} \approx \453.13

c. $(\$375)\left(1 + \frac{0.095}{365}\right)^{730} \approx \453.46

d. $(\$375)e^{0.095 \cdot 2} \approx \453.47

18. a. $(\$375)(1.144)^2 \approx \490.78

b. $(\$375)\left(1 + \frac{0.144}{12}\right)^{24} \approx \499.30

c. $(\$375)\left(1 + \frac{0.144}{365}\right)^{730} \approx \500.13

d. $(\$375)e^{0.144 \cdot 2} \approx \500.16

$$T = 75 + 225e^{kt}$$

19. a. $\left(1 + \frac{0.12}{12}\right)^{12t} = 2$

$$1.01^{12t} = 2$$

$$12t = \log_{1.01} 2$$

$$t = \frac{1}{12} \log_{1.01} 2 = \frac{\ln 2}{12 \ln 1.01} \approx 5.805$$

It will take about 5.805 years or 5 years, 10 months.

b. $e^{0.12t} = 2 \Rightarrow t = \frac{\ln 2}{0.12} \approx 5.776$

It will take about 5.776 years or 5 years, 9 months, and 9 days.

20. $\$4000(1.115)^4 \approx \6182

21. 1626 to 2000 is 374 years.

$$y = 24e^{0.06 \cdot 374} \approx \$133.6 \text{ billion}$$

22. $\$100(1.08)^{969} \approx \2.441×10^{34}

23. If t is the doubling time, then

$$\left(1 + \frac{p}{100}\right)^t = 2$$

$$t \ln\left(1 + \frac{p}{100}\right) = \ln 2$$

$$t = \frac{\ln 2}{\ln\left(1 + \frac{p}{100}\right)} \approx \frac{\ln 2}{\frac{p}{100}} = \frac{100 \ln 2}{p} \approx \frac{70}{p}$$

24. $\frac{dy}{dt} = ky(L - y)$

$$\frac{1}{y(L - y)} dy = k dt$$

$$\left[\frac{1}{Ly} + \frac{1}{L(L - y)}\right] dy = k dt$$

$$\frac{1}{L} \int \left(\frac{1}{y} + \frac{1}{L - y}\right) dy = \int k dt$$

$$\frac{1}{L} [\ln|y| - \ln|L - y|] = kt + C_1$$

$$\ln\left|\frac{y}{L - y}\right| = Lkt + LC_1$$

$$\left|\frac{y}{L - y}\right| = e^{Lkt + LC_1} = e^{LC_1} \cdot e^{Lkt}, \text{ so } \frac{y}{L - y} = Ce^{Lkt}$$

$$\left(\begin{array}{l} \text{Note that: } C = Ce^0 = Ce^{Lk \cdot 0} \\ = \frac{y(0)}{L - y(0)} = \frac{y_0}{L - y_0} \end{array}\right)$$

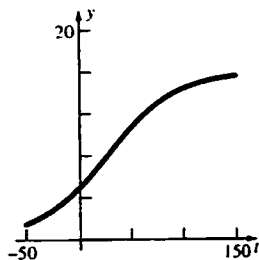
$$y = LCe^{Lkt} - yCe^{Lkt}$$

$$y + yCe^{Lkt} = LCe^{Lkt}$$

$$y = \frac{LCe^{Lkt}}{1 + Ce^{Lkt}} = \frac{LC}{\frac{1}{e^{Lkt}} + C} = \frac{LC}{C + e^{-Lkt}}$$

$$= \frac{L \cdot \frac{y_0}{L - y_0}}{\frac{y_0}{L - y_0} + e^{-Lkt}} = \frac{Ly_0}{y_0 + (L - y_0)e^{-Lkt}}$$

$$25. \quad y = \frac{16.5}{5 + (16 - 5)e^{-16(0.00186)t}} = \frac{80}{5 + 11e^{-0.02976t}}$$



$$26. \quad a. \quad \lim_{x \rightarrow 0} (1 + x)^{1000} = 1^{1000} = 1$$

$$b. \quad \lim_{x \rightarrow 0} 1^{1/x} = \lim_{x \rightarrow 0} 1 = 1$$

$$c. \quad \lim_{x \rightarrow 0^+} (1 + \varepsilon)^{1/x} = \lim_{n \rightarrow \infty} (1 + \varepsilon)^n = \infty$$

$$d. \quad \lim_{x \rightarrow 0^-} (1 + \varepsilon)^{1/x} = \lim_{n \rightarrow \infty} \frac{1}{(1 + \varepsilon)^n} = 0$$

$$e. \quad \lim_{x \rightarrow 0} (1 + x)^{1/x} = e$$

$$27. \quad a. \quad \lim_{x \rightarrow 0} (1 - x)^{1/x} = \lim_{x \rightarrow 0} \frac{1}{[1 + (-x)]^{1/(-x)}} = \frac{1}{e}$$

$$b. \quad \lim_{x \rightarrow 0} (1 + 3x)^{1/x} = \lim_{x \rightarrow 0} \left[(1 + 3x)^{\frac{1}{3x}} \right]^3 = e^3$$

$$\begin{aligned} c. \quad \lim_{n \rightarrow \infty} \left(\frac{n+2}{n} \right)^n &= \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n} \right)^n \\ &= \lim_{x \rightarrow 0^+} (1 + 2x)^{1/x} \\ &= \lim_{x \rightarrow 0^+} \left[(1 + 2x)^{\frac{1}{2x}} \right]^2 = e^2 \end{aligned}$$

$$\begin{aligned} d. \quad \lim_{n \rightarrow \infty} \left(\frac{n-1}{n} \right)^{2n} &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right)^{2n} \\ &= \lim_{x \rightarrow 0^+} (1 - x)^{2/x} \\ &= \lim_{x \rightarrow 0^+} \left[(1 - x)^{\frac{1}{-x}} \right]^{-2} = \frac{1}{e^2} \end{aligned}$$

$$28. \quad \frac{dy}{dt} = ay + b$$

$$\int \frac{dy}{y + \frac{b}{a}} = \int a \, dt$$

$$\ln \left| y + \frac{b}{a} \right| = at + C$$

$$\left| y + \frac{b}{a} \right| = e^{at+C}; \quad y + \frac{b}{a} = Ae^{at}$$

$$y = Ae^{at} - \frac{b}{a}$$

$$y_0 = A - \frac{b}{a} \Rightarrow A = y_0 + \frac{b}{a}$$

$$y = \left(y_0 + \frac{b}{a} \right) e^{at} - \frac{b}{a}$$

$$29. \quad \text{Let } y = \text{population in millions, } t = 0 \text{ in 1985, } a = 0.012, b = 0.06, y_0 = 10$$

$$\frac{dy}{dt} = 0.012y + 0.06$$

$$y = \left(10 + \frac{0.06}{0.012} \right) e^{0.012t} - \frac{0.06}{0.012} = 15e^{0.012t} - 5$$

From 1985 to 2010 is 25 years. At $t = 25$,
 $y = 15e^{0.012 \cdot 25} - 5 \approx 15.25$. The population in 2010 will be about 15.25 million.

$$30. \quad \text{Let } N(t) \text{ be the number of people who have heard the news after } t \text{ days. Then } \frac{dN}{dt} = k(L - N).$$

$$\int \frac{1}{L - N} dN = \int k \, dt$$

$$-\ln(L - N) = kt + C$$

$$L - N = e^{-kt-C}$$

$$N = L - Ae^{-kt}$$

$$N(0) = 0, \Rightarrow A = L$$

$$N(t) = L(1 - e^{-kt})$$

$$N(5) = \frac{L}{2} \Rightarrow \frac{L}{2} = L(1 - e^{-5k})$$

$$\frac{1}{2} = e^{-5k}$$

$$k = \frac{\ln \frac{1}{2}}{-5} \approx 0.1386$$

$$N(t) = L(1 - e^{-0.1386t})$$

$$0.99L = L(1 - e^{-0.1386t})$$

$$0.01 = e^{-0.1386t}$$

$$t = \frac{\ln 0.01}{-0.1386} \approx 33$$

99% of the people will have heard about the scandal after 33 days.

31. Maximum population:

$$13,500,000 \text{ mi}^2 \cdot \frac{640 \text{ acres}}{1 \text{ mi}^2} \cdot \frac{1 \text{ person}}{\frac{1}{2} \text{ acre}}$$

$$= 1.728 \times 10^{10} \text{ people}$$

Let $t = 0$ be in 1998.

$$(5.9 \times 10^9)e^{0.0132t} = 1.728 \times 10^{10}$$

$$t = \frac{\ln\left(\frac{1.728 \cdot 10^{10}}{5.9 \cdot 10^9}\right)}{0.0132} \approx 81.4 \text{ years from 1998, or}$$

sometime in the year 2079.

32. a. $k = 0.0132 - 0.0002t$

b. $y' = (0.0132 - 0.0002t)y$

c. $\frac{dy}{dt} = (0.0132 - 0.0002t)y$

$$\frac{dy}{y} = (0.0132 - 0.0002t) dt$$

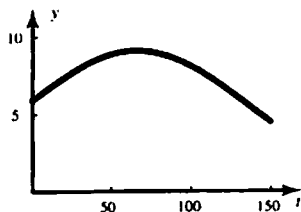
$$\ln y = 0.0132t - 0.0001t^2 + C_0$$

$$y = C_1 e^{0.0132t - 0.0001t^2}$$

The initial condition $y(0) = 5.9$ implies that

$$C_1 = 5.9. \text{ Thus } y = 5.9e^{0.0132t - 0.0001t^2}$$

d.



e. The maximum population will occur when

$$\frac{d}{dt}(0.0132t - 0.0001t^2) = 0$$

$$0.0132 = 0.0002t$$

$$t = 0.0132 / 0.0002 = 66$$

$$t = 66, \text{ which is year 2064.}$$

The population will equal the 1998 value of

$$5.9 \text{ billion when } 0.0132t - 0.0001t^2 = 0$$

$$t = 0 \text{ or } t = 132.$$

The model predicts that the population will return to the 1998 level in year 2130.

33. a. $k = 0.0132 - 0.0001t$

b. $y' = (0.0132 - 0.0001t)y$

c. $\frac{dy}{dt} = (0.0132 - 0.0001t)y$

$$\frac{dy}{y} = (0.0132 - 0.0001t) dt$$

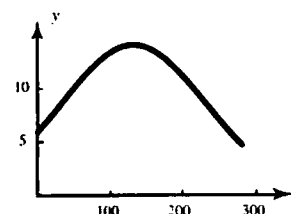
$$\ln y = 0.0132t - 0.00005t^2 + C_0$$

$$y = C_1 e^{0.0132t - 0.00005t^2}$$

The initial condition $y(0) = 5.9$ implies that

$$C_1 = 5.9. \text{ Thus } y = 5.9e^{0.0132t - 0.00005t^2}$$

d.



e. The maximum population will occur when

$$\frac{d}{dt}(0.0132t - 0.00005t^2) = 0$$

$$0.0132 = 0.0001t$$

$$t = 0.0132 / 0.0001 = 132$$

$$t = 132, \text{ which is year 2130.}$$

The population will equal the 1998 value of

$$5.9 \text{ billion when } 0.0132t - 0.00005t^2 = 0$$

$$t = 0 \text{ or } t = 264.$$

The model predicts that the population will return to the 1998 level in year 2262.

34. $E'(x) = \lim_{h \rightarrow 0} \frac{E(x+h) - E(x)}{h}$

$$= \lim_{h \rightarrow 0} \frac{E(x)E(h) - E(x)}{h}$$

$$= \lim_{h \rightarrow 0} E(x) \cdot \frac{E(h) - 1}{h} = E(x) \lim_{h \rightarrow 0} \frac{E(h) - 1}{h}$$

$$E(x) = E(x+0) = E(x) \cdot E(0)$$

$$\text{so } E(0) = 1.$$

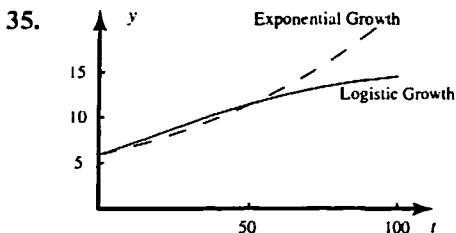
$$\text{Thus, } E'(x) = E(x) \lim_{h \rightarrow 0} \frac{E(h) - E(0)}{h}$$

$$= E(x) \lim_{h \rightarrow 0} \frac{E(0+h) - E(0)}{h} = E(x) \cdot E'(0)$$

$$= kE(x) \text{ where } k = E'(0).$$

Hence, $E(x) = E_0 e^{kx} = E(0)e^{kx} = 1 \cdot e^{kx} = e^{kx}$.

Check: $E(u+v) = e^{k(u+v)} = e^{ku+kv}$
 $= e^{ku} \cdot e^{kv} = E(u) \cdot E(v)$



7.6 Concepts Review

- $\exp\left(\int P(x)dx\right)$
- $y \exp\left(\int P(x)dx\right)$
- $\frac{1}{x}; \frac{d}{dx}\left(\frac{y}{x}\right) = 1; x^2 + Cx$
- particular

Problem Set 7.6

- Integrating factor is e^x .
 $D(ye^x) = 1$
 $y = e^{-x}(x + C)$
- The left-hand side is already an exact derivative.
 $D[y(x+1)] = x^2 - 1$
 $y = \frac{x^3 - 3x + C}{3(x+1)}$
- $y' + \frac{x}{1-x^2}y = \frac{ax}{1-x^2}$
Integrating factor:
 $\exp \int \frac{x}{1-x^2} dx = \exp[\ln(1-x^2)^{-1/2}]$
 $= (1-x^2)^{-1/2}$
 $D[y(1-x^2)^{-1/2}] = ax(1-x^2)^{-3/2}$ (see note 2 above.)

Exponential growth:

In 2010 ($t = 12$): 6.91 billion

In 2040 ($t = 42$): 10.27 billion

In 2090 ($t = 92$): 19.87 billion

Logistic growth:

In 2010 ($t = 12$): 7.29 billion

In 2040 ($t = 42$): 10.74 billion

In 2090 ($t = 92$): 14.28 billion

36. a. $\lim_{x \rightarrow 0} (1+x)^{1/x} = e$

b. $\lim_{x \rightarrow 0} (1-x)^{1/x} = \frac{1}{e}$

Then $y(1-x^2)^{-1/2} = a(1-x^2)^{-1/2} + C$, so
 $y = a + C(1-x^2)^{1/2}$.

4. Integrating factor is $\sec x$.

$$D[y \sec x] = \sec^2 x$$

$$y = \sin x + C \cos x$$

5. Integrating factor is $\frac{1}{x}$.

$$D\left[\frac{y}{x}\right] = e^x$$

$$y = xe^x + Cx$$

6. $y' - ay = f(x)$

Integrating factor: $e^{\int -a dx} = e^{-ax}$

$$D[ye^{-ax}] = e^{-ax} f(x)$$

Then $ye^{-ax} = \int e^{-ax} f(x) dx$, so

$$y = e^{ax} \int e^{-ax} f(x) dx.$$

7. Integrating factor is x . $D[yx] = 1$; $y = 1 + Cx^{-1}$

8. Integrating factor is $(x+1)^2$.

$$D[y(x+1)^2] = (x+1)^5$$

$$y = \left(\frac{1}{6}\right)(x+1)^4 + C(x+1)^{-2}$$

9. $y' + f(x)y = f(x)$

Integrating factor: $e^{\int f(x) dx}$

$$D\left[ye^{\int f(x) dx}\right] = f(x)e^{\int f(x) dx}$$

Then $ye^{\int f(x) dx} = e^{\int f(x) dx} + C$, so

$$y = 1 + Ce^{-\int f(x) dx}.$$

10. Integrating factor is e^{2x} .

$$D[ye^{2x}] = xe^{2x}$$

$$y = \left(\frac{1}{2}\right)x - \left(\frac{1}{4}\right) + Ce^{-2x}$$

11. Integrating factor is $\frac{1}{x}$. $D\left[\frac{y}{x}\right] = 3x^2$; $y = x^4 + Cx$

$$y = x^4 + 2x \text{ goes through } (1, 3).$$

12. $y' + 3y = e^{2x}$

$$\text{Integrating factor: } e^{\int 3dx} = e^{3x}$$

$$D[ye^{3x}] = e^{5x}$$

$$\text{Then } ye^{3x} = \frac{e^{5x}}{5} + C. x = 0, y = 1 \Rightarrow C = \frac{4}{5}, \text{ so}$$

$$ye^{3x} = \frac{e^{5x}}{5} + \frac{4}{5}.$$

$$\text{Therefore, } y = \frac{e^{2x} + 4e^{-3x}}{5} \text{ is the particular solution through } (0, 1).$$

13. Integrating factor: xe^x

$$d[ye^x] = 1; y = e^{-x}(1 + Cx^{-1}); y = e^{-x}(1 - x^{-1}) \text{ through } (1, 0).$$

14. Integrating factor is $\sin^2 x$.

$$D[y \sin^2 x] = 2 \sin^2 x \cos x$$

$$y \sin^2 x = \frac{2}{3} \sin^3 x + C$$

$$y = \frac{2}{3} \sin x + \frac{C}{\sin^2 x}$$

$$y = \frac{2}{3} \sin x + \frac{5}{12} \csc^2 x$$

$$\text{goes through } \left(\frac{\pi}{6}, 2\right).$$

15. Let y denote the number of pounds of chemical A after t minutes.

$$\frac{dy}{dt} = \left(2 \frac{\text{lbs}}{\text{gal}}\right) \left(3 \frac{\text{gal}}{\text{min}}\right) - \left(\frac{y \text{ lbs}}{20 \text{ gal}}\right) \left(3 \frac{\text{gal}}{\text{min}}\right)$$

$$= 6 - \frac{3y}{20} \text{ lb/min}$$

$$y' + \frac{3}{20}y = 6$$

$$\text{Integrating factor: } e^{\int (3/20)dt} = e^{3t/20}$$

$$D[ye^{3t/20}] = 6e^{3t/20}$$

$$\text{Then } ye^{3t/20} = 40e^{3t/20} + C. t = 0, y = 10 \Rightarrow C = -30.$$

$$\text{Therefore, } y(t) = 40 - 30e^{-3t/20}, \text{ so}$$

$$y(20) = 40 - 30e^{-3} \approx 38.506 \text{ lb.}$$

16. $\frac{dy}{dt} = (2)(4) - \left(\frac{y}{200}\right)(4)$ or $y' + \frac{y}{50} = 8$

$$\text{Integrating factor is } e^{t/50}.$$

$$D[ye^{t/50}] = 8e^{t/50}$$

$$y(t) = 400 + Ce^{-t/50}$$

$$y(t) = 400 - 350e^{-t/50} \text{ goes through } (0, 50).$$

$$y(40) = 400 - 350e^{-0.8} \approx 242.735 \text{ lb of salt}$$

17. $\frac{dy}{dt} = 4 - \left[\frac{y}{(120 - 2t)}\right](6)$ or $y' + \left[\frac{3}{(60 - t)}\right]y = 4$

$$\text{Integrating factor is } (60 - t)^{-3}.$$

$$D[y(60 - t)^{-3}] = 4(60 - t)^{-3}$$

$$y(t) = 2(60 - t) + C(60 - t)^3$$

$$y(t) = 2(60 - t) - \left(\frac{1}{1800}\right)(60 - t)^3 \text{ goes through } (0, 0).$$

18. $\frac{dy}{dt} = \frac{-2y}{50 + t}$ or $y' + \frac{2}{50 + t}y = 0$.

$$\text{Integrating factor:}$$

$$\exp\left(\int \frac{2}{50 + t} dt\right) = e^{2 \ln(50 + t)} = (50 + t)^2$$

$$D[y(50 + t)^2] = 0$$

$$\text{Then } y(50 + t)^2 = C. t = 0, y = 30 \Rightarrow C = 75000$$

$$\text{Thus, } y(50 + t)^2 = 75,000.$$

$$\text{If } y = 25, 25(50 + t)^2 = 75,000, \text{ so}$$

$$t = \sqrt{3000} - 50 \approx 4.772 \text{ min.}$$

19. $I' + 10^6 I = 1$

$$\text{Integrating factor} = \exp(10^6 t)$$

$$D[I \exp(10^6 t)] = \exp(10^6 t)$$

$$I(t) = 10^{-6} + C \exp(-10^6 t)$$

$$I(t) = 10^{-6}[1 - \exp(-10^6 t)] \text{ goes through } (0, 0).$$

20. $3.5I' = 120 \sin 377t$

$$I' = \left(\frac{240}{7}\right) \sin 377t$$

$$I = \left(-\frac{240}{2639}\right) \cos 377t + C$$

$$I(t) = \left(\frac{240}{2639}\right)(1 - \cos 377t) \text{ through } (0, 0).$$

21. $1000 I = 120 \sin 377t$
 $I(t) = 0.12 \sin 377t$

22. $\frac{dx}{dt} = -\frac{2x}{100}$
 $x' + \left(\frac{1}{50}\right)x = 0$

Integrating factor is $e^{t/50}$.

$$D[xe^{t/50}] = 0$$

$$x = Ce^{-t/50}$$

$$x(t) = 50e^{-t/50} \text{ satisfies } t = 0, x = 50.$$

$$\frac{dy}{dt} = 2\left(\frac{50e^{-t/50}}{100}\right) - 2\left(\frac{y}{200}\right)$$

$$y' + \left(\frac{1}{100}\right)y = e^{-t/50}$$

Integrating factor is $e^{t/100}$.

$$D[ye^{t/100}] = e^{-t/100}$$

$$y(t) = e^{-t/100}(C - 100e^{-t/100})$$

$$y(t) = e^{-t/100}(250 - 100e^{-t/100}) \text{ satisfies } t = 0, y = 150.$$

23. Let y be the number of gallons of pure alcohol in the tank at time t .

a. $y' = \frac{dy}{dt} = 5(0.25) - \left(\frac{5}{100}\right)y = 1.25 - 0.05y$

Integrating factor is $e^{0.05t}$.

$$y(t) = 25 + Ce^{-0.05t}; y = 100, t = 0, C = 75$$

$$y(t) = 25 + 75e^{-0.05t}; y = 50, t = T,$$

$$T = 20(\ln 3) \approx 21.97 \text{ min}$$

- b. Let A be the number of gallons of pure alcohol drained away.

$$(100 - A) + 0.25A = 50 \Rightarrow A = \frac{200}{3}$$

It took $\frac{200}{5}$ minutes for the draining and the

same amount of time to refill, so

$$T = \frac{2\left(\frac{200}{3}\right)}{5} = \frac{80}{3} \approx 26.67 \text{ min.}$$

- c. c would need to satisfy

$$\frac{200}{3} + \frac{200}{c} < 20(\ln 3).$$

$$c > \frac{10}{(3 \ln 3 - 2)} \approx 7.7170$$

d. $y' = 4(0.25) - 0.05y = 1 - 0.05y$

Solving for y , as in part a, yields

$y = 20 + 80e^{-0.05t}$. The drain is closed when $t = 0.8T$. We require that

$$(20 + 80e^{-0.05 \cdot 0.8T}) + 4 \cdot 0.25 \cdot 0.25 \cdot 0.8T = 50,$$

$$\text{or } 400e^{-0.04T} + T = 150.$$

24. a. $v' + av = -g$

Integrating factor: e^{at}

$$e^{at}(v' + av) = -ge^{at}; \frac{d}{dt}(ve^{at}) = -ge^{at}$$

$$ve^{at} = \int -ge^{at} dt = \frac{-g}{a}e^{at} + C; v = \frac{-g}{a} + Ce^{-at}$$

$$v = v_0, t = 0$$

$$v_0 = \frac{-g}{a} + C \Rightarrow C = v_0 + \frac{g}{a}$$

$$\text{Therefore, } v = \frac{-g}{a} + \left(v_0 + \frac{g}{a}\right)e^{-at}, \text{ so}$$

$$v(t) = v_\infty + (v_0 - v_\infty)e^{-at}.$$

b. $\frac{dy}{dt} = v_\infty + (v_0 - v_\infty)e^{-at}$, so

$$y = v_\infty - \frac{(v_0 - v_\infty)e^{-at}}{a} + C.$$

$$y = y_0, t = 0 \Rightarrow y_0 = \frac{-(v_0 - v_\infty)}{a} + C$$

$$\Rightarrow C = y_0 + \frac{v_0 - v_\infty}{a}$$

$$y = v_\infty t - \frac{(v_0 - v_\infty)e^{-at}}{a} + \left(y_0 + \frac{v_0 - v_\infty}{a}\right)$$

$$= y_0 + v_\infty t + \frac{v_0 - v_\infty}{a}(1 - e^{-at})$$

25. a. $v_\infty = -\frac{32}{0.05} = -640$

$$v(t) = [120 - (-640)]e^{-0.05t} + (-640) = 0 \text{ if}$$

$$t = 20 \ln\left(\frac{19}{16}\right).$$

$$y(t) = 0 + (-640)t$$

$$+ \left(\frac{1}{0.05}\right)[120 - (-640)](1 - e^{-0.05t})$$

$$= -640t + 15,200(1 - e^{-0.05t})$$

Therefore, the maximum altitude is

$$y\left(20 \ln\left(\frac{19}{16}\right)\right) = -12,800 \ln\left(\frac{19}{16}\right) + \frac{45,600}{19} \approx 200.32$$

$$\begin{aligned} \text{b. } -640T + 15,200(1 - e^{-0.05T}) &= 0; \\ 95 - 4T - 95e^{-0.05T} &= 0 \end{aligned}$$

26. For t in $[0, 15]$,

$$v_{\infty} = \frac{-32}{0.10} = -320.$$

$$v(t) = (0 + 320)e^{-0.1t} - 320 = 320(e^{-0.1t} - 1);$$

$$v(15) = 320(e^{-1.5} - 1) \approx -248.6$$

$$y(t) = 8000 - 320t + 10(320)(1 - e^{-0.1t});$$

$$y(15) = 3200(2 - e^{-1.5}) \approx 5686$$

Let t be the number of seconds after the parachute opens that it takes Megan to reach the ground.

$$\text{For } t \text{ in } [15, 15+T], \quad v_{\infty} = -\frac{32}{1.6} = -20.$$

$$0 = y(T + 15)$$

$$= [3200(2 - e^{-1.5})]$$

$$-20T + (0.625)[320(e^{-1.5} - 1) + 20](1 - e^{-1.6T})$$

$$\approx 5543 - 20T - 142.9e^{-1.6T} \approx 5543 - 20T \quad [\text{since}$$

$$T > 50, \text{ so } e^{-1.6T} < 10^{-35} \text{ (very small)}]$$

Therefore, $T \approx 277$, so it takes Megan about 292 s (4 min, 52 s) to reach the ground.

$$27. \text{ a. } e^{-\ln x + C} \left(\frac{dy}{dx} - \frac{y}{x} \right) = x^2 e^{-\ln x + C}$$

$$e^{-\ln x} e^C \left(\frac{dy}{dx} - \frac{y}{x} \right) = x^2 e^C e^{-\ln x}$$

$$\frac{1}{x} e^C \frac{dy}{dx} - y e^C \frac{1}{x^2} = x^2 e^C \frac{1}{x}$$

$$\frac{d}{dx} \left(e^C \frac{1}{x} y \right) = x e^C$$

$$\text{b. } e^C \frac{y}{x} = e^C \int x dx$$

$$\frac{y}{x} = \frac{x^2}{2} + C_1$$

$$y = \frac{x^3}{2} + C_1 x$$

$$28. \quad e^{\int P(x) dx + C} \frac{dy}{dx} + P(x) e^{\int P(x) dx + C} y$$

$$= Q(x) e^{\int P(x) dx + C}$$

$$\frac{d}{dx} \left(e^{\int P(x) dx + C} y \right) = Q(x) e^{\int P(x) dx + C}$$

$$y e^{\int P(x) dx + C} = \int Q(x) e^{\int P(x) dx} e^C dx + C_1$$

$$y = e^{-\int P(x) dx} \int Q(x) e^{\int P(x) dx} dx$$

$$+ C_2 e^{-\int P(x) dx}$$

7.7 Concepts Review

$$1. \quad \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]; \arcsin$$

$$2. \quad \left(-\frac{\pi}{2}, \frac{\pi}{2} \right); \arctan$$

$$3. \quad 1$$

$$4. \quad \pi$$

Problem Set 7.7

$$1. \quad \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \text{ since } \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$2. \quad \arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3} \text{ since } \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$3. \quad \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3} \text{ since } \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$4. \quad \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4} \text{ since } \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$5. \quad \arctan(\sqrt{3}) = \frac{\pi}{3} \text{ since } \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$6. \quad \operatorname{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3} \text{ since } \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \text{ so}$$

$$\sec\left(\frac{\pi}{3}\right) = 2$$

$$7. \quad \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \text{ since } \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$8. \quad \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6} \text{ since } \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

$$9. \sin(\sin^{-1} 0.4567) = 0.4567 \text{ by definition}$$

$$10. \cos(\sin^{-1} 0.56) = \sqrt{1 - \sin^2(\sin^{-1} 0.56)} \\ = \sqrt{1 - (0.56)^2} \approx 0.828$$

$$11. \sin^{-1}(0.1113) \approx 0.1115$$

$$12. \arccos(0.6341) \approx 0.8840$$

$$13. \cos(\operatorname{arccot} 3.212) = \cos\left(\arctan \frac{1}{3.212}\right) \\ \approx \cos 0.3018 \approx 0.9548$$

$$14. \sec(\arccos 0.5111) = \frac{1}{\cos(\arccos 0.5111)} \\ = \frac{1}{0.5111} \approx 1.957$$

$$15. \sec^{-1}(-2.222) = \cos^{-1}\left(\frac{1}{-2.222}\right) \approx 2.038$$

$$16. \tan^{-1}(-60.11) \approx -1.554$$

$$17. \cos(\sin(\tan^{-1} 2.001)) \approx 0.6259$$

$$18. \sin^2(\ln(\cos 0.5555)) \approx 0.02632$$

$$19. \theta = \sin^{-1} \frac{x}{8}$$

$$20. \theta = \tan^{-1} \frac{x}{6}$$

$$21. \theta = \sin^{-1} \frac{5}{x}$$

$$22. \theta = \cos^{-1} \frac{9}{x} \text{ or } \theta = \sec^{-1} \frac{x}{9}$$

$$23. \text{ Let } \theta_1 \text{ be the angle opposite the side of length 3,} \\ \text{and } \theta_2 = \theta_1 - \theta, \text{ so } \theta = \theta_1 - \theta_2. \text{ Then } \tan \theta_1 = \frac{3}{x} \\ \text{and } \tan \theta_2 = \frac{1}{x}. \theta = \tan^{-1} \frac{3}{x} - \tan^{-1} \frac{1}{x}.$$

$$24. \text{ Let } \theta_1 \text{ be the angle opposite the side of length 5,} \\ \text{and } \theta_2 = \theta_1 - \theta, \text{ and } y \text{ the length of the unlabeled} \\ \text{side. Then } \theta = \theta_1 - \theta_2 \text{ and } y = \sqrt{x^2 - 25}. \\ \tan \theta_1 = \frac{5}{y} = \frac{5}{\sqrt{x^2 - 25}}, \tan \theta_2 = \frac{2}{y} = \frac{2}{\sqrt{x^2 - 25}}, \\ \theta = \tan^{-1} \frac{5}{\sqrt{x^2 - 25}} - \tan^{-1} \frac{2}{\sqrt{x^2 - 25}}$$

$$25. \cos\left[2 \sin^{-1}\left(-\frac{2}{3}\right)\right] = 1 - 2 \sin^2\left[\sin^{-1}\left(-\frac{2}{3}\right)\right] \\ = 1 - 2\left(-\frac{2}{3}\right)^2 = \frac{1}{9}$$

$$26. \tan\left[2 \tan^{-1}\left(\frac{1}{3}\right)\right] = \frac{2 \tan\left[\tan^{-1}\left(\frac{1}{3}\right)\right]}{1 - \tan^2\left[\tan^{-1}\left(\frac{1}{3}\right)\right]} \\ = \frac{2 \cdot \frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2} = \frac{3}{4}$$

$$27. \sin\left[\cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{5}{13}\right)\right] = \sin\left[\cos^{-1}\left(\frac{3}{5}\right)\right] \cos\left[\cos^{-1}\left(\frac{5}{13}\right)\right] + \cos\left[\cos^{-1}\left(\frac{3}{5}\right)\right] \sin\left[\cos^{-1}\left(\frac{5}{13}\right)\right] \\ = \sqrt{1 - \left(\frac{3}{5}\right)^2} \cdot \frac{5}{13} + \frac{3}{5} \sqrt{1 - \left(\frac{5}{13}\right)^2} = \frac{56}{65}$$

$$28. \cos\left[\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right] = \cos\left[\cos^{-1}\left(\frac{4}{5}\right)\right] \cos\left[\sin^{-1}\left(\frac{12}{13}\right)\right] - \sin\left[\cos^{-1}\left(\frac{4}{5}\right)\right] \sin\left[\sin^{-1}\left(\frac{12}{13}\right)\right] \\ = \frac{4}{5} \cdot \sqrt{1 - \left(\frac{12}{13}\right)^2} - \sqrt{1 - \left(\frac{4}{5}\right)^2} \cdot \frac{12}{13} = -\frac{16}{65}$$

$$29. \tan(\sin^{-1} x) = \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)} = \frac{x}{\sqrt{1 - x^2}}$$

$$30. \sin(\tan^{-1} x) = \frac{1}{\csc(\tan^{-1} x)} = \frac{1}{\sqrt{1 + \cot^2(\tan^{-1} x)}} \\ = \frac{1}{\sqrt{1 + \frac{1}{\tan^2(\tan^{-1} x)}}} = \frac{1}{\sqrt{1 + \frac{1}{x^2}}} = \frac{x}{\sqrt{x^2 + 1}}$$

$$31. \cos(2\sin^{-1} x) = 1 - 2\sin^2(\sin^{-1} x) = 1 - 2x^2$$

$$32. \tan(2\tan^{-1} x) = \frac{2\tan(\tan^{-1} x)}{1 - \tan^2(\tan^{-1} x)} = \frac{2x}{1 - x^2}$$

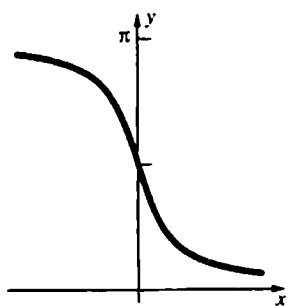
$$33. a. \lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2} \text{ since } \lim_{\theta \rightarrow \pi/2^-} \tan \theta = \infty$$

$$b. \lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2} \text{ since } \\ \lim_{\theta \rightarrow -\pi/2^+} \tan \theta = -\infty$$

$$34. a. \lim_{x \rightarrow \infty} \sec^{-1} x = \lim_{x \rightarrow \infty} \cos^{-1}\left(\frac{1}{x}\right) \\ = \lim_{z \rightarrow 0^+} \cos^{-1} z = \frac{\pi}{2}$$

$$b. \lim_{x \rightarrow -\infty} \sec^{-1} x = \lim_{x \rightarrow -\infty} \cos^{-1}\left(\frac{1}{x}\right) \\ = \lim_{z \rightarrow 0^-} \cos^{-1} z = \frac{\pi}{2}$$

35.



$$36. \frac{d}{dx} e^{\tan x} = e^{\tan x} \frac{d}{dx} \tan x = e^{\tan x} \sec^2 x$$

$$37. \frac{d}{dx} \ln(\sec x + \tan x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \\ = \frac{(\sec x)(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

$$38. \frac{d}{dx} [-\ln(\csc x + \cot x)] = -\frac{-\csc x \cot x - \csc^2 x}{\csc x + \cot x} \\ = \frac{\csc x(\cot x + \csc x)}{\cot x + \csc x} = \csc x$$

$$39. \frac{d}{dx} \sin^{-1}(2x^2) = \frac{1}{\sqrt{1 - (2x^2)^2}} \cdot 4x = \frac{4x}{\sqrt{1 - 4x^4}}$$

$$40. \frac{d}{dx} \arccos(e^x) = -\frac{1}{\sqrt{1 - (e^x)^2}} \cdot e^x = -\frac{e^x}{\sqrt{1 - e^{2x}}}$$

$$41. \frac{d}{dx} [x^3 \tan^{-1}(e^x)] = x^3 \cdot \frac{e^x}{1 + (e^x)^2} + 3x^2 \tan^{-1}(e^x) \\ = x^2 \left[\frac{xe^x}{1 + e^{2x}} + 3 \tan^{-1}(e^x) \right]$$

$$42. \frac{d}{dx} (e^x \arcsin x^2) = e^x \cdot \frac{2x}{\sqrt{1 - (x^2)^2}} + e^x \arcsin x^2 \\ = e^x \left(\frac{2x}{\sqrt{1 - x^4}} + \arcsin x^2 \right)$$

$$43. \frac{d}{dx} (\tan^{-1} x)^3 = 3(\tan^{-1} x)^2 \cdot \frac{1}{1 + x^2} = \frac{3(\tan^{-1} x)^2}{1 + x^2}$$

$$44. \frac{d}{dx} \tan(\cos^{-1} x) = \frac{d}{dx} \frac{\sin(\cos^{-1} x)}{\cos(\cos^{-1} x)} = \frac{d}{dx} \frac{\sqrt{1 - x^2}}{x} \\ = \frac{x \cdot \frac{1}{2} \cdot \frac{-1}{\sqrt{1 - x^2}} (-2x) - \sqrt{1 - x^2} \cdot 1}{x^2} \\ = \frac{-x^2 - (1 - x^2)}{x^2 \sqrt{1 - x^2}} = -\frac{1}{x^2 \sqrt{1 - x^2}}$$

$$45. \frac{d}{dx} \sec^{-1}(x^3) = \frac{1}{|x^3| \sqrt{(x^3)^2 - 1}} \cdot 3x^2 = \frac{3}{|x| \sqrt{x^6 - 1}}$$

$$46. \frac{d}{dx} (\sec^{-1} x)^3 = 3(\sec^{-1} x)^2 \cdot \frac{1}{|x| \sqrt{x^2 - 1}} \\ = \frac{3(\sec^{-1} x)^2}{|x| \sqrt{x^2 - 1}}$$

$$47. \frac{d}{dx} (1 + \sin^{-1} x)^3 = 3(1 + \sin^{-1} x)^2 \cdot \frac{1}{\sqrt{1 - x^2}} \\ = \frac{3(1 + \sin^{-1} x)^2}{\sqrt{1 - x^2}}$$

$$48. \text{ Let } u = x^2, \text{ so } du = 2x dx.$$

$$\int x \sin(x^2) dx = \frac{1}{2} \int \sin(u) \cdot 2x dx$$

$$\begin{aligned}
 &= \frac{1}{2} \int \sin u \, du = -\frac{1}{2} \cos u + C \\
 &= -\frac{1}{2} \cos(x^2) + C
 \end{aligned}$$

49. Let $u = \sin 2x$, so $du = 2 \cos 2x \, dx$.

$$\begin{aligned}
 \int \sin 2x \cos 2x \, dx &= \frac{1}{2} \int \sin 2x (2 \cos 2x) \, dx \\
 &= \frac{1}{2} \int u \, du \\
 &= \frac{u^2}{4} + C = \frac{1}{4} \sin^2 2x + C
 \end{aligned}$$

50. Let $u = \cos x$, so $du = -\sin x \, dx$.

$$\begin{aligned}
 \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = -\int \frac{1}{\cos x} (-\sin x) \, dx \\
 &= -\int \frac{1}{u} \, du = -\ln|u| + C = -\ln|\cos x| + C
 \end{aligned}$$

51. Let $u = e^{2x}$, so $du = 2e^{2x} \, dx$.

$$\begin{aligned}
 \int e^{2x} \cos(e^{2x}) \, dx &= \frac{1}{2} \int \cos(e^{2x}) (2e^{2x}) \, dx \\
 &= \frac{1}{2} \int \cos u \, du \\
 &= \frac{1}{2} \sin u + C = \frac{1}{2} \sin(e^{2x}) + C \\
 \int_0^1 e^{2x} \cos(e^{2x}) \, dx &= \left[\frac{1}{2} \sin(e^{2x}) \right]_0^1 \\
 &= \left[\frac{1}{2} \sin(e^2) - \frac{1}{2} \sin(e^0) \right] \\
 &= \frac{\sin e^2 - \sin 1}{2} \approx 0.0262
 \end{aligned}$$

52. Let $u = \sin x$, so $du = \cos x \, dx$.

$$\begin{aligned}
 \int \sin^2 x \cos x \, dx &= \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C \\
 \int_0^{\pi/2} \sin^2 x \cos x \, dx &= \left[\frac{\sin^3 x}{3} \right]_0^{\pi/2} = \frac{1}{3} - 0 = \frac{1}{3}
 \end{aligned}$$

53.
$$\begin{aligned}
 \int_0^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}} \, dx &= [\arcsin x]_0^{\sqrt{2}/2} \\
 &= \arcsin \frac{\sqrt{2}}{2} - \arcsin 0 = \frac{\pi}{4}
 \end{aligned}$$

54.
$$\begin{aligned}
 \int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}} &= \int_{\sqrt{2}}^2 \frac{dx}{|x|\sqrt{x^2-1}} = \left[\sec^{-1} x \right]_{\sqrt{2}}^2 \\
 &= \sec^{-1} 2 - \sec^{-1} \sqrt{2} \\
 &= \cos^{-1} \left(\frac{1}{2} \right) - \cos^{-1} \left(\frac{\sqrt{2}}{2} \right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}
 \end{aligned}$$

55.
$$\begin{aligned}
 \int_{-1}^1 \frac{1}{1+x^2} \, dx &= \left[\tan^{-1} x \right]_{-1}^1 = \tan^{-1} 1 - \tan^{-1}(-1) \\
 &= \frac{\pi}{4} - \left(-\frac{\pi}{4} \right) = \frac{\pi}{2}
 \end{aligned}$$

56. Let $u = \cos \theta$, so $du = -\sin \theta \, d\theta$.

$$\begin{aligned}
 \int \frac{\sin \theta}{1+\cos^2 \theta} \, d\theta &= -\int \frac{1}{1+\cos^2 \theta} (-\sin \theta) \, d\theta \\
 &= -\int \frac{1}{1+u^2} \, du = -\tan^{-1} u + C \\
 &= -\tan^{-1}(\cos \theta) + C \\
 \int_0^{\pi/2} \frac{\sin \theta}{1+\cos^2 \theta} \, d\theta &= \left[-\tan^{-1}(\cos \theta) \right]_0^{\pi/2} \\
 &= -\tan^{-1} 0 + \tan^{-1} 1 = -0 + \frac{\pi}{4} = \frac{\pi}{4}
 \end{aligned}$$

57. Let $u = 2x$, so $du = 2 \, dx$.

$$\begin{aligned}
 \int \frac{1}{1+4x^2} \, dx &= \frac{1}{2} \int \frac{1}{1+(2x)^2} 2 \, dx \\
 &= \frac{1}{2} \int \frac{1}{1+u^2} \, du = \frac{1}{2} \arctan u + C \\
 &= \frac{1}{2} \arctan 2x + C
 \end{aligned}$$

58. Let $u = e^x$, so $du = e^x \, dx$.

$$\begin{aligned}
 \int \frac{e^x}{1+e^{2x}} \, dx &= \int \frac{e^x}{1+(e^x)^2} \, dx = \int \frac{1}{1+u^2} \, du \\
 &= \arctan u + C = \arctan e^x + C
 \end{aligned}$$

59. The top of the picture is 7.6 ft above eye level, and the bottom of the picture is 2.6 ft above eye level. Let θ_1 be the angle between the viewer's line of sight to the top of the picture and the horizontal. Then call $\theta_2 = \theta_1 - \theta$, so $\theta = \theta_1 - \theta_2$.

$$\begin{aligned}
 \tan \theta_1 &= \frac{7.6}{b}; \quad \tan \theta_2 = \frac{2.6}{b}; \\
 \theta &= \tan^{-1} \frac{7.6}{b} - \tan^{-1} \frac{2.6}{b}
 \end{aligned}$$

If $b = 12.9$, $\theta \approx 0.3335$ or 19.1° .

60. a. Restrict $2x$ to $[0, \pi]$, i.e., restrict x to $\left[0, \frac{\pi}{2}\right]$.

Then $y = 3 \cos 2x$

$$\frac{y}{3} = \cos 2x$$

$$2x = \arccos \frac{y}{3}$$

$$x = f^{-1}(y) = \frac{1}{2} \arccos \frac{y}{3}$$

$$f^{-1}(x) = \frac{1}{2} \arccos \frac{x}{3}$$

b. Restrict $3x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, i.e., restrict x to

$$\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$$

$$\text{Then } y = 2 \sin 3x$$

$$\frac{y}{2} = \sin 3x$$

$$3x = \arcsin \frac{y}{2}$$

$$x = f^{-1}(y) = \frac{1}{3} \arcsin \frac{y}{2}$$

$$f^{-1}(x) = \frac{1}{3} \arcsin \frac{x}{2}$$

c. Restrict x to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$y = \frac{1}{2} \tan x$$

$$2y = \tan x$$

$$x = f^{-1}(y) = \arctan 2y$$

$$f^{-1}(x) = \arctan 2x$$

d. Restrict x to $\left(-\infty, -\frac{2}{\pi}\right) \cup \left(\frac{2}{\pi}, \infty\right)$ so $\frac{1}{x}$ is

$$\text{restricted to } \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$$

$$\text{then } y = \sin \frac{1}{x}$$

$$\frac{1}{x} = \arcsin y$$

$$x = f^{-1}(y) = \frac{1}{\arcsin y}$$

$$f^{-1}(x) = \frac{1}{\arcsin x}$$

$$61. \tan \left[2 \tan^{-1} \left(\frac{1}{4} \right) \right] = \frac{2 \tan \left[\tan^{-1} \left(\frac{1}{4} \right) \right]}{1 - \tan^2 \left[\tan^{-1} \left(\frac{1}{4} \right) \right]}$$

$$= \frac{2 \cdot \frac{1}{4}}{1 - \left(\frac{1}{4} \right)^2} = \frac{8}{15}$$

$$\tan \left[3 \tan^{-1} \left(\frac{1}{4} \right) \right] = \tan \left[2 \tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{1}{4} \right) \right]$$

$$= \frac{\tan \left[2 \tan^{-1} \left(\frac{1}{4} \right) \right] + \tan \left[\tan^{-1} \left(\frac{1}{4} \right) \right]}{1 - \tan \left[2 \tan^{-1} \left(\frac{1}{4} \right) \right] \tan \left[\tan^{-1} \left(\frac{1}{4} \right) \right]}$$

$$= \frac{\frac{8}{15} + \frac{1}{4}}{1 - \frac{8}{15} \cdot \frac{1}{4}} = \frac{47}{52}$$

$$\tan \left[3 \tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{5}{99} \right) \right]$$

$$= \frac{\tan \left[3 \tan^{-1} \left(\frac{1}{4} \right) \right] + \tan \left[\tan^{-1} \left(\frac{5}{99} \right) \right]}{1 - \tan \left[3 \tan^{-1} \left(\frac{1}{4} \right) \right] \tan \left[\tan^{-1} \left(\frac{5}{99} \right) \right]}$$

$$= \frac{\frac{47}{52} + \frac{5}{99}}{1 - \frac{47}{52} \cdot \frac{5}{99}} = \frac{4913}{4913} = 1 = \tan \frac{\pi}{4}$$

$$\text{Thus, } 3 \tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{5}{99} \right) = \tan^{-1}(1) = \frac{\pi}{4}.$$

$$62. \tan \left[2 \tan^{-1} \left(\frac{1}{5} \right) \right] = \frac{2 \tan \left[\tan^{-1} \left(\frac{1}{5} \right) \right]}{1 - \tan^2 \left[\tan^{-1} \left(\frac{1}{5} \right) \right]}$$

$$= \frac{2 \cdot \frac{1}{5}}{1 - \left(\frac{1}{5} \right)^2} = \frac{5}{12}$$

$$\tan \left[4 \tan^{-1} \left(\frac{1}{5} \right) \right] = \tan \left[2 \cdot 2 \tan^{-1} \left(\frac{1}{5} \right) \right]$$

$$= \frac{2 \tan \left[2 \tan^{-1} \left(\frac{1}{5} \right) \right]}{1 - \tan^2 \left[2 \tan^{-1} \left(\frac{1}{5} \right) \right]} = \frac{2 \cdot \frac{5}{12}}{1 - \left(\frac{5}{12} \right)^2} = \frac{120}{119}$$

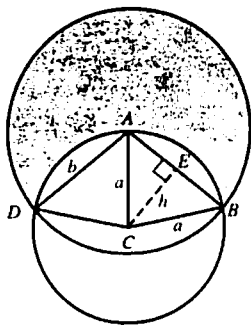
$$\tan \left[4 \tan^{-1} \left(\frac{1}{5} \right) - \tan^{-1} \left(\frac{1}{239} \right) \right]$$

$$= \frac{\tan \left[4 \tan^{-1} \left(\frac{1}{5} \right) \right] - \tan \left[\tan^{-1} \left(\frac{1}{239} \right) \right]}{1 + \tan \left[4 \tan^{-1} \left(\frac{1}{5} \right) \right] \tan \left[\tan^{-1} \left(\frac{1}{239} \right) \right]}$$

$$= \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = \frac{28,561}{28,561} = 1 = \tan \frac{\pi}{4}$$

$$\text{Thus, } 4 \tan^{-1} \left(\frac{1}{5} \right) - \tan^{-1} \left(\frac{1}{239} \right) = \tan^{-1}(1) = \frac{\pi}{4}$$

63.



Let θ represent $\angle DAB$, then $\angle CAB$ is $\frac{\theta}{2}$. Since

$\triangle ABC$ is isosceles, $|AE| = \frac{b}{2}$, $\cos \frac{\theta}{2} = \frac{\frac{b}{2}}{a} = \frac{b}{2a}$ and

$\theta = 2 \cos^{-1} \frac{b}{2a}$. Thus sector ADB has area

$\frac{1}{2} \left(2 \cos^{-1} \frac{b}{2a} \right) b^2 = b^2 \cos^{-1} \frac{b}{2a}$. Let ϕ represent

$\angle DCB$, then $\angle ACB$ is $\frac{\phi}{2}$ and $\angle ECA$ is $\frac{\phi}{4}$, so

$\sin \frac{\phi}{4} = \frac{\frac{b}{2}}{a} = \frac{b}{2a}$ and $\phi = 4 \sin^{-1} \frac{b}{2a}$. Thus sector

DCB has area $\frac{1}{2} \left(4 \sin^{-1} \frac{b}{2a} \right) a^2 = 2a^2 \sin^{-1} \frac{b}{2a}$.

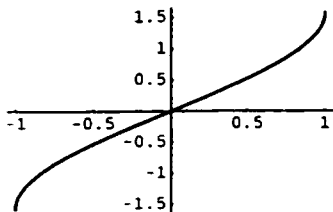
These sectors overlap on the triangles $\triangle DAC$ and $\triangle CAB$, each of which has area

$$\frac{1}{2} |AB| h = \frac{1}{2} b \sqrt{a^2 - \left(\frac{b}{2}\right)^2} = \frac{1}{2} b \frac{\sqrt{4a^2 - b^2}}{2}.$$

The large circle has area πb^2 , hence the shaded region has area

$$\pi b^2 - b^2 \cos^{-1} \frac{b}{2a} - 2a^2 \sin^{-1} \frac{b}{2a} + \frac{1}{2} b \sqrt{4a^2 - b^2}$$

64.



They have the same graph.

Conjecture: $\arcsin x = \arctan \frac{x}{\sqrt{1-x^2}}$ for

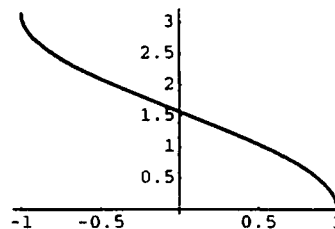
$-1 < x < 1$

Proof: Let $\theta = \arcsin x$, so $x = \sin \theta$.

Then $\frac{x}{\sqrt{1-x^2}} = \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} = \frac{\sin \theta}{\cos \theta} = \tan \theta$

so $\theta = \arctan \frac{x}{\sqrt{1-x^2}}$.

65.



It is the same graph as $y = \arccos x$.

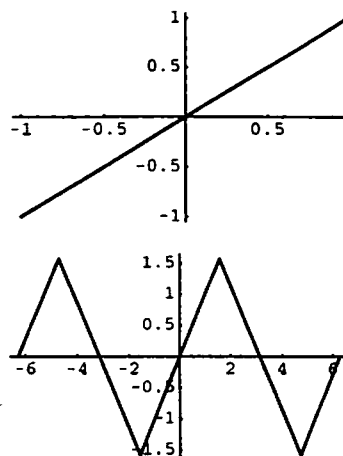
Conjecture: $\frac{\pi}{2} - \arcsin x = \arccos x$

Proof: Let $\theta = \frac{\pi}{2} - \arcsin x$

Then $x = \sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta$

so $\theta = \arccos x$.

66.



$y = \sin(\arcsin x)$ is the line $y = x$, but only defined for $-1 \leq x \leq 1$.

$y = \arcsin(\sin x)$ is defined for all x , but only the portion for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ is the line $y = x$.

$$\begin{aligned} 67. \quad \int \frac{dx}{\sqrt{a^2 - x^2}} &= \int \frac{dx}{\sqrt{a^2 \left[1 - \left(\frac{x}{a}\right)^2 \right]}} \\ &= \int \frac{1}{|a|} \cdot \frac{dx}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \int \frac{1}{a} \cdot \frac{dx}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \text{ since } a > 0 \end{aligned}$$

Let $u = \frac{x}{a}$, so $du = \frac{1}{a} dx$.

$$\int \frac{1}{a} \cdot \frac{dx}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + C$$

$$= \sin^{-1} \frac{x}{a} + C$$

$$\begin{aligned}
 68. \quad D_x \sin^{-1} \frac{x}{a} &= \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a} \\
 &= \frac{1}{\sqrt{\frac{a^2 - x^2}{a^2}}} \cdot \frac{1}{a} = \frac{|a|}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a} \\
 &= \frac{a}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a}, \text{ since } a > 0 \\
 &= \frac{1}{\sqrt{a^2 - x^2}}
 \end{aligned}$$

$$\begin{aligned}
 69. \quad \text{Let } u = \frac{x}{a}, \text{ so } du &= \frac{1}{a} dx \\
 \int \frac{dx}{a^2 + x^2} &= \frac{1}{a} \int \frac{1}{1 + \left(\frac{x}{a}\right)^2} \frac{1}{a} dx \\
 &= \frac{1}{a} \int \frac{1}{1 + u^2} du = \frac{1}{a} \tan^{-1} u + C \\
 &= \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 70. \quad \text{Let } u = \frac{x}{a}, \text{ so } du &= (1/a) dx. \text{ Since } a > 0, \\
 \int \frac{dx}{x\sqrt{x^2 - a^2}} &= \frac{1}{a} \int \frac{1}{\left(\frac{x}{a}\right)\sqrt{\left(\frac{x}{a}\right)^2 - 1}} \frac{1}{a} dx \\
 &= \frac{1}{a} \int \frac{1}{u\sqrt{u^2 - 1}} du \\
 &= \frac{1}{a} \sec^{-1} |u| + C = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 71. \quad \text{Note that } \frac{d}{dx} \sin^{-1} \left(\frac{x}{a} \right) &= \frac{1}{\sqrt{a^2 - x^2}} \text{ (See} \\
 &\text{Problem 67).} \\
 \frac{d}{dx} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C \right]
 \end{aligned}$$

$$\begin{aligned}
 74. \quad \text{a. } \theta &= \cos^{-1} \left(\frac{x}{b} \right) - \cos^{-1} \left(\frac{x}{a} \right) \quad \frac{d\theta}{dt} = \left(\frac{-1}{\sqrt{1 - \left(\frac{x}{b}\right)^2}} \right) \left(\frac{1}{b} \right) \left(\frac{dx}{dt} \right) - \left(\frac{-1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \right) \left(\frac{1}{a} \right) \left(\frac{dx}{dt} \right) \\
 &= \left(\frac{1}{\sqrt{a^2 - x^2}} - \frac{1}{\sqrt{b^2 - x^2}} \right) \frac{dx}{dt}
 \end{aligned}$$

$$\text{b. } \theta = \tan^{-1} \left(\frac{a+x}{\sqrt{b^2 - x^2}} \right) - \sin^{-1} \left(\frac{x}{b} \right)$$

$$\begin{aligned}
 &= \frac{1}{2} \sqrt{a^2 - x^2} + \frac{x}{2} \frac{1}{\sqrt{a^2 - x^2}} (-2x) \\
 &\quad + \frac{a^2}{2} \frac{1}{\sqrt{a^2 - x^2}} + 0 \\
 &= \frac{1}{2} \sqrt{a^2 - x^2} + \frac{1 - x^2 + a^2}{2 \sqrt{a^2 - x^2}} = \sqrt{a^2 - x^2}
 \end{aligned}$$

$$\begin{aligned}
 72. \quad \int_{-a}^a \sqrt{a^2 - x^2} dx &= 2 \int_0^a \sqrt{a^2 - x^2} dx \\
 &= 2 \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a \\
 &= 2 \left[\frac{a}{2} (0) + \frac{a^2}{2} \sin^{-1}(1) - \frac{0}{2} \sqrt{a^2} - \frac{a^2}{2} \sin^{-1}(0) \right] \\
 &= a^2 \sin^{-1}(1) = \frac{\pi a^2}{2}
 \end{aligned}$$

This result is expected because the integral should be half the area of a circle with radius a .

73. Let θ be the angle subtended by the viewer's eye.

$$\begin{aligned}
 \theta &= \tan^{-1} \left(\frac{12}{b} \right) - \tan^{-1} \left(\frac{2}{b} \right) \\
 \frac{d\theta}{db} &= \frac{1}{1 + \left(\frac{12}{b}\right)^2} \left(-\frac{12}{b^2} \right) - \frac{1}{1 + \left(\frac{2}{b}\right)^2} \left(-\frac{2}{b^2} \right) \\
 &= \frac{2}{b^2 + 4} - \frac{12}{b^2 + 144} = \frac{10(24 - b^2)}{(b^2 + 4)(b^2 + 144)}
 \end{aligned}$$

Since $\frac{d\theta}{db} > 0$ for b in $[0, 2\sqrt{6})$

and $\frac{d\theta}{db} < 0$ for b in $(2\sqrt{6}, \infty)$, the angle is

maximized for $b = 2\sqrt{6} \approx 4.899$.

The ideal distance is about 4.9 ft from the wall.

$$\begin{aligned}
\frac{d\theta}{dt} &= \left(\frac{1}{1 + \left(\frac{a+x}{\sqrt{b^2-x^2}} \right)^2} \right) \left(\frac{\frac{\sqrt{b^2-x^2} + \frac{(a+x)x}{\sqrt{b^2-x^2}}}{b^2-x^2} \right) \left(\frac{dx}{dt} \right) - \left(\frac{1}{\sqrt{1 - \left(\frac{x}{b} \right)^2}} \right) \left(\frac{1}{b} \right) \left(\frac{dx}{dt} \right) \\
&= \left[\left(\frac{b^2-x^2}{b^2-x^2 + (a+x)^2} \right) \left(\frac{b^2+ax}{(b^2-x^2)^{3/2}} \right) - \frac{1}{\sqrt{b^2-x^2}} \right] \frac{dx}{dt} \\
&= \left[\frac{b^2+ax}{(b^2+a^2+2ax)\sqrt{b^2-x^2}} - \frac{1}{\sqrt{b^2-x^2}} \right] \frac{dx}{dt} \\
&= \left[-\frac{a^2+ax}{(b^2+a^2+2ax)\sqrt{b^2-x^2}} \right] \frac{dx}{dt}
\end{aligned}$$

75. Let $h(t)$ represent the height of the elevator (the number of feet above the spectator's line of sight) t seconds after the line of sight passes horizontal, and let $\theta(t)$ denote the angle of elevation.

Then $h(t) = 15t$, so $\theta(t) = \tan^{-1} \left(\frac{15t}{60} \right) = \tan^{-1} \left(\frac{t}{4} \right)$.

$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{t}{4} \right)^2} \left(\frac{1}{4} \right) = \frac{4}{16 + t^2}$$

At $t = 6$, $\frac{d\theta}{dt} = \frac{4}{16 + 6^2} = \frac{1}{13}$ radians per second or about 4.41° per second.

76. Let $x(t)$ be the horizontal distance from the observer to the plane, in miles, t minutes before the plane is overhead. Let $t = 0$ when the distance to the plane is 3 miles. Then

$x(0) = \sqrt{3^2 - 2^2} = \sqrt{5}$. The speed of the plane is 10 miles per minute, so $x(t) = \sqrt{5} - 10t$. The angle of

elevation is $\theta(t) = \tan^{-1} \left(\frac{2}{x(t)} \right) = \tan^{-1} \left(\frac{2}{\sqrt{5} - 10t} \right)$.

$$\begin{aligned}
\text{so } \frac{d\theta}{dt} &= \frac{1}{1 + \left(\frac{2}{\sqrt{5} - 10t} \right)^2} \left(\frac{-2}{(\sqrt{5} - 10t)^2} \right) (-10) \\
&= \frac{20}{(\sqrt{5} - 10t)^2 + 4}.
\end{aligned}$$

When $t = 0$, $\frac{d\theta}{dt} = \frac{20}{9} \approx 2.22$ radians per minute.

77. Let x represent the position on the shoreline and let θ represent the angle of the beam ($x = 0$ and $\theta = 0$ when the light is pointed at P). Then

$$\theta = \tan^{-1} \left(\frac{x}{2} \right), \text{ so } \frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{x}{2} \right)^2} \cdot \frac{1}{2} \frac{dx}{dt} = \frac{2}{4 + x^2} \frac{dx}{dt}$$

When $x = 1$,

$\frac{dx}{dt} = 5\pi$, so $\frac{d\theta}{dt} = \frac{2}{4 + 1^2} (5\pi) = 2\pi$. The beacon revolves at a rate of 2π radians per minute or 1 revolution per minute.

78. Let x represent the length of the rope and let θ represent the angle of depression of the rope.

Then $\theta = \sin^{-1} \left(\frac{8}{x} \right)$, so

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1 - \left(\frac{8}{x} \right)^2}} \cdot \frac{8}{x^2} \frac{dx}{dt} = -\frac{8}{x\sqrt{x^2 - 64}} \frac{dx}{dt}.$$

When $x = 17$ and $\frac{dx}{dt} = -5$, we obtain

$$\frac{d\theta}{dt} = -\frac{8}{17\sqrt{17^2 - 64}} (-5) = \frac{8}{51}.$$

The angle of depression is increasing at a rate of $8/51 \approx 0.16$ radians per second.

79. Let x represent the distance to the center of the earth and let θ represent the angle subtended by the

earth. Then $\theta = 2 \sin^{-1} \left(\frac{6376}{x} \right)$, so

$$\begin{aligned}
\frac{d\theta}{dt} &= 2 \frac{1}{\sqrt{1 - \left(\frac{6376}{x} \right)^2}} \left(-\frac{6376}{x^2} \right) \frac{dx}{dt} \\
&= -\frac{12,752}{x\sqrt{x^2 - 6376^2}} \frac{dx}{dt}
\end{aligned}$$

When she is 3000 km from the surface

$x = 3000 + 6376 = 9376$ and $\frac{dx}{dt} = -2$. Substituting

these values, we obtain $\frac{d\theta}{dt} \approx 3.96 \times 10^{-4}$ radians per second.

7.8 Concepts Review

1. $\frac{e^x - e^{-x}}{2}; \frac{e^x + e^{-x}}{2}$
2. $\cosh^2 x - \sinh^2 x = 1$
3. the graph of $x^2 - y^2 = 1$, a hyperbola
4. catenary; a hanging chain

Problem Set 7.8

1. $\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$
 $= \frac{2e^x}{2} = e^x$
2. $\cosh 2x + \sinh 2x = \frac{e^{2x} + e^{-2x}}{2} + \frac{e^{2x} - e^{-2x}}{2}$
 $= \frac{2e^{2x}}{2} = e^{2x}$
3. $\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}$
 $= \frac{2e^{-x}}{2} = e^{-x}$
4. $\cosh 2x - \sinh 2x = \frac{e^{2x} + e^{-2x}}{2} - \frac{e^{2x} - e^{-2x}}{2} = \frac{2e^{-2x}}{2} = e^{-2x}$
5. $\sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$
 $= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} + \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}}{4}$
 $= \frac{2e^{x+y} - 2e^{-(x+y)}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$
6. $\sinh x \cosh y - \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} - \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$
 $= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} - \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}}{4}$
 $= \frac{2e^{x-y} - 2e^{-x+y}}{4} = \frac{e^{x-y} - e^{-(x-y)}}{2} = \sinh(x-y)$

$$\begin{aligned}
 7. \quad \cosh x \cosh y + \sinh x \sinh y &= \frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} \\
 &= \frac{e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}}{4} + \frac{e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}}{4} \\
 &= \frac{2e^{x+y} + 2e^{-x-y}}{4} = \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x+y)
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \cosh x \cosh y - \sinh x \sinh y &= \frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} - \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} \\
 &= \frac{e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}}{4} - \frac{e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}}{4} \\
 &= \frac{2e^{x-y} + 2e^{-x+y}}{4} = \frac{e^{x-y} + e^{-(x-y)}}{2} = \cosh(x-y)
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} &= \frac{\frac{\sinh x}{\cosh x} + \frac{\sinh y}{\cosh y}}{1 + \frac{\sinh x}{\cosh x} \cdot \frac{\sinh y}{\cosh y}} \\
 &= \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} = \frac{\sinh(x+y)}{\cosh(x+y)} \\
 &= \tanh(x+y)
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y} &= \frac{\frac{\sinh x}{\cosh x} - \frac{\sinh y}{\cosh y}}{1 - \frac{\sinh x}{\cosh x} \cdot \frac{\sinh y}{\cosh y}} \\
 &= \frac{\sinh x \cosh y - \cosh x \sinh y}{\cosh x \cosh y - \sinh x \sinh y} = \frac{\sinh(x-y)}{\cosh(x-y)} \\
 &= \tanh(x-y)
 \end{aligned}$$

$$\begin{aligned}
 11. \quad 2 \sinh x \cosh x &= \sinh x \cosh x + \cosh x \sinh x \\
 &= \sinh(x+x) = \sinh 2x
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \cosh^2 x + \sinh^2 x &= \cosh x \cosh x + \sinh x \sinh x \\
 &= \cosh(x+x) = \cosh 2x
 \end{aligned}$$

$$13. \quad D_x \sinh^2 x = 2 \sinh x \cosh x = \sinh 2x$$

$$14. \quad D_x \cosh^2 x = 2 \cosh x \sinh x = \sinh 2x$$

$$15. \quad D_x (5 \sinh^2 x) = 10 \sinh x \cdot \cosh x = 5 \sinh 2x$$

$$16. \quad D_x \cosh^3 x = 3 \cosh^2 x \sinh x$$

$$17. \quad D_x \cosh(3x+1) = \sinh(3x+1) \cdot 3 = 3 \sinh(3x+1)$$

$$\begin{aligned}
 18. \quad D_x \sinh(x^2 + x) &= \cosh(x^2 + x) \cdot (2x+1) \\
 &= (2x+1) \cosh(x^2 + x)
 \end{aligned}$$

$$\begin{aligned}
 19. \quad D_x \ln(\sinh x) &= \frac{1}{\sinh x} \cdot \cosh x = \frac{\cosh x}{\sinh x} \\
 &= \coth x
 \end{aligned}$$

$$\begin{aligned}
 20. \quad D_x \ln(\coth x) &= \frac{1}{\coth x} (-\operatorname{csch}^2 x) \\
 &= -\frac{\sinh x}{\cosh x} \cdot \frac{1}{\sinh^2 x} = -\frac{1}{\sinh x \cosh x} \\
 &= -\operatorname{csch} x \operatorname{sech} x
 \end{aligned}$$

$$\begin{aligned}
 21. \quad D_x (x^2 \cosh x) &= x^2 \cdot \sinh x + \cosh x \cdot 2x \\
 &= x^2 \sinh x + 2x \cosh x
 \end{aligned}$$

$$\begin{aligned}
 22. \quad D_x (x^{-2} \sinh x) &= x^{-2} \cdot \cosh x + \sinh x \cdot (-2x^{-3}) \\
 &= x^{-2} \cosh x - 2x^{-3} \sinh x
 \end{aligned}$$

$$23. \quad D_x (\cosh 3x \sinh x) = \cosh 3x \cdot \cosh x + \sinh x \cdot \sinh 3x \cdot 3 = \cosh 3x \cosh x + 3 \sinh 3x \sinh x$$

$$24. \quad D_x (\sinh x \cosh 4x) = \sinh x \cdot \sinh 4x \cdot 4 + \cosh 4x \cdot \cosh x = 4 \sinh x \sinh 4x + \cosh x \cosh 4x$$

$$25. \quad D_x (\tanh x \sinh 2x) = \tanh x \cdot \cosh 2x \cdot 2 + \sinh 2x \cdot \operatorname{sech}^2 x = 2 \tanh x \cosh 2x + \sinh 2x \operatorname{sech}^2 x$$

$$26. \quad D_x (\coth 4x \sinh x) = \coth 4x \cdot \cosh x + \sinh x (-\operatorname{csch}^2 4x) \cdot 4 = \cosh x \coth 4x - 4 \sinh x \operatorname{csch}^2 4x$$

$$27. D_x \sinh^{-1}(x^2) = \frac{1}{\sqrt{(x^2)^2 + 1}} \cdot 2x = \frac{2x}{\sqrt{x^4 + 1}}$$

$$28. D_x \cosh^{-1}(x^3) = \frac{1}{\sqrt{(x^3)^2 - 1}} \cdot 3x^2 = \frac{3x^2}{\sqrt{x^6 - 1}}$$

$$29. D_x \tanh^{-1}(2x - 3) = \frac{1}{1 - (2x - 3)^2} \cdot 2 = \frac{2}{1 - (4x^2 - 12x + 9)} = \frac{2}{-4x^2 + 12x - 8} = -\frac{1}{2(x^2 - 3x + 2)}$$

$$30. D_x \coth^{-1}(x^5) = D_x \tanh^{-1}\left(\frac{1}{x^5}\right) = \frac{1}{1 - \left(\frac{1}{x^5}\right)^2} \cdot \left(-\frac{5}{x^6}\right) = \frac{x^{10}}{x^{10} - 1} \cdot \left(-\frac{5}{x^6}\right) = -\frac{5x^4}{x^{10} - 1}$$

$$31. D_x [x \cosh^{-1}(3x)] = x \cdot \frac{1}{\sqrt{(3x)^2 - 1}} \cdot 3 + \cosh^{-1}(3x) \cdot 1 = \frac{3x}{\sqrt{9x^2 - 1}} + \cosh^{-1} 3x$$

$$32. D_x (x^2 \sinh^{-1} x^5) = x^2 \cdot \frac{1}{\sqrt{(x^5)^2 + 1}} \cdot 5x^4 + \sinh^{-1} x^5 \cdot 2x = \frac{5x^6}{\sqrt{x^{10} + 1}} + 2x \sinh^{-1} x^5$$

$$33. D_x \ln(\cosh^{-1} x) = \frac{1}{\cosh^{-1} x} \cdot \frac{1}{\sqrt{x^2 - 1}} \\ = \frac{1}{\sqrt{x^2 - 1} \cosh^{-1} x}$$

34. $\cosh^{-1}(\cos x)$ does not have a derivative, since $D_u \cosh^{-1} u$ is only defined for $u > 1$ while $\cos x \leq 1$ for all x .

$$35. D_x \tanh(\cot x) = \operatorname{sech}^2(\cot x) \cdot (-\csc^2 x) \\ = -\csc^2 x \operatorname{sech}^2(\cot x)$$

$$36. D_x \coth^{-1}(\tanh x) = D_x \tanh^{-1}\left(\frac{1}{\tanh x}\right) \\ = D_x \tanh^{-1}(\coth x) \\ = \frac{1}{1 - (\coth x)^2} (-\operatorname{csch}^2 x) = \frac{-\operatorname{csch}^2 x}{-\operatorname{csch}^2 x} = 1$$

$$37. \text{Area} = \int_0^{\ln 3} \cosh 2x dx = \left[\frac{1}{2} \sinh 2x \right]_0^{\ln 3} \\ = \frac{1}{2} \left(\frac{e^{2 \ln 3} - e^{-2 \ln 3}}{2} - \frac{e^0 - e^{-0}}{2} \right) \\ = \frac{1}{4} (e^{\ln 9} - e^{-\ln 9}) = \frac{1}{4} \left(9 - \frac{1}{9} \right) = \frac{20}{9}$$

38. Let $u = 3x + 2$, so $du = 3 dx$.

$$\int \sinh(3x + 2) dx = \frac{1}{3} \int \sinh u du = \frac{1}{3} \cosh u + C \\ = \frac{1}{3} \cosh(3x + 2) + C$$

39. Let $u = \pi x^2 + 5$, so $du = 2\pi x dx$.

$$\int x \cosh(\pi x^2 + 5) dx = \frac{1}{2\pi} \int \cosh u du \\ = \frac{1}{2\pi} \sinh u + C = \frac{1}{2\pi} \sinh(\pi x^2 + 5) + C$$

40. Let $u = \sqrt{z}$, so $du = \frac{1}{2\sqrt{z}} dz$.

$$\int \frac{\cosh \sqrt{z}}{\sqrt{z}} dz = 2 \int \cosh u du = 2 \sinh u + C \\ = 2 \sinh \sqrt{z} + C$$

41. Let $u = 2z^{1/4}$, so $du = \frac{1}{4} \cdot 2z^{-3/4} dz = \frac{1}{2\sqrt[4]{z^3}} dz$.

$$\int \frac{\sinh(2z^{1/4})}{\sqrt[4]{z^3}} dz = 2 \int \sinh u du = 2 \cosh u + C \\ = 2 \cosh(2z^{1/4}) + C$$

42. Let $u = e^x$, so $du = e^x dx$.

$$\int e^x \sinh e^x dx = \int \sinh u du = \cosh u + C \\ = \cosh e^x + C$$

43. Let $u = \sin x$, so $du = \cos x dx$

$$\int \cos x \sinh(\sin x) dx = \int \sinh u du = \cosh u + C \\ = \cosh(\sin x) + C$$

44. Let $u = \ln(\cosh x)$, so

$$du = \frac{1}{\cosh x} \cdot \sinh x = \tanh x dx.$$

$$\int \tanh x \ln(\cosh x) dx = \int u du = \frac{u^2}{2} + C \\ = \frac{1}{2} [\ln(\cosh x)]^2 + C$$

45. Let $u = \ln(\sinh x^2)$, so

$$du = \frac{1}{\sinh x^2} \cdot \cosh x^2 \cdot 2x dx = 2x \coth x^2 dx.$$

$$\int x \coth x^2 \ln(\sinh x^2) dx = \frac{1}{2} \int u du = \frac{1}{2} \cdot \frac{u^2}{2} + C \\ = \frac{1}{4} [\ln(\sinh x^2)]^2 + C$$

46. Area = $\int_{-\ln 5}^{\ln 5} \cosh 2x dx = 2 \int_0^{\ln 5} \cosh 2x dx$

$$= 2 \left[\frac{1}{2} \sinh 2x \right]_0^{\ln 5} \\ = \sinh(2 \ln 5) = \frac{1}{2} (e^{2 \ln 5} - e^{-2 \ln 5}) \\ = \frac{1}{2} (e^{\ln 25} - e^{-\ln 25}) = \frac{1}{2} \left(25 - \frac{1}{25} \right) \\ = \frac{312}{25} = 12.48$$

47. Note that the graphs of $y = \sinh x$ and $y = 0$ intersect at the origin.

$$\text{Area} = \int_0^{\ln 2} \sinh x dx = [\cosh x]_0^{\ln 2} \\ = \frac{e^{\ln 2} + e^{-\ln 2}}{2} - \frac{e^0 + e^0}{2} = \frac{1}{2} \left(2 + \frac{1}{2} \right) - 1 = \frac{1}{4}$$

48. $\tanh x = 0$ when $\sinh x = 0$, which is when $x = 0$.

$$\text{Area} = \int_{-8}^0 (-\tanh x) dx + \int_0^8 \tanh x dx$$

$$= 2 \int_0^8 \tanh x dx = 2 \int_0^8 \frac{\sinh x}{\cosh x} dx$$

Let $u = \cosh x$, so $du = \sinh x dx$.

$$2 \int \frac{\sinh x}{\cosh x} dx = 2 \int \frac{1}{u} du = 2 \ln |u| + C$$

$$2 \int_0^8 \frac{\sinh x}{\cosh x} dx = [2 \ln |\cosh x|]_0^8 \\ = 2(\ln |\cosh 8| - \ln 1) = 2 \ln(\cosh 8) \approx 14.61$$

49. Volume = $\int_0^1 \pi \cosh^2 x dx = \frac{\pi}{2} \int_0^1 (1 + \cosh 2x) dx$

$$= \frac{\pi}{2} \left[x + \frac{\sinh 2x}{2} \right]_0^1 \\ = \frac{\pi}{2} \left(1 + \frac{\sinh 2}{2} - 0 \right) \\ = \frac{\pi}{2} + \frac{\pi \sinh 2}{4} \approx 4.42$$

50. Volume = $\int_0^{\ln 10} \pi \sinh^2 x dx$

$$= \pi \int_0^{\ln 10} \left(\frac{e^x - e^{-x}}{2} \right)^2 dx \\ = \pi \int_0^{\ln 10} \frac{e^{2x} - 2 + e^{-2x}}{4} dx = \frac{\pi}{4} \int_0^{\ln 10} (e^{2x} - 2 + e^{-2x}) dx \\ = \frac{\pi}{4} \left[\frac{1}{2} e^{2x} - 2x - \frac{1}{2} e^{-2x} \right]_0^{\ln 10} \\ = \frac{\pi}{8} [e^{2x} - 4x - e^{-2x}]_0^{\ln 10} \\ = \frac{\pi}{8} \left(100 - 4 \ln 10 - \frac{1}{100} \right) \approx 35.65$$

51. Note that $1 + \sinh^2 x = \cosh^2 x$ and

$$\cosh^2 x = \frac{1 + \cosh 2x}{2}$$

$$\text{Surface area} = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

$$= \int_0^1 2\pi \cosh x \sqrt{1 + \sinh^2 x} dx$$

$$= \int_0^1 2\pi \cosh x \cosh x dx$$

$$= \int_0^1 \pi (1 + \cosh 2x) dx$$

$$= \left[\pi x + \frac{\pi}{2} \sinh 2x \right]_0^1 = \pi + \frac{\pi}{2} \sinh 2 \approx 8.84$$

$$52. \text{ Surface area} = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 2\pi \sinh x \sqrt{1 + \cosh^2 x} dx$$

Let $u = \cosh x$, so $du = \sinh x dx$

$$\int 2\pi \sinh x \sqrt{1 + \cosh^2 x} dx = 2\pi \int \sqrt{1 + u^2} du = 2\pi \left[\frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}| + C \right]$$

$= \pi \cosh x \sqrt{1 + \cosh^2 x} + \pi \ln |\cosh x + \sqrt{1 + \cosh^2 x}| + C$ (The integration of $\int \sqrt{1 + u^2} du$ is shown in Formula 44 of the Tables in the back of the text, which is covered in Chapter 8.)

$$\int_0^1 2\pi \sinh x \sqrt{1 + \cosh^2 x} dx = \pi \left[\cosh x \sqrt{1 + \cosh^2 x} + \ln |\cosh x + \sqrt{1 + \cosh^2 x}| \right]_0^1$$

$$= \pi \left[\cosh 1 \sqrt{1 + \cosh^2 1} + \ln |\cosh 1 + \sqrt{1 + \cosh^2 1}| - (\sqrt{2} + \ln |1 + \sqrt{2}|) \right] \approx 5.53$$

$$53. y = a \cosh\left(\frac{x}{a}\right) + C$$

$$\frac{dy}{dx} = \sinh\left(\frac{x}{a}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{a} \cosh\left(\frac{x}{a}\right)$$

We need to show that $\frac{d^2y}{dx^2} = \frac{1}{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

Note that $1 + \sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$ and $\cosh\left(\frac{x}{a}\right) > 0$. Therefore,

$$\frac{1}{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{a} \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)} = \frac{1}{a} \sqrt{\cosh^2\left(\frac{x}{a}\right)} = \frac{1}{a} \cosh\left(\frac{x}{a}\right) = \frac{d^2y}{dx^2}$$

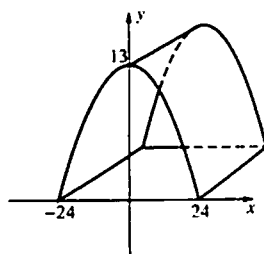
54. a. The graph of $y = b - a \cosh\left(\frac{x}{a}\right)$ is symmetric about the y -axis, so if its width along the

x -axis is $2a$, its x -intercepts are $(\pm a, 0)$. Therefore, $y(a) = b - a \cosh\left(\frac{a}{a}\right) = 0$, so $b = a \cosh 1 \approx 1.54308a$.

b. The height is $y(0) \approx 1.54308a - a \cosh 0 = 0.54308a$.

c. If $2a = 48$, the height is about $0.54308a = (0.54308)(24) \approx 13$.

55. a.



b. Area under the curve is

$$\int_{-24}^{24} \left[37 - 24 \cosh\left(\frac{x}{24}\right) \right] dx = \left[37x - 576 \sinh\left(\frac{x}{24}\right) \right]_{-24}^{24} \approx 422$$

Volume is about $(422)(100) = 42,200 \text{ ft}^3$.

c. Length of the curve is

$$\int_{-24}^{24} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{-24}^{24} \sqrt{1 + \sinh^2\left(\frac{x}{24}\right)} dx = \int_{-24}^{24} \cosh\left(\frac{x}{24}\right) dx = \left[24 \sinh\left(\frac{x}{24}\right) \right]_{-24}^{24} = 48 \sinh 1 \approx 56.4$$

$$\text{Surface area} \approx (56.4)(100) = 5640 \text{ ft}^2$$

$$\begin{aligned} 56. \text{ Area} &= \frac{1}{2} \cosh t \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1} dx = \frac{1}{2} \cosh t \sinh t - \left[\frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| \right]_1^{\cosh t} \\ &= \frac{1}{2} \cosh t \sinh t - \left[\frac{1}{2} \cosh t \sqrt{\cosh^2 t - 1} - \frac{1}{2} \ln |\cosh t + \sqrt{\cosh^2 t - 1}| - 0 \right] \\ &= \frac{1}{2} \cosh t \sinh t - \frac{1}{2} \cosh t \sinh t + \frac{1}{2} \ln |\cosh t + \sinh t| = \frac{1}{2} \ln e^t = \frac{t}{2} \end{aligned}$$

$$\begin{aligned} 57. \text{ a. } (\sinh x + \cosh x)^r &= \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} \right)^r = \left(\frac{2e^x}{2} \right)^r = e^{rx} \\ \sinh rx + \cosh rx &= \frac{e^{rx} - e^{-rx}}{2} + \frac{e^{rx} + e^{-rx}}{2} = \frac{2e^{rx}}{2} = e^{rx} \end{aligned}$$

$$\begin{aligned} \text{b. } (\cosh x - \sinh x)^r &= \left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \right)^r = \left(\frac{2e^{-x}}{2} \right)^r = e^{-rx} \\ \cosh rx - \sinh rx &= \frac{e^{rx} + e^{-rx}}{2} - \frac{e^{rx} - e^{-rx}}{2} = \frac{2e^{-rx}}{2} = e^{-rx} \end{aligned}$$

$$\begin{aligned} \text{c. } (\cos x + i \sin x)^r &= \left(\frac{e^{ix} + e^{-ix}}{2} + i \frac{e^{ix} - e^{-ix}}{2i} \right)^r = \left(\frac{2e^{ix}}{2} \right)^r = e^{irx} \\ \cos rx + i \sin rx &= \frac{e^{irx} + e^{-irx}}{2} + i \frac{e^{irx} - e^{-irx}}{2i} = \frac{2e^{irx}}{2} = e^{irx} \end{aligned}$$

$$\begin{aligned} \text{d. } (\cos x - i \sin x)^r &= \left(\frac{e^{ix} + e^{-ix}}{2} - i \frac{e^{ix} - e^{-ix}}{2i} \right)^r = \left(\frac{2e^{-ix}}{2} \right)^r = e^{-irx} \\ \cos rx - i \sin rx &= \frac{e^{irx} + e^{-irx}}{2} - i \frac{e^{irx} - e^{-irx}}{2i} = \frac{2e^{-irx}}{2} = e^{-irx} \end{aligned}$$

$$\begin{aligned} 58. \text{ a. } gd(-t) &= \tan^{-1}[\sinh(-t)] \\ &= \tan^{-1}(-\sinh t) = -\tan^{-1}(\sinh t) = -gd(t) \end{aligned}$$

so gd is odd.

$$\begin{aligned} D_t[gd(t)] &= \frac{1}{1 + \sinh^2 t} \cdot \cosh t = \frac{\cosh t}{\cosh^2 t} \\ &= \text{sech } t > 0 \text{ for all } t, \text{ so } gd \text{ is increasing.} \end{aligned}$$

$$D_t^2[gd(t)] = D_t(\text{sech } t) = -\text{sech } t \tanh t$$

$$\begin{aligned} D_t^2[gd(t)] &= 0 \text{ when } \tanh t = 0, \text{ since} \\ \text{sech } t > 0 \text{ for all } t. \tanh t &= 0 \text{ at } t = 0 \text{ and} \\ \tanh t < 0 \text{ for } t < 0, \text{ thus } D_t^2[gd(t)] > 0 &\text{ for} \\ t < 0 \text{ and } D_t^2[gd(t)] < 0 &\text{ for } t > 0. \text{ Hence} \end{aligned}$$

$gd(t)$ has an inflection point at
(0, $gd(0)$) = (0, $\tan^{-1} 0$) = (0, 0).

b. If $y = \tan^{-1}(\sinh t)$ then $\tan y = \sinh t$ so

$$\sin y = \frac{\tan y}{\sqrt{\tan^2 y + 1}} = \frac{\sinh t}{\sqrt{\sinh^2 t + 1}}$$

$$= \frac{\sinh t}{\cosh t} = \tanh t \text{ so } y = \sin^{-1}(\tanh t)$$

$$\text{Also, } D_t y = \frac{1}{1 + \sinh^2 t} \cdot \cosh t$$

$$= \frac{\cosh t}{\cosh^2 t} = \frac{1}{\cosh t} = \text{sech } t,$$

so $y = \int_0^x \operatorname{sech} u \, du$ by the Fundamental Theorem of Calculus.

59. Area = $\int_0^x \cosh t \, dt = [\sinh t]_0^x = \sinh x$

Arc length =

$$\int_0^x \sqrt{1 + [D_t \cosh t]^2} \, dt = \int_0^x \sqrt{1 + \sinh^2 t} \, dt$$

$$= \int_0^x \cosh t \, dt = [\sinh t]_0^x = \sinh x$$

60. From Problem 54, the equation of an inverted catenary is $y = b - a \cosh \frac{x}{a}$. Given the information about the Gateway Arch, the curve passes through the points $(\pm 315, 0)$ and $(0, 630)$.

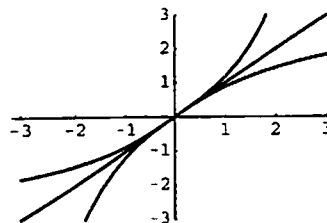
Thus, $b = a \cosh \frac{315}{a}$ and $630 = b - a$, so

$$b = a + 630.$$

$$a + 630 = a \cosh \frac{315}{a} \Rightarrow a \approx 128, \text{ so } b \approx 758.$$

The equation is $y = 758 - 128 \cosh \frac{x}{128}$.

61.



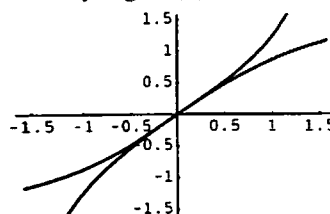
The functions $y = \sinh x$ and $y = \ln(x + \sqrt{x^2 + 1})$ are inverse functions.

62. $y = gd(x) = \tan^{-1}(\sinh x)$

$$\tan y = \sinh x$$

$$x = gd^{-1}(y) = \sinh^{-1}(\tan y)$$

Thus, $y = gd^{-1}(x) = \sinh^{-1}(\tan x)$



7.9 Chapter Review

Concepts Test

1. False: $\ln 0$ is undefined.

2. True: $\frac{d^2 y}{dx^2} = -\frac{1}{x^2} < 0$ for all $x > 0$.

3. True: $\int_1^{e^3} \frac{1}{t} \, dt = [\ln |t|]_1^{e^3} = \ln e^3 - \ln 1 = 3$

4. False: The graph is intersected *at most* once by every horizontal line.

5. True: The range of $y = \ln x$ is the set of all real numbers.

6. False: $\ln x - \ln y = \ln\left(\frac{x}{y}\right)$

7. False: $4 \ln x = \ln(x^4)$

8. True: $\ln(2e^{x+1}) - \ln(2e^x) = \ln \frac{2e^{x+1}}{2e^x}$
 $= \ln e = 1$

9. True: $f(g(x)) = 4 + e^{\ln(x-4)}$
 $= 4 + (x - 4) = x$
 and
 $g(f(x)) = \ln(4 + e^x - 4) = \ln e^x = x$

10. False: $\exp(x + y) = \exp x \exp y$

11. True: $\ln x$ is an increasing function.

12. False: Only true for $x > 1$, or $\ln x > 0$.

13. True: $e^z > 0$ for all z .

14. True: e^x is an increasing function.

15. True: $\lim_{x \rightarrow 0^+} (\ln \sin x - \ln x)$
 $= \lim_{x \rightarrow 0^+} \ln\left(\frac{\sin x}{x}\right) = \ln 1 = 0$

16. True: $\pi^{\sqrt{2}} = e^{\sqrt{2} \ln \pi}$

17. False: $\ln \pi$ is a constant so $\frac{d}{dx} \ln \pi = 0$.

18. True: $\frac{d}{dx}(\ln 3|x| + C)$
 $= \frac{d}{dx}(\ln|x| + \ln 3 + C) = \frac{1}{x}$

19. True: e is a number.

20. True: $\exp[g(x)] \neq 0$ because 0 is not in the range of the function $y = e^x$.

21. False: $D_x(x^x) = x^x(1 + \ln x)$

22. True: $2(\tan x + \sec x)' - (\tan x + \sec x)^2$
 $= 2(\sec^2 x + \sec x \tan x)$
 $- \tan^2 x - 2 \tan x \sec x - \sec^2 x$
 $= \sec^2 x - \tan^2 x = 1$

23. True: The integrating factor is
 $e^{\int 4/x dx} = e^{4 \ln x} = (e^{\ln x})^4 = x^4$

24. False: $\sin(\arcsin(2))$ is undefined

25. False: $\arcsin(\sin 2\pi) = \arcsin 0 = 0$

26. True: $\sinh x$ is increasing.

27. False: $\cosh x$ is not increasing.

28. True: $\cosh(0) = 1 = e^0$
 If $x > 0$, $e^x > 1$ while $e^{-x} < 1 < e^x$ so
 $\cosh x = \frac{1}{2}(e^x + e^{-x}) < \frac{1}{2}(2e^x)$
 $= e^x = e^{|x|}$. If $x < 0$, $-x > 0$ and
 $e^{-x} > 1$ while $e^x < 1 < e^{-x}$ so
 $\cosh x = \frac{1}{2}(e^x + e^{-x}) < \frac{1}{2}(2e^{-x})$
 $= e^{-x} = e^{|x|}$.

29. True: $|\sinh x| \leq \frac{1}{2}e^{|x|}$ is equivalent to
 $|e^x - e^{-x}| \leq e^{|x|}$. When $x = 0$,
 $\sinh x = 0 < \frac{1}{2}e^0 = \frac{1}{2}$. If $x > 0$,
 $e^x > 1$ and $e^{-x} < 1 < e^x$, thus
 $|e^x - e^{-x}| = e^x - e^{-x} < e^x = e^{|x|}$.
 If $x < 0$, $e^{-x} > 1$ and $e^x < 1 < e^{-x}$,
 thus

$$|e^x - e^{-x}| = -(e^x - e^{-x})$$

$$= e^{-x} - e^x < e^{-x} = e^{|x|}.$$

30. False: $\tan^{-1}\left(\frac{1}{2}\right) \approx 0.4636$

but $\frac{\sin^{-1}\left(\frac{1}{2}\right)}{\cos^{-1}\left(\frac{1}{2}\right)} = \frac{1}{2}$

31. False: $\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2}$
 $= \frac{1}{2}\left(3 + \frac{1}{3}\right) = \frac{5}{3}$

32. False: $\lim_{x \rightarrow 0} \ln\left(\frac{\sin x}{x}\right) = \ln 1 = 0$

33. True: $\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$, since
 $\lim_{x \rightarrow -\frac{\pi}{2}^+} \tan x = -\infty$.

34. False: $\cosh x > 1$ for $x \neq 0$, while $\sin^{-1} u$ is only defined for $-1 \leq u \leq 1$.

35. True: $\tanh x = \frac{\sinh x}{\cosh x}$; $\sinh x$ is an odd function and $\cosh x$ is an even function.

36. False: Both functions satisfy $y'' - y = 0$.

37. True: $\ln 3^{100} = 100 \ln 3 > 100 \cdot 1$ since $\ln 3 > 1$.

38. False: $\ln(x - 3)$ is not defined for $x < 3$.

39. True: y triples every time t increases by t_1 .

40. False: $x(0) = C$; $\frac{1}{2}C = Ce^{-kt}$ when
 $\frac{1}{2} = e^{-kt}$, so $\ln \frac{1}{2} = -kt$ or
 $t = \frac{\ln \frac{1}{2}}{-k} = \frac{-\ln 2}{-k} = \frac{\ln 2}{k}$

41. True: $(y(t) + z(t))' = y'(t) + z'(t)$
 $= ky(t) + kz(t) = k(y(t) + z(t))$

42. False: Only true if $C = 0$;
 $(y_1(t) + y_2(t))' = y_1'(t) + y_2'(t)$

$$= ky_1(t) + C + ky_2(t) + C$$

$$= k(y_1(t) + y_2(t)) + 2C.$$

43. False: Use the substitution $u = -h$.
 $\lim_{h \rightarrow 0} (1-h)^{-1/h} = \lim_{u \rightarrow 0} (1+u)^{1/u} = e$
 by Theorem 7.5.A.

44. False: $e^{0.05} \approx 1.051 < \left(1 + \frac{0.06}{12}\right)^{12} \approx 1.062$

45. True: If $D_x(a^x) = a^x \ln a = a^x$, then
 $\ln a = 1$, so $a = e$.

Sample Test Problems

- $\ln \frac{x^4}{2} = 4 \ln x - \ln 2$
 $\frac{d}{dx} \ln \frac{x^4}{2} = \frac{d}{dx} (4 \ln x - \ln 2) = \frac{4}{x}$
- $\frac{d}{dx} \sin^2(x^3) = 2 \sin(x^3) \frac{d}{dx} \sin(x^3)$
 $= 2 \sin(x^3) \cos(x^3) \frac{d}{dx} x^3 = 6x^2 \sin(x^3) \cos(x^3)$
- $\frac{d}{dx} e^{x^2-4x} = e^{x^2-4x} \frac{d}{dx} (x^2-4x)$
 $= (2x-4)e^{x^2-4x}$
- $\frac{d}{dx} \log_{10}(x^5-1) = \frac{1}{(x^5-1) \ln 10} \frac{d}{dx} (x^5-1)$
 $= \frac{5x^4}{(x^5-1) \ln 10}$
- $\frac{d}{dx} \tan(\ln e^x) = \frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} e^{\ln \cot x} = \frac{d}{dx} \cot x = -\csc^2 x$
- $\frac{d}{dx} 2 \tanh \sqrt{x} = 2 \operatorname{sech}^2 \sqrt{x} \frac{d}{dx} \sqrt{x} = \frac{\operatorname{sech}^2 \sqrt{x}}{\sqrt{x}}$
- $\frac{d}{dx} \tanh^{-1}(\sin x) = \frac{1}{1-\sin^2 x} \frac{d}{dx} \sin x = \frac{\cos x}{1-\sin^2 x}$
 $= \frac{\cos x}{\cos^2 x} = \sec x$
- $\frac{d}{dx} \sinh^{-1}(\tan x) = \frac{1}{\sqrt{\tan^2 x + 1}} \frac{d}{dx} \tan x$
 $= \frac{\sec^2 x}{\sqrt{\tan^2 x + 1}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = |\sec x|$
- $\frac{d}{dx} 2 \sin^{-1} \sqrt{3x} = \frac{2}{\sqrt{1-(\sqrt{3x})^2}} \frac{d}{dx} \sqrt{3x}$
 $= \frac{2}{\sqrt{1-3x}} \frac{3}{2\sqrt{3x}} = \frac{3}{\sqrt{3x-9x^2}}$
- $\frac{d}{dx} \sec^{-1} e^x = \frac{1}{|e^x| \sqrt{(e^x)^2 - 1}} \frac{d}{dx} e^x$
 $= \frac{e^x}{e^x \sqrt{e^{2x} - 1}} = \frac{1}{\sqrt{e^{2x} - 1}}$
- $\frac{d}{dx} \ln \sin^2\left(\frac{x}{2}\right) = \frac{1}{\sin^2\left(\frac{x}{2}\right)} \frac{d}{dx} \sin^2\left(\frac{x}{2}\right)$
 $= \frac{1}{\sin^2\left(\frac{x}{2}\right)} 2 \sin\left(\frac{x}{2}\right) \frac{d}{dx} \sin\left(\frac{x}{2}\right)$
 $= \frac{1}{\sin^2\left(\frac{x}{2}\right)} \left[2 \sin\left(\frac{x}{2}\right)\right] \frac{1}{2} \cos\left(\frac{x}{2}\right) = \cot\left(\frac{x}{2}\right)$
- $\frac{d}{dx} 3 \ln(e^{5x} + 1) = \frac{3}{e^{5x} + 1} (5e^{5x}) = \frac{15e^{5x}}{e^{5x} + 1}$
- $\frac{d}{dx} \ln(2x^3 - 4x + 5)$
 $= \frac{1}{2x^3 - 4x + 5} \frac{d}{dx} (2x^3 - 4x + 5) = \frac{6x^2 - 4}{2x^3 - 4x + 5}$
- $\frac{d}{dx} \cos e^{\sqrt{x}} = -\sin e^{\sqrt{x}} \frac{d}{dx} e^{\sqrt{x}}$
 $= (-\sin e^{\sqrt{x}}) e^{\sqrt{x}} \frac{d}{dx} \sqrt{x}$
 $= -\frac{e^{\sqrt{x}} \sin e^{\sqrt{x}}}{2\sqrt{x}}$
- $\frac{d}{dx} \ln(\tanh x) = \frac{1}{\tanh x} \frac{d}{dx} \tanh x$
 $= \frac{1}{\tanh x} \operatorname{sech}^2 x = \operatorname{csch} x \operatorname{sech} x$

$$17. \frac{d}{dx} 2 \cos^{-1} \sqrt{x} = \frac{-2}{\sqrt{1-(\sqrt{x})^2}} \frac{d}{dx} \sqrt{x}$$

$$= \frac{-2}{\sqrt{1-x}} \frac{1}{2\sqrt{x}} = -\frac{1}{\sqrt{x-x^2}}$$

$$18. \frac{d}{dx} [4^{3x} + (3x)^4] = \frac{d}{dx} (64^x + 81x^4)$$

$$= 64^x \ln 64 + 324x^3$$

$$19. \frac{d}{dx} 2 \csc e^{\ln \sqrt{x}} = \frac{d}{dx} 2 \csc \sqrt{x}$$

$$= -2 \csc \sqrt{x} \cot \sqrt{x} \frac{d}{dx} \sqrt{x}$$

$$= -\frac{\csc \sqrt{x} \cot \sqrt{x}}{\sqrt{x}}$$

$$20. \frac{d}{dx} (\log_{10} 2x)^{2/3}$$

$$= \frac{2}{3} (\log_{10} 2x)^{-1/3} \frac{d}{dx} (\log_{10} 2 + \log_{10} x)$$

$$= \frac{2}{3} (\log_{10} 2x)^{-1/3} \frac{1}{x \ln 10}$$

$$= \frac{2}{3x \ln 10 \sqrt[3]{\log_{10} 2x}}$$

$$21. \frac{d}{dx} 4 \tan 5x \sec 5x$$

$$= 20 \sec^2 5x \sec 5x + 20 \tan 5x \sec 5x \tan 5x$$

$$= 20 \sec 5x (\sec^2 5x + \tan^2 5x)$$

$$= 20 \sec 5x (2 \sec^2 5x - 1)$$

$$22. \frac{d}{dx} \tan^{-1} \left(\frac{x^2}{2} \right) = \frac{1}{\left(\frac{x^2}{2} \right)^2 + 1} \frac{d}{dx} \left(\frac{x^2}{2} \right)$$

$$= \frac{x}{\left(\frac{x^4}{4} \right) + 1} = \frac{4x}{x^4 + 4}$$

$$\frac{d}{dx} \left[x \tan^{-1} \left(\frac{x^2}{2} \right) \right]$$

$$= (1) \tan^{-1} \left(\frac{x^2}{2} \right) + (x) \left(\frac{4x}{x^4 + 4} \right)$$

$$= \tan^{-1} \left(\frac{x^2}{2} \right) + \frac{4x^2}{x^4 + 4}$$

$$23. \frac{d}{dx} x^{1+x} = \frac{d}{dx} e^{(1+x) \ln x}$$

$$= e^{(1+x) \ln x} \frac{d}{dx} [(1+x) \ln x]$$

$$= x^{1+x} \left[(1)(\ln x) + (1+x) \left(\frac{1}{x} \right) \right]$$

$$= x^{1+x} \left(\ln x + 1 + \frac{1}{x} \right)$$

$$24. \frac{d}{dx} (1+x^2)^e = e(1+x^2)^{e-1} \frac{d}{dx} (1+x^2)$$

$$= 2xe(1+x^2)^{e-1}$$

$$25. \text{ Let } u = 3x - 1, \text{ so } du = 3 dx.$$

$$\int e^{3x-1} dx = \frac{1}{3} \int e^{3x-1} 3 dx = \frac{1}{3} \int e^u du$$

$$= \frac{1}{3} e^u + C = \frac{1}{3} e^{3x-1} + C$$

Check:

$$\frac{d}{dx} \left(\frac{1}{3} e^{3x-1} + C \right) = \frac{1}{3} e^{3x-1} \frac{d}{dx} (3x-1) = e^{3x-1}$$

$$26. \text{ Let } u = \sin 3x, \text{ so } du = 3 \cos 3x dx.$$

$$\int 6 \cot 3x dx = 2 \int \frac{1}{\sin 3x} 3 \cos 3x dx = 2 \int \frac{1}{u} du$$

$$= 2 \ln |u| + C = 2 \ln |\sin 3x| + C$$

Check:

$$\frac{d}{dx} (2 \ln |\sin 3x| + C) = \frac{2}{\sin 3x} \frac{d}{dx} \sin 3x$$

$$= \frac{2(3 \cos 3x)}{\sin 3x} = 6 \cot 3x$$

$$27. \text{ Let } u = e^x, \text{ so } du = e^x dx.$$

$$\int e^x \sin e^x dx = \int \sin u du = -\cos u + C$$

$$= -\cos e^x + C$$

Check:

$$\frac{d}{dx} (-\cos e^x + C) = (\sin e^x) \frac{d}{dx} e^x = e^x \sin e^x$$

$$28. \text{ Let } u = x^2 + x - 5, \text{ so } du = (2x+1) dx.$$

$$\int \frac{6x+3}{x^2+x-5} dx = 3 \int \frac{1}{x^2+x-5} (2x+1) dx$$

$$= 3 \int \frac{1}{u} du = 3 \ln |u| + C = 3 \ln |x^2+x-5| + C$$

Check:

$$\frac{d}{dx} (3 \ln |x^2+x-5| + C)$$

$$= \frac{3}{x^2 + x - 5} \frac{d}{dx}(x^2 + x - 5)$$

$$= \frac{6x + 3}{x^2 + x - 5}$$

29. Let $u = e^{x+3} + 1$, so $du = e^{x+3} dx$.

$$\int \frac{e^{x+2}}{e^{x+3} + 1} dx = \frac{1}{e} \int \frac{1}{e^{x+3} + 1} e^{x+3} dx = \frac{1}{e} \int \frac{1}{u} du$$

$$= \frac{1}{e} \ln|u| + C = \frac{\ln(e^{x+3} + 1)}{e} + C$$

Check:

$$\frac{d}{dx} \left(\frac{\ln(e^{x+3} + 1)}{e} + C \right) = \frac{1}{e} \frac{1}{e^{x+3} + 1} \frac{d}{dx}(e^{x+3} + 1)$$

$$= \frac{e^{x+3} e^{-1}}{e^{x+3} + 1} = \frac{e^{x+2}}{e^{x+3} + 1}$$

30. Let $u = x^2$, so $du = 2x dx$.

$$\int 4x \cos x^2 dx = 2 \int (\cos x^2) 2x dx = 2 \int \cos u du$$

$$= 2 \sin u + C = 2 \sin x^2 + C$$

Check:

$$\frac{d}{dx}(2 \sin x^2 + C) = 2 \cos x^2 \frac{d}{dx} x^2 = 4x \cos x^2$$

31. Let $u = 2x$, so $du = 2 dx$.

$$\int \frac{4}{\sqrt{1-4x^2}} dx = 2 \int \frac{1}{\sqrt{1-(2x)^2}} 2dx$$

$$= 2 \int \frac{1}{\sqrt{1-u^2}} du$$

$$= 2 \sin^{-1} u + C = 2 \sin^{-1} 2x + C$$

Check:

$$\frac{d}{dx}(2 \sin^{-1} 2x + C) = 2 \left(\frac{1}{\sqrt{1-(2x)^2}} \right) \frac{d}{dx} 2x$$

$$= \frac{4}{\sqrt{1-4x^2}}$$

32. Let $u = \sin x$, so $du = \cos x dx$.

$$\int \frac{\cos x}{1 + \sin^2 x} dx = \int \frac{1}{1 + u^2} du = \tan^{-1} u + C$$

$$= \tan^{-1}(\sin x) + C$$

Check:

$$\frac{d}{dx} [\tan^{-1}(\sin x) + C] = \frac{1}{1 + \sin^2 x} \frac{d}{dx} \sin x$$

$$= \frac{\cos x}{1 + \sin^2 x}$$

33. Let $u = \ln x$, so $du = \frac{1}{x} dx$.

$$\int \frac{-1}{x + x(\ln x)^2} dx = - \int \frac{1}{1 + (\ln x)^2} \cdot \frac{1}{x} dx$$

$$= - \int \frac{1}{1 + u^2} du = - \tan^{-1} u + C = - \tan^{-1}(\ln x) + C$$

Check:

$$\frac{d}{dx} [- \tan^{-1}(\ln x) + C] = - \frac{1}{1 + (\ln x)^2} \frac{d}{dx} \ln x$$

$$= \frac{-1}{x + x(\ln x)^2}$$

34. Let $u = x - 3$, so $du = dx$.

$$\int \operatorname{sech}^2(x-3) dx = \int \operatorname{sech}^2 u du = \tanh u + C$$

$$= \tanh(x-3) + C$$

Check:

$$\frac{d}{dx} [\tanh(x-3)] = \operatorname{sech}^2(x-3) \frac{d}{dx}(x-3)$$

$$= \operatorname{sech}^2(x-3)$$

35. $f'(x) = \cos x - \sin x$; $f'(x) = 0$ when $\tan x = 1$,

$$x = \frac{\pi}{4}$$

$f'(x) > 0$ when $\cos x > \sin x$ which occurs when

$$-\frac{\pi}{2} \leq x < \frac{\pi}{4}.$$

$f''(x) = -\sin x - \cos x$; $f''(x) = 0$ when

$$\tan x = -1, \quad x = -\frac{\pi}{4}$$

$f''(x) > 0$ when $\cos x < -\sin x$ which occurs

$$\text{when } -\frac{\pi}{2} \leq x < -\frac{\pi}{4}.$$

Increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{4}\right]$

Decreasing on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

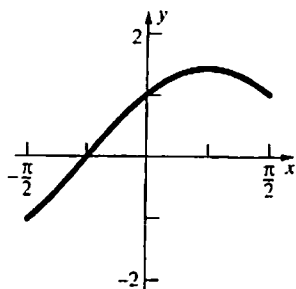
Concave up on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$

Concave down on $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$

Inflection point at $\left(-\frac{\pi}{4}, 0\right)$

Global maximum at $\left(\frac{\pi}{4}, \sqrt{2}\right)$

Global minimum at $\left(-\frac{\pi}{2}, -1\right)$



36. $f(x) = \frac{x^2}{e^x}$

$$f'(x) = \frac{e^x(2x) - x^2(e^x)}{(e^x)^2} = \frac{2x - x^2}{e^x}$$

f is increasing on $[0, 2]$ because $f'(x) > 0$ on $(0, 2)$.

f is decreasing on $(-\infty, 0] \cup [2, \infty)$ because $f'(x) < 0$ on $(-\infty, 0) \cup (2, \infty)$.

$$f''(x) = \frac{e^x(2 - 2x) - (2x - x^2)e^x}{(e^x)^2} = \frac{x^2 - 4x + 2}{e^x}$$

Inflection points are at

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 2}}{2} = 2 \pm \sqrt{2}.$$

The graph of f is concave up on $(-\infty, 2 - \sqrt{2}) \cup (2 + \sqrt{2}, \infty)$ because $f''(x) > 0$ on these intervals.

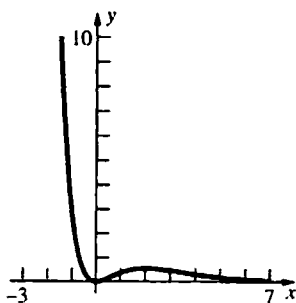
The graph of f is concave down on $(2 - \sqrt{2}, 2 + \sqrt{2})$ because $f''(x) < 0$ on this interval.

The absolute minimum value is $f(0) = 0$.

The relative maximum value is $f(2) = \frac{4}{e^2}$.

The inflection points are

$$\left(2 - \sqrt{2}, \frac{6 - 4\sqrt{2}}{e^{2-\sqrt{2}}}\right) \text{ and } \left(2 + \sqrt{2}, \frac{6 + 4\sqrt{2}}{e^{2+\sqrt{2}}}\right).$$



37. a. $f'(x) = 5x^4 + 6x^2 + 4 \geq 4 > 0$ for all x , so $f(x)$ is increasing.

b. $f(1) = 7$, so $g(7) = f^{-1}(7) = 1$.

c. $g'(7) = \frac{1}{f'(1)} = \frac{1}{15}$

38. $\frac{1}{2} = e^{10k}$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{10} \approx -0.06931$$

$$y = 100e^{-0.06931t}$$

$$1 = 100e^{-0.06931t}$$

$$t = \frac{\ln\left(\frac{1}{100}\right)}{-0.06931} \approx 66.44$$

It will take about 66.44 years.

39. a. $(\$100)(1.12) = \112

b. $(\$100)\left(1 + \frac{0.12}{12}\right)^{12} \approx \112.68

c. $(\$100)\left(1 + \frac{0.12}{365}\right)^{365} \approx \112.75

d. $(\$100)e^{(0.12)(1)} \approx \112.75

40. Let x be the horizontal distance from the airplane to the searchlight, $\frac{dx}{dt} = 300$.

$$\tan \theta = \frac{500}{x}, \text{ so } \theta = \tan^{-1} \frac{500}{x}.$$

$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{500}{x}\right)^2} \left(-\frac{500}{x^2}\right) \frac{dx}{dt}$$

$$= -\frac{500}{x^2 + 250,000} \frac{dx}{dt}$$

When $\theta = 30^\circ$, $x = \frac{500}{\tan 30^\circ} = 500\sqrt{3}$ and

$$\frac{d\theta}{dt} = -\frac{500}{(500\sqrt{3})^2 + (500)^2} (300)$$

$$= -\frac{300}{2000} = -\frac{3}{20}. \text{ The angle is decreasing at the rate of } 0.15 \text{ rad/s} \approx 8.59^\circ/\text{s}.$$

41. $y = (\cos x)^{\sin x} = e^{\sin x \ln(\cos x)}$

$$\frac{dy}{dx} = e^{\sin x \ln(\cos x)} \frac{d}{dx} [\sin x \ln(\cos x)]$$

$$= e^{\sin x \ln(\cos x)} \left[\cos x \ln(\cos x) + (\sin x) \left(\frac{1}{\cos x} \right) (-\sin x) \right]$$

$$= (\cos x)^{\sin x} \left[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x} \right]$$

At $x = 0$, $\frac{dy}{dx} = 1^0(1 \ln 1 - 0) = 0$.

The tangent line has slope 0, so it is horizontal:
 $y = 1$.

42. Let t represent the number of years since 1990.

$$14,000 = 10,000e^{10k}$$

$$k = \frac{\ln(1.4)}{10} \approx 0.03365$$

$$y = 10,000e^{0.03365t}$$

$$y(20) = 10,000e^{(0.03365)(20)} \approx 19,601$$

The population will be about 19,600.

43. Integrating factor is $|x|$. $D[y|x] = 0$; $y = Cx^{-1}$

44. Integrating factor is x^2 .

$$D[yx^2] = x^3; y = \left(\frac{1}{4}\right)x^2 + Cx^{-2}$$

45. (Linear first-order) $y' + 2xy = 2x$

Integrating factor: $e^{\int 2x dx} = e^{x^2}$

$$D[ye^{x^2}] = 2xe^{x^2}; ye^{x^2} = e^{x^2} + C;$$

$$y = 1 + Ce^{-x^2}$$

If $x = 0$, $y = 3$, then $3 = 1 + C$, so $C = 2$.

Therefore, $y = 1 + 2e^{-x^2}$.

46. Integrating factor is e^{-ax} .

$$D[ye^{-ax}] = 1; y = e^{ax}(x + C)$$

47. Integrating factor is e^{-2x} .

$$D[ye^{-2x}] = e^{-x}; y = -e^x + Ce^{2x}$$

48. a. $Q'(t) = 3 - 0.02Q$

b. $Q'(t) + 0.02Q = 3$

Integrating factor is $e^{0.02t}$

$$D[Qe^{0.02t}] = 3e^{0.02t}$$

$$Q(t) = 150 + Ce^{-0.02t}$$

$$Q(t) = 150 - 30e^{-0.02t} \text{ goes through } (0, 120).$$

- c. $Q \rightarrow 150$ g, as $t \rightarrow \infty$.

7.10 Additional Problem Set

1. $1000e^{(0.05)(1)} = \$1051.27$

2. $A_0e^{(0.05)(1)} = 1000$

$$A_0 = 1000e^{-0.05} \approx \$951.23$$

3. a. $100e^{0.05(360/365)} + 100e^{0.05(330/365)}$
 $+ 100e^{0.05(300/365)} + 100e^{0.05(270/365)}$
 $+ 100e^{0.05(240/365)} + 100e^{0.05(210/365)} + 100e^{0.05(180/365)} + 100e^{0.05(150/365)}$
 $+ 100e^{0.05(120/365)} + 100e^{0.05(90/365)} + 100e^{0.05(60/365)} + 100e^{0.05(30/365)} \approx \1232.61

- b. The formula in part a. is a geometric series $\sum_{k=1}^n ar^{k-1}$ with $a = 100e^{0.05(30/365)}$, $r = e^{0.05(30/365)}$,

and $n = 12$. The sum is $\frac{100e^{0.05(30/365)} - 100e^{0.05(390/365)}}{1 - e^{0.05(30/365)}}$.

4. a. Let $q = 1 + 0.05 \cdot \left(\frac{30}{365}\right)$.

The amount of money after 30 days is $A_1 = A_0q - 144$.

The amount of money after 60 days is $A_2 = A_1q - 144 = A_0q^2 - 144q - 144$.

The amount of money after 90 days is $A_3 = A_2q - 144 = A_0q^3 - 144q^2 - 144q - 144$. Thus, the amount of money after $30n$ days is $A_n = A_0q^n - 144 \sum_{k=0}^{n-1} q^k = A_0q^n - 144 \left(\frac{q^n - 1}{q - 1} \right)$. If the amount of money after 3600 = 30 · 120 days is 0, we have $0 = A_0q^{120} - 144 \left(\frac{q^{120} - 1}{q - 1} \right)$.

$$A_0q^{120} = 144 \left(\frac{q^{120} - 1}{q - 1} \right)$$

$$A_0 = 144 \left(\frac{1 - q^{-120}}{q - 1} \right) = 144 \left[\frac{1 - \left(1 + 0.05 \cdot \left(\frac{30}{365} \right) \right)^{-120}}{0.05 \cdot \left(\frac{30}{365} \right)} \right]$$

Note that $-10 \cdot \left(\frac{365}{30} \right) \approx -120$.

$$\left(\text{Or ten years} = 10 \cdot \frac{365}{30} \cdot 30 \text{ days, so } 0 = A_0q^{10 \cdot (365/30)} - 144 \left(\frac{q^{10 \cdot (365/30)} - 1}{q - 1} \right) \right)$$

$$\text{Thus, } A_0 = 144 \left[\frac{1 - \left(1 + 0.05 \cdot \left(\frac{30}{365} \right) \right)^{-10 \cdot (365/30)}}{0.05 \cdot \left(\frac{30}{365} \right)} \right]$$

b. Let $q = e^{-0.05(30/365)}$. Following the same reasoning as in part a., the amount of money after $30n$ days is

$$A_n = A_0q^n - 144 \left(\frac{q^n - 1}{q - 1} \right), \text{ and if the amount of money after 3600 days is 0,}$$

$$A_0 = 144 \left(\frac{1 - q^{-120}}{q - 1} \right) = 144 \left[\frac{1 - e^{0.05(3600/365)}}{e^{-0.05(30/365)} - 1} \right]$$

$$\text{Or, by using 10 years, } A_0 = 144 \left[\frac{1 - e^{0.05 \cdot 10}}{e^{-0.05(30/365)} - 1} \right]$$

5. a. If you deposit A_0 into an account earning an interest rate of $100r$ percent, compounded continuously, then the amount you have at time t is $A(t) = A_0e^{rt}$. The money has doubled when $A(t) = 2A_0$ or $A_0e^{rt} = 2A_0$, so T satisfies the equation $A_0e^{rT} = 2A_0$.

$$A_0e^{r(70/100r)} = A_0e^{0.7} \approx 2A_0$$

b. Solving $A_0e^{rT} = 2A_0$ for T to get $T = \frac{1}{r} \ln 2 \approx \frac{1}{r} (0.7) = \frac{70}{100r}$

$$\text{c. } T \approx \frac{70}{100(0.07)} = 10$$

Ten years

6. If $f(t) = e^{kt}$, then $\frac{f'(t)}{f(t)} = \frac{ke^{kt}}{e^{kt}} = k$.

7. $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

$$\lim_{x \rightarrow \infty} \frac{f'(x)}{f(x)} = \lim_{x \rightarrow \infty} \frac{n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1}{a_n x^n + a_{n-1} x + \cdots + a_1 x + a_0} = \lim_{x \rightarrow \infty} \frac{\frac{n a_n}{x} + \frac{(n-1) a_{n-1}}{x^2} + \cdots + \frac{a_1}{x^n}}{a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}} = 0$$

8. $\frac{f'(x)}{f(x)} = k > 0$ can be written as

$$\frac{1}{y} \frac{dy}{dx} = k \text{ where } y = f(x).$$

$$\frac{dy}{y} = k dx \text{ has the solution } y = C e^{kx}.$$

Thus, the equation $f(x) = C e^{kx}$ represents exponential growth since $k > 0$.

9. $\frac{f'(x)}{f(x)} = k < 0$ can be written as $\frac{1}{y} \frac{dy}{dx} = k$ where $y = f(x)$. $\frac{dy}{y} = k dx$ has the solution $y = C e^{kx}$.

Thus, $f(x) = C e^{kx}$ which represents exponential decay since $k < 0$.

10. a. The error in $f(x)$ due to a relative error in the input of size Δx is $f(x + \Delta x) - f(x)$, thus the relative error in $f(x)$ due to this relative error in input is

$$\frac{f(x + \Delta x) - f(x)}{f(x)} = \frac{f(x + \Delta x) - f(x)}{f(x)} \cdot \frac{\Delta x}{\Delta x} \cdot \frac{x}{x} = \frac{f(x + \Delta x) - f(x)}{\Delta x \cdot f(x)} \cdot \frac{\Delta x}{x} = \frac{f(x + \Delta x) - f(x)}{\Delta x \cdot f(x)} \cdot \frac{(x + \Delta x) - x}{x} \cdot x$$

b. $\rho_x = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{x}$, while $\frac{d \ln(f(x))}{dx} = \frac{f'(x)}{f(x)}$, so $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{f(x)}$

$$= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x \cdot f(x)} \cdot \frac{(x + \Delta x) - x}{x} \cdot x \right] = \left[\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \cdot \frac{1}{f(x)} \right] \cdot \left[\lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{x} \right] \cdot x$$

$$= f'(x) \cdot \frac{1}{f(x)} \cdot \rho_x \cdot x = x \cdot \frac{d \ln(f(x))}{dx} \cdot \rho_x$$

c. $x \cdot \frac{d}{dx} [\ln(e^x)] = x \cdot \frac{d}{dx} (x) = x \cdot 1 = x$

11. a. In order of increasing slope, the graphs represent the curves $y = 2^x$, $y = 3^x$, and $y = 4^x$.

- b. $\ln y$ is linear with respect to x , and at $x = 0$, $y = 1$ since $C = 1$.

- c. The graph passes through the points (0.2, 4) and (0.6, 8). Thus, $4 = C b^{0.2}$ and $8 = C b^{0.6}$.
Dividing the second equation by the first, gets $2 = b^{0.4}$ so $b = 2^{5/2}$.
Therefore $C = 2^{3/2}$.

12. The graph of the equation whose log-log plot has negative slope contains the points (2, 7) and (7, 2).

$$\text{Thus, } 7 = C 2^r \text{ and } 2 = C 7^r, \text{ so } \frac{7}{2} = \left(\frac{2}{7} \right)^r. \ln \frac{7}{2} = r \ln \frac{2}{7} \Rightarrow r = \frac{\ln 7 - \ln 2}{\ln 2 - \ln 7} = -1 \text{ and } C = 14.$$

Hence, one equation is $y = 14x^{-1}$.

The graph of one equation contains the points

$$(7, 30) \text{ and } (10, 70). \text{ Thus, } 30 = C 7^r \text{ and } 70 = C 10^r, \text{ so } \frac{3}{7} = \left(\frac{7}{10} \right)^r$$

$$\ln \frac{3}{7} = r \ln \frac{7}{10} \Rightarrow r = \frac{\ln 3 - \ln 7}{\ln 7 - \ln 10} \approx 2.38 \text{ and } C \approx 30 \cdot 7^{-2.38} \approx 0.29. \text{ Hence, another equation is } y = 0.29x^{2.38}.$$

The graph of another equation contains the points (1, 2) and (7, 5). Thus, $2 = C1^r$ and $5 = C7^r$, so $C = 2$ and

$\ln 5 - \ln 2 = r \ln 7 \Rightarrow r = \frac{\ln 5 - \ln 2}{\ln 7} \approx 0.47$. Hence, the last equation is $y = x^{0.47}$. Other answers are possible depending on selection of points.

13. a. $\eta = 3 \operatorname{sech}^2 \frac{x}{2}$

$$\eta' = -3 \operatorname{sech}^2 \frac{x}{2} \tanh \frac{x}{2}$$

$$3\eta^2 = 3 \cdot 9 \operatorname{sech}^4 \frac{x}{2} = 27 \operatorname{sech}^4 \frac{x}{2}$$

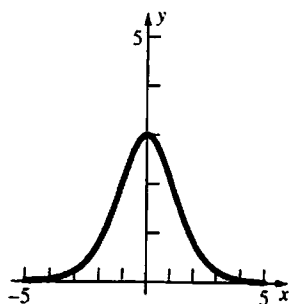
$$\eta^3 + 3(\eta')^2 = 27 \operatorname{sech}^6 \frac{x}{2} + 3 \left(9 \operatorname{sech}^4 \frac{x}{2} \tanh^2 \frac{x}{2} \right)$$

$$= 27 \operatorname{sech}^4 \frac{x}{2} \left(\operatorname{sech}^2 \frac{x}{2} + \tanh^2 \frac{x}{2} \right)$$

$$= 27 \operatorname{sech}^4 \frac{x}{2} = 3\eta^2$$

Thus, $\eta = 3 \operatorname{sech}^2 \frac{x}{2}$ satisfies the equation.

b.



8.1 Concepts Review

1. $\int u^5 du$

2. e^x

3. $\int_1^2 u^3 du$

4. complete the square.

Problem Set 8.1

1. $\int (x-2)^5 dx = \frac{1}{6}(x-2)^6 + C$

2. $\int \sqrt{3x} dx = \frac{1}{3} \int \sqrt{3x} 3 dx = \frac{2}{9} (3x)^{3/2} + C$

3. $u = x^2 + 1, du = 2x dx$

When $x = 0, u = 1$ and when $x = 1, u = 5$.

$$\int_0^1 x(x^2 + 1)^5 dx = \frac{1}{2} \int_1^5 (x^2 + 1)(2x dx)$$

$$= \frac{1}{2} \int_1^5 u^5 du$$

$$= \left[\frac{u^6}{12} \right]_1^5 = \frac{5^6 - 1^6}{12} = \frac{15624}{12} = 1302$$

4. $u = 1 - x^2, du = -2x dx$

When $x = 0, u = 1$ and when $x = 1, u = 0$.

$$\int_0^1 x\sqrt{1-x^2} dx = -\frac{1}{2} \int_1^0 \sqrt{1-x^2} (-2x dx)$$

$$= -\frac{1}{2} \int_1^0 u^{1/2} du = \frac{1}{2} \int_0^1 u^{1/2} du$$

$$= \left[\frac{1}{3} u^{3/2} \right]_0^1 = \frac{1}{3}$$

5. $\int \frac{dx}{x^2 + 4} = \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + C$

6. $u = 2 + e^x, du = e^x dx$

$$\int \frac{e^x}{2 + e^x} dx = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|2 + e^x| + C = \ln(2 + e^x) + C$$

7. $u = x^2 + 4, du = 2x dx$

$$\int \frac{x}{x^2 + 4} dx = \frac{1}{2} \int \frac{du}{u}$$

$$= \frac{1}{2} \ln|u| + C$$

$$= \frac{1}{2} \ln|x^2 + 4| + C$$

$$= \frac{1}{2} \ln(x^2 + 4) + C$$

8. $\int \frac{2t^2}{2t^2 + 1} dt = \int \frac{2t^2 + 1 - 1}{2t^2 + 1} dt$

$$= \int dt - \int \frac{1}{2t^2 + 1} dt$$

$$u = \sqrt{2}t, du = \sqrt{2}dt$$

$$t - \int \frac{1}{2t^2 + 1} dt = t - \frac{1}{\sqrt{2}} \int \frac{du}{1 + u^2}$$

$$= t - \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}t) + C$$

9. $u = 4 + z^2, du = 2z dz$

$$\int 6z\sqrt{4 + z^2} dz = 3 \int \sqrt{u} du$$

$$= 2u^{3/2} + C$$

$$= 2(4 + z^2)^{3/2} + C$$

10. $u = 2t + 1, du = 2dt$

$$\int \frac{5}{\sqrt{2t + 1}} dt = \frac{5}{2} \int \frac{du}{\sqrt{u}}$$

$$= 5\sqrt{u} + C$$

$$= 5\sqrt{2t + 1} + C$$

$$11. \int \frac{\tan z}{\cos^2 z} dz = \int \tan z \sec^2 z dz$$

$$u = \tan z, du = \sec^2 z dz$$

$$\int \tan z \sec^2 z dz = \int u du = \frac{1}{2} u^2 + C$$

$$= \frac{1}{2} \tan^2 z + C$$

$$12. u = \cos z, du = -\sin z dz$$

$$\int e^{\cos z} \sin z dz = -\int e^{\cos z} (-\sin z dz)$$

$$= -\int e^u du = -e^u + C$$

$$= -e^{\cos z} + C$$

$$13. u = \sqrt{t}, du = \frac{1}{2\sqrt{t}} dt$$

$$\int \frac{\sin \sqrt{t}}{\sqrt{t}} dt = 2 \int \sin u du$$

$$= -2 \cos u + C$$

$$= -2 \cos \sqrt{t} + C$$

$$14. u = x^2, du = 2x dx$$

$$\int \frac{2x dx}{\sqrt{1-x^4}} = \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}(x^2) + C$$

$$15. u = \sin x, du = \cos x dx$$

$$\int_0^{\pi/4} \frac{\cos x}{1+\sin^2 x} dx = \int_0^{\sqrt{2}/2} \frac{du}{1+u^2}$$

$$= [\tan^{-1} u]_0^{\sqrt{2}/2} = \tan^{-1} \frac{\sqrt{2}}{2}$$

$$\approx 0.6155$$

$$16. u = \sqrt{1-x}, du = -\frac{1}{2\sqrt{1-x}} dx$$

$$\int_0^{3/4} \frac{\sin \sqrt{1-x}}{\sqrt{1-x}} dx = -2 \int_1^{1/2} \sin u du$$

$$= 2 \int_{1/2}^1 \sin u du$$

$$= [-2 \cos u]_{1/2}^1 = -2 \left(\cos 1 - \cos \frac{1}{2} \right)$$

$$\approx 0.6746$$

$$17. \int \frac{3x^2 + 2x}{x+1} dx = \int (3x-1) dx + \int \frac{1}{x+1} dx$$

$$= \frac{3}{2} x^2 - x + \ln|x+1| + C$$

$$18. \int \frac{x^3 + 7x}{x-1} dx = \int (x^2 + x + 8) dx + 8 \int \frac{1}{x-1} dx$$

$$= \frac{1}{3} x^3 + \frac{1}{2} x^2 + 8x + 8 \ln|x-1| + C$$

$$19. u = \ln 4x^2, du = \frac{2}{x} dx$$

$$\int \frac{\sin(\ln 4x^2)}{x} dx = \frac{1}{2} \int \sin u du$$

$$= -\frac{1}{2} \cos u + C$$

$$= -\frac{1}{2} \cos(\ln 4x^2) + C$$

$$20. u = \ln x, du = \frac{1}{x} dx$$

$$\int \frac{\sec^2(\ln x)}{2x} dx = \frac{1}{2} \int \sec^2 u du$$

$$= \frac{1}{2} \tan u + C$$

$$= \frac{1}{2} \tan(\ln x) + C$$

$$21. \int \frac{6e^x}{\sqrt{1-e^{2x}}} dx = 6 \sin^{-1}(e^x) + C$$

$$22. u = x^2, du = 2x dx$$

$$\int \frac{x}{x^4 + 4} dx = \frac{1}{2} \int \frac{du}{4+u^2} = \frac{1}{4} \tan^{-1} \frac{u}{2} + C$$

$$= \frac{1}{4} \tan^{-1} \left(\frac{x^2}{2} \right) + C$$

$$23. u = 1 - e^{2x}, du = -2e^{2x} dx$$

$$\int \frac{3e^{2x}}{\sqrt{1-e^{2x}}} dx = -\frac{3}{2} \int \frac{du}{\sqrt{u}}$$

$$= -3\sqrt{u} + C$$

$$= -3\sqrt{1-e^{2x}} + C$$

$$24. \int \frac{x^3}{x^4 + 4} dx = \frac{1}{4} \int \frac{4x^3}{x^4 + 4} dx$$

$$= \frac{1}{4} \ln|x^4 + 4| + C$$

$$= \frac{1}{4} \ln(x^4 + 4) + C$$

$$\begin{aligned}
 25. \quad \int_0^1 t 3^{t^2} dt &= \frac{1}{2} \int_0^1 2t 3^{t^2} dt \\
 &= \left[\frac{3^{t^2}}{2 \ln 3} \right]_0^1 = \frac{3}{2 \ln 3} - \frac{1}{2 \ln 3} \\
 &= \frac{1}{\ln 3} \approx 0.9102
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \int_0^{\pi/6} 2^{\cos x} \sin x dx &= - \int_0^{\pi/6} 2^{\cos x} (-\sin x dx) \\
 &= \left[-\frac{2^{\cos x}}{\ln 2} \right]_0^{\pi/6} \\
 &= -\frac{1}{\ln 2} (2^{\sqrt{3}/2} - 2) \\
 &= \frac{2 - 2^{\sqrt{3}/2}}{\ln 2} \approx 0.2559
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \int \frac{\sin x - \cos x}{\sin x} dx &= \int \left(1 - \frac{\cos x}{\sin x} \right) dx \\
 u = \sin x, \quad du &= \cos x dx \\
 \int \frac{\sin x - \cos x}{\sin x} dx &= x - \int \frac{du}{u} \\
 &= x - \ln|u| + C \\
 &= x - \ln|\sin x| + C
 \end{aligned}$$

$$\begin{aligned}
 28. \quad u = \cos(4t - 1), \quad du &= -4 \sin(4t - 1) dt \\
 \int \frac{\sin(4t - 1)}{1 - \sin^2(4t - 1)} dt &= \int \frac{\sin(4t - 1)}{\cos^2(4t - 1)} dt \\
 &= -\frac{1}{4} \int \frac{1}{u^2} du \\
 &= \frac{1}{4} u^{-1} + C = \frac{1}{4} \sec(4t - 1) + C
 \end{aligned}$$

$$\begin{aligned}
 29. \quad u = e^x, \quad du &= e^x dx \\
 \int e^x \sec e^x dx &= \int \sec u du \\
 &= \ln|\sec u + \tan u| + C \\
 &= \ln|\sec e^x + \tan e^x| + C
 \end{aligned}$$

$$\begin{aligned}
 30. \quad u = e^x, \quad du &= e^x dx \\
 \int e^x \sec^2(e^x) dx &= \int \sec^2 u du = \tan u + C \\
 &= \tan(e^x) + C
 \end{aligned}$$

$$\begin{aligned}
 31. \quad \int \frac{\sec^3 x + e^{\sin x}}{\sec x} dx &= \int (\sec^2 x + e^{\sin x} \cos x) dx \\
 &= \tan x + \int e^{\sin x} \cos x dx \\
 u = \sin x, \quad du &= \cos x dx
 \end{aligned}$$

$$\begin{aligned}
 \tan x + \int e^{\sin x} \cos x dx &= \tan x + \int e^u du \\
 &= \tan x + e^u + C = \tan x + e^{\sin x} + C
 \end{aligned}$$

$$\begin{aligned}
 32. \quad u &= \sqrt{3t^2 - t - 1}, \\
 du &= \frac{1}{2} (3t^2 - t - 1)^{-1/2} (6t - 1) dt \\
 \int \frac{(6t - 1) \sin \sqrt{3t^2 - t - 1}}{\sqrt{3t^2 - t - 1}} dt &= 2 \int \sin u du \\
 &= -2 \cos u + C \\
 &= -2 \cos \sqrt{3t^2 - t - 1} + C
 \end{aligned}$$

$$\begin{aligned}
 33. \quad u &= t^3 - 2, \quad du = 3t^2 dt \\
 \int \frac{t^2 \cos(t^3 - 2)}{\sin^2(t^3 - 2)} dt &= \frac{1}{3} \int \frac{\cos u}{\sin^2 u} du \\
 v = \sin u, \quad dv &= \cos u du \\
 \frac{1}{3} \int \frac{\cos u}{\sin^2 u} du &= \frac{1}{3} \int v^{-2} dv = -\frac{1}{3} v^{-1} + C \\
 &= -\frac{1}{3 \sin u} + C \\
 &= -\frac{1}{3 \sin(t^3 - 2)} + C
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \int \frac{1 + \cos 2x}{\sin^2 2x} dx &= \int \frac{1}{\sin^2 2x} dx + \int \frac{\cos 2x}{\sin^2 2x} dx \\
 &= \int \csc^2 2x dx + \int \cot 2x \csc 2x dx \\
 &= -\frac{1}{2} \cot 2x - \frac{1}{2} \csc 2x + C
 \end{aligned}$$

$$\begin{aligned}
 35. \quad u &= t^3 - 2, \quad du = 3t^2 dt \\
 \int \frac{t^2 \cos^2(t^3 - 2)}{\sin^2(t^3 - 2)} dt &= \frac{1}{3} \int \frac{\cos^2 u}{\sin^2 u} du \\
 &= \frac{1}{3} \int \cot^2 u du = \frac{1}{3} \int (\csc^2 u - 1) du \\
 &= \frac{1}{3} [-\cot u - u] + C_1 \\
 &= \frac{1}{3} [-\cot(t^3 - 2) - (t^3 - 2)] + C_1 \\
 &= -\frac{1}{3} [\cot(t^3 - 2) + t^3] + C
 \end{aligned}$$

$$\begin{aligned}
 36. \quad u &= 1 + \cot 2t, \quad du = -2 \csc^2 2t \\
 \int \frac{\csc^2 2t}{\sqrt{1 + \cot 2t}} dt &= -\frac{1}{2} \int \frac{1}{\sqrt{u}} du \\
 &= -\sqrt{u} + C \\
 &= -\sqrt{1 + \cot 2t} + C
 \end{aligned}$$

$$37. u = \tan^{-1} 2t, du = \frac{2}{1+4t^2} dt$$

$$\begin{aligned} \int \frac{e^{\tan^{-1} 2t}}{1+4t^2} dt &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C = \frac{1}{2} e^{\tan^{-1} 2t} + C \end{aligned}$$

$$\begin{aligned} 38. u &= -t^2 - 2t - 5, \\ du &= (-2t - 2)dt = -2(t+1)dt \\ \int (t+1)e^{-t^2-2t-5} dt &= -\frac{1}{2} \int e^u du \\ &= -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-t^2-2t-5} + C \end{aligned}$$

$$\begin{aligned} 39. u &= 3y^2, du = 6y dy \\ \int \frac{y}{\sqrt{16-9y^4}} dy &= \frac{1}{6} \int \frac{1}{\sqrt{4^2-u^2}} du \\ &= \frac{1}{6} \sin^{-1} \left(\frac{u}{4} \right) + C \\ &= \frac{1}{6} \sin^{-1} \left(\frac{3y^2}{4} \right) + C \end{aligned}$$

$$\begin{aligned} 40. u &= 3x, du = 3 dx \\ \int \cosh 3x dx &= \frac{1}{3} \int (\cosh u) du = \frac{1}{3} \sinh u + C \\ &= \frac{1}{3} \sinh 3x + C \end{aligned}$$

$$\begin{aligned} 41. u &= x^3, du = 3x^2 dx \\ \int x^2 \sinh x^3 dx &= \frac{1}{3} \int \sinh u du \\ &= \frac{1}{3} \cosh u + C \\ &= \frac{1}{3} \cosh x^3 + C \end{aligned}$$

$$\begin{aligned} 42. u &= 2x, du = 2 dx \\ \int \frac{5}{\sqrt{9-4x^2}} dx &= \frac{5}{2} \int \frac{1}{\sqrt{3^2-u^2}} du \\ &= \frac{5}{2} \sin^{-1} \left(\frac{u}{3} \right) + C \\ &= \frac{5}{2} \sin^{-1} \left(\frac{2x}{3} \right) + C \end{aligned}$$

$$\begin{aligned} 43. u &= e^{3t}, du = 3e^{3t} dt \\ \int \frac{e^{3t}}{\sqrt{4-e^{6t}}} dt &= \frac{1}{3} \int \frac{1}{\sqrt{2^2-u^2}} du \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3} \sin^{-1} \left(\frac{u}{2} \right) + C \\ &= \frac{1}{3} \sin^{-1} \left(\frac{e^{3t}}{2} \right) + C \end{aligned}$$

$$\begin{aligned} 44. u &= 2t, du = 2dt \\ \int \frac{dt}{2t\sqrt{4t^2-1}} &= \frac{1}{2} \int \frac{1}{u\sqrt{u^2-1}} du \\ &= \frac{1}{2} \left[\sec^{-1} |u| \right] + C \\ &= \frac{1}{2} \sec^{-1} (2t) + C \end{aligned}$$

$$\begin{aligned} 45. u &= \cos x, du = -\sin x dx \\ \int_0^{\pi/2} \frac{\sin x}{16+\cos^2 x} dx &= -\int_1^0 \frac{1}{16+u^2} du \\ &= \int_0^1 \frac{1}{16+u^2} du \\ &= \left[\frac{1}{4} \tan^{-1} \left(\frac{u}{4} \right) \right]_0^1 \\ &= \left[\frac{1}{4} \tan^{-1} \left(\frac{1}{4} \right) - \frac{1}{4} \tan^{-1} 0 \right] \\ &= \frac{1}{4} \tan^{-1} \left(\frac{1}{4} \right) \approx 0.0612 \end{aligned}$$

$$\begin{aligned} 46. u &= e^{2x} + e^{-2x}, du = (2e^{2x} - 2e^{-2x})dx \\ &= 2(e^{2x} - e^{-2x})dx \\ \int_0^1 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx &= \frac{1}{2} \int_2^{e^2+e^{-2}} \frac{1}{u} du \\ &= \frac{1}{2} \left[\ln |u| \right]_2^{e^2+e^{-2}} \\ &= \frac{1}{2} \ln |e^2 + e^{-2}| - \frac{1}{2} \ln 2 \\ &= \frac{1}{2} \ln \left| \frac{e^4 + 1}{e^2} \right| - \frac{1}{2} \ln 2 \\ &= \frac{1}{2} \ln(e^4 + 1) - \frac{1}{2} \ln(e^2) - \frac{1}{2} \ln 2 \\ &= \frac{1}{2} \left(\ln \left(\frac{e^4 + 1}{2} \right) - 2 \right) \approx 0.6625 \end{aligned}$$

$$\begin{aligned} 47. \int \frac{1}{x^2 + 2x + 5} dx &= \int \frac{1}{x^2 + 2x + 1 + 4} dx \\ &= \int \frac{1}{(x+1)^2 + 2^2} d(x+1) \\ &= \frac{1}{2} \tan^{-1} \left(\frac{x+1}{2} \right) + C \end{aligned}$$

$$\begin{aligned}
 48. \int \frac{1}{x^2 - 4x + 9} dx &= \int \frac{1}{x^2 - 4x + 4 + 5} dx \\
 &= \int \frac{1}{(x-2)^2 + (\sqrt{5})^2} d(x-2) \\
 &= \frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{x-2}{\sqrt{5}} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 49. \int \frac{dx}{9x^2 + 18x + 10} &= \int \frac{dx}{9x^2 + 18x + 9 + 1} \\
 &= \int \frac{dx}{(3x+3)^2 + 1^2} \\
 u = 3x+3, du = 3 dx \\
 \int \frac{dx}{(3x+3)^2 + 1^2} &= \frac{1}{3} \int \frac{du}{u^2 + 1^2} \\
 &= \frac{1}{3} \tan^{-1}(3x+3) + C
 \end{aligned}$$

$$\begin{aligned}
 50. \int \frac{dx}{\sqrt{16+6x-x^2}} &= \int \frac{dx}{\sqrt{-(x^2-6x+9-25)}} \\
 &= \int \frac{dx}{\sqrt{-(x-3)^2 + 5^2}} \\
 &= \int \frac{dx}{\sqrt{5^2 - (x-3)^2}} \\
 &= \sin^{-1} \left(\frac{x-3}{5} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 51. \int \frac{x+1}{9x^2 + 18x + 10} dx &= \frac{1}{18} \int \frac{18x+18}{9x^2 + 18x + 10} dx \\
 &= \frac{1}{18} \ln |9x^2 + 18x + 10| + C
 \end{aligned}$$

$$\begin{aligned}
 52. \int \frac{3-x}{\sqrt{16+6x-x^2}} dx &= \frac{1}{2} \int \frac{6-2x}{\sqrt{16+6x-x^2}} dx \\
 &= \sqrt{16+6x-x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 53. u = \sqrt{2}t, du = \sqrt{2}dt \\
 \int \frac{dt}{t\sqrt{2t^2-9}} &= \int \frac{du}{u\sqrt{u^2-3^2}} \\
 &= \frac{1}{3} \sec^{-1} \left(\frac{|\sqrt{2}t|}{3} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 54. \int \frac{\tan x}{\sqrt{\sec^2 x - 4}} dx &= \int \frac{\cos x}{\cos x} \frac{\tan x}{\sqrt{\sec^2 x - 4}} dx \\
 &= \int \frac{\sin x}{\sqrt{1-4\cos^2 x}} dx \\
 u = 2 \cos x, du = -2 \sin x dx
 \end{aligned}$$

$$\begin{aligned}
 &\int \frac{\sin x}{\sqrt{1-4\cos^2 x}} dx \\
 &= -\frac{1}{2} \int \frac{1}{\sqrt{1-u^2}} du = -\frac{1}{2} \sin^{-1} u + C \\
 &= -\frac{1}{2} \sin^{-1}(2 \cos x) + C
 \end{aligned}$$

$$\begin{aligned}
 55. \int x\sqrt{3x+2} dx \\
 &= \frac{2}{15 \cdot 3^2} (3 \cdot 3x - 2 \cdot 2)(3x+2)^{3/2} + C \\
 &= \frac{2}{135} (9x-4)(3x+2)^{3/2} + C
 \end{aligned}$$

Use Formula 96 with $a = 3$, $b = 2$, and $u = x$ for $\int u\sqrt{3u+2} du$.

$$\begin{aligned}
 56. \int 2t\sqrt{3-4t} dt &= 2 \int t\sqrt{-4t+3} dt \\
 &= 2 \left[\frac{2}{15(-4)^2} (3(-4)t - 2 \cdot 3)(-4t+3)^{3/2} + C \right] \\
 &= \frac{1}{60} (-12t-6)(3-4t)^{3/2} + C \\
 &= -\frac{1}{10} (2t+1)(3-4t)^{3/2} + C
 \end{aligned}$$

Use Formula 96 with $a = -4$, $b = 3$, and $u = t$ for $\int u\sqrt{-4u+3} du$.

$$\begin{aligned}
 57. u = 4x, du = 4 dx \\
 \int \frac{dx}{9-16x^2} &= \frac{1}{4} \int \frac{du}{3^2-u^2} du \\
 &= \frac{1}{4} \left[\frac{1}{2(3)} \ln \left| \frac{u+3}{u-3} \right| \right] + C \\
 &= \frac{1}{24} \ln \left| \frac{4x+3}{4x-3} \right| + C
 \end{aligned}$$

Use Formula 18 with $a = 3$ for $\int \frac{du}{3^2-u^2}$.

$$\begin{aligned}
 58. \int \frac{dx}{5x^2-11} &= -\int \frac{dx}{11-5x^2} \\
 u = \sqrt{5}x, du = \sqrt{5} dx \\
 -\int \frac{dx}{11-5x^2} &= -\frac{1}{\sqrt{5}} \int \frac{du}{(\sqrt{11})^2-u^2} \\
 &= -\frac{1}{\sqrt{5}} \left[\frac{1}{2\sqrt{11}} \ln \left| \frac{u+\sqrt{11}}{u-\sqrt{11}} \right| \right] + C \\
 &= -\frac{1}{2\sqrt{55}} \ln \left| \frac{\sqrt{5}x+\sqrt{11}}{\sqrt{5}x-\sqrt{11}} \right| + C
 \end{aligned}$$

Use Formula 18 with $a = \sqrt{11}$ for

$$\int \frac{du}{(\sqrt{11})^2 - u^2}.$$

$$\begin{aligned} 59. \int x^2 \sqrt{9-2x^2} dx &= \int \sqrt{2} x^2 \sqrt{\frac{9-2x^2}{2}} dx \\ &= \sqrt{2} \int x^2 \sqrt{\frac{9}{2} - x^2} dx = \sqrt{2} \int x^2 \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 - x^2} dx \\ &= \sqrt{2} \left[\frac{x}{8} \left(2x^2 - \frac{9}{2} \right) \sqrt{\frac{9}{2} - x^2} + \frac{\left(\frac{81}{4}\right)}{8} \sin^{-1} \left(\frac{x}{\frac{3}{\sqrt{2}}} \right) \right] + C \\ &= \frac{x}{16} (4x^2 - 9) \sqrt{9-2x^2} + \frac{81\sqrt{2}}{32} \sin^{-1} \left(\frac{\sqrt{2}x}{3} \right) + C \end{aligned}$$

Use Formula 57 with $a = \frac{3}{\sqrt{2}}$ and $u = x$ for

$$\int x^2 \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 - x^2} dx.$$

$$\begin{aligned} 60. u = \sqrt{3}t, du = \sqrt{3}dt \\ \int \frac{\sqrt{16-3t^2}}{t} dt &= \int \frac{\sqrt{4^2 - u^2}}{u} du \\ &= \sqrt{16-u^2} - 4 \ln \left| \frac{4 + \sqrt{16-u^2}}{u} \right| + C \\ &= \sqrt{16-3t^2} - 4 \ln \left| \frac{4 + \sqrt{16-3t^2}}{\sqrt{3}t} \right| + C \end{aligned}$$

Use Formula 55 with $a = 4$ for $\int \frac{\sqrt{4^2 - u^2}}{u} du$.

$$\begin{aligned} 61. u = \sqrt{3}x, du = \sqrt{3}dx \\ \int \frac{dx}{\sqrt{5+3x^2}} &= \frac{1}{\sqrt{3}} \int \frac{du}{\sqrt{(\sqrt{5})^2 + u^2}} \\ &= \frac{1}{\sqrt{3}} \ln \left| u + \sqrt{u^2 + 5} \right| + C \\ &= \frac{1}{\sqrt{3}} \ln \left| \sqrt{3}x + \sqrt{3x^2 + 5} \right| + C \end{aligned}$$

Use Formula 45 with $a = \sqrt{5}$ for

$$\int \frac{du}{\sqrt{(\sqrt{5})^2 + u^2}}.$$

$$\begin{aligned} 62. u = \sqrt{5}t, du = \sqrt{5}dt \\ \int t^2 \sqrt{3+5t^2} dt &= \frac{1}{\sqrt{5}} \int \frac{u^2}{5} \sqrt{(\sqrt{3})^2 + u^2} du \end{aligned}$$

$$\begin{aligned} &= \frac{1}{5\sqrt{5}} \int u^2 \sqrt{(\sqrt{3})^2 + u^2} du \\ &= \frac{1}{5\sqrt{5}} \left[\frac{u}{8} (3+2u^2) \sqrt{3+u^2} \right. \\ &\quad \left. - \frac{9}{8} \ln \left(u + \sqrt{3+u^2} \right) \right] + C \\ &= \frac{1}{5\sqrt{5}} \left[\frac{\sqrt{5}t}{8} (3+10t^2) \sqrt{3+5t^2} \right. \\ &\quad \left. - \frac{9}{8} \ln \left(\sqrt{5}t + \sqrt{3+5t^2} \right) \right] + C \end{aligned}$$

Use Formula 48 with $a = \sqrt{3}$ for

$$\int u^2 \sqrt{(\sqrt{3})^2 + u^2} du.$$

$$63. u = t + 1, du = dt$$

$$\begin{aligned} \int \frac{dt}{\sqrt{t^2 + 2t - 3}} &= \int \frac{dt}{\sqrt{t^2 + 2t + 1 - 4}} \\ &= \int \frac{dt}{\sqrt{(t+1)^2 - 4}} \\ &= \int \frac{du}{\sqrt{u^2 - 2^2}} = \ln \left| u + \sqrt{u^2 - 4} \right| + C \\ &= \ln \left| t + 1 + \sqrt{t^2 + 2t - 3} \right| + C \end{aligned}$$

Use Formula 45 with $a = 2$ for $\int \frac{du}{\sqrt{u^2 - 2^2}}$.

$$\begin{aligned} 64. u = x + 1, du = dx \\ \int \frac{\sqrt{x^2 + 2x - 3}}{x+1} dx &= \int \frac{\sqrt{x^2 + 2x + 1 - 4}}{x+1} dx \\ &= \int \frac{\sqrt{(x+1)^2 - 4}}{x+1} dx = \int \frac{\sqrt{u^2 - 2^2}}{u} du \\ &= \sqrt{u^2 - 4} - 2 \sec^{-1} \frac{u}{2} + C \\ &= \sqrt{x^2 + 2x - 3} - 2 \sec^{-1} \left(\frac{x+1}{2} \right) + C \end{aligned}$$

Use Formula 47 with $a = 2$ for $\int \frac{\sqrt{u^2 - 2^2}}{u} du$

$$\begin{aligned} 65. u = \sin t, du = \cos t dt \\ \int \frac{\sin t \cos t}{\sqrt{3 \sin t + 5}} dt \\ &= \int \frac{u}{\sqrt{3u + 5}} du = \frac{2}{3 \cdot 3^2} (3u - 2 \cdot 5) \sqrt{3u + 5} + C \\ &= \frac{2}{27} (3 \sin t - 10) \sqrt{3 \sin t + 5} + C \end{aligned}$$

Use Formula 98 with $a = 3$, and $b = 5$ for

$$\int \frac{u}{\sqrt{3u+5}} du.$$

66. $u = \cos x$, $du = -\sin x dx$

$$\begin{aligned} \int \frac{\sin x}{\cos x \sqrt{5-4\cos x}} dx &= - \int \frac{du}{u \sqrt{5-4u}} \\ &= - \int \frac{du}{u \sqrt{-4u+5}} \\ &= -\frac{1}{\sqrt{5}} \ln \left| \frac{\sqrt{5-4u}-\sqrt{5}}{\sqrt{5-4u}+\sqrt{5}} \right| + C \\ &= -\frac{1}{\sqrt{5}} \ln \left| \frac{\sqrt{5-4\cos x}-\sqrt{5}}{\sqrt{5-4\cos x}+\sqrt{5}} \right| + C \end{aligned}$$

Use Formula 100a with $a = -4$, and $b = 5$ for

$$\int \frac{du}{u \sqrt{-4u+5}}.$$

67. The length is given by

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^{\pi/4} \sqrt{1 + \left[\frac{1}{\cos x}(-\sin x)\right]^2} dx \\ &= \int_0^{\pi/4} \sqrt{1 + \tan^2 x} dx \\ &= \int_0^{\pi/4} \sqrt{\sec^2 x} dx \\ &= \int_0^{\pi/4} \sec x dx \\ &= \left[\ln |\sec x + \tan x| \right]_0^{\pi/4} \\ &= \ln |\sqrt{2} + 1| - \ln |1| \\ &= \ln |\sqrt{2} + 1| \approx 0.881 \end{aligned}$$

68. $\sec x = \frac{1}{\cos x} = \frac{1 + \sin x}{\cos x(1 + \sin x)}$

$$\begin{aligned} &= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1 + \sin x)} = \frac{\sin x(1 + \sin x) + \cos^2 x}{\cos x(1 + \sin x)} \\ &= \frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x} \\ \int \sec x &= \int \left(\frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x} \right) dx \end{aligned}$$

70. $V = 2\pi \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(x + \frac{\pi}{4}\right) |\sin x - \cos x| dx$

$$u = x - \frac{\pi}{4}, du = dx$$

$$= \int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{1 + \sin x} dx$$

For the first integral use $u = \cos x$, $du = -\sin x dx$, and for the second integral use $v = 1 + \sin x$, $dv = \cos x dx$.

$$\begin{aligned} \int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{1 + \sin x} dx &= - \int \frac{du}{u} + \int \frac{dv}{v} \\ &= -\ln |u| + \ln |v| + C \\ &= -\ln |\cos x| + \ln |1 + \sin x| + C \\ &= \ln \left| \frac{1 + \sin x}{\cos x} \right| + C \\ &= \ln |\sec x + \tan x| + C \end{aligned}$$

69. $u = x - \pi$, $du = dx$

$$\begin{aligned} \int_0^{2\pi} \frac{x |\sin x|}{1 + \cos^2 x} dx &= \int_{-\pi}^{\pi} \frac{(u + \pi) |\sin(u + \pi)|}{1 + \cos^2(u + \pi)} du \\ &= \int_{-\pi}^{\pi} \frac{(u + \pi) |\sin u|}{1 + \cos^2 u} du \\ &= \int_{-\pi}^{\pi} \frac{u |\sin u|}{1 + \cos^2 u} du + \int_{-\pi}^{\pi} \frac{\pi |\sin u|}{1 + \cos^2 u} du \\ \int_{-\pi}^{\pi} \frac{u |\sin u|}{1 + \cos^2 u} du &= 0 \text{ by symmetry.} \\ \int_{-\pi}^{\pi} \frac{\pi |\sin u|}{1 + \cos^2 u} du &= 2 \int_0^{\pi} \frac{\pi \sin u}{1 + \cos^2 u} du \\ v = \cos u, dv &= -\sin u du \\ -2 \int_1^{-1} \frac{\pi}{1 + v^2} dv &= 2\pi \int_{-1}^1 \frac{1}{1 + v^2} dv \\ &= 2\pi [\tan^{-1} v]_{-1}^1 = 2\pi \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) \right] \\ &= 2\pi \left(\frac{\pi}{2} \right) = \pi^2 \end{aligned}$$

$$\begin{aligned}
V &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(u + \frac{\pi}{2} \right) \left| \sin \left(u + \frac{\pi}{4} \right) - \cos \left(u + \frac{\pi}{4} \right) \right| du \\
&= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(u + \frac{\pi}{2} \right) \left| \frac{\sqrt{2}}{2} \sin u + \frac{\sqrt{2}}{2} \cos u - \frac{\sqrt{2}}{2} \cos u + \frac{\sqrt{2}}{2} \sin u \right| du \\
&= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(u + \frac{\pi}{2} \right) \left| \frac{\sqrt{2}}{2} \sin u \right| du = 2\sqrt{2}\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u |\sin u| du + \sqrt{2}\pi^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin u| du \\
&= 2\sqrt{2}\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u |\sin u| du = 0 \text{ by symmetry.} \\
V &= \sqrt{2}\pi^2 \int_0^{\frac{\pi}{2}} \sin u du = 2\sqrt{2}\pi^2 [-\cos u]_0^{\frac{\pi}{2}} = 2\sqrt{2}\pi^2
\end{aligned}$$

8.2 Concepts Review

1. $\int \frac{1 + \cos 2x}{2} dx$

2. $\int (1 - \sin^2 x) \cos x dx$

3. $\int \sin^2 x (1 - \sin^2 x) \cos x dx$

4. $\cos mx \cos nx = \frac{1}{2} [\cos(m+n)x + \cos(m-n)x]$

$$= \frac{3}{48} \int du - \frac{1}{24} \int 2 \cos 2u du + \frac{1}{192} \int 4 \cos 4u du$$

$$= \frac{3}{48} (6x) - \frac{1}{24} \sin 12x + \frac{1}{192} \sin 24x + C$$

$$= \frac{3}{8} x - \frac{1}{24} \sin 12x + \frac{1}{192} \sin 24x + C$$

3. $\int \sin^3 x dx = \int \sin x (1 - \cos^2 x) dx$
 $= \int \sin x dx - \int \sin x \cos^2 x dx$
 $= -\cos x + \frac{1}{3} \cos^3 x + C$

Problem Set 8.2

1. $\int \sin^2 x dx = \int \frac{1 - \cos 2x}{2} dx$
 $= \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x dx$
 $= \frac{1}{2} x - \frac{1}{4} \sin 2x + C$

2. $u = 6x, du = 6 dx$
 $\int \sin^4 6x dx = \frac{1}{6} \int \sin^4 u du$
 $= \frac{1}{6} \int \left(\frac{1 - \cos 2u}{2} \right)^2 du$
 $= \frac{1}{24} \int (1 - 2 \cos 2u + \cos^2 2u) du$
 $= \frac{1}{24} \int du - \frac{1}{24} \int 2 \cos 2u du + \frac{1}{48} \int (1 + \cos 4u) du$

6. $\int_0^{\pi/2} \sin^6 \theta d\theta = \int_0^{\pi/2} \left(\frac{1 - \cos 2\theta}{2} \right)^3 d\theta$
 $= \frac{1}{8} \int_0^{\pi/2} (1 - 3 \cos 2\theta + 3 \cos^2 2\theta - \cos^3 2\theta) d\theta$

4. $\int \cos^3 x dx =$
 $= \int \cos x (1 - \sin^2 x) dx$
 $= \int \cos x dx - \int \cos x \sin^2 x dx$
 $= \sin x - \frac{1}{3} \sin^3 x + C$

5. $\int_0^{\pi/2} \cos^5 \theta d\theta = \int_0^{\pi/2} (1 - \sin^2 \theta)^2 \cos \theta d\theta$
 $= \int_0^{\pi/2} (1 - 2 \sin^2 \theta + \sin^4 \theta) \cos \theta d\theta$
 $= \left[\sin \theta - \frac{2}{3} \sin^3 \theta + \frac{1}{5} \sin^5 \theta \right]_0^{\pi/2}$
 $= \left(1 - \frac{2}{3} + \frac{1}{5} \right) - 0 = \frac{8}{15}$

$$\begin{aligned}
&= \frac{1}{8} \int_0^{\pi/2} d\theta - \frac{3}{16} \int_0^{\pi/2} 2 \cos 2\theta d\theta + \frac{3}{8} \int_0^{\pi/2} \cos^2 2\theta d\theta - \frac{1}{8} \int_0^{\pi/2} \cos^3 2\theta d\theta \\
&= \frac{1}{8} [\theta]_0^{\pi/2} - \frac{3}{16} [\sin 2\theta]_0^{\pi/2} + \frac{3}{8} \int_0^{\pi/2} \left(\frac{1 + \cos 4\theta}{2} \right) d\theta - \frac{1}{8} \int_0^{\pi/2} (1 - \sin^2 2\theta) \cos 2\theta d\theta \\
&= \frac{1}{8} \cdot \frac{\pi}{2} + \frac{3}{16} \int_0^{\pi/2} d\theta + \frac{3}{64} \int_0^{\pi/2} 4 \cos 4\theta d\theta - \frac{1}{16} \int_0^{\pi/2} 2 \cos 2\theta d\theta + \frac{1}{16} \int_0^{\pi/2} \sin^2 2\theta \cdot 2 \cos 2\theta d\theta \\
&= \frac{\pi}{16} + \frac{3\pi}{32} + \frac{3}{64} [\sin 4\theta]_0^{\pi/2} - \frac{1}{16} [\sin 2\theta]_0^{\pi/2} + \frac{1}{48} [\sin^3 2\theta]_0^{\pi/2} = \frac{5\pi}{32}
\end{aligned}$$

$$\begin{aligned}
7. \quad \int \sin^5 4x \cos^2 4x dx &= \int (1 - \cos^2 4x)^2 \cos^2 4x \sin 4x dx = \int (1 - 2 \cos^2 4x + \cos^4 4x) \cos^2 4x \sin 4x dx \\
&= -\frac{1}{4} \int (\cos^2 4x - 2 \cos^4 4x + \cos^6 4x) (-4 \sin 4x) dx = -\frac{1}{12} \cos^3 4x + \frac{1}{10} \cos^5 4x - \frac{1}{28} \cos^7 4x + C
\end{aligned}$$

$$\begin{aligned}
8. \quad \int (\sin^3 2t) \sqrt{\cos 2t} dt &= \int (1 - \cos^2 2t) (\cos 2t)^{1/2} \sin 2t dt = -\frac{1}{2} \int [(\cos 2t)^{1/2} - (\cos 2t)^{5/2}] (-2 \sin 2t) dt \\
&= -\frac{1}{3} (\cos 2t)^{3/2} + \frac{1}{7} (\cos 2t)^{7/2} + C
\end{aligned}$$

$$\begin{aligned}
9. \quad \int \cos^3 3\theta \sin^{-2} 3\theta d\theta &= \int (1 - \sin^2 3\theta) \sin^{-2} 3\theta \cos 3\theta d\theta = \frac{1}{3} \int (\sin^{-2} 3\theta - 1) 3 \cos 3\theta d\theta \\
&= -\frac{1}{3} \csc 3\theta - \frac{1}{3} \sin 3\theta + C
\end{aligned}$$

$$\begin{aligned}
10. \quad \int \sin^{1/2} 2z \cos^3 2z dz &= \int (1 - \sin^2 2z) \sin^{1/2} 2z \cos 2z dz \\
&= \frac{1}{2} \int (\sin^{1/2} 2z - \sin^{5/2} 2z) 2 \cos 2z dz = \frac{1}{3} \sin^{3/2} 2z - \frac{1}{7} \sin^{7/2} 2z + C
\end{aligned}$$

$$\begin{aligned}
11. \quad \int \sin^4 3t \cos^4 3t dt &= \int \left(\frac{1 - \cos 6t}{2} \right)^2 \left(\frac{1 + \cos 6t}{2} \right)^2 dt = \frac{1}{16} \int (1 - 2 \cos^2 6t + \cos^4 6t) dt \\
&= \frac{1}{16} \int \left[1 - (1 + \cos 12t) + \frac{1}{4} (1 + \cos 12t)^2 \right] dt = -\frac{1}{16} \int \cos 12t dt + \frac{1}{64} \int (1 + 2 \cos 12t + \cos^2 12t) dt \\
&= -\frac{1}{192} \int 12 \cos 12t dt + \frac{1}{64} \int dt + \frac{1}{384} \int 12 \cos 12t dt + \frac{1}{128} \int (1 + \cos 24t) dt \\
&= -\frac{1}{192} \sin 12t + \frac{1}{64} t + \frac{1}{384} \sin 12t + \frac{1}{128} t + \frac{1}{3072} \sin 24t + C = \frac{3}{128} t - \frac{1}{384} \sin 12t + \frac{1}{3072} \sin 24t + C
\end{aligned}$$

$$\begin{aligned}
12. \quad \int \cos^6 \theta \sin^2 \theta d\theta &= \int \left(\frac{1 + \cos 2\theta}{2} \right)^3 \left(\frac{1 - \cos 2\theta}{2} \right) d\theta = \frac{1}{16} \int (1 + 2 \cos 2\theta - 2 \cos^3 2\theta - \cos^4 2\theta) d\theta \\
&= \frac{1}{16} \int d\theta + \frac{1}{16} \int 2 \cos 2\theta d\theta - \frac{1}{8} \int (1 - \sin^2 2\theta) \cos 2\theta d\theta - \frac{1}{64} \int (1 + \cos 4\theta)^2 d\theta \\
&= \frac{1}{16} \int d\theta + \frac{1}{16} \int 2 \cos 2\theta d\theta - \frac{1}{16} \int 2 \cos 2\theta d\theta + \frac{1}{16} \int 2 \sin^2 2\theta \cos 2\theta d\theta - \frac{1}{64} \int (1 + 2 \cos 4\theta + \cos^2 4\theta) d\theta \\
&= \frac{1}{16} \int d\theta + \frac{1}{16} \int \sin^2 2\theta \cdot 2 \cos 2\theta d\theta - \frac{1}{64} \int d\theta - \frac{1}{128} \int 4 \cos 4\theta d\theta - \frac{1}{128} \int (1 + \cos 8\theta) d\theta \\
&= \frac{1}{16} \theta + \frac{1}{48} \sin^3 2\theta - \frac{1}{64} \theta - \frac{1}{128} \sin 4\theta - \frac{1}{128} \theta - \frac{1}{1024} \sin 8\theta + C \\
&= \frac{5}{128} \theta + \frac{1}{48} \sin^3 2\theta - \frac{1}{128} \sin 4\theta - \frac{1}{1024} \sin 8\theta + C
\end{aligned}$$

$$13. \int \sin 4y \cos 5y \, dy = \frac{1}{2} \int [\sin 9y + \sin(-y)] \, dy = \frac{1}{2} \int (\sin 9y - \sin y) \, dy$$

$$= \frac{1}{2} \left(-\frac{1}{9} \cos 9y + \cos y \right) + C = \frac{1}{2} \cos y - \frac{1}{18} \cos 9y + C$$

$$14. \int \cos y \cos 4y \, dy = \frac{1}{2} \int [\cos 5y + \cos(-3y)] \, dy = \frac{1}{10} \sin 5y - \frac{1}{6} \sin(-3y) + C = \frac{1}{10} \sin 5y + \frac{1}{6} \sin 3y + C$$

$$15. \int \sin^4 \left(\frac{w}{2} \right) \cos^2 \left(\frac{w}{2} \right) dw = \int \left(\frac{1 - \cos w}{2} \right)^2 \left(\frac{1 + \cos w}{2} \right) dw = \frac{1}{8} \int (1 - \cos w - \cos^2 w + \cos^3 w) dw$$

$$= \frac{1}{8} \int \left[1 - \cos w - \frac{1}{2}(1 + \cos 2w) + (1 - \sin^2 w) \cos w \right] dw = \frac{1}{8} \int \left[\frac{1}{2} - \frac{1}{2} \cos 2w - \sin^2 w \cos w \right] dw$$

$$= \frac{1}{16} w - \frac{1}{32} \sin 2w - \frac{1}{24} \sin^3 w + C$$

$$16. \int \sin 3t \sin t \, dt = \int -\frac{1}{2} [\cos 4t - \cos 2t] \, dt$$

$$= -\frac{1}{2} \left(\int \cos 4t \, dt - \int \cos 2t \, dt \right)$$

$$= -\frac{1}{2} \left(\frac{1}{4} \sin 4t - \frac{1}{2} \sin 2t \right) + C$$

$$= -\frac{1}{8} \sin 4t + \frac{1}{4} \sin 2t + C$$

$$= \int (\cot^2 x \csc^2 x - \cot^2 x) \, dx$$

$$= \int \cot^2 x \csc^2 x \, dx - \int (\csc^2 x - 1) \, dx$$

$$= -\frac{1}{3} \cot^3 x + \cot x + x + C$$

$$17. \int \tan^4 x \, dx = \int (\tan^2 x)(\tan^2 x) \, dx$$

$$= \int (\tan^2 x)(\sec^2 x - 1) \, dx$$

$$= \int (\tan^2 x \sec^2 x - \tan^2 x) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

$$= \frac{1}{3} \tan^3 x - \tan x + x + C$$

$$18. \int \cot^4 x \, dx = \int (\cot^2 x)(\cot^2 x) \, dx$$

$$= \int (\cot^2 x)(\csc^2 x - 1) \, dx$$

$$19. \int \tan^3 x \, dx = \int (\tan x)(\tan^2 x) \, dx$$

$$= \int (\tan x)(\sec^2 x - 1) \, dx$$

$$= \frac{1}{2} \tan^2 x + \ln |\cos x| + C$$

$$20. \int \cot^3 x \, dx = \int (\cot x)(\cot^2 x) \, dx$$

$$= \int (\cot x)(\csc^2 x - 1) \, dx$$

$$= \int \cot x \csc^2 x \, dx - \int \cot x \, dx$$

$$= -\frac{1}{2} \cot^2 x - \ln |\sin x| + C$$

$$21. \int \tan^5 \left(\frac{\theta}{2} \right) d\theta$$

$$u = \left(\frac{\theta}{2} \right); \, du = \frac{d\theta}{2}$$

$$\int \tan^5 \left(\frac{\theta}{2} \right) d\theta = 2 \int \tan^5 u \, du$$

$$= 2 \int (\tan^3 u)(\sec^2 u - 1) \, du$$

$$= 2 \int \tan^3 u \sec^2 u \, du - 2 \int \tan^3 u \, du$$

$$\begin{aligned}
&= 2 \int \tan^3 u \sec^2 u \, du - 2 \int \tan u (\sec^2 u - 1) \, du \\
&= 2 \int \tan^3 u \sec^2 u \, du - 2 \int \tan u \sec^2 u \, du + 2 \int \tan u \, du \\
&= \frac{1}{2} \tan^4 \left(\frac{\theta}{2} \right) - \tan^2 \left(\frac{\theta}{2} \right) - 2 \ln \left| \cos \frac{\theta}{2} \right| + C
\end{aligned}$$

22. $\int \cot^5 2t \, dt$
 $u = 2t; du = 2dt$

$$\begin{aligned}
\int \cot^5 2t \, dt &= \frac{1}{2} \int \cot^5 u \, du \\
&= \frac{1}{2} \int (\cot^3 u) (\cot^2 u) \, du = \frac{1}{2} \int (\cot^3 u) (\csc^2 u - 1) \, du \\
&= \frac{1}{2} \int (\cot^3 u) (\csc^2 u) \, du - \frac{1}{2} \int \cot^3 u \, du \\
&= \frac{1}{2} \int (\cot^3 u) (\csc^2 u) \, du - \frac{1}{2} \int (\cot u) (\csc^2 u - 1) \, du \\
&= \frac{1}{2} \int (\cot^3 u) (\csc^2 u) \, du - \frac{1}{2} \int (\cot u) (\csc^2 u) \, du + \frac{1}{2} \int \cot u \, du \\
&= -\frac{1}{8} \cot^4 u + \frac{1}{4} \cot^2 u + \frac{1}{2} \ln |\sin u| + C \\
&= -\frac{1}{8} \cot^4 2t + \frac{1}{4} \cot^2 2t + \frac{1}{2} \ln |\sin 2t| + C
\end{aligned}$$

23. $\int \tan^{-3} x \sec^4 x \, dx = \int (\tan^{-3} x) (\sec^2 x) (\sec^2 x) \, dx$
 $= \int (\tan^{-3} x) (1 + \tan^2 x) (\sec^2 x) \, dx$
 $= \int \tan^{-3} x \sec^2 x \, dx + \int (\tan x)^{-1} \sec^2 x \, dx$
 $= -\frac{1}{2} \tan^{-2} x + \ln |\tan x| + C$

24. $\int \tan^{-3/2} x \sec^4 x \, dx = \int (\tan^{-3/2} x) (\sec^2 x) (\sec^2 x) \, dx$
 $= \int (\tan^{-3/2} x) (1 + \tan^2 x) (\sec^2 x) \, dx$
 $= \int \tan^{-3/2} x \sec^2 x \, dx + \int \tan^{1/2} x \sec^2 x \, dx$
 $= -2 \tan^{-1/2} x + \frac{2}{3} \tan^{3/2} x + C$

25. $\int \tan^3 x \sec^2 x \, dx = \int (\tan^2 x) (\sec x) (\sec x \tan x) \, dx$
 $= \int (\sec^2 x - 1) (\sec x) (\sec x \tan x) \, dx$
 $= \int \sec^3 x \sec x \tan x \, dx - \int \sec x (\sec x \tan x) \, dx$
 $= \frac{1}{4} \sec^4 x - \frac{1}{2} \sec^2 x + C$

26. $\int \tan^3 x \sec^{-1/2} x \, dx = \int \tan^2 x \sec^{-3/2} x (\sec x \tan x) \, dx$
 $= \int (\sec^2 x - 1) (\sec^{-3/2} x) (\sec x \tan x) \, dx$

$$= \int \sec^{1/2} x (\sec x \tan x) dx - \int \sec^{-3/2} x (\sec x \tan x) dx$$

$$= \frac{2}{3} \sec^{3/2} x + 2 \sec^{-1/2} x + C$$

$$27. \int_{-\pi}^{\pi} \cos mx \cos nx dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos[(m+n)x] + \cos[(m-n)x]) dx = \frac{1}{2} \left[\frac{1}{m+n} \sin[(m+n)x] + \frac{1}{m-n} \sin[(m-n)x] \right]_{-\pi}^{\pi}$$

$= 0$ for $m \neq n$, since $\sin k\pi = 0$ for all integers k .

$$28. \text{ If we let } u = \frac{\pi x}{L} \text{ then } du = \frac{\pi}{L} dx. \text{ Making the substitution and changing the limits as necessary, we get}$$

$$\int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = \frac{L}{\pi} \int_{-\pi}^{\pi} \cos mu \cos nu du = 0 \quad (\text{See Problem 27})$$

$$29. \int_0^{\pi} \pi(x + \sin x)^2 dx = \pi \int_0^{\pi} (x^2 + 2x \sin x + \sin^2 x) dx = \pi \int_0^{\pi} x^2 dx + 2\pi \int_0^{\pi} x \sin x dx + \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) dx$$

$$= \pi \left[\frac{1}{3} x^3 \right]_0^{\pi} + 2\pi [\sin x - x \cos x]_0^{\pi} + \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \frac{1}{3} \pi^4 + 2\pi(0 + \pi - 0) + \frac{\pi}{2}(\pi - 0 - 0) = \frac{1}{3} \pi^4 + \frac{5}{2} \pi^2 \approx 57.1437$$

Use Formula 40 with $u = x$ for $\int x \sin x dx$

$$30. V = 2\pi \int_0^{\sqrt{\pi/2}} x \sin^2(x^2) dx$$

$$u = x^2, \quad du = 2x dx$$

$$V = \pi \int_0^{\pi/2} \sin^2 u du = \pi \int_0^{\pi/2} \frac{1 - \cos 2u}{2} du = \pi \left[\frac{1}{2} u - \frac{1}{4} \sin 2u \right]_0^{\pi/2} = \frac{\pi^2}{4} \approx 2.4674$$

$$31. \text{ a. } \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\sum_{n=1}^N a_n \sin(nx) \right) \sin(mx) dx = \frac{1}{\pi} \sum_{n=1}^N a_n \int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx$$

From Example 6,

$$\int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases} \text{ so every term in the sum is 0 except for when } n = m.$$

If $m > N$, there is no term where $n = m$, while if $m \leq N$, then $n = m$ occurs. When $n = m$

$$a_n \int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx = a_m \pi \text{ so when } m \leq N,$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) dx = \frac{1}{\pi} \cdot a_m \cdot \pi = a_m.$$

$$\text{b. } \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\sum_{n=1}^N a_n \sin(nx) \right) \left(\sum_{m=1}^N a_m \sin(mx) \right) dx = \frac{1}{\pi} \sum_{n=1}^N a_n \sum_{m=1}^N a_m \int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx$$

From Example 6, the integral is 0 except when $m = n$. When $m = n$, we obtain

$$\frac{1}{\pi} \sum_{n=1}^N a_n (a_n \pi) = \sum_{n=1}^N a_n^2.$$

$$32. \text{ a. Proof by induction}$$

$$n = 1: \cos \frac{x}{2} = \cos \frac{x}{2}$$

Assume true for $k \leq n$.

$$\cos \frac{x}{2} \cos \frac{x}{4} \cdots \cos \frac{x}{2^n} \cdot \cos \frac{x}{2^{n+1}} = \left[\cos \frac{1}{2^n} x + \cos \frac{3}{2^n} x + \cdots + \cos \frac{2^n - 1}{2^n} x \right] \frac{1}{2^{n-1}} \cos \frac{x}{2^{n+1}}$$

Note that

$$\left(\cos \frac{k}{2^n} x\right) \left(\cos \frac{1}{2^{n+1}} x\right) = \frac{1}{2} \left[\cos \frac{2k+1}{2^{n+1}} x + \cos \frac{2k-1}{2^{n+1}} x \right], \text{ so}$$

$$\left[\cos \frac{1}{2^n} x + \cos \frac{3}{2^n} x + \cdots + \cos \frac{2^n-1}{2^n} x \right] \left(\cos \frac{1}{2^{n+1}} x \right) \frac{1}{2^{n-1}} = \left[\cos \frac{1}{2^{n+1}} x + \cos \frac{3}{2^{n+1}} x + \cdots + \cos \frac{2^{n+1}-1}{2^{n+1}} x \right] \frac{1}{2^n}$$

$$\begin{aligned} \text{b. } \lim_{n \rightarrow \infty} \left[\cos \frac{1}{2^n} x + \cos \frac{3}{2^n} x + \cdots + \cos \frac{2^n-1}{2^n} x \right] \frac{1}{2^{n-1}} &= \frac{1}{x} \lim_{n \rightarrow \infty} \left[\cos \frac{1}{2^n} x + \cos \frac{3}{2^n} x + \cdots + \cos \frac{2^n-1}{2^n} x \right] \frac{x}{2^{n-1}} \\ &= \frac{1}{x} \int_0^x \cos t \, dt \end{aligned}$$

$$\text{c. } \frac{1}{x} \int_0^x \cos t \, dt = \frac{1}{x} [\sin t]_0^x = \frac{\sin x}{x}$$

33. Using the half-angle identity $\cos \frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$, we see that since

$$\cos \frac{\pi}{4} = \cos \frac{\pi}{2} = \frac{\sqrt{2}}{2}$$

$$\cos \frac{\pi}{8} = \cos \frac{\pi}{4} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{2}},$$

$$\cos \frac{\pi}{16} = \cos \frac{\pi}{8} = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2}}, \text{ etc.}$$

$$\text{Thus, } \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots = \cos \left(\frac{\pi}{2} \right) \cos \left(\frac{\pi}{4} \right) \cos \left(\frac{\pi}{8} \right) \cdots$$

$$= \lim_{n \rightarrow \infty} \cos \left(\frac{\pi}{2} \right) \cos \left(\frac{\pi}{4} \right) \cdots \cos \left(\frac{\pi}{2^n} \right) = \frac{\sin \left(\frac{\pi}{2} \right)}{\frac{\pi}{2}} = \frac{2}{\pi}$$

34. Since $(k - \sin x)^2 = (\sin x - k)^2$, the volume of S is $\int_0^\pi \pi(k - \sin x)^2 = \pi \int_0^\pi (k^2 - 2k \sin x + \sin^2 x) dx$

$$= \pi k^2 \int_0^\pi dx - 2k\pi \int_0^\pi \sin x \, dx + \frac{\pi}{2} \int_0^\pi (1 - \cos 2x) dx = \pi k^2 [x]_0^\pi + 2k\pi [\cos x]_0^\pi + \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^\pi$$

$$= \pi^2 k^2 + 2k\pi(-1-1) + \frac{\pi}{2}(\pi-0) = \pi^2 k^2 - 4k\pi + \frac{\pi^2}{2}$$

Let $f(k) = \pi^2 k^2 - 4k\pi + \frac{\pi^2}{2}$, then $f'(k) = 2\pi^2 k - 4\pi$ and $f'(k) = 0$ when $k = \frac{2}{\pi}$.

The critical points of $f(k)$ on $0 \leq k \leq 1$ are 0 , $\frac{2}{\pi}$, 1 .

$$f(0) = \frac{\pi^2}{2} \approx 4.93, f\left(\frac{2}{\pi}\right) = 4 - 8 + \frac{\pi^2}{2} \approx 0.93, f(1) = \pi^2 - 4\pi + \frac{\pi^2}{2} \approx 2.24$$

a. S has minimum volume when $k = \frac{2}{\pi}$.

b. S has maximum volume when $k = 0$.

8.3 Concepts Review

1. $\sqrt{x-3}$
2. $2 \sin t$
3. $2 \tan t$
4. $2 \sec t$

Problem Set 8.3

1. $u = \sqrt{x+1}, u^2 = x+1, 2u du = dx$
 $\int x\sqrt{x+1} dx = \int (u^2 - 1)u(2u du)$
 $= \int (2u^4 - 2u^2) du = \frac{2}{5}u^5 - \frac{2}{3}u^3 + C$
 $= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$
2. $u = \sqrt[3]{x+\pi}, u^3 = x+\pi, 3u^2 du = dx$
 $\int x\sqrt[3]{x+\pi} dx = \int (u^3 - \pi)u(3u^2 du)$
 $= \int (3u^6 - 3\pi u^3) du = \frac{3}{7}u^7 - \frac{3\pi}{4}u^4 + C$
 $= \frac{3}{7}(x+\pi)^{7/3} - \frac{3\pi}{4}(x+\pi)^{4/3} + C$
3. $u = \sqrt{3t+4}, u^2 = 3t+4, 2u du = 3 dt$
 $\int \frac{t dt}{\sqrt{3t+4}} = \int \frac{\frac{1}{3}(u^2 - 4) \frac{2}{3}u du}{u} = \frac{2}{9} \int (u^2 - 4) du$
 $= \frac{2}{27}u^3 - \frac{8}{9}u + C$
 $= \frac{2}{27}(3t+4)^{3/2} - \frac{8}{9}(3t+4)^{1/2} + C$
4. $u = \sqrt{x+4}, u^2 = x+4, 2u du = dx$
 $\int \frac{x^2 + 3x}{\sqrt{x+4}} dx = \int \frac{(u^2 - 4)^2 + 3(u^2 - 4)u}{u} 2u du$
 $= 2 \int (u^4 - 5u^2 + 4) du = \frac{2}{5}u^5 - \frac{10}{3}u^3 + 8u + C$
 $= \frac{2}{5}(x+4)^{5/2} - \frac{10}{3}(x+4)^{3/2} + 8(x+4)^{1/2} + C$
5. $u = \sqrt{t}, u^2 = t, 2u du = dt$
 $\int_1^2 \frac{dt}{\sqrt{t+e}} = \int_1^{\sqrt{2}} \frac{2u du}{u+e} = 2 \int_1^{\sqrt{2}} \frac{u+e-e}{u+e} du$
 $= 2 \int_1^{\sqrt{2}} du - 2 \int_1^{\sqrt{2}} \frac{e}{u+e} du$
 $= 2[u]_1^{\sqrt{2}} - 2e[\ln|u+e|]_1^{\sqrt{2}}$

$$= 2(\sqrt{2} - 1) - 2e[\ln(\sqrt{2} + e) - \ln(1 + e)]$$

$$= 2\sqrt{2} - 2 - 2e \ln \left(\frac{\sqrt{2} + e}{1 + e} \right)$$

6. $u = \sqrt{t}, u^2 = t, 2u du = dt$
 $\int_0^1 \frac{\sqrt{t}}{t+1} dt = \int_0^1 \frac{u}{u^2+1} (2u du)$
 $= 2 \int_0^1 \frac{u^2}{u^2+1} du = 2 \int_0^1 \frac{u^2+1-1}{u^2+1} du$
 $= 2 \int_0^1 du - 2 \int_0^1 \frac{1}{u^2+1} du = 2[u]_0^1 - 2[\tan^{-1} u]_0^1$
 $= 2 - 2 \tan^{-1} 1 = 2 - \frac{\pi}{2} \approx 0.4292$
7. $u = (3t+2)^{1/2}, u^2 = 3t+2, 2u du = 3 dt$
 $\int t(3t+2)^{3/2} dt = \int \frac{1}{3}(u^2 - 2)u^3 \left(\frac{2}{3}u du \right)$
 $= \frac{2}{9} \int (u^6 - 2u^4) du = \frac{2}{63}u^7 - \frac{4}{45}u^5 + C$
 $= \frac{2}{63}(3t+2)^{7/2} - \frac{4}{45}(3t+2)^{5/2} + C$
8. $u = (1-x)^{1/3}, u^3 = 1-x, 3u^2 du = -dx$
 $\int x(1-x)^{2/3} dx = \int (1-u^3)u^2(-3u^2) du$
 $= 3 \int (u^7 - u^4) du = \frac{3}{8}u^8 - \frac{3}{5}u^5 + C$
 $= \frac{3}{8}(1-x)^{8/3} - \frac{3}{5}(1-x)^{5/3} + C$
9. $x = 2 \sin t, dx = 2 \cos t dt$
 $\int \frac{\sqrt{4-x^2}}{x} dx = \int \frac{2 \cos t}{2 \sin t} (2 \cos t dt)$
 $= 2 \int \frac{1 - \sin^2 t}{\sin t} dt = 2 \int \csc t dt - 2 \int \sin t dt$
 $= 2 \ln |\csc t - \cot t| + 2 \cos t + C$
 $= 2 \ln \left| \frac{2 - \sqrt{4-x^2}}{x} \right| + \sqrt{4-x^2} + C$
10. $x = 4 \sin t, dx = 4 \cos t dt$
 $\int \frac{x^2 dx}{\sqrt{16-x^2}} = 16 \int \frac{\sin^2 t \cos t}{\cos t} dt$
 $= 16 \int \sin^2 t dt = 8 \int (1 - \cos 2t) dt$
 $= 8t - 4 \sin 2t + C = 8t - 8 \sin t \cos t + C$
 $= 8 \sin^{-1} \left(\frac{x}{4} \right) - \frac{x\sqrt{16-x^2}}{2} + C$

$$11. x = 2 \tan t, dx = 2 \sec^2 t dt$$

$$\begin{aligned} \int \frac{dx}{(x^2 + 4)^{3/2}} &= \int \frac{2 \sec^2 t dt}{(4 \sec^2 t)^{3/2}} = \frac{1}{4} \int \cos t dt \\ &= \frac{1}{4} \sin t + C = \frac{x}{4\sqrt{x^2 + 4}} + C \end{aligned}$$

$$12. t = \sec x, dt = \sec x \tan x dx$$

$$\text{Note that } 0 \leq x < \frac{\pi}{2}.$$

$$\sqrt{t^2 - 1} = |\tan x| = \tan x$$

$$\begin{aligned} \int_2^3 \frac{dt}{t^2 \sqrt{t^2 - 1}} &= \int_{\pi/3}^{\sec^{-1}(3)} \frac{\sec x \tan x}{\sec^2 x \tan x} dx \\ &= \int_{\pi/3}^{\sec^{-1}(3)} \cos x dx \\ &= [\sin x]_{\pi/3}^{\sec^{-1}(3)} = \sin[\sec^{-1}(3)] - \sin \frac{\pi}{3} \\ &= \sin \left[\cos^{-1} \left(\frac{1}{3} \right) \right] - \frac{\sqrt{3}}{2} = \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2} \approx 0.0768 \end{aligned}$$

$$13. t = \sec x, dt = \sec x \tan x dx$$

$$\text{Note that } \frac{\pi}{2} < x \leq \pi.$$

$$\sqrt{t^2 - 1} = |\tan x| = -\tan x$$

$$\begin{aligned} \int_2^3 \frac{\sqrt{t^2 - 1}}{t^3} dt &= \int_{2\pi/3}^{\sec^{-1}(-3)} \frac{-\tan x}{\sec^3 x} \sec x \tan x dx \\ &= \int_{2\pi/3}^{\sec^{-1}(-3)} -\sin^2 x dx = \int_{2\pi/3}^{\sec^{-1}(-3)} \left(\frac{1}{2} \cos 2x - \frac{1}{2} \right) dx \\ &= \left[\frac{1}{4} \sin 2x - \frac{1}{2} x \right]_{2\pi/3}^{\sec^{-1}(-3)} \\ &= \left[\frac{1}{2} \sin x \cos x - \frac{1}{2} x \right]_{2\pi/3}^{\sec^{-1}(-3)} \\ &= -\frac{\sqrt{2}}{9} - \frac{1}{2} \sec^{-1}(-3) + \frac{\sqrt{3}}{8} + \frac{\pi}{3} \approx 0.151252 \end{aligned}$$

$$14. t = \sin x, dt = \cos x dx$$

$$\begin{aligned} \int \frac{t}{\sqrt{1-t^2}} dt &= \int \sin x dx = -\cos x + C \\ &= -\sqrt{1-t^2} + C \end{aligned}$$

$$15. z = \sin t, dz = \cos t dt$$

$$\begin{aligned} \int \frac{2z-3}{\sqrt{1-z^2}} dz &= \int (2 \sin t - 3) dt \\ &= -2 \cos t - 3t + C \\ &= -2\sqrt{1-z^2} - 3 \sin^{-1} z + C \end{aligned}$$

$$16. x = \pi \tan t, dx = \pi \sec^2 t dt$$

$$\begin{aligned} \int \frac{\pi x - 1}{\sqrt{x^2 + \pi^2}} dx &= \int (\pi^2 \tan t - 1) \sec t dt \\ &= \pi^2 \int \tan t \sec t dt - \int \sec t dt \\ &= \pi^2 \sec t - \ln |\sec t + \tan t| + C \\ &= \pi \sqrt{x^2 + \pi^2} - \ln \left| \frac{1}{\pi} \sqrt{x^2 + \pi^2} + \frac{x}{\pi} \right| + C \\ \int_0^{\pi} \frac{\pi x - 1}{\sqrt{x^2 + \pi^2}} dx &= \left[\pi \sqrt{x^2 + \pi^2} - \ln \left| \frac{\sqrt{x^2 + \pi^2}}{\pi} + \frac{x}{\pi} \right| \right]_0^{\pi} \\ &= [\sqrt{2}\pi^2 - \ln(\sqrt{2} + 1)] - [\pi^2 - \ln 1] \\ &= (\sqrt{2} - 1)\pi^2 - \ln(\sqrt{2} + 1) \approx 3.207 \end{aligned}$$

$$17. x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x+1)^2 + 4$$

$$u = x + 1, du = dx$$

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + 2x + 5}} &= \int \frac{du}{\sqrt{u^2 + 4}} \\ u = 2 \tan t, du &= 2 \sec^2 t dt \\ \int \frac{du}{\sqrt{u^2 + 4}} &= \int \sec t dt = \ln |\sec t + \tan t| + C \\ \ln \left| \frac{\sqrt{u^2 + 4}}{2} + \frac{u}{2} \right| + C_1 &= \ln \left| \frac{\sqrt{x^2 + 2x + 5} + x + 1}{2} \right| + C_1 \\ &= \ln |\sqrt{x^2 + 2x + 5} + x + 1| + C \end{aligned}$$

$$18. x^2 + 4x + 5 = x^2 + 4x + 4 + 1 = (x+2)^2 + 1$$

$$u = x + 2, du = dx$$

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + 4x + 5}} &= \int \frac{du}{\sqrt{u^2 + 1}} \\ u = \tan t, du &= \sec^2 t dt \\ \int \frac{du}{\sqrt{u^2 + 1}} &= \int \sec t dt = \ln |\sec t + \tan t| + C \end{aligned}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+4x+5}} &= \ln \left| \sqrt{u^2+1} + u \right| + C \\ &= \ln \left| \sqrt{x^2+4x+5} + x+2 \right| + C\end{aligned}$$

19. $x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x+1)^2 + 4$

$u = x + 1, du = dx$

$$\int \frac{3x}{\sqrt{x^2+2x+5}} dx = \int \frac{3u-3}{\sqrt{u^2+4}} du$$

$$= 3 \int \frac{u}{\sqrt{u^2+4}} du - 3 \int \frac{du}{\sqrt{u^2+4}}$$

(Use the result of Problem 17.)

$$= 3\sqrt{u^2+4} - 3 \ln \left| \sqrt{u^2+4} + u \right| + C$$

$$= 3\sqrt{x^2+2x+5} - 3 \ln \left| \sqrt{x^2+2x+5} + x+1 \right| + C$$

20. $x^2 + 4x + 5 = x^2 + 4x + 4 + 1 = (x+2)^2 + 1$

$u = x + 2, du = dx$

$$\int \frac{2x-1}{\sqrt{x^2+4x+5}} dx = \int \frac{2u-5}{\sqrt{u^2+1}} du$$

$$= \int \frac{2u du}{\sqrt{u^2+1}} - 5 \int \frac{du}{\sqrt{u^2+1}}$$

(Use the result of Problem 18.)

$$= 2\sqrt{u^2+1} - 5 \ln \left| \sqrt{u^2+1} + u \right| + C$$

$$= 2\sqrt{x^2+4x+5} - 5 \ln \left| \sqrt{x^2+4x+5} + x+2 \right| + C$$

21. $5 - 4x - x^2 = 9 - (4 + 4x + 4x^2) = 9 - (x+2)^2$

$u = x + 2, du = dx$

$$\int \sqrt{5-4x-x^2} dx = \int \sqrt{9-u^2} du$$

$u = 3 \sin t, du = 3 \cos t dt$

$$\int \sqrt{9-u^2} du = 9 \int \cos^2 t dt = \frac{9}{2} \int (1 + \cos 2t) dt$$

$$= \frac{9}{2} \left(t + \frac{1}{2} \sin 2t \right) + C = \frac{9}{2} (t + \sin t \cos t) + C$$

$$= \frac{9}{2} \sin^{-1} \left(\frac{u}{3} \right) + \frac{1}{2} u \sqrt{9-u^2} + C$$

$$= \frac{9}{2} \sin^{-1} \left(\frac{x+2}{3} \right) + \frac{x+2}{2} \sqrt{5-4x-x^2} + C$$

22. $16 + 6x - x^2 = 25 - (9 - 6x + x^2) = 25 - (x-3)^2$

$u = x - 3, du = dx$

$$\int \frac{dx}{\sqrt{16+6x-x^2}} = \int \frac{du}{\sqrt{25-u^2}}$$

$u = 5 \sin t, du = 5 \cos t dt$

$$\int \frac{du}{\sqrt{25-u^2}} = \int dt = t + C = \sin^{-1} \left(\frac{u}{5} \right) + C$$

$$= \sin^{-1} \left(\frac{x-3}{5} \right) + C$$

23. $4x - x^2 = 4 - (4 - 4x + x^2) = 4 - (x-2)^2$

$u = x - 2, du = dx$

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{du}{\sqrt{4-u^2}}$$

$u = 2 \sin t, du = 2 \cos t dt$

$$\int \frac{du}{\sqrt{4-u^2}} = \int dt = t + C = \sin^{-1} \left(\frac{u}{2} \right) + C$$

$$= \sin^{-1} \left(\frac{x-2}{2} \right) + C$$

24. $4x - x^2 = 4 - (4 - 4x + x^2) = 4 - (x-2)^2$

$u = x - 2, du = dx$

$$\int \frac{x}{\sqrt{4x-x^2}} dx = \int \frac{u+2}{\sqrt{4-u^2}} du$$

$$= - \int \frac{-u du}{\sqrt{4-u^2}} + 2 \int \frac{du}{\sqrt{4-u^2}}$$

(Use the result of Problem 23.)

$$= -\sqrt{4-u^2} + 2 \sin^{-1} \left(\frac{u}{2} \right) + C$$

$$= -\sqrt{4x-x^2} + 2 \sin^{-1} \left(\frac{x-2}{2} \right) + C$$

25. $x^2 + 2x + 2 = x^2 + 2x + 1 + 1 = (x+1)^2 + 1$

$u = x + 1, du = dx$

$$\int \frac{2x+1}{x^2+2x+2} dx = \int \frac{2u-1}{u^2+1} du$$

$$= \int \frac{2u}{u^2+1} du - \int \frac{du}{u^2+1}$$

$$= \ln |u^2+1| - \tan^{-1} u + C$$

$$= \ln |x^2+2x+2| - \tan^{-1} (x+1) + C$$

26. $x^2 - 6x + 18 = x^2 - 6x + 9 + 9 = (x-3)^2 + 9$

$u = x - 3, du = dx$

$$\int \frac{2x-1}{x^2-6x+18} dx = \int \frac{2u+5}{u^2+9} du$$

$$= \int \frac{2u du}{u^2+9} + 5 \int \frac{du}{u^2+9}$$

$$= \ln |u^2+9| + \frac{5}{3} \tan^{-1} \left(\frac{u}{3} \right) + C$$

$$= \ln |x^2-6x+18| + \frac{5}{3} \tan^{-1} \left(\frac{x-3}{3} \right) + C$$

$$27. V = \pi \int_0^1 \left(\frac{1}{x^2 + 2x + 5} \right)^2 dx$$

$$= \pi \int_0^1 \left[\frac{1}{(x+1)^2 + 4} \right]^2 dx$$

$$x + 1 = 2 \tan t, \quad dx = 2 \sec^2 t \, dt$$

$$V = \pi \int_{\tan^{-1}(1/2)}^{\pi/4} \left(\frac{1}{4 \sec^2 t} \right)^2 2 \sec^2 t \, dt$$

$$= \frac{\pi}{8} \int_{\tan^{-1}(1/2)}^{\pi/4} \frac{1}{\sec^2 t} dt = \frac{\pi}{8} \int_{\tan^{-1}(1/2)}^{\pi/4} \cos^2 t \, dt$$

$$= \frac{\pi}{8} \int_{\tan^{-1}(1/2)}^{\pi/4} \left(\frac{1}{2} + \frac{1}{2} \cos 2t \right) dt$$

$$= \frac{\pi}{8} \left[\frac{1}{2} t + \frac{1}{4} \sin 2t \right]_{\tan^{-1}(1/2)}^{\pi/4}$$

$$= \frac{\pi}{8} \left[\frac{1}{2} t + \frac{1}{2} \sin t \cos t \right]_{\tan^{-1}(1/2)}^{\pi/4}$$

$$= \frac{\pi}{8} \left[\left(\frac{\pi}{8} + \frac{1}{4} \right) - \left(\frac{1}{2} \tan^{-1} \frac{1}{2} + \frac{1}{5} \right) \right]$$

$$= \frac{\pi}{16} \left(\frac{1}{10} + \frac{\pi}{4} - \tan^{-1} \frac{1}{2} \right) \approx 0.082811$$

$$28. V = 2\pi \int_0^1 \frac{1}{x^2 + 2x + 5} x \, dx$$

$$= 2\pi \int_0^1 \frac{x}{(x+1)^2 + 4} dx$$

$$= 2\pi \int_0^1 \frac{x+1}{(x+1)^2 + 4} dx - 2\pi \int_0^1 \frac{1}{(x+1)^2 + 4} dx$$

$$= 2\pi \left[\frac{1}{2} \ln[(x+1)^2 + 4] \right]_0^1 - 2\pi \left[\frac{1}{2} \tan^{-1} \left(\frac{x+1}{2} \right) \right]_0^1$$

$$= \pi [\ln 8 - \ln 5] - \pi \left[\tan^{-1} 1 - \tan^{-1} \frac{1}{2} \right]$$

$$= \pi \left(\ln \frac{8}{5} - \frac{\pi}{4} + \tan^{-1} \frac{1}{2} \right) \approx 0.465751$$

$$29. \text{ a. } u = x^2 + 9, \quad du = 2x \, dx$$

$$\int \frac{x \, dx}{x^2 + 9} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C$$

$$= \frac{1}{2} \ln |x^2 + 9| + C$$

$$\text{ b. } x = 3 \tan t, \quad dx = 3 \sec^2 t \, dt$$

$$\int \frac{x \, dx}{x^2 + 9} = \int \tan t \, dt = -\ln |\cos t| + C$$

$$= -\ln \left| \frac{3}{\sqrt{x^2 + 9}} \right| + C_1$$

$$= \ln \left| \sqrt{x^2 + 9} \right| - \ln |3| + C_1$$

$$= \ln |(x^2 + 9)^{1/2}| + C = \frac{1}{2} \ln |x^2 + 9| + C$$

$$30. u = \sqrt{9 + x^2}, \quad u^2 = 9 + x^2, \quad 2u \, du = 2x \, dx$$

$$\int_0^3 \frac{x^3 \, dx}{\sqrt{9 + x^2}} = \int_0^3 \frac{x^2}{\sqrt{9 + x^2}} x \, dx = \int_3^{3\sqrt{2}} \frac{u^2 - 9}{u} u \, du$$

$$= \int_3^{3\sqrt{2}} (u^2 - 9) \, du = \left[\frac{u^3}{3} - 9u \right]_3^{3\sqrt{2}} = 18 - 9\sqrt{2}$$

$$\approx 5.272$$

$$31. \text{ a. } u = \sqrt{4 - x^2}, \quad u^2 = 4 - x^2, \quad 2u \, du = -2x \, dx$$

$$\int \frac{\sqrt{4 - x^2}}{x} dx = \int \frac{\sqrt{4 - x^2}}{x^2} x \, dx = -\int \frac{u^2 \, du}{4 - u^2}$$

$$= \int \frac{-4 + 4 - u^2}{4 - u^2} du = -4 \int \frac{1}{4 - u^2} du + \int du$$

$$= -4 \cdot \frac{1}{4} \ln \left| \frac{u+2}{u-2} \right| + u + C$$

$$= -\ln \left| \frac{\sqrt{4 - x^2} + 2}{\sqrt{4 - x^2} - 2} \right| + \sqrt{4 - x^2} + C$$

$$\text{ b. } x = 2 \sin t, \quad dx = 2 \cos t \, dt$$

$$\int \frac{\sqrt{4 - x^2}}{x} dx = 2 \int \frac{\cos^2 t}{\sin t} dt$$

$$= 2 \int \frac{(1 - \sin^2 t)}{\sin t} dt$$

$$= 2 \int \csc t \, dt - 2 \int \sin t \, dt$$

$$= 2 \ln |\csc t - \cot t| + 2 \cos t + C$$

$$= 2 \ln \left| \frac{2}{x} - \frac{\sqrt{4 - x^2}}{x} \right| + \sqrt{4 - x^2} + C$$

$$= 2 \ln \left| \frac{2 - \sqrt{4 - x^2}}{x} \right| + \sqrt{4 - x^2} + C$$

To reconcile the answers, note that

$$-\ln \left| \frac{\sqrt{4 - x^2} + 2}{\sqrt{4 - x^2} - 2} \right| = \ln \left| \frac{\sqrt{4 - x^2} - 2}{\sqrt{4 - x^2} + 2} \right|$$

$$= \ln \left| \frac{(\sqrt{4 - x^2} - 2)^2}{(\sqrt{4 - x^2} + 2)(\sqrt{4 - x^2} - 2)} \right|$$

$$\begin{aligned}
 &= \ln \left| \frac{(2 - \sqrt{4 - x^2})^2}{4 - x^2 - 4} \right| = \ln \left| \frac{(2 - \sqrt{4 - x^2})^2}{-x^2} \right| \\
 &= \ln \left(\left(\frac{2 - \sqrt{4 - x^2}}{x} \right)^2 \right) = 2 \ln \left| \frac{2 - \sqrt{4 - x^2}}{x} \right|
 \end{aligned}$$

32. The equation of the circle with center $(-a, 0)$ is $(x + a)^2 + y^2 = b^2$, so $y = \pm \sqrt{b^2 - (x + a)^2}$. By symmetry, the area of the overlap is four times the area of the region bounded by $x = 0$, $y = 0$, and $y = \sqrt{b^2 - (x + a)^2}$.

$$A = 4 \int_0^{b-a} \sqrt{b^2 - (x + a)^2} dx$$

$$x + a = b \sin t, dx = b \cos t dt$$

$$A = 4 \int_{\sin^{-1}(a/b)}^{\pi/2} b^2 \cos^2 t dt$$

$$= 2b^2 \int_{\sin^{-1}(a/b)}^{\pi/2} (1 + \cos 2t) dt$$

$$= 2b^2 \left[t + \frac{1}{2} \sin 2t \right]_{\sin^{-1}(a/b)}^{\pi/2}$$

$$= 2b^2 [t + \sin t \cos t]_{\sin^{-1}(a/b)}^{\pi/2}$$

$$= 2b^2 \left[\frac{\pi}{2} - \left(\sin^{-1} \left(\frac{a}{b} \right) + \frac{a}{b} \frac{\sqrt{b^2 - a^2}}{b} \right) \right]$$

$$= \pi b^2 - 2b^2 \sin^{-1} \left(\frac{a}{b} \right) - 2a\sqrt{b^2 - a^2}$$

33. a. The coordinate of C is $(0, -a)$. The lower arc of the lune lies on the circle given by the equation $x^2 + (y + a)^2 = 2a^2$ or

$$y = \pm \sqrt{2a^2 - x^2} - a. \text{ The upper arc of the lune lies on the circle given by the equation } x^2 + y^2 = a^2 \text{ or } y = \pm \sqrt{a^2 - x^2}.$$

$$A = \int_{-a}^a \sqrt{a^2 - x^2} dx - \int_{-a}^a (\sqrt{2a^2 - x^2} - a) dx$$

$$= \int_{-a}^a \sqrt{a^2 - x^2} dx - \int_{-a}^a \sqrt{2a^2 - x^2} dx + 2a^2$$

Note that $\int_{-a}^a \sqrt{a^2 - x^2} dx$ is the area of a semicircle with radius a , so

$$\int_{-a}^a \sqrt{a^2 - x^2} dx = \frac{\pi a^2}{2}.$$

For $\int_{-a}^a \sqrt{2a^2 - x^2} dx$, let

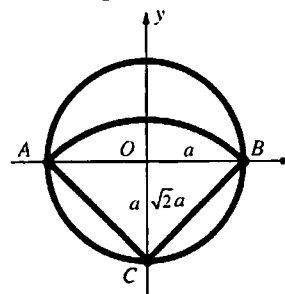
$$x = \sqrt{2}a \sin t, dx = \sqrt{2}a \cos t dt$$

$$\int_{-a}^a \sqrt{2a^2 - x^2} dx = \int_{-\pi/4}^{\pi/4} 2a^2 \cos^2 t dt$$

$$\begin{aligned}
 &= a^2 \int_{-\pi/4}^{\pi/4} (1 + \cos 2t) dt = a^2 \left[t + \frac{1}{2} \sin 2t \right]_{-\pi/4}^{\pi/4} \\
 &= \frac{\pi a^2}{2} + a^2 \\
 A &= \frac{\pi a^2}{2} - \left(\frac{\pi a^2}{2} + a^2 \right) + 2a^2 = a^2
 \end{aligned}$$

Thus, the area of the lune is equal to the area of the square.

- b. Without using calculus, consider the following labels on the figure.



Area of the lune = Area of the semicircle of radius a at O + Area $(\triangle ABC)$ - Area of the sector ABC .

$$A = \frac{1}{2} \pi a^2 + a^2 - \frac{1}{2} \left(\frac{\pi}{2} \right) (\sqrt{2}a)^2$$

$$= \frac{1}{2} \pi a^2 + a^2 - \frac{1}{2} \pi a^2 = a^2$$

Note that since BC has length $\sqrt{2}a$, the measure of angle OCB is $\frac{\pi}{4}$, so the measure of angle ACB is $\frac{\pi}{2}$.

34. Using reasoning similar to Problem 33 b, the area is

$$\begin{aligned}
 &\frac{1}{2} \pi a^2 + \frac{1}{2} (2a) \sqrt{b^2 - a^2} - \frac{1}{2} \left(2 \sin^{-1} \frac{a}{b} \right) b^2 \\
 &= \frac{1}{2} \pi a^2 + a \sqrt{b^2 - a^2} - b^2 \sin^{-1} \frac{a}{b}.
 \end{aligned}$$

$$35. \frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}$$

$$y = \int -\frac{\sqrt{a^2 - x^2}}{x} dx$$

$$x = a \sin t, dx = a \cos t dt$$

$$y = \int -\frac{a \cos t}{a \sin t} a \cos t dt = -a \int \frac{\cos^2 t}{\sin t} dt$$

$$= -a \int \frac{1 - \sin^2 t}{\sin t} dt = a \int (\sin t - \csc t) dt$$

$$= a(-\cos t - \ln |\csc t - \cot t|) + C$$

$$\cos t = \frac{\sqrt{a^2 - x^2}}{a}, \csc t = \frac{a}{x}, \cot t = \frac{\sqrt{a^2 - x^2}}{x}$$

$$y = a \left(-\frac{\sqrt{a^2 - x^2}}{a} - \ln \left| \frac{a}{x} - \frac{\sqrt{a^2 - x^2}}{x} \right| \right) + C$$

$$= -\sqrt{a^2 - x^2} - a \ln \left| \frac{a - \sqrt{a^2 - x^2}}{x} \right| + C$$

Since $y = 0$ when $x = a$,
 $0 = 0 - a \ln 1 + C$, so $C = 0$.

$$y = -\sqrt{a^2 - x^2} - a \ln \left| \frac{a - \sqrt{a^2 - x^2}}{x} \right|$$

8.4 Concepts Review

1. $uv - \int v du$
2. $x; \sin x dx$
3. 1
4. reduction

Problem Set 8.4

$$1. \quad u = x \quad dv = e^x dx$$

$$du = dx \quad v = e^x$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$$

$$2. \quad u = x \quad dv = e^{3x} dx$$

$$du = dx \quad v = \frac{1}{3} e^{3x}$$

$$\int x e^{3x} dx = \frac{1}{3} x e^{3x} - \int \frac{1}{3} e^{3x} dx$$

$$= \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C$$

$$3. \quad u = t \quad dv = e^{5t+\pi} dt$$

$$du = dt \quad v = \frac{1}{5} e^{5t+\pi}$$

$$\int t e^{5t+\pi} dt = \frac{1}{5} t e^{5t+\pi} - \int \frac{1}{5} e^{5t+\pi} dt$$

$$= \frac{1}{5} t e^{5t+\pi} - \frac{1}{25} e^{5t+\pi} + C$$

$$4. \quad u = t + 7 \quad dv = e^{2t+3} dt$$

$$du = dt \quad v = \frac{1}{2} e^{2t+3}$$

$$\int (t+7) e^{2t+3} dt = \frac{1}{2} (t+7) e^{2t+3} - \int \frac{1}{2} e^{2t+3} dt$$

$$= \frac{1}{2} (t+7) e^{2t+3} - \frac{1}{4} e^{2t+3} + C$$

$$= \frac{t}{2} e^{2t+3} + \frac{13}{4} e^{2t+3} + C$$

$$5. \quad u = x \quad dv = \cos x dx$$

$$du = dx \quad v = \sin x$$

$$\int x \cos x dx = x \sin x - \int \sin x dx$$

$$= x \sin x + \cos x + C$$

$$6. \quad u = x \quad dv = \sin 2x dx$$

$$du = dx \quad v = -\frac{1}{2} \cos 2x$$

$$\int x \sin 2x dx = -\frac{1}{2} x \cos 2x - \int -\frac{1}{2} \cos 2x dx$$

$$= -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$$

$$7. \quad u = t - 3 \quad dv = \cos(t-3) dt$$

$$du = dt \quad v = \sin(t-3)$$

$$\int (t-3) \cos(t-3) dt = (t-3) \sin(t-3) - \int \sin(t-3) dt$$

$$= (t-3) \sin(t-3) + \cos(t-3) + C$$

$$8. \quad u = x - \pi \quad dv = \sin(x) dx$$

$$du = dx \quad v = -\cos x$$

$$\int (x - \pi) \sin(x) dx = -(x - \pi) \cos x + \int \cos x dx$$

$$= (\pi - x) \cos x + \sin x + C$$

$$9. \quad u = t \quad dv = \sqrt{t+1} dt$$

$$du = dt \quad v = \frac{2}{3} (t+1)^{3/2}$$

$$\int t \sqrt{t+1} dt = \frac{2}{3} t (t+1)^{3/2} - \int \frac{2}{3} (t+1)^{3/2} dt$$

$$= \frac{2}{3} t (t+1)^{3/2} - \frac{4}{15} (t+1)^{5/2} + C$$

$$10. \quad u = t \quad dv = \sqrt[3]{2t+7} dt$$

$$du = dt \quad v = \frac{3}{8} (2t+7)^{4/3}$$

$$\int t \sqrt[3]{2t+7} dt = \frac{3}{8} t (2t+7)^{4/3} - \int \frac{3}{8} (2t+7)^{4/3} dt$$

$$= \frac{3}{8} t (2t+7)^{4/3} - \frac{9}{112} (2t+7)^{7/3} + C$$

$$11. u = \ln 3x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = x$$

$$\int \ln 3x dx = x \ln 3x - \int x \frac{1}{x} dx = x \ln 3x - x + C$$

$$12. u = \ln(7x^5) \quad dv = dx$$

$$du = \frac{5}{x} dx \quad v = x$$

$$\begin{aligned} \int \ln(7x^5) dx &= x \ln(7x^5) - \int x \frac{5}{x} dx \\ &= x \ln(7x^5) - 5x + C \end{aligned}$$

$$13. u = \arctan x \quad dv = dx$$

$$du = \frac{1}{1+x^2} dx \quad v = x$$

$$\begin{aligned} \int \arctan x &= x \arctan x - \int \frac{x}{1+x^2} dx \\ &= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} dx \\ &= x \arctan x - \frac{1}{2} \ln(1+x^2) + C \end{aligned}$$

$$14. u = \arctan 5x \quad dv = dx$$

$$du = \frac{5}{1+25x^2} dx \quad v = x$$

$$\begin{aligned} \int \arctan 5x dx &= x \arctan 5x - \int \frac{5x}{1+25x^2} dx \\ &= x \arctan 5x - \frac{1}{10} \int \frac{50x dx}{1+25x^2} \\ &= x \arctan 5x - \frac{1}{10} \ln(1+25x^2) + C \end{aligned}$$

$$15. u = \ln x \quad dv = \frac{dx}{x^2}$$

$$du = \frac{1}{x} dx \quad v = -\frac{1}{x}$$

$$\begin{aligned} \int \frac{\ln x}{x^2} dx &= -\frac{\ln x}{x} - \int -\frac{1}{x} \left(\frac{1}{x} \right) dx \\ &= -\frac{\ln x}{x} - \frac{1}{x} + C \end{aligned}$$

$$16. u = \ln 2x^5 \quad dv = \frac{1}{x^2} dx$$

$$du = \frac{5}{x} dx \quad v = -\frac{1}{x}$$

$$\int_2^3 \frac{\ln 2x^5}{x^2} dx = -\frac{1}{x} \ln 2x^5 + 5 \int_2^3 \frac{1}{x^2} dx$$

$$\begin{aligned} &= \left[-\frac{1}{x} \ln 2x^5 - \frac{5}{x} \right]_2^3 \\ &= \left(-\frac{1}{3} \ln(2 \cdot 3^5) - \frac{5}{3} \right) - \left(-\frac{1}{2} \ln(2 \cdot 2^5) - \frac{5}{2} \right) \\ &= \frac{8}{3} \ln 2 - \frac{5}{3} \ln 3 + \frac{5}{6} \ln 0.85 \approx -0.11806 \end{aligned}$$

$$17. u = \ln t \quad dv = \sqrt{t} dt$$

$$du = \frac{1}{t} dt \quad v = \frac{2}{3} t^{3/2}$$

$$\begin{aligned} \int_1^e \sqrt{t} \ln t dt &= \left[\frac{2}{3} t^{3/2} \ln t \right]_1^e - \int_1^e \frac{2}{3} t^{1/2} dt \\ &= \frac{2}{3} e^{3/2} \ln e - \frac{2}{3} \cdot 1 \ln 1 - \left[\frac{4}{9} t^{3/2} \right]_1^e \\ &= \frac{2}{3} e^{3/2} - 0 - \frac{4}{9} e^{3/2} + \frac{4}{9} = \frac{2}{9} e^{3/2} + \frac{4}{9} \end{aligned}$$

$$18. u = \ln x^3 \quad dv = \sqrt{2x} dx$$

$$du = \frac{3}{x} dx \quad v = \frac{1}{3} (2x)^{3/2}$$

$$\begin{aligned} \int_5^1 \sqrt{2x} \ln x^3 dx &= \left[\frac{1}{3} (2x)^{3/2} \ln x^3 \right]_5^1 - \int_5^1 2^{3/2} \sqrt{x} dx \\ &= \left[\frac{1}{3} (2x)^{3/2} \ln x^3 - \frac{2^{5/2}}{3} x^{3/2} \right]_5^1 \\ &= \frac{1}{3} (2)^{3/2} \ln 1 - \frac{2^{5/2}}{3} - \left(\frac{1}{3} (10)^{3/2} \ln 5^3 - \frac{2^{5/2}}{3} 5^{3/2} \right) \\ &= \frac{4\sqrt{2}}{3} 5^{3/2} - \frac{4\sqrt{2}}{3} - 10^{3/2} \ln 5 \approx -31.699 \end{aligned}$$

$$19. u = \ln z \quad dv = z^3 dz$$

$$du = \frac{1}{z} dz \quad v = \frac{1}{4} z^4$$

$$\begin{aligned} \int z^3 \ln z dz &= \frac{1}{4} z^4 \ln z - \int \frac{1}{4} z^4 \cdot \frac{1}{z} dz \\ &= \frac{1}{4} z^4 \ln z - \frac{1}{4} \int z^3 dz \\ &= \frac{1}{4} z^4 \ln z - \frac{1}{16} z^4 + C \end{aligned}$$

$$20. u = \arctan t \quad dv = t dt$$

$$du = \frac{1}{1+t^2} dt \quad v = \frac{1}{2} t^2$$

$$\begin{aligned} \int t \arctan t dt &= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int \frac{t^2}{1+t^2} dt \\ &= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int \frac{1+t^2-1}{1+t^2} dt \end{aligned}$$

$$= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int dt + \frac{1}{2} \int \frac{1}{1+t^2} dt$$

$$= \frac{1}{2} t^2 \arctan t - \frac{1}{2} t + \frac{1}{2} \arctan t + C$$

21. $u = \arctan\left(\frac{1}{t}\right) \quad dv = dt$

$$du = -\frac{1}{1+t^2} dt \quad v = t$$

$$\int \arctan\left(\frac{1}{t}\right) dt = t \arctan\left(\frac{1}{t}\right) + \int \frac{t}{1+t^2} dt$$

$$= t \arctan\left(\frac{1}{t}\right) + \frac{1}{2} \ln(1+t^2) + C$$

22. $u = \ln(t^7) \quad dv = t^5 dt$

$$du = \frac{7}{t} dt \quad v = \frac{1}{6} t^6$$

$$\int t^5 \ln(t^7) dt = \frac{1}{6} t^6 \ln(t^7) - \frac{7}{6} \int t^5 dt$$

$$= \frac{1}{6} t^6 \ln(t^7) - \frac{7}{36} t^6 + C$$

23. $u = x \cos^2 x \quad dv = \sin x dx$

$$du = (\cos^2 x - 2x \cos x \sin x) dx \quad v = -\cos x$$

$$\int x \cos^2 x \sin x dx = -x \cos^3 x + \int (\cos^3 x - 2x \cos^2 x \sin x) dx = -x \cos^3 x + \int \cos^3 x dx - 2 \int x \cos^2 x \sin x dx$$

$$3 \int x \cos^2 x \sin x dx = -x \cos^3 x + \int \cos x (1 - \sin^2 x) dx = -x \cos^3 x + \sin x - \frac{1}{3} \sin^3 x + C$$

$$\int x \cos^2 x \sin x dx = -\frac{x}{3} \cos^3 x + \frac{1}{3} \sin x - \frac{1}{9} \sin^3 x + C$$

24. $u = x \sin^3 \pi x \quad v = \frac{1}{\pi} \sin \pi x \quad dv = \cos \pi x dx$

$$du = (3\pi x \sin^2 \pi x \cos \pi x + \sin^3 \pi x) dx$$

$$\int x \sin^3 \pi x \cos \pi x dx = \frac{x}{\pi} \sin^4 \pi x - \frac{1}{\pi} \int (3\pi x \sin^3 \pi x \cos \pi x + \sin^4 \pi x) dx$$

$$= \frac{x}{\pi} \sin^4 \pi x - 3 \int x \sin^3 \pi x \cos \pi x dx - \frac{1}{\pi} \int \sin^4 \pi x dx$$

$$4 \int x \sin^3 \pi x \cos \pi x dx = \frac{x}{\pi} \sin^4 \pi x - \frac{1}{\pi} \int \left(\frac{1 - \cos 2\pi x}{2} \right)^2 dx = \frac{x}{\pi} \sin^4 \pi x - \frac{1}{4\pi} \int (1 - 2\cos 2\pi x + \cos^2 2\pi x) dx$$

$$= \frac{x}{\pi} \sin^4 \pi x - \frac{x}{4\pi} + \frac{1}{4\pi^2} \sin 2\pi x - \frac{1}{4\pi} \int \left(\frac{1 + \cos 4\pi x}{2} \right) dx = \frac{x}{\pi} \sin^4 \pi x - \frac{3x}{8\pi} + \frac{1}{4\pi^2} \sin 2\pi x - \frac{1}{32\pi^2} \sin 4\pi x + C$$

$$\int x \sin^3 \pi x \cos \pi x dx = \frac{x}{4\pi} \sin^4 \pi x - \frac{3x}{32\pi} + \frac{1}{16\pi^2} \sin 2\pi x - \frac{1}{128\pi^2} \sin 4\pi x + C$$

25. $u = x \quad dv = \csc^2 x dx$

$$du = dx \quad v = -\cot x$$

$$\int_{\pi/6}^{\pi/2} x \csc^2 x dx = [-x \cot x]_{\pi/6}^{\pi/2} + \int_{\pi/6}^{\pi/2} \cot x dx = [-x \cot x + \ln |\sin x|]_{\pi/6}^{\pi/2}$$

$$= -\frac{\pi}{2} \cdot 0 + \ln 1 + \frac{\pi}{6} \sqrt{3} - \ln \frac{1}{2} = \frac{\pi}{2\sqrt{3}} + \ln 2 \approx 1.60$$

26. $\int_{\pi/4}^{\pi/2} \csc^2 x dx = [-\cot x]_{\pi/4}^{\pi/2} = -\cot \frac{\pi}{2} + \cot \frac{\pi}{4} = 0 + 1 = 1$

27. $u = x \quad dv = \sec^2 x dx$

$$du = dx \quad v = \tan x$$

$$\int_{\pi/6}^{\pi/4} x \sec^2 x \, dx = [x \tan x]_{\pi/6}^{\pi/4} - \int_{\pi/6}^{\pi/4} \tan x \, dx = [x \tan x + \ln |\cos x|]_{\pi/6}^{\pi/4} = \frac{\pi}{4} + \ln \frac{\sqrt{2}}{2} - \left(\frac{\pi}{6\sqrt{3}} + \ln \frac{\sqrt{3}}{2} \right)$$

$$= \frac{\pi}{4} + \ln \frac{\sqrt{2}}{2} - \frac{\pi}{6\sqrt{3}} - \ln \frac{\sqrt{3}}{2} = \frac{\pi}{4} - \frac{\pi}{6\sqrt{3}} + \frac{1}{2} \ln \frac{2}{3} \approx 0.28$$

28. $u = \sec^{-1} \sqrt{x} \quad dv = dx$

$$du = \frac{1}{2x\sqrt{x-1}} dx \quad v = x$$

$$\int \sec^{-1} \sqrt{x} \, dx = x \sec^{-1} \sqrt{x} - \frac{1}{2} \int \frac{dx}{\sqrt{x-1}} = x \sec^{-1} \sqrt{x} - \sqrt{x-1} + C$$

29. $u = x^3 \quad dv = x^2 \sqrt{x^3 + 4} \, dx$

$$du = 3x^2 \, dx \quad v = \frac{2}{9} (x^3 + 4)^{3/2}$$

$$\int x^5 \sqrt{x^3 + 4} \, dx = \frac{2}{9} x^3 (x^3 + 4)^{3/2} - \int \frac{2}{3} x^2 (x^3 + 4)^{3/2} \, dx = \frac{2}{9} x^3 (x^3 + 4)^{3/2} - \frac{4}{45} (x^3 + 4)^{5/2} + C$$

30. $u = x^7 \quad dv = x^6 \sqrt{x^7 + 1} \, dx$

$$du = 7x^6 \, dx \quad v = \frac{2}{21} (x^7 + 1)^{3/2}$$

$$\int x^{13} \sqrt{x^7 + 1} \, dx = \frac{2}{21} x^7 (x^7 + 1)^{3/2} - \int \frac{2}{3} x^6 (x^7 + 1)^{3/2} \, dx = \frac{2}{21} x^7 (x^7 + 1)^{3/2} - \frac{4}{105} (x^7 + 1)^{5/2} + C$$

31. $u = t^4 \quad dv = \frac{t^3}{(7 - 3t^4)^{3/2}} dt$

$$du = 4t^3 \, dt \quad v = \frac{1}{6(7 - 3t^4)^{1/2}}$$

$$\int \frac{t^7}{(7 - 3t^4)^{3/2}} dt = \frac{t^4}{6(7 - 3t^4)^{1/2}} - \frac{2}{3} \int \frac{t^3}{(7 - 3t^4)^{1/2}} dt = \frac{t^4}{6(7 - 3t^4)^{1/2}} + \frac{1}{9} (7 - 3t^4)^{1/2} + C$$

32. $u = x^2 \quad dv = x \sqrt{4 - x^2} \, dx$

$$du = 2x \, dx \quad v = -\frac{1}{3} (4 - x^2)^{3/2}$$

$$\int x^3 \sqrt{4 - x^2} \, dx = -\frac{1}{3} x^2 (4 - x^2)^{3/2} + \frac{2}{3} \int x (4 - x^2)^{3/2} \, dx = -\frac{1}{3} x^2 (4 - x^2)^{3/2} - \frac{2}{15} (4 - x^2)^{5/2} + C$$

33. $u = z^4 \quad dv = \frac{z^3}{(4 - z^4)^2} dz$

$$du = 4z^3 \, dz \quad v = \frac{1}{4(4 - z^4)}$$

$$\int \frac{z^7}{(4 - z^4)^2} dz = \frac{z^4}{4(4 - z^4)} - \int \frac{z^3}{4 - z^4} dz = \frac{z^4}{4(4 - z^4)} + \frac{1}{4} \ln |4 - z^4| + C$$

34. $u = x \quad dv = \cosh x \, dx$

$$du = dx \quad v = \sinh x$$

$$\int x \cosh x \, dx = x \sinh x - \int \sinh x \, dx = x \sinh x - \cosh x + C$$

$$\begin{aligned}
 35. \quad u &= x & dv &= \sinh x \, dx \\
 du &= dx & v &= \cosh x \\
 \int x \sinh x \, dx &= x \cosh x - \int \cosh x \, dx = x \cosh x - \sinh x + C
 \end{aligned}$$

$$\begin{aligned}
 36. \quad u &= \sec x & dv &= \sec^2 x \, dx \\
 du &= \sec x \tan x \, dx & v &= \tan x \\
 \int \sec^3 x \, dx &= \sec x \tan x - \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\
 2 \int \sec^3 x \, dx &= \sec x \tan x + \int \sec x \, dx = \sec x \tan x + \ln |\sec x + \tan x| + C \\
 \int \sec^3 x \, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C
 \end{aligned}$$

$$\begin{aligned}
 37. \quad u &= \ln x & dv &= x^{-1/2} \, dx \\
 du &= \frac{1}{x} \, dx & v &= 2x^{1/2} \\
 \int \frac{\ln x}{\sqrt{x}} \, dx &= 2\sqrt{x} \ln x - 2 \int \frac{1}{x^{1/2}} \, dx = 2\sqrt{x} \ln x - 4\sqrt{x} + C
 \end{aligned}$$

$$\begin{aligned}
 38. \quad u &= x & dv &= (3x+10)^{49} \, dx \\
 du &= dx & v &= \frac{1}{150} (3x+10)^{50} \\
 \int x(3x+10)^{49} \, dx &= \frac{x}{150} (3x+10)^{50} - \frac{1}{150} \int (3x+10)^{50} \, dx = \frac{x}{150} (3x+10)^{50} - \frac{1}{22,950} (3x+10)^{51} + C
 \end{aligned}$$

$$\begin{aligned}
 39. \quad u &= x & dv &= 2^x \, dx \\
 du &= dx & v &= \frac{1}{\ln 2} 2^x \\
 \int x 2^x \, dx &= \frac{x}{\ln 2} 2^x - \frac{1}{\ln 2} \int 2^x \, dx \\
 &= \frac{x}{\ln 2} 2^x - \frac{1}{(\ln 2)^2} 2^x + C
 \end{aligned}$$

$$\begin{aligned}
 \int x^2 e^x \, dx &= x^2 e^x - 2 \left(x e^x - \int e^x \, dx \right) \\
 &= x^2 e^x - 2x e^x + 2e^x + C
 \end{aligned}$$

$$\begin{aligned}
 40. \quad u &= z & dv &= a^z \, dz \\
 du &= dz & v &= \frac{1}{\ln a} a^z \\
 \int z a^z \, dz &= \frac{z}{\ln a} a^z - \frac{1}{\ln a} \int a^z \, dz \\
 &= \frac{z}{\ln a} a^z - \frac{1}{(\ln a)^2} a^z + C
 \end{aligned}$$

$$\begin{aligned}
 42. \quad u &= x^4 & dv &= x e^{x^2} \, dx \\
 du &= 4x^3 \, dx & v &= \frac{1}{2} e^{x^2} \\
 \int x^5 e^{x^2} \, dx &= \frac{1}{2} x^4 e^{x^2} - \int 2x^3 e^{x^2} \, dx \\
 u &= x^2 & dv &= 2x e^{x^2} \, dx \\
 du &= 2x \, dx & v &= e^{x^2} \\
 \int x^5 e^{x^2} \, dx &= \frac{1}{2} x^4 e^{x^2} - \left(x^2 e^{x^2} - \int 2x e^{x^2} \, dx \right) \\
 &= \frac{1}{2} x^4 e^{x^2} - x^2 e^{x^2} + e^{x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 41. \quad u &= x^2 & dv &= e^x \, dx \\
 du &= 2x \, dx & v &= e^x \\
 \int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\
 u &= x & dv &= e^x \, dx \\
 du &= dx & v &= e^x
 \end{aligned}$$

$$\begin{aligned}
 43. \quad u &= \ln^2 z & dv &= dz \\
 du &= \frac{2 \ln z}{z} \, dz & v &= z \\
 \int \ln^2 z \, dz &= z \ln^2 z - 2 \int \ln z \, dz \\
 u &= \ln z & dv &= dz \\
 du &= \frac{1}{z} \, dz & v &= z
 \end{aligned}$$

$$\begin{aligned}\int \ln^2 z \, dz &= z \ln^2 z - 2 \left(z \ln z - \int dz \right) \\ &= z \ln^2 z - 2z \ln z + 2z + C\end{aligned}$$

$$\begin{aligned}44. \quad u &= \ln^2 x^{20} & dv &= dx \\ du &= \frac{40 \ln x^{20}}{x} dx & v &= x \\ \int \ln^2 x^{20} dx &= x \ln^2 x^{20} - 40 \int \ln x^{20} dx \\ u &= \ln x^{20} & dv &= dx \\ du &= \frac{20}{x} dx & v &= x \\ \int \ln^2 x^{20} dx &= x \ln^2 x^{20} - 40 \left(x \ln x^{20} - 20 \int dx \right) \\ &= x \ln^2 x^{20} - 40x \ln x^{20} + 800x + C\end{aligned}$$

$$\begin{aligned}45. \quad u &= e^t & dv &= \cos t \, dt \\ du &= e^t dt & v &= \sin t \\ \int e^t \cos t \, dt &= e^t \sin t - \int e^t \sin t \, dt \\ u &= e^t & dv &= \sin t \, dt \\ du &= e^t dt & v &= -\cos t \\ \int e^t \cos t \, dt &= e^t \sin t - \left[-e^t \cos t + \int e^t \cos t \, dt \right] \\ \int e^t \cos t \, dt &= e^t \sin t + e^t \cos t - \int e^t \cos t \, dt \\ 2 \int e^t \cos t \, dt &= e^t \sin t + e^t \cos t + C \\ \int e^t \cos t \, dt &= \frac{1}{2} e^t (\sin t + \cos t) + C\end{aligned}$$

$$\begin{aligned}48. \quad u &= r^2 & dv &= \sin r \, dr \\ du &= 2r \, dr & v &= -\cos r \\ \int r^2 \sin r \, dr &= -r^2 \cos r + 2 \int r \cos r \, dr \\ u &= r & dv &= \cos r \, dr \\ du &= dr & v &= \sin r \\ \int r^2 \sin r \, dr &= -r^2 \cos r + 2 \left(r \sin r - \int \sin r \, dr \right) = -r^2 \cos r + 2r \sin r + 2 \cos r + C\end{aligned}$$

$$\begin{aligned}49. \quad u &= \sin(\ln x) & dv &= dx \\ du &= \cos(\ln x) \cdot \frac{1}{x} dx & v &= x \\ \int \sin(\ln x) dx &= x \sin(\ln x) - \int \cos(\ln x) dx \\ u &= \cos(\ln x) & dv &= dx \\ du &= -\sin(\ln x) \cdot \frac{1}{x} dx & v &= x \\ \int \sin(\ln x) dx &= x \sin(\ln x) - \left[x \cos(\ln x) - \int -\sin(\ln x) dx \right] \\ \int \sin(\ln x) dx &= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx\end{aligned}$$

$$\begin{aligned}46. \quad u &= e^{at} & dv &= \sin t \, dt \\ du &= ae^{at} dt & v &= -\cos t \\ \int e^{at} \sin t \, dt &= -e^{at} \cos t + a \int e^{at} \cos t \, dt \\ u &= e^{at} & dv &= \cos t \, dt \\ du &= ae^{at} dt & v &= \sin t \\ \int e^{at} \sin t \, dt &= -e^{at} \cos t + a \left(e^{at} \sin t - a \int e^{at} \sin t \, dt \right) \\ \int e^{at} \sin t \, dt &= -e^{at} \cos t + ae^{at} \sin t - a^2 \int e^{at} \sin t \, dt \\ (1 + a^2) \int e^{at} \sin t \, dt &= -e^{at} \cos t + ae^{at} \sin t + C \\ \int e^{at} \sin t \, dt &= \frac{-e^{at} \cos t}{a^2 + 1} + \frac{ae^{at} \sin t}{a^2 + 1} + C\end{aligned}$$

$$\begin{aligned}47. \quad u &= x^2 & dv &= \cos x \, dx \\ du &= 2x \, dx & v &= \sin x \\ \int x^2 \cos x \, dx &= x^2 \sin x - \int 2x \sin x \, dx \\ u &= 2x & dv &= \sin x \, dx \\ du &= 2 \, dx & v &= -\cos x \\ \int x^2 \cos x \, dx &= x^2 \sin x - \left(-2x \cos x + \int 2 \cos x \, dx \right) \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C\end{aligned}$$

$$2 \int \sin(\ln x) dx = x \sin(\ln x) - x \cos(\ln x) + C$$

$$\int \sin(\ln x) dx = \frac{x}{2} [\sin(\ln x) - \cos(\ln x)] + C$$

50. $u = \cos(\ln x) \quad dv = dx$

$$du = -\sin(\ln x) \frac{1}{x} dx \quad v = x$$

$$\int \cos(\ln x) dx = x \cos(\ln x) + \int \sin(\ln x) dx$$

$$u = \sin(\ln x) \quad dv = dx$$

$$du = \cos(\ln x) \frac{1}{x} dx \quad v = x$$

$$\int \cos(\ln x) dx = x \cos(\ln x) + \left[x \sin(\ln x) - \int \cos(\ln x) dx \right]$$

$$2 \int \cos(\ln x) dx = x [\cos(\ln x) + \sin(\ln x)] + C$$

$$\int \cos(\ln x) dx = \frac{x}{2} [\cos(\ln x) + \sin(\ln x)] + C$$

51. $u = (\ln x)^3 \quad dv = dx$

$$du = \frac{3 \ln^2 x}{x} dx \quad v = x$$

$$\int (\ln x)^3 dx = x (\ln x)^3 - 3 \int \ln^2 x dx$$

$$= x \ln^3 x - 3(x \ln^2 x - 2x \ln x + 2x + C)$$

$$= x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x + C$$

52. $u = (\ln x)^4 \quad dv = dx$

$$du = \frac{4 \ln^3 x}{x} dx \quad v = x$$

$$\int (\ln x)^4 dx = x (\ln x)^4 - 4 \int \ln^3 x dx = x \ln^4 x - 4(x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x + C)$$

$$= x \ln^4 x - 4x \ln^3 x + 12x \ln^2 x - 24x \ln x + 24x + C$$

53. $u = \sin x \quad dv = \sin(3x) dx$

$$du = \cos x dx \quad v = -\frac{1}{3} \cos(3x)$$

$$\int \sin x \sin(3x) dx = -\frac{1}{3} \sin x \cos(3x) + \frac{1}{3} \int \cos x \cos(3x) dx$$

$$u = \cos x \quad dv = \cos(3x) dx$$

$$du = -\sin x dx \quad v = \frac{1}{3} \sin(3x)$$

$$\int \sin x \sin(3x) dx = -\frac{1}{3} \sin x \cos(3x) + \frac{1}{3} \left[\frac{1}{3} \cos x \sin(3x) + \frac{1}{3} \int \sin x \sin(3x) dx \right]$$

$$= -\frac{1}{3} \sin x \cos(3x) + \frac{1}{9} \cos x \sin(3x) + \frac{1}{9} \int \sin x \sin(3x) dx$$

$$\frac{8}{9} \int \sin x \sin(3x) dx = -\frac{1}{3} \sin x \cos(3x) + \frac{1}{9} \cos x \sin(3x) + C$$

$$\int \sin x \sin(3x) dx = -\frac{3}{8} \sin x \cos(3x) + \frac{1}{8} \cos x \sin(3x) + C$$

$$54. \quad u = \cos(5x) \quad dv = \sin(7x)dx$$

$$du = -5 \sin(5x)dx \quad v = -\frac{1}{7}\cos(7x)$$

$$\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{7} \int \sin(5x)\cos(7x)dx$$

$$u = \sin(5x) \quad dv = \cos(7x)dx$$

$$du = 5 \cos(5x)dx \quad v = \frac{1}{7}\sin(7x)$$

$$\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{7} \left[\frac{1}{7}\sin(5x)\sin(7x) - \frac{5}{7} \int \cos(5x)\sin(7x)dx \right]$$

$$= -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{49}\sin(5x)\sin(7x) + \frac{25}{49} \int \cos(5x)\sin(7x)dx$$

$$\frac{24}{49} \int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{49}\sin(5x)\sin(7x) + C$$

$$\int \cos(5x)\sin(7x)dx = -\frac{7}{24}\cos(5x)\cos(7x) - \frac{5}{24}\sin(5x)\sin(7x) + C$$

$$55. \quad u = e^{\alpha z} \quad dv = \sin \beta z \, dz$$

$$du = \alpha e^{\alpha z} dz \quad v = -\frac{1}{\beta}\cos \beta z$$

$$\int e^{\alpha z} \sin \beta z \, dz = -\frac{1}{\beta}e^{\alpha z} \cos \beta z + \frac{\alpha}{\beta} \int e^{\alpha z} \cos \beta z \, dz$$

$$u = e^{\alpha z} \quad dv = \cos \beta z \, dz$$

$$du = \alpha e^{\alpha z} dz \quad v = \frac{1}{\beta}\sin \beta z$$

$$\int e^{\alpha z} \sin \beta z \, dz = -\frac{1}{\beta}e^{\alpha z} \cos \beta z + \frac{\alpha}{\beta} \left[\frac{1}{\beta}e^{\alpha z} \sin \beta z - \frac{\alpha}{\beta} \int e^{\alpha z} \sin \beta z \, dz \right]$$

$$= -\frac{1}{\beta}e^{\alpha z} \cos \beta z + \frac{\alpha}{\beta^2}e^{\alpha z} \sin \beta z - \frac{\alpha^2}{\beta^2} \int e^{\alpha z} \sin \beta z \, dz$$

$$\frac{\beta^2 + \alpha^2}{\beta^2} \int e^{\alpha z} \sin \beta z \, dz = -\frac{1}{\beta}e^{\alpha z} \cos \beta z + \frac{\alpha}{\beta^2}e^{\alpha z} \sin \beta z + C$$

$$\int e^{\alpha z} \sin \beta z \, dz = \frac{-\beta}{\alpha^2 + \beta^2}e^{\alpha z} \cos \beta z + \frac{\alpha}{\alpha^2 + \beta^2}e^{\alpha z} \sin \beta z + C = \frac{e^{\alpha z}(\alpha \sin \beta z - \beta \cos \beta z)}{\alpha^2 + \beta^2} + C$$

$$56. \quad u = e^{\alpha z} \quad dv = \cos \beta z \, dz$$

$$du = \alpha e^{\alpha z} dz \quad v = \frac{1}{\beta}\sin \beta z$$

$$\int e^{\alpha z} \cos \beta z \, dz = \frac{1}{\beta}e^{\alpha z} \sin \beta z - \frac{\alpha}{\beta} \int e^{\alpha z} \sin \beta z \, dz$$

$$u = e^{\alpha z} \quad dv = \sin \beta z \, dz$$

$$du = \alpha e^{\alpha z} dz \quad v = -\frac{1}{\beta}\cos \beta z$$

$$\int e^{\alpha z} \cos \beta z \, dz = \frac{1}{\beta}e^{\alpha z} \sin \beta z - \frac{\alpha}{\beta} \left[-\frac{1}{\beta}e^{\alpha z} \cos \beta z + \frac{\alpha}{\beta} \int e^{\alpha z} \cos \beta z \, dz \right]$$

$$= \frac{1}{\beta}e^{\alpha z} \sin \beta z + \frac{\alpha}{\beta^2}e^{\alpha z} \cos \beta z - \frac{\alpha^2}{\beta^2} \int e^{\alpha z} \cos \beta z \, dz$$

$$\frac{\alpha^2 + \beta^2}{\beta^2} \int e^{\alpha z} \cos \beta z \, dz = \frac{\alpha}{\beta^2} e^{\alpha z} \cos \beta z + \frac{1}{\beta} e^{\alpha z} \sin \beta z + C$$

$$\int e^{\alpha z} \cos \beta z \, dz = \frac{e^{\alpha z} (\alpha \cos \beta z + \beta \sin \beta z)}{\alpha^2 + \beta^2} + C$$

$$57. \quad u = \ln x \quad dv = x^\alpha dx$$

$$du = \frac{1}{x} dx \quad v = \frac{x^{\alpha+1}}{\alpha+1}, \quad \alpha \neq -1$$

$$\int x^\alpha \ln x \, dx = \frac{x^{\alpha+1}}{\alpha+1} \ln x - \frac{1}{\alpha+1} \int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} \ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2} + C, \quad \alpha \neq -1$$

$$58. \quad u = (\ln x)^2 \quad dv = x^\alpha dx$$

$$du = \frac{2 \ln x}{x} dx \quad v = \frac{x^{\alpha+1}}{\alpha+1}, \quad \alpha \neq -1$$

$$\begin{aligned} \int x^\alpha (\ln x)^2 dx &= \frac{x^{\alpha+1}}{\alpha+1} (\ln x)^2 - \frac{2}{\alpha+1} \int x^\alpha \ln x \, dx = \frac{x^{\alpha+1}}{\alpha+1} (\ln x)^2 - \frac{2}{\alpha+1} \left[\frac{x^{\alpha+1}}{\alpha+1} \ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2} \right] + C \\ &= \frac{x^{\alpha+1}}{\alpha+1} (\ln x)^2 - 2 \frac{x^{\alpha+1}}{(\alpha+1)^2} \ln x + 2 \frac{x^{\alpha+1}}{(\alpha+1)^3} + C, \quad \alpha \neq -1 \end{aligned}$$

Problem 57 was used for $\int x^\alpha \ln x \, dx$.

$$59. \quad u = x^\alpha \quad dv = e^{\beta x} dx$$

$$du = \alpha x^{\alpha-1} dx \quad v = \frac{1}{\beta} e^{\beta x}$$

$$\int x^\alpha e^{\beta x} dx = \frac{x^\alpha e^{\beta x}}{\beta} - \frac{\alpha}{\beta} \int x^{\alpha-1} e^{\beta x} dx$$

$$60. \quad u = x^\alpha \quad dv = \sin \beta x \, dx$$

$$du = \alpha x^{\alpha-1} dx \quad v = -\frac{1}{\beta} \cos \beta x$$

$$\int x^\alpha \sin \beta x \, dx = -\frac{x^\alpha \cos \beta x}{\beta} + \frac{\alpha}{\beta} \int x^{\alpha-1} \cos \beta x \, dx$$

$$61. \quad u = x^\alpha \quad dv = \cos \beta x \, dx$$

$$du = \alpha x^{\alpha-1} dx \quad v = \frac{1}{\beta} \sin \beta x$$

$$\int x^\alpha \cos \beta x \, dx = \frac{x^\alpha \sin \beta x}{\beta} - \frac{\alpha}{\beta} \int x^{\alpha-1} \sin \beta x \, dx$$

$$62. \quad u = (\ln x)^\alpha \quad dv = dx$$

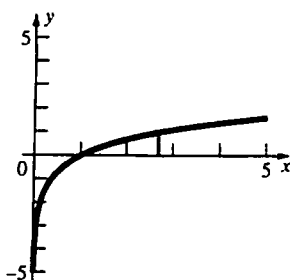
$$du = \frac{\alpha (\ln x)^{\alpha-1}}{x} dx \quad v = x$$

$$\int (\ln x)^\alpha dx = x (\ln x)^\alpha - \alpha \int (\ln x)^{\alpha-1} dx$$

63. $u = (a^2 - x^2)^n$
 $dv = dx$
 $du = -2ax(a^2 - x^2)^{n-1} dx$
 $v = x$
 $\int (a^2 - x^2)^n dx = x(a^2 - x^2)^n - \int x^2(a^2 - x^2)^{n-1} dx$
64. $u = \cos^{-1} x$
 $dv = \cos x dx$
 $du = -(a - 1) \cos^{-1} x dx$
 $v = \sin x$
 $\int \cos^{-1} x dx = \sin x - (a - 1) \int \cos^{-1} x dx$
 $\int \cos^{-1} x dx = \sin x + (a - 1) \int \cos^{-1} x dx$
 $\int \cos^{-1} x dx = \sin x + (a - 1) \int \cos^{-1} x dx$
 $\int \cos^{-1} x dx = \sin x + (a - 1) \int \cos^{-1} x dx$
65. $u = \cos^{-1} \beta x$
 $dv = \cos \beta x dx$
 $du = -\beta \cos^{-1} \beta x dx$
 $v = \frac{1}{\beta} \sin \beta x$
 $\int \cos^{-1} \beta x dx = \frac{1}{\beta} \sin \beta x - \beta \int \cos^{-1} \beta x dx$
 $\int \cos^{-1} \beta x dx = \frac{1}{\beta} \sin \beta x + \beta \int \cos^{-1} \beta x dx$
 $\int \cos^{-1} \beta x dx = \frac{1}{\beta} \sin \beta x + \beta \int \cos^{-1} \beta x dx$
 $\int \cos^{-1} \beta x dx = \frac{1}{\beta} \sin \beta x + \beta \int \cos^{-1} \beta x dx$
66. $\int x^4 e^{3x} dx = \frac{1}{4} x^4 e^{3x} - \frac{3}{4} \int x^3 e^{3x} dx$
 $= \frac{1}{4} x^4 e^{3x} - \frac{3}{4} \left[\frac{1}{3} x^3 e^{3x} - \frac{3}{4} \int x^2 e^{3x} dx \right]$
 $= \frac{1}{4} x^4 e^{3x} - \frac{1}{4} x^3 e^{3x} + \frac{9}{16} \int x^2 e^{3x} dx$
 $= \frac{1}{4} x^4 e^{3x} - \frac{1}{4} x^3 e^{3x} + \frac{9}{16} \left[\frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x e^{3x} dx \right]$
 $= \frac{1}{4} x^4 e^{3x} - \frac{1}{4} x^3 e^{3x} + \frac{3}{8} x^2 e^{3x} - \frac{1}{2} \int x e^{3x} dx$
 $= \frac{1}{4} x^4 e^{3x} - \frac{1}{4} x^3 e^{3x} + \frac{3}{8} x^2 e^{3x} - \frac{1}{2} \left[\frac{1}{3} x e^{3x} - \frac{1}{9} \int e^{3x} dx \right]$
 $= \frac{1}{4} x^4 e^{3x} - \frac{1}{4} x^3 e^{3x} + \frac{3}{8} x^2 e^{3x} - \frac{1}{6} x e^{3x} + \frac{1}{18} e^{3x} + C$
67. $\int x^4 \cos 3x dx = \frac{1}{4} x^4 \sin 3x - \frac{3}{4} \int x^3 \sin 3x dx$
 $= \frac{1}{4} x^4 \sin 3x - \frac{3}{4} \left[-\frac{1}{3} x^3 \cos 3x + \frac{3}{4} \int x^2 \cos 3x dx \right]$
 $= \frac{1}{4} x^4 \sin 3x + \frac{1}{4} x^3 \cos 3x - \frac{9}{16} \int x^2 \cos 3x dx$
 $= \frac{1}{4} x^4 \sin 3x + \frac{1}{4} x^3 \cos 3x - \frac{9}{16} \left[\frac{1}{3} x^2 \sin 3x - \frac{2}{3} \int x \sin 3x dx \right]$
 $= \frac{1}{4} x^4 \sin 3x + \frac{1}{4} x^3 \cos 3x - \frac{3}{8} x^2 \sin 3x + \frac{1}{2} \int x \sin 3x dx$
 $= \frac{1}{4} x^4 \sin 3x + \frac{1}{4} x^3 \cos 3x - \frac{3}{8} x^2 \sin 3x + \frac{1}{2} \left[-\frac{1}{3} x \cos 3x + \frac{1}{9} \int \cos 3x dx \right]$
 $= \frac{1}{4} x^4 \sin 3x + \frac{1}{4} x^3 \cos 3x - \frac{3}{8} x^2 \sin 3x - \frac{1}{6} x \cos 3x + \frac{1}{18} \sin 3x + C$

$$\begin{aligned}
 68. \quad \int \cos^6 3x \, dx &= \frac{1}{18} \cos^5 3x \sin 3x + \frac{5}{6} \int \cos^4 3x \, dx = \frac{1}{18} \cos^5 3x \sin 3x + \frac{5}{6} \left[\frac{1}{12} \cos^3 3x \sin 3x + \frac{3}{4} \int \cos^2 3x \, dx \right] \\
 &= \frac{1}{18} \cos^5 3x \sin 3x + \frac{5}{72} \cos^3 3x \sin 3x + \frac{5}{8} \left[\frac{1}{6} \cos 3x \sin 3x + \frac{1}{2} \int dx \right] \\
 &= \frac{1}{18} \cos^5 3x \sin 3x + \frac{5}{72} \cos^3 3x \sin 3x + \frac{5}{48} \cos 3x \sin 3x + \frac{5x}{16} + C
 \end{aligned}$$

69. First make a sketch.



From the sketch, the area is given by

$$\int_1^e \ln x \, dx$$

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = x$$

$$\int_1^e \ln x \, dx = [x \ln x]_1^e - \int_1^e dx = [x \ln x - x]_1^e = (e - e) - (1 \cdot 0 - 1) = 1$$

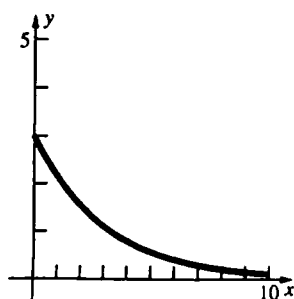
$$70. \quad V = \int_1^e \pi (\ln x)^2 \, dx$$

$$u = (\ln x)^2 \quad dv = dx$$

$$du = \frac{2 \ln x}{x} dx \quad v = x$$

$$\begin{aligned}
 \pi \int_1^e (\ln x)^2 \, dx &= \pi \left([x (\ln x)^2]_1^e - 2 \int_1^e \ln x \, dx \right) = \pi [x (\ln x)^2 - 2(x \ln x - x)]_1^e \\
 &= \pi [(e - 2e + 2e) - (0 - 0 + 2)] = \pi(e - 2) \approx 2.26
 \end{aligned}$$

71.



$$\int_0^9 3e^{-x/3} \, dx = -9 \int_0^9 e^{-x/3} \left(-\frac{1}{3} dx \right) = -9 [e^{-x/3}]_0^9 = -\frac{9}{e^3} + 9 \approx 8.55$$

$$72. \quad V = \int_0^9 \pi (3e^{-x/3})^2 \, dx = 9\pi \int_0^9 e^{-2x/3} \, dx$$

$$= 9\pi \left(-\frac{3}{2} \right) \int_0^9 e^{-2x/3} \left(-\frac{2}{3} dx \right) = -\frac{27\pi}{2} [e^{-2x/3}]_0^9 = -\frac{27\pi}{2e^6} + \frac{27\pi}{2} \approx 42.31$$

$$\begin{aligned}
 73. \quad \int_0^{\pi/4} (x \cos x - x \sin x) dx &= \int_0^{\pi/4} x \cos x dx - \int_0^{\pi/4} x \sin x dx \\
 &= \left([x \sin x]_0^{\pi/4} - \int_0^{\pi/4} \sin x dx \right) - \left([-x \cos x]_0^{\pi/4} + \int_0^{\pi/4} \cos x dx \right) \\
 &= [x \sin x + \cos x + x \cos x - \sin x]_0^{\pi/4} = \frac{\sqrt{2}\pi}{4} - 1 \approx 0.11
 \end{aligned}$$

Use Problems 60 and 61 for $\int x \sin x dx$ and $\int x \cos x dx$.

$$\begin{aligned}
 74. \quad V &= 2\pi \int_0^{2\pi} x \sin\left(\frac{x}{2}\right) dx \\
 u &= x & dv &= \sin \frac{x}{2} dx \\
 du &= dx & v &= -2 \cos \frac{x}{2} \\
 V &= 2\pi \left(\left[-2x \cos \frac{x}{2} \right]_0^{2\pi} + \int_0^{2\pi} 2 \cos \frac{x}{2} dx \right) = 2\pi \left(4\pi + \left[4 \sin \frac{x}{2} \right]_0^{2\pi} \right) = 8\pi^2
 \end{aligned}$$

$$\begin{aligned}
 75. \quad \int_1^e \ln x^2 dx &= 2 \int_1^e \ln x dx \\
 u &= \ln x & dv &= dx \\
 du &= \frac{1}{x} dx & v &= x \\
 2 \int_1^e \ln x dx &= 2 \left([x \ln x]_1^e - \int_1^e dx \right) = 2(e - [x]_1^e) = 2 \\
 \int_1^e x \ln x^2 dx &= 2 \int_1^e x \ln x dx \\
 u &= \ln x & dv &= x dx \\
 du &= \frac{1}{x} dx & v &= \frac{1}{2} x^2 \\
 2 \int_1^e x \ln x dx &= 2 \left(\left[\frac{1}{2} x^2 \ln x \right]_1^e - \int_1^e \frac{1}{2} x dx \right) = 2 \left(\frac{1}{2} e^2 - \left[\frac{1}{4} x^2 \right]_1^e \right) = \frac{1}{2} (e^2 + 1) \\
 \frac{1}{2} \int_1^e (\ln x)^2 dx & \\
 u &= (\ln x)^2 & dv &= dx \\
 du &= \frac{2 \ln x}{x} dx & v &= x \\
 \frac{1}{2} \int_1^e (\ln x)^2 dx &= \frac{1}{2} \left([x (\ln x)^2]_1^e - 2 \int_1^e \ln x dx \right) = \frac{1}{2} (e - 2) \\
 \bar{x} &= \frac{\frac{1}{2} (e^2 + 1)}{2} = \frac{e^2 + 1}{4} \\
 \bar{y} &= \frac{\frac{1}{2} (e - 2)}{2} = \frac{e - 2}{4}
 \end{aligned}$$

$$\begin{aligned}
 76. \quad \text{a.} \quad u &= \cot x & dv &= \csc^2 x dx \\
 du &= -\csc^2 x dx & v &= -\cot x \\
 \int \cot x \csc^2 x dx &= -\cot^2 x - \int \cot x \csc^2 x dx = -\frac{1}{2} \cot^2 x + C
 \end{aligned}$$

$$\begin{aligned} \text{b. } u &= \csc x & dv &= \cot x \csc x \, dx \\ du &= -\cot x \csc x \, dx & v &= -\csc x \\ \int \cot x \csc^2 x \, dx &= -\csc^2 x - \int \cot x \csc^2 x \, dx = -\frac{1}{2} \csc^2 x + C \end{aligned}$$

$$\text{c. } -\frac{1}{2} \cot^2 x = -\frac{1}{2} (\csc^2 x - 1) = -\frac{1}{2} \csc^2 x + \frac{1}{2}$$

$$77. \text{ a. } p(x) = x^3 - 2x$$

$$g(x) = e^x$$

All antiderivatives of $g(x) = e^x$

$$\int (x^3 - 2x)e^x \, dx = (x^3 - 2x)e^x - (3x^2 - 2)e^x + 6xe^x - 6e^x + C$$

$$\text{b. } p(x) = x^2 - 3x + 1$$

$$g(x) = \sin x$$

$$G_1(x) = -\cos x$$

$$G_2(x) = -\sin x$$

$$G_3(x) = \cos x$$

$$\int (x^2 - 3x + 1)\sin x \, dx = (x^2 - 3x + 1)(-\cos x) - (2x - 3)(-\sin x) + 2\cos x + C$$

$$\begin{aligned} 78. \quad 2\sec^3 x \, dx &= [\sec^3 x + \sec x(1 + \tan^2 x)] \, dx \\ &= (\sec x + \sec^3 x + \sec x \tan^2 x) \, dx = \sec x \, dx + (\sec^3 x + \sec x \tan^2 x) \, dx \\ &= \sec x \, dx + d(\sec x \tan x) \\ \int \sec^3 x \, dx &= \frac{1}{2} \int [\sec x \, dx + d(\sec x \tan x)] = \frac{1}{2} \int \sec x \, dx + \frac{1}{2} \int d(\sec x \tan x) \\ &= \frac{1}{2} \ln |\sec x + \tan x| + \frac{1}{2} (\sec x \tan x) + C \end{aligned}$$

$$79. \quad \int_0^{2\pi} \sin^n x \, dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 4 \int_0^{\pi/2} \sin^n x \, dx & \text{if } n \text{ is even} \end{cases}$$

From Formula 113,

$$4 \int_0^{\pi/2} \sin^n x \, dx = 4 \left(\frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \frac{\pi}{2} \right) \text{ if } n \text{ is even}$$

$$\int_0^{2\pi} \sin^n x \, dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} 2\pi & \text{if } n \text{ is even} \end{cases}$$

$$80. \text{ a. } \int_{2\pi n - 2\pi}^{2\pi n - \pi} x \sin x \, dx$$

$$\text{b. } V = 2\pi \int_{2\pi}^{3\pi} x^2 \sin x \, dx$$

$$u = x^2$$

$$du = 2x \, dx$$

$$dv = \sin x \, dx$$

$$v = -\cos x$$

$$V = 2\pi \left(\left[-x^2 \cos x \right]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} 2x \cos x \, dx \right) = 2\pi \left(9\pi^2 + 4\pi^2 + \int_{2\pi}^{3\pi} 2x \cos x \, dx \right)$$

$$u = 2x$$

$$du = 2 \, dx$$

$$dv = \cos x \, dx$$

$$v = \sin x$$

$$\begin{aligned} V &= 2\pi \left(13\pi^2 + [2x \sin x]_{2\pi}^{3\pi} - \int_{2\pi}^{3\pi} 2 \sin x \, dx \right) \\ &= 2\pi \left(13\pi^2 + [2 \cos x]_{2\pi}^{3\pi} \right) = 2\pi(13\pi^2 - 4) \approx 781 \end{aligned}$$

81. $u = f(x) \quad dv = \sin nx \, dx$

$$du = f'(x)dx \quad v = -\frac{1}{n} \cos nx$$

$$a_n = \frac{1}{\pi} \left[\underbrace{\left[-\frac{1}{n} \cos(nx) f(x) \right]_{-\pi}^{\pi}}_{\text{Term 1}} + \underbrace{\frac{1}{n} \int_{-\pi}^{\pi} \cos(nx) f'(x) dx}_{\text{Term 2}} \right]$$

$$\text{Term 1} = \frac{1}{n} \cos(n\pi)(f(-\pi) - f(\pi)) = \pm \frac{1}{n} (f(-\pi) - f(\pi))$$

Since $f'(x)$ is continuous on $[-\infty, \infty]$, it is bounded. Thus, $\int_{-\pi}^{\pi} \cos(nx) f'(x) dx$ is bounded so

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\pi n} \left[\pm(f(-\pi) - f(\pi)) + \int_{-\pi}^{\pi} \cos(nx) f'(x) dx \right] = 0.$$

82. $\frac{G_n}{n} = \frac{[(n+1)(n+2)\cdots(n+n)]^{1/n}}{[n^n]^{1/n}} = \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \cdots \left(1 + \frac{n}{n}\right) \right]^{1/n}$

$$\ln \left(\frac{G_n}{n} \right) = \frac{1}{n} \ln \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \cdots \left(1 + \frac{n}{n}\right) \right]$$

$$= \frac{1}{n} \left[\ln \left(1 + \frac{1}{n}\right) + \ln \left(1 + \frac{2}{n}\right) + \cdots + \ln \left(1 + \frac{n}{n}\right) \right]$$

$$\lim_{n \rightarrow \infty} \ln \left(\frac{G_n}{n} \right) = \int_1^2 \ln x \, dx = 2 \ln 2 - 1$$

$$\lim_{n \rightarrow \infty} \left(\frac{G_n}{n} \right) = e^{2 \ln 2 - 1} = 4e^{-1} = \frac{4}{e}$$

83. The proof fails to consider the constants when integrating $\frac{1}{t}$.

$$\int_t^1 \frac{1}{t} dt = 1 + \int_t^1 \frac{1}{t} dt$$

$$\ln t + C_1 = 1 + \ln t + C_2$$

$$\text{Thus } C_1 - C_2 = 1.$$

84. $\frac{d}{dx} [e^{5x} (C_1 \cos 7x + C_2 \sin 7x) + C_3] = 5e^{5x} (C_1 \cos 7x + C_2 \sin 7x) + e^{5x} (-7C_1 \sin 7x + 7C_2 \cos 7x)$

$$= e^{5x} [(5C_1 + 7C_2) \cos 7x + (5C_2 - 7C_1) \sin 7x]$$

$$\text{Thus, } 5C_1 + 7C_2 = 4 \text{ and } 5C_2 - 7C_1 = 6.$$

$$\text{Solving, } C_1 = -\frac{11}{37}; C_2 = \frac{29}{37}$$

85. $u = f(x) \quad dv = dx$

$$du = f'(x)dx \quad v = x$$

$$\int_a^b f(x) dx = [xf(x)]_a^b - \int_a^b xf'(x) dx$$

Starting with the same integral,

$$\begin{aligned}
 u &= f(x) & dv &= dx \\
 du &= f'(x)dx & v &= x - a \\
 \int_a^b f(x) dx &= [(x-a)f(x)]_a^b - \int_a^b (x-a)f'(x)dx
 \end{aligned}$$

$$\begin{aligned}
 86. \quad u &= f'(x) & dv &= dx \\
 du &= f''(x)dx & v &= x - a \\
 f(b) - f(a) &= \int_a^b f'(x)dx = [(x-a)f'(x)]_a^b - \int_a^b (x-a)f''(x)dx = f'(b)(b-a) - \int_a^b (x-a)f''(x)dx
 \end{aligned}$$

Starting with the same integral,

$$\begin{aligned}
 u &= f'(x) & dv &= dx \\
 du &= f''(x)dx & v &= x - b \\
 f(b) - f(a) &= \int_a^b f'(x)dx = [(x-b)f'(x)]_a^b - \int_a^b (x-b)f''(x)dx = f'(a)(b-a) - \int_a^b (x-b)f''(x)dx
 \end{aligned}$$

87. Use proof by induction.

$$\begin{aligned}
 n = 1: \quad f(a) + f'(a)(t-a) + \int_a^t (t-x)f''(x)dx &= f(a) + f'(a)(t-a) + [f'(x)(t-x)]_a^t + \int_a^t f'(x)dx \\
 &= f(a) + f'(a)(t-a) - f'(a)(t-a) + [f(x)]_a^t = f(t)
 \end{aligned}$$

Thus, the statement is true for $n = 1$. Note that integration by parts was used with $u = (t-x)$, $dv = f''(x)dx$.

Suppose the statement is true for n .

$$f(t) = f(a) + \sum_{i=1}^n \frac{f^{(i)}(a)}{i!} (t-a)^i + \int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx$$

Integrate $\int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx$ by parts.

$$u = f^{(n+1)}(x) \quad dv = \frac{(t-x)^n}{n!} dx$$

$$du = f^{(n+2)}(x) \quad v = -\frac{(t-x)^{n+1}}{(n+1)!}$$

$$\begin{aligned}
 \int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx &= \left[-\frac{(t-x)^{n+1}}{(n+1)!} f^{(n+1)}(x) \right]_a^t + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx \\
 &= \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx
 \end{aligned}$$

$$\begin{aligned}
 \text{Thus } f(t) &= f(a) + \sum_{i=1}^n \frac{f^{(i)}(a)}{i!} (t-a)^i + \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx \\
 &= f(a) + \sum_{i=1}^{n+1} \frac{f^{(i)}(a)}{i!} (t-a)^i + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx
 \end{aligned}$$

Thus, the statement is true for $n+1$.

$$88. \quad a. \quad B(\alpha, \beta) = \int_0^1 x^\alpha (1-x)^\beta dx$$

$$x = 1 - u, \quad dx = -du$$

$$\int_0^1 x^\alpha (1-x)^\beta dx = \int_1^0 (1-u)^\alpha (u)^\beta (-du) = \int_0^1 (1-u)^\alpha u^\beta du = B(\beta, \alpha)$$

Thus, $B(\alpha, \beta) = B(\beta, \alpha)$.

$$\text{b. } B(\alpha, \beta) = \int_0^1 x^\alpha (1-x)^\beta dx$$

$$u = x^\alpha \quad dv = (1-x)^\beta dx$$

$$du = \alpha x^{\alpha-1} dx \quad v = -\frac{1}{\beta+1} (1-x)^{\beta+1}$$

$$B(\alpha, \beta) = \left[-\frac{1}{\beta+1} x^\alpha (1-x)^{\beta+1} \right]_0^1 + \frac{\alpha}{\beta+1} \int_0^1 x^{\alpha-1} (1-x)^{\beta+1} dx$$

$$= \frac{\alpha}{\beta+1} \int_0^1 x^{\alpha-1} (1-x)^{\beta+1} dx = \frac{\alpha}{\beta+1} B(\alpha-1, \beta+1)$$

$$B(\alpha, \beta) = \int_0^1 x^\alpha (1-x)^\beta dx$$

$$u = (1-x)^\beta \quad dv = x^\alpha dx$$

$$du = -\beta(1-x)^{\beta-1} dx \quad v = \frac{1}{\alpha+1} x^{\alpha+1}$$

$$B(\alpha, \beta) = \left[\frac{1}{\alpha+1} x^{\alpha+1} (1-x)^\beta \right]_0^1 + \frac{\beta}{\alpha+1} \int_0^1 x^{\alpha+1} (1-x)^{\beta-1} dx = \frac{\beta}{\alpha+1} \int_0^1 x^{\alpha+1} (1-x)^{\beta-1} dx = \frac{\beta}{\alpha+1} B(\alpha+1, \beta-1)$$

c. Assume that $n \leq m$. Using part b. n times,

$$B(n, m) = \frac{n}{m+1} B(n-1, m+1) = \frac{n(n-1)}{(m+1)(m+2)} B(n-2, m+2) = \dots = \frac{n(n-1)\dots 2 \cdot 1}{(m+1)(m+2)\dots(m+n)} B(0, m+n).$$

$$B(0, m+n) = \int_0^1 (1-x)^{m+n} dx = -\frac{1}{m+n+1} [(1-x)^{m+n+1}]_0^1 = \frac{1}{m+n+1}$$

$$\text{Thus, } B(n, m) = \frac{n(n-1)\dots 2 \cdot 1}{(m+1)(m+2)\dots(m+n)(m+n+1)} = \frac{n!m!}{(m+n+1)!}$$

$$\text{If } n > m, \text{ then } B(n, m) = B(m, n) = \frac{n!m!}{(m+n+1)!} \text{ by the above reasoning.}$$

$$89. \quad u = f(t) \quad dv = f''(t) dt$$

$$du = f'(t) dt \quad v = f'(t)$$

$$\int_a^b f''(t) f(t) dt = [f(t) f'(t)]_a^b - \int_a^b [f'(t)]^2 dt$$

$$= f(b) f'(b) - f(a) f'(a) - \int_a^b [f'(t)]^2 dt = - \int_a^b [f'(t)]^2 dt$$

$$[f'(t)]^2 \geq 0, \text{ so } - \int_a^b [f'(t)]^2 \leq 0.$$

$$90. \quad \int_0^x \left(\int_0^t f(z) dz \right) dt$$

$$u = \int_0^t f(z) dz \quad dv = dt$$

$$du = f(t) dt \quad v = t$$

$$\int_0^x \left(\int_0^t f(z) dz \right) dt = \left[t \int_0^t f(z) dz \right]_0^x - \int_0^x t f(t) dt = \int_0^x x f(z) dz - \int_0^x t f(t) dt$$

$$\text{By letting } z = t, \int_0^x x f(z) dz = \int_0^x x f(t) dt, \text{ so}$$

$$\int_0^x \left(\int_0^t f(z) dz \right) dt = \int_0^x x f(t) dt - \int_0^x t f(t) dt = \int_0^x (x-t) f(t) dt$$

91. Let $I = \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_n) dt_n \cdots dt_2 dt_1$ be the iterated integral. Note that for $i \geq 2$, the limits of integration of the integral with respect to t_i are 0 to t_{i-1} so that any change of variables in an outer integral affects the limits, and hence the variables in all interior integrals.

$$I = \int_0^x \left[\int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n) dt_n \cdots dt_3 dt_2 \right] dt_1$$

$$u = \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n) dt_n \cdots dt_3 dt_2 \quad dv = dt_1$$

$$du = \left(\int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n) dt_n \cdots dt_3 \right) dt_1 \quad v = t_1$$

$$I = \left[t_1 \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n) dt_n \cdots dt_3 dt_2 \right]_{t_1=0}^{t_1=x} - \int_0^x t_1 \left(\int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n) dt_n \cdots dt_3 \right) dt_1$$

$$= x \int_0^x \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n) dt_n \cdots dt_3 dt_2 - \int_0^x t_1 \left(\int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n) dt_n \cdots dt_3 \right) dt_1$$

Make the change of variables $t_i = t_{i-1}$, so $dt_i = dt_{i-1}$ for $i = 2$ to n .

$$\text{Hence, } I = \int_0^x x \int_0^{t_1} \cdots \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \cdots dt_2 dt_1 - \int_0^x t_1 \left(\int_0^{t_1} \cdots \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \cdots dt_2 \right) dt_1$$

$$= \int_0^x (x - t_1) \left(\int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \cdots dt_3 dt_2 \right) dt_1$$

Use integration by parts again.

$$u = \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \cdots dt_3 dt_2 \quad dv = (x - t_1) dt_1$$

$$du = \left(\int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \cdots dt_3 \right) dt_1 \quad v = -\frac{1}{2}(x - t_1)^2$$

$$I = \left[-\frac{1}{2}(x - t_1)^2 \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \cdots dt_3 dt_2 \right]_{t_1=0}^{t_1=x} + \int_0^x \frac{1}{2}(x - t_1)^2 \left(\int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \cdots dt_3 \right) dt_1$$

$$= \int_0^x \frac{1}{2}(x - t_1)^2 \left(\int_0^{t_2} \cdots \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \cdots dt_3 \right) dt_1$$

Since $(x - t_1)^2 = 0$ when $t_1 = x$ while $\int_0^{t_1} g(t_2) dt_2 = 0$ when $t_1 = 0$ for any integrand $g(t_2)$.

Again change variables so that $t_i = t_{i-1}$ for $i = 2$ to $n - 1$.

$$I = \int_0^x \frac{1}{2}(x - t_1)^2 \left(\int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-3}} f(t_{n-2}) dt_{n-2} \cdots dt_3 dt_2 \right) dt_1$$

$$u = \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-3}} f(t_{n-2}) dt_{n-2} \cdots dt_3 dt_2 \quad dv = \frac{1}{2}(x - t_1)^2 dt_1$$

$$du = \left(\int_0^{t_2} \cdots \int_0^{t_{n-3}} f(t_{n-2}) dt_{n-2} \cdots dt_3 \right) dt_1 \quad v = -\frac{1}{3!}(x - t_1)^3$$

$$I = \left[-\frac{1}{3!}(x - t_1)^3 \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-3}} f(t_{n-2}) dt_{n-2} \cdots dt_3 dt_2 \right]_{t_1=0}^{t_1=x} + \int_0^x \frac{1}{3!}(x - t_1)^3 \left(\int_0^{t_2} \cdots \int_0^{t_{n-3}} f(t_{n-2}) dt_{n-2} \cdots dt_3 \right) dt_1$$

Changing variables and using integration by parts as before, then changing variables again yields

$$I = \int_0^x \frac{1}{4!}(x - t_1)^4 \left(\int_0^{t_1} \cdots \int_0^{t_{n-5}} f(t_{n-4}) dt_{n-4} \cdots dt_2 \right) dt_1$$

Reducing the n -fold iterated integral to a single integral requires $n - 1$ integrations by parts, each one contributing a factor of $x - t_1$. The integral is multiplied by $\frac{1}{k!}$ after k integrations by parts, hence

$$I = \frac{1}{(n-1)!} \int_0^x f(t_1)(x - t_1)^{n-1} dt_1.$$

92. Proof by induction.

$n = 1$:

$$u = P_1(x) \quad dv = e^x dx$$

$$du = \frac{dP_1(x)}{dx} dx \quad v = e^x$$

$$\int e^x P_1(x) dx = e^x P_1(x) - \int e^x \frac{dP_1(x)}{dx} dx = e^x P_1(x) - \frac{dP_1(x)}{dx} \int e^x dx = e^x P_1(x) - e^x \frac{dP_1(x)}{dx}$$

Note that $\frac{dP_1(x)}{dx}$ is a constant.

Suppose the formula is true for n . By using integration by parts with $u = P_{n+1}(x)$ and $dv = e^x dx$,

$$\int e^x P_{n+1}(x) dx = e^x P_{n+1}(x) - \int e^x \frac{dP_{n+1}(x)}{dx} dx$$

Note that $\frac{dP_{n+1}(x)}{dx}$ is a polynomial of degree n , so

$$\begin{aligned} \int e^x P_{n+1}(x) dx &= e^x P_{n+1}(x) - \left[e^x \sum_{j=0}^n (-1)^j \frac{d^j}{dx^j} \left(\frac{dP_{n+1}(x)}{dx} \right) \right] \\ &= e^x P_{n+1}(x) + e^x \sum_{j=1}^{n+1} (-1)^j \frac{d^j P_{n+1}(x)}{dx^{j+1}} = e^x \sum_{j=0}^{n+1} (-1)^j \frac{d^j P_{n+1}(x)}{dx^{j+1}} \end{aligned}$$

$$\begin{aligned} 93. \quad \int (3x^4 + 2x^2) e^x dx &= e^x \sum_{j=0}^4 (-1)^j \frac{d^j (3x^4 + 2x^2)}{dx^j} \\ &= e^x [3x^4 + 2x^2 - 12x^3 - 4x + 36x^2 + 4 - 72x + 72] \\ &= e^x (3x^4 - 12x^3 + 38x^2 - 76x + 76) \end{aligned}$$

8.5 Concepts Review

1. proper

$$2. \quad x - 1 + \frac{5}{x+1}$$

$$3. \quad 2x^2 + 3x - 1 = ax^2 + bx + c$$

$a = 2; b = 3; c = -1$

$$4. \quad \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$2. \quad \frac{2}{x^2+3x} = \frac{2}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$$

$$2 = A(x+3) + Bx$$

$$A = \frac{2}{3}, B = -\frac{2}{3}$$

$$\int \frac{2}{x^2+3x} dx = \frac{2}{3} \int \frac{1}{x} dx - \frac{2}{3} \int \frac{1}{x+3} dx$$

$$= \frac{2}{3} \ln|x| - \frac{2}{3} \ln|x+3| + C$$

$$3. \quad \frac{3}{x^2-1} = \frac{3}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$3 = A(x-1) + B(x+1)$$

$$A = -\frac{3}{2}, B = \frac{3}{2}$$

$$\int \frac{3}{x^2-1} dx = -\frac{3}{2} \int \frac{1}{x+1} dx + \frac{3}{2} \int \frac{1}{x-1} dx$$

$$= -\frac{3}{2} \ln|x+1| + \frac{3}{2} \ln|x-1| + C$$

Problem Set 8.5

$$1. \quad \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$1 = A(x+1) + Bx$$

$$A = 1, B = -1$$

$$\int \frac{1}{x(x+1)} dx = \int \frac{1}{x} dx - \int \frac{1}{x+1} dx$$

$$= \ln|x| - \ln|x+1| + C$$

$$4. \frac{5x}{2x^3+6x^2} = \frac{5x}{2x^2(x+3)} = \frac{5}{2x(x+3)}$$

$$= \frac{A}{x} + \frac{B}{x+3}$$

$$\frac{5}{2} = A(x+3) + Bx$$

$$A = \frac{5}{6}, B = -\frac{5}{6}$$

$$\int \frac{5x}{2x^3+6x^2} dx = \frac{5}{6} \int \frac{1}{x} dx - \frac{5}{6} \int \frac{1}{x+3} dx$$

$$= \frac{5}{6} \ln|x| - \frac{5}{6} \ln|x+3| + C$$

$$5. \frac{x-11}{x^2+3x-4} = \frac{x-11}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

$$x-11 = A(x-1) + B(x+4)$$

$$A=3, B=-2$$

$$\int \frac{x-11}{x^2+3x-4} dx = 3 \int \frac{1}{x+4} dx - 2 \int \frac{1}{x-1} dx$$

$$= 3 \ln|x+4| - 2 \ln|x-1| + C$$

$$6. \frac{x-7}{x^2-x-12} = \frac{x-7}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}$$

$$x-7 = A(x+3) + B(x-4)$$

$$A = -\frac{3}{7}, B = \frac{10}{7}$$

$$\int \frac{x-7}{x^2-x-12} dx = -\frac{3}{7} \int \frac{1}{x-4} dx + \frac{10}{7} \int \frac{1}{x+3} dx$$

$$= -\frac{3}{7} \ln|x-4| + \frac{10}{7} \ln|x+3| + C$$

$$7. \frac{3x-13}{x^2+3x-10} = \frac{3x-13}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

$$3x-13 = A(x-2) + B(x+5)$$

$$A=4, B=-1$$

$$\int \frac{3x-13}{x^2+3x-10} dx = 4 \int \frac{1}{x+5} dx - \int \frac{1}{x-2} dx$$

$$= 4 \ln|x+5| - \ln|x-2| + C$$

$$11. \frac{17x-3}{3x^2+x-2} = \frac{17x-3}{(3x-2)(x+1)} = \frac{A}{3x-2} + \frac{B}{x+1}$$

$$17x-3 = A(x+1) + B(3x-2)$$

$$A=5, B=4$$

$$\int \frac{17x-3}{3x^2+x-2} dx = \int \frac{5}{3x-2} dx + \int \frac{4}{x+1} dx = \frac{5}{3} \ln|3x-2| + 4 \ln|x+1| + C$$

$$8. \frac{x+\pi}{x^2-3\pi x+2\pi^2} = \frac{x+\pi}{(x-2\pi)(x-\pi)} = \frac{A}{x-2\pi} + \frac{B}{x-\pi}$$

$$x+\pi = A(x-\pi) + B(x-2\pi)$$

$$A=3, B=-2$$

$$\int \frac{x+\pi}{x^2-3\pi x+2\pi^2} dx = \int \frac{3}{x-2\pi} dx - \int \frac{2}{x-\pi} dx$$

$$= 3 \ln|x-2\pi| - 2 \ln|x-\pi| + C$$

$$9. \frac{2x+21}{2x^2+9x-5} = \frac{2x+21}{(2x-1)(x+5)} = \frac{A}{2x-1} + \frac{B}{x+5}$$

$$2x+21 = A(x+5) + B(2x-1)$$

$$A=4, B=-1$$

$$\int \frac{2x+21}{2x^2+9x-5} dx = \int \frac{4}{2x-1} dx - \int \frac{1}{x+5} dx$$

$$= 2 \ln|2x-1| - \ln|x+5| + C$$

$$10. \frac{2x^2-x-20}{x^2+x-6} = \frac{2(x^2+x-6)-3x-8}{x^2+x-6}$$

$$= 2 - \frac{3x+8}{x^2+x-6}$$

$$\frac{3x+8}{x^2+x-6} = \frac{3x+8}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

$$3x+8 = A(x-2) + B(x+3)$$

$$A = \frac{1}{5}, B = \frac{14}{5}$$

$$\int \frac{2x^2-x-20}{x^2+x-6} dx$$

$$= \int 2 dx - \frac{1}{5} \int \frac{1}{x+3} dx - \frac{14}{5} \int \frac{1}{x-2} dx$$

$$= 2x - \frac{1}{5} \ln|x+3| - \frac{14}{5} \ln|x-2| + C$$

$$12. \frac{5-x}{x^2-x(\pi+4)+4\pi} = \frac{5-x}{(x-\pi)(x-4)} = \frac{A}{x-\pi} + \frac{B}{x-4}$$

$$5-x = A(x-4) + B(x-\pi)$$

$$A = \frac{5-\pi}{\pi-4}, B = \frac{1}{4-\pi}$$

$$\int \frac{5-x}{x^2-x(\pi+4)+4\pi} dx = \frac{5-\pi}{\pi-4} \int \frac{1}{x-\pi} dx + \frac{1}{4-\pi} \int \frac{1}{x-4} dx = \frac{5-\pi}{\pi-4} \ln|x-\pi| + \frac{1}{4-\pi} \ln|x-4| + C$$

$$13. \frac{2x^2+x-4}{x^3-x^2-2x} = \frac{2x^2+x-4}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$2x^2+x-4 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

$$A = 2, B = -1, C = 1$$

$$\int \frac{2x^2+x-4}{x^3-x^2-2x} dx = \int \frac{2}{x} dx - \int \frac{1}{x+1} dx + \int \frac{1}{x-2} dx = 2\ln|x| - \ln|x+1| + \ln|x-2| + C$$

$$14. \frac{7x^2+2x-3}{(2x-1)(3x+2)(x-3)} = \frac{A}{2x-1} + \frac{B}{3x+2} + \frac{C}{x-3}$$

$$7x^2+2x-3 = A(3x+2)(x-3) + B(2x-1)(x-3) + C(2x-1)(3x+2)$$

$$A = \frac{1}{35}, B = -\frac{1}{7}, C = \frac{6}{5}$$

$$\int \frac{7x^2+2x-3}{(2x-1)(3x+2)(x-3)} dx = \frac{1}{35} \int \frac{1}{2x-1} dx - \frac{1}{7} \int \frac{1}{3x+2} dx + \frac{6}{5} \int \frac{1}{x-3} dx$$

$$= \frac{1}{70} \ln|2x-1| - \frac{1}{21} \ln|3x+2| + \frac{6}{5} \ln|x-3| + C$$

$$15. \frac{6x^2+22x-23}{(2x-1)(x^2+x-6)} = \frac{6x^2+22x-23}{(2x-1)(x+3)(x-2)} = \frac{A}{2x-1} + \frac{B}{x+3} + \frac{C}{x-2}$$

$$6x^2+22x-23 = A(x+3)(x-2) + B(2x-1)(x-2) + C(2x-1)(x+3) \quad A = 2, B = -1, C = 3$$

$$\int \frac{6x^2+22x-23}{(2x-1)(x^2+x-6)} dx = \int \frac{2}{2x-1} dx - \int \frac{1}{x+3} dx + \int \frac{3}{x-2} dx = \ln|2x-1| - \ln|x+3| + 3\ln|x-2| + C$$

$$16. \frac{x^3-6x^2+11x-6}{4x^3-28x^2+56x-32} = \frac{1}{4} \left(\frac{x^3-6x^2+11x-6}{x^3-7x^2+14x-8} \right) = \frac{1}{4} \left(1 + \frac{x^2-3x+2}{x^3-7x^2+14x-8} \right)$$

$$= \frac{1}{4} \left(1 + \frac{(x-1)(x-2)}{(x-1)(x-2)(x-4)} \right) = \frac{1}{4} \left(1 + \frac{1}{x-4} \right)$$

$$\int \frac{x^3-6x^2+11x-6}{4x^3-28x^2+56x-32} dx = \int \frac{1}{4} dx + \frac{1}{4} \int \frac{1}{x-4} dx = \frac{1}{4}x + \frac{1}{4} \ln|x-4| + C$$

$$17. \frac{x^3}{x^2+x-2} = x-1 + \frac{3x-2}{x^2+x-2}$$

$$\frac{3x-2}{x^2+x-2} = \frac{3x-2}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$3x-2 = A(x-1) + B(x+2)$$

$$A = \frac{8}{3}, B = \frac{1}{3}$$

$$\int \frac{x^3}{x^2 + x - 2} dx = \int (x-1) dx + \frac{8}{3} \int \frac{1}{x+2} dx + \frac{1}{3} \int \frac{1}{x-1} dx = \frac{1}{2}x^2 - x + \frac{8}{3} \ln|x+2| + \frac{1}{3} \ln|x-1| + C$$

$$18. \frac{x^3 + x^2}{x^2 + 5x + 6} = x - 4 + \frac{14x + 24}{(x+3)(x+2)}$$

$$\frac{14x + 24}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2}$$

$$14x + 24 = A(x+2) + B(x+3)$$

$$A = 18, B = -4$$

$$\int \frac{x^3 + x^2}{x^2 + 5x + 6} dx = \int (x-4) dx + \int \frac{18}{x+3} dx - \int \frac{4}{x+2} dx = \frac{1}{2}x^2 - 4x + 18 \ln|x+3| - 4 \ln|x+2| + C$$

$$19. \frac{x^4 + 8x^2 + 8}{x^3 - 4x} = x + \frac{12x^2 + 8}{x(x+2)(x-2)}$$

$$\frac{12x^2 + 8}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

$$12x^2 + 8 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$$

$$A = -2, B = 7, C = 7$$

$$\int \frac{x^4 + 8x^2 + 8}{x^3 - 4x} dx = \int x dx - 2 \int \frac{1}{x} dx + 7 \int \frac{1}{x+2} dx + 7 \int \frac{1}{x-2} dx = \frac{1}{2}x^2 - 2 \ln|x| + 7 \ln|x+2| + 7 \ln|x-2| + C$$

$$20. \frac{x^6 + 4x^3 + 4}{x^3 - 4x^2} = x^3 + 4x^2 + 16x + 68 + \frac{272x^2 + 4}{x^3 - 4x^2}$$

$$\frac{272x^2 + 4}{x^2(x-4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$$

$$272x^2 + 4 = Ax(x-4) + B(x-4) + Cx^2$$

$$A = -\frac{1}{4}, B = -1, C = \frac{1089}{4}$$

$$\begin{aligned} \int \frac{x^6 + 4x^3 + 4}{x^3 - 4x^2} dx &= \int (x^3 + 4x^2 + 16x + 68) dx - \frac{1}{4} \int \frac{1}{x} dx - \int \frac{1}{x^2} dx + \frac{1089}{4} \int \frac{1}{x-4} dx \\ &= \frac{1}{4}x^4 + \frac{4}{3}x^3 + 8x^2 + 68x - \frac{1}{4} \ln|x| + \frac{1}{x} + \frac{1089}{4} \ln|x-4| + C \end{aligned}$$

$$21. \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

$$x+1 = A(x-3) + B$$

$$A = 1, B = 4$$

$$\int \frac{x+1}{(x-3)^2} dx = \int \frac{1}{x-3} dx + \int \frac{4}{(x-3)^2} dx = \ln|x-3| - \frac{4}{x-3} + C$$

$$22. \frac{5x+7}{x^2+4x+4} = \frac{5x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$5x+7 = A(x+2) + B$$

$$A = 5, B = -3$$

$$\int \frac{5x+7}{x^2+4x+4} dx = \int \frac{5}{x+2} dx - \int \frac{3}{(x+2)^2} dx = 5 \ln|x+2| + \frac{3}{x+2} + C$$

$$23. \frac{3x+2}{x^3+3x^2+3x+1} = \frac{3x+2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$3x+2 = A(x+1)^2 + B(x+1) + C$$

$$A=0, B=3, C=-1$$

$$\int \frac{3x+2}{x^3+3x^2+3x+1} dx = \int \frac{3}{(x+1)^2} dx - \int \frac{1}{(x+1)^3} dx = -\frac{3}{x+1} + \frac{1}{2(x+1)^2} + C$$

$$24. \frac{x^6}{(x-2)^2(1-x)^5} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{1-x} + \frac{D}{(1-x)^2} + \frac{E}{(1-x)^3} + \frac{F}{(1-x)^4} + \frac{G}{(1-x)^5}$$

$$A=128, B=-64, C=129, D=-72, E=30, F=-8, G=1$$

$$\int \frac{x^6}{(x-2)^2(1-x)^5} dx = \int \left[\frac{128}{x-2} - \frac{64}{(x-2)^2} + \frac{129}{1-x} - \frac{72}{(1-x)^2} + \frac{30}{(1-x)^3} - \frac{8}{(1-x)^4} + \frac{1}{(1-x)^5} \right] dx$$

$$= 128 \ln|x-2| + \frac{64}{x-2} - 129 \ln|1-x| + \frac{72}{1-x} - \frac{15}{(1-x)^2} + \frac{8}{3(1-x)^3} - \frac{1}{4(1-x)^4} + C$$

$$25. \frac{3x^2-21x+32}{x^3-8x^2+16x} = \frac{3x^2-21x+32}{x(x-4)^2} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{(x-4)^2}$$

$$3x^2-21x+32 = A(x-4)^2 + Bx(x-4) + Cx$$

$$A=2, B=1, C=-1$$

$$\int \frac{3x^2-21x+32}{x^3-8x^2+16x} dx = \int \frac{2}{x} dx + \int \frac{1}{x-4} dx - \int \frac{1}{(x-4)^2} dx = 2 \ln|x| + \ln|x-4| + \frac{1}{x-4} + C$$

$$26. \frac{x^2+19x+10}{2x^4+5x^3} = \frac{x^2+19x+10}{x^3(2x+5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{2x+5}$$

$$A=-1, B=3, C=2, D=2$$

$$\int \frac{x^2+19x+10}{2x^4+5x^3} dx = \int \left(-\frac{1}{x} + \frac{3}{x^2} + \frac{2}{x^3} + \frac{2}{2x+5} \right) dx = -\ln|x| - \frac{3}{x} - \frac{1}{x^2} + \ln|2x+5| + C$$

$$27. \frac{2x^2+x-8}{x^3+4x} = \frac{2x^2+x-8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$A=-2, B=4, C=1$$

$$\int \frac{2x^2+x-8}{x^3+4x} dx = -2 \int \frac{1}{x} dx + \int \frac{4x+1}{x^2+4} dx = -2 \int \frac{1}{x} dx + 2 \int \frac{2x}{x^2+4} dx + \int \frac{1}{x^2+4} dx$$

$$= -2 \ln|x| + 2 \ln|x^2+4| + \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$28. \frac{3x+2}{x(x+2)^2+16x} = \frac{3x+2}{x(x^2+4x+20)} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+20}$$

$$A=\frac{1}{10}, B=-\frac{1}{10}, C=\frac{13}{5}$$

$$\int \frac{3x+2}{x(x+2)^2+16x} dx = \frac{1}{10} \int \frac{1}{x} dx + \int \frac{-\frac{1}{10}x+\frac{13}{5}}{x^2+4x+20} dx = \frac{1}{10} \int \frac{1}{x} dx + \frac{14}{5} \int \frac{1}{(x+2)^2+16} dx - \frac{1}{20} \int \frac{2x+4}{x^2+4x+20} dx$$

$$= \frac{1}{10} \ln|x| + \frac{7}{10} \tan^{-1}\left(\frac{x+2}{4}\right) - \frac{1}{20} \ln|x^2+4x+20| + C$$

$$29. \frac{2x^2 - 3x - 36}{(2x-1)(x^2+9)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+9}$$

$$A = -4, B = 3, C = 0$$

$$\int \frac{2x^2 - 3x - 36}{(2x-1)(x^2+9)} dx = -4 \int \frac{1}{2x-1} dx + \int \frac{3x}{x^2+9} dx = -2 \ln|2x-1| + \frac{3}{2} \ln|x^2+9| + C$$

$$30. \frac{1}{x^4 - 16} = \frac{1}{(x-2)(x+2)(x^2+4)}$$

$$= \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$

$$A = \frac{1}{32}, B = -\frac{1}{32}, C = 0, D = -\frac{1}{8}$$

$$\int \frac{1}{x^4 - 16} dx = \frac{1}{32} \int \frac{1}{x-2} dx - \frac{1}{32} \int \frac{1}{x+2} dx - \frac{1}{8} \int \frac{1}{x^2+4} dx = \frac{1}{32} \ln|x-2| - \frac{1}{32} \ln|x+2| - \frac{1}{16} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$31. \frac{1}{(x-1)^2(x+4)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+4} + \frac{D}{(x+4)^2}$$

$$A = -\frac{2}{125}, B = \frac{1}{25}, C = \frac{2}{125}, D = \frac{1}{25}$$

$$\int \frac{1}{(x-1)^2(x+4)^2} dx = -\frac{2}{125} \int \frac{1}{x-1} dx + \frac{1}{25} \int \frac{1}{(x-1)^2} dx + \frac{2}{125} \int \frac{1}{x+4} dx + \frac{1}{25} \int \frac{1}{(x+4)^2} dx$$

$$= -\frac{2}{125} \ln|x-1| - \frac{1}{25(x-1)} + \frac{2}{125} \ln|x+4| - \frac{1}{25(x+4)} + C$$

$$32. \frac{x^3 - 8x^2 - 1}{(x+3)(x^2 - 4x + 5)} = 1 + \frac{-7x^2 + 7x - 16}{(x+3)(x^2 - 4x + 5)}$$

$$\frac{-7x^2 + 7x - 16}{(x+3)(x^2 - 4x + 5)} = \frac{A}{x+3} + \frac{Bx+C}{x^2 - 4x + 5}$$

$$A = -\frac{50}{13}, B = -\frac{41}{13}, C = \frac{14}{13}$$

$$\int \frac{x^3 - 8x^2 - 1}{(x+3)(x^2 - 4x + 5)} dx = \int \left[1 - \frac{50}{13} \left(\frac{1}{x+3} \right) + \frac{-\frac{41}{13}x + \frac{14}{13}}{x^2 - 4x + 5} \right] dx$$

$$= \int dx - \frac{50}{13} \int \frac{1}{x+3} dx - \frac{68}{13} \int \frac{1}{(x-2)^2 + 1} dx - \frac{41}{26} \int \frac{2x-4}{x^2 - 4x + 5} dx$$

$$= x - \frac{50}{13} \ln|x+3| - \frac{68}{13} \tan^{-1}(x-2) - \frac{41}{26} \ln|x^2 - 4x + 5| + C$$

$$33. x = \sin t, dx = \cos t dt$$

$$\int \frac{(\sin^3 t - 8\sin^2 t - 1)\cos t}{(\sin t + 3)(\sin^2 t - 4\sin t + 5)} dt = \int \frac{x^3 - 8x^2 - 1}{(x+3)(x^2 - 4x + 5)} dx$$

$$= x - \frac{50}{13} \ln|x+3| - \frac{68}{13} \tan^{-1}(x-2) - \frac{41}{26} \ln|x^2 - 4x + 5| + C$$

which is the result of Problem 32.

$$\int \frac{(\sin^3 t - 8\sin^2 t - 1)\cos t}{(\sin t + 3)(\sin^2 t - 4\sin t + 5)} dt = \sin t - \frac{50}{13} \ln|\sin t + 3| - \frac{68}{13} \tan^{-1}(\sin t - 2) - \frac{41}{26} \ln|\sin^2 t - 4\sin t + 5| + C$$

34. $x = \sin t, dx = \cos t dt$

$$\int \frac{\cos t}{\sin^4 t - 16} dt = \int \frac{1}{x^4 - 16} dx = \frac{1}{32} \ln|x-2| - \frac{1}{32} \ln|x+2| - \frac{1}{16} \tan^{-1}\left(\frac{x}{2}\right) + C$$

which is the result of Problem 30.

$$\int \frac{\cos t}{\sin^4 t - 16} dt = \frac{1}{32} \ln|\sin t - 2| - \frac{1}{32} \ln|\sin t + 2| - \frac{1}{16} \tan^{-1}\left(\frac{\sin t}{2}\right) + C$$

35. $\frac{x^3 - 4x}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$

$A = 1, B = 0, C = -5, D = 0$

$$\int \frac{x^3 - 4x}{(x^2 + 1)^2} dx = \int \frac{x}{x^2 + 1} dx - 5 \int \frac{x}{(x^2 + 1)^2} dx = \frac{1}{2} \ln|x^2 + 1| + \frac{5}{2(x^2 + 1)} + C$$

36. $x = \cos t, dx = -\sin t dt$

$$\int \frac{(\sin t)(4\cos^2 t - 1)}{(\cos t)(1 + 2\cos^2 t + \cos^4 t)} dt = - \int \frac{4x^2 - 1}{x(1 + 2x^2 + x^4)} dx$$

$$\frac{4x^2 - 1}{x(1 + 2x^2 + x^4)} = \frac{4x^2 - 1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

$A = -1, B = 1, C = 0, D = 5, E = 0$

$$- \int \left[-\frac{1}{x} + \frac{x}{x^2 + 1} + \frac{5x}{(x^2 + 1)^2} \right] dx = \ln|x| - \frac{1}{2} \ln|x^2 + 1| + \frac{5}{2(x^2 + 1)} + C = \ln|\cos t| - \frac{1}{2} \ln|\cos^2 t + 1| + \frac{5}{2(\cos^2 t + 1)} + C$$

37. $\frac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x} = \frac{x(2x^2 + 5x + 16)}{x(x^4 + 8x^2 + 16)} = \frac{2x^2 + 5x + 16}{(x^2 + 4)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2}$

$A = 0, B = 2, C = 5, D = 8$

$$\int \frac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x} dx = \int \frac{2}{x^2 + 4} dx + \int \frac{5x + 8}{(x^2 + 4)^2} dx = \int \frac{2}{x^2 + 4} dx + \int \frac{5x}{(x^2 + 4)^2} dx + \int \frac{8}{(x^2 + 4)^2} dx$$

To integrate $\int \frac{8}{(x^2 + 4)^2} dx$, let $x = 2 \tan \theta, dx = 2 \sec^2 \theta d\theta$.

$$\int \frac{8}{(x^2 + 4)^2} dx = \int \frac{16 \sec^2 \theta}{16 \sec^4 \theta} d\theta = \int \cos^2 \theta d\theta = \int \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta$$

$$= \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta + C = \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C = \frac{1}{2} \tan^{-1} \frac{x}{2} + \frac{x}{x^2 + 4} + C$$

$$\int \frac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x} dx = \tan^{-1} \frac{x}{2} - \frac{5}{2(x^2 + 4)} + \frac{1}{2} \tan^{-1} \frac{x}{2} + \frac{x}{x^2 + 4} + C = \frac{3}{2} \tan^{-1} \frac{x}{2} + \frac{2x - 5}{2(x^2 + 4)} + C$$

38. $\frac{x - 17}{x^2 + x - 12} = \frac{x - 17}{(x + 4)(x - 3)} = \frac{A}{x + 4} + \frac{B}{x - 3}$

$A = 3, B = -2$

$$\int_4^6 \frac{x - 17}{x^2 + x - 12} dx = \int_4^6 \left(\frac{3}{x + 4} - \frac{2}{x - 3} \right) dx = [3 \ln|x + 4| - 2 \ln|x - 3|]_4^6 = (3 \ln 10 - 2 \ln 3) - (3 \ln 8 - 2 \ln 1)$$

$= 3 \ln 10 - 2 \ln 3 - 3 \ln 8 \approx -1.53$

39. $u = \sin \theta, du = \cos \theta d\theta$

$$\int_0^{\pi/4} \frac{\cos \theta}{(1 - \sin^2 \theta)(\sin^2 \theta + 1)^2} d\theta = \int_0^{1/\sqrt{2}} \frac{1}{(1 - u^2)(u^2 + 1)^2} du = \int_0^{1/\sqrt{2}} \frac{1}{(1 - u)(1 + u)(u^2 + 1)^2} du$$

$$\frac{1}{(1-u^2)(u^2+1)^2} = \frac{A}{1-u} + \frac{B}{1+u} + \frac{Cu+D}{u^2+1} + \frac{Eu+F}{(u^2+1)^2}$$

$$A = \frac{1}{8}, B = \frac{1}{8}, C = 0, D = \frac{1}{4}, E = 0, F = \frac{1}{2}$$

$$\begin{aligned} \int_0^{1/\sqrt{2}} \frac{1}{(1-u^2)(u^2+1)^2} du &= \frac{1}{8} \int_0^{1/\sqrt{2}} \frac{1}{1-u} du + \frac{1}{8} \int_0^{1/\sqrt{2}} \frac{1}{1+u} du + \frac{1}{4} \int_0^{1/\sqrt{2}} \frac{1}{u^2+1} du + \frac{1}{2} \int_0^{1/\sqrt{2}} \frac{1}{(u^2+1)^2} du \\ &= \left[-\frac{1}{8} \ln|1-u| + \frac{1}{8} \ln|1+u| + \frac{1}{4} \tan^{-1} u + \frac{1}{4} \left(\tan^{-1} u + \frac{u}{u^2+1} \right) \right]_0^{1/\sqrt{2}} \\ &= \left[\frac{1}{8} \ln \left| \frac{1+u}{1-u} \right| + \frac{1}{2} \tan^{-1} u + \frac{u}{4(u^2+1)} \right]_0^{1/\sqrt{2}} \\ &= \frac{1}{8} \ln \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right| + \frac{1}{2} \tan^{-1} \frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}} \approx 0.65 \end{aligned}$$

(To integrate $\int \frac{1}{(u^2+1)^2} du$, let $u = \tan t$.)

40. $\frac{3x+13}{x^2+4x+3} = \frac{3x+13}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$
 $A = -2, B = 5$
 $\int_1^5 \frac{3x+13}{x^2+4x+3} dx = [-2 \ln|x+3| + 5 \ln|x+1|]_1^5$
 $= -2 \ln 8 + 5 \ln 6 + 2 \ln 4 - 5 \ln 2 \approx 4.11$

41. a. Separating variables, we obtain

$$\frac{dx}{(a-x)(b-x)} = k dt$$

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$$

$$A = -\frac{1}{a-b}, B = \frac{1}{a-b}$$

$$\int \frac{dx}{(a-x)(b-x)}$$

$$= \frac{1}{a-b} \int \left(-\frac{1}{a-x} + \frac{1}{b-x} \right) dx = \int k dt$$

$$\frac{\ln|a-x| - \ln|b-x|}{a-b} = kt + C$$

$$\frac{1}{a-b} \ln \left| \frac{a-x}{b-x} \right| = kt + C$$

$$\frac{a-x}{b-x} = C e^{(a-b)kt}$$

Since $x = 0$ when $t = 0$, $C = \frac{a}{b}$, so

$$a-x = (b-x) \frac{a}{b} e^{(a-b)kt}$$

$$a(1 - e^{(a-b)kt}) = x \left(1 - \frac{a}{b} e^{(a-b)kt} \right)$$

$$x(t) = \frac{a(1 - e^{(a-b)kt})}{1 - \frac{a}{b} e^{(a-b)kt}} = \frac{ab(1 - e^{(a-b)kt})}{b - a e^{(a-b)kt}}$$

b. Since $b > a$ and $k > 0$, $e^{(a-b)kt} \rightarrow 0$ as $t \rightarrow \infty$. Thus,

$$x \rightarrow \frac{ab(1)}{b-0} = a.$$

c. $x(t) = \frac{8(1 - e^{-2kt})}{4 - 2e^{-2kt}}$

$$x(20) = 1, \text{ so } 4 - 2e^{-40k} = 8 - 8e^{-40k}$$

$$6e^{-40k} = 4$$

$$k = -\frac{1}{40} \ln \frac{2}{3}$$

$$e^{-2kt} = e^{t/20 \ln 2/3} = e^{\ln(2/3)^{t/20}} = \left(\frac{2}{3} \right)^{t/20}$$

$$x(t) = \frac{4 \left(1 - \left(\frac{2}{3} \right)^{t/20} \right)}{2 - \left(\frac{2}{3} \right)^{t/20}}$$

$$x(60) = \frac{4 \left(1 - \left(\frac{2}{3} \right)^3 \right)}{2 - \left(\frac{2}{3} \right)^3} = \frac{38}{23} \approx 1.65 \text{ grams}$$

d. If $a = b$, the differential equation is, after separating variables

$$\frac{dx}{(a-x)^2} = k dt$$

$$\int \frac{dx}{(a-x)^2} = \int k dt$$

$$\frac{1}{a-x} = kt + C$$

$$\frac{1}{kt+C} = a-x$$

$$x(t) = a - \frac{1}{kt+C}$$

Since $x = 0$ when $t = 0$, $C = \frac{1}{a}$, so

$$\begin{aligned} x(t) &= a - \frac{1}{kt + \frac{1}{a}} = a - \frac{a}{akt + 1} \\ &= a \left(1 - \frac{1}{akt + 1} \right) = a \left(\frac{akt}{akt + 1} \right). \end{aligned}$$

42. a. $\frac{dy}{dt} = ky(16 - y)$

$$\frac{dy}{y(16 - y)} = kdt$$

$$\int \frac{dy}{y(16 - y)} = \int kdt$$

$$\frac{1}{16} \int \left(\frac{1}{y} + \frac{1}{16 - y} \right) dy = kt + C$$

$$\frac{1}{16} (\ln|y| - \ln|16 - y|) = kt + C$$

$$\ln \left| \frac{y}{16 - y} \right| = 16kt + C$$

$$\frac{y}{16 - y} = Ce^{16kt}$$

$$y(0) = 2: \frac{1}{7} = C; \frac{y}{16 - y} = \frac{1}{7} e^{16kt}$$

$$y(50) = 4: \frac{1}{3} = \frac{1}{7} e^{800k}, \text{ so } k = \frac{1}{800} \ln \frac{7}{3}$$

$$\frac{y}{16 - y} = \frac{1}{7} e^{\left(\frac{1}{50} \ln \frac{7}{3}\right)t}$$

$$7y = 16e^{\left(\frac{1}{50} \ln \frac{7}{3}\right)t} - ye^{\left(\frac{1}{50} \ln \frac{7}{3}\right)t}$$

$$y = \frac{16e^{\left(\frac{1}{50} \ln \frac{7}{3}\right)t}}{7 + e^{\left(\frac{1}{50} \ln \frac{7}{3}\right)t}} = \frac{16}{1 + 7e^{-\left(\frac{1}{50} \ln \frac{7}{3}\right)t}}$$

b. $y(90) = \frac{16}{1 + 7e^{-\left(\frac{1}{50} \ln \frac{7}{3}\right)90}} \approx 6.34 \text{ billion}$

c. $9 = \frac{16}{1 + 7e^{-\left(\frac{1}{50} \ln \frac{7}{3}\right)t}}$

$$7e^{-\left(\frac{1}{50} \ln \frac{7}{3}\right)t} = \frac{16}{9} - 1$$

$$e^{-\left(\frac{1}{50} \ln \frac{7}{3}\right)t} = \frac{1}{9}$$

$$-\left(\frac{1}{50} \ln \frac{7}{3}\right)t = \ln \frac{1}{9}$$

$$t = -50 \left(\frac{\ln \frac{1}{9}}{\ln \frac{7}{3}} \right) \approx 129.66$$

The population will be 9 billion in 2055.

43. a. $\frac{dy}{dt} = ky(10 - y)$

$$\frac{dy}{y(10 - y)} = kdt$$

$$\frac{1}{10} \int \left(\frac{1}{y} + \frac{1}{10 - y} \right) dy = \int kdt$$

$$\ln \left| \frac{y}{10 - y} \right| = 10kt + C$$

$$\frac{y}{10 - y} = Ce^{10kt}$$

$$y(0) = 2: \frac{1}{4} = C; \frac{y}{10 - y} = \frac{1}{4} e^{10kt}$$

$$y(50) = 4: \frac{2}{3} = \frac{1}{4} e^{500k}, k = \frac{1}{500} \ln \frac{8}{3}$$

$$\frac{y}{10 - y} = \frac{1}{4} e^{\left(\frac{1}{50} \ln \frac{8}{3}\right)t}$$

$$4y = 10e^{\left(\frac{1}{50} \ln \frac{8}{3}\right)t} - ye^{\left(\frac{1}{50} \ln \frac{8}{3}\right)t}$$

$$y = \frac{10e^{\left(\frac{1}{50} \ln \frac{8}{3}\right)t}}{4 + e^{\left(\frac{1}{50} \ln \frac{8}{3}\right)t}} = \frac{10}{1 + 4e^{-\left(\frac{1}{50} \ln \frac{8}{3}\right)t}}$$

b. $y(90) = \frac{10}{1 + 4e^{-\left(\frac{1}{50} \ln \frac{8}{3}\right)90}} \approx 5.94 \text{ billion}$

c. $9 = \frac{10}{1 + 4e^{-\left(\frac{1}{50} \ln \frac{8}{3}\right)t}}$

$$4e^{-\left(\frac{1}{50} \ln \frac{8}{3}\right)t} = \frac{10}{9} - 1$$

$$e^{-\left(\frac{1}{50} \ln \frac{8}{3}\right)t} = \frac{1}{36}$$

$$-\left(\frac{1}{50} \ln \frac{8}{3}\right)t = \ln \frac{1}{36}$$

$$t = -50 \left(\frac{\ln \frac{1}{36}}{\ln \frac{8}{3}} \right) \approx 182.68$$

The population will be 9 billion in 2108.

44. Separating variables, we obtain

$$\frac{dy}{(y - m)(M - y)} = k dt.$$

$$\begin{aligned}\frac{1}{(y-m)(M-y)} &= \frac{A}{y-m} + \frac{B}{M-y} \\ A &= \frac{1}{M-m}, B = \frac{1}{M-m} \\ \int \frac{dy}{(y-m)(M-y)} &= \frac{1}{M-m} \int \left(\frac{1}{y-m} + \frac{1}{M-y} \right) dy \\ &= \int k dt \\ \frac{\ln|y-m| - \ln|M-y|}{M-m} &= kt + C \\ \frac{1}{M-m} \ln \left| \frac{y-m}{M-y} \right| &= kt + C \\ \frac{y-m}{M-y} &= Ce^{(M-m)kt} \\ y-m &= (M-y)Ce^{(M-m)kt} \\ y(1 + Ce^{(M-m)kt}) &= m + MCe^{(M-m)kt} \\ y &= \frac{m + MCe^{(M-m)kt}}{1 + Ce^{(M-m)kt}} = \frac{me^{-(M-m)kt} + MC}{e^{-(M-m)kt} + C} \\ \text{as } t \rightarrow \infty, e^{-(M-m)kt} &\rightarrow 0 \text{ since } M > m. \\ \text{Thus } y &\rightarrow \frac{MC}{C} = M \text{ as } t \rightarrow \infty.\end{aligned}$$

45. Separating variables, we obtain

$$\begin{aligned}\frac{dy}{(A-y)(B+y)} &= k dt \\ \frac{1}{(A-y)(B+y)} &= \frac{C}{A-y} + \frac{D}{B+y} \\ C &= \frac{1}{A+B}, D = \frac{1}{A+B} \\ \int \frac{dy}{(A-y)(B+y)} &= \frac{1}{A+B} \int \left(\frac{1}{A-y} + \frac{1}{B+y} \right) dy\end{aligned}$$

$$\begin{aligned}&= \int k dt \\ \frac{-\ln(A-y) + \ln(B+y)}{A+B} &= kt + C \\ \frac{1}{A+B} \ln \left| \frac{B+y}{A-y} \right| &= kt + C \\ \frac{B+y}{A-y} &= Ce^{(A+B)kt} \\ B+y &= (A-y)Ce^{(A+B)kt} \\ y(1 + Ce^{(A+B)kt}) &= ACe^{(A+B)kt} - B \\ y(t) &= \frac{ACe^{(A+B)kt} - B}{1 + Ce^{(A+B)kt}}\end{aligned}$$

46. $u = \sin x, du = \cos x dx$

$$\begin{aligned}\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(\sin^2 x + 1)^2} dx &= \int_{1/2}^1 \frac{1}{x(x^2 + 1)^2} dx \\ \frac{1}{x(x^2 + 1)^2} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} \\ A = 1, B = -1, C = 0, D = -1, E = 0 \\ \int_{1/2}^1 \frac{1}{x(x^2 + 1)^2} dx &= \int_{1/2}^1 \frac{1}{x} dx - \int_{1/2}^1 \frac{x}{x^2 + 1} dx - \int_{1/2}^1 \frac{x}{(x^2 + 1)^2} dx \\ &= \left[\ln x - \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2(x^2 + 1)} \right]_{1/2}^1 \\ &= 0 - \frac{1}{2} \ln 2 + \frac{1}{4} - \left(\ln \frac{1}{2} - \frac{1}{2} \ln \frac{5}{4} + \frac{2}{5} \right) \approx 0.308\end{aligned}$$

8.6 Chapter Review

Concepts Test

- True: The resulting integrand will be of the form $\sin u$.
- True: The resulting integrand will be of the form $\frac{1}{a^2 + u^2}$.
- False: Try the substitution $u = x^4, du = 4x^3 dx$

- False: Use the substitution $u = x^2 - 3x + 5, du = (2x - 3)dx$.
- True: The resulting integrand will be of the form $\frac{1}{a^2 + u^2}$.
- True: The resulting integrand will be of the form $\frac{1}{\sqrt{a^2 - x^2}}$.
- True: This integral is most easily solved with a partial fraction decomposition.

8. False: This improper fraction should be reduced first, then a partial fraction decomposition can be used.
9. True: Because both exponents are even positive integers, half-angle formulas are used.
10. False: Use the substitution
 $u = 1 + e^x$, $du = e^x dx$
11. False: Use the substitution
 $u = -x^2 - 4x$, $du = (-2x - 4)dx$
12. True: This substitution eliminates the radical.
13. True: Then expand and use the substitution
 $u = \sin x$, $du = \cos x dx$
14. True: The trigonometric substitution
 $x = 3\sin t$ will eliminate the radical.
15. True: Let $u = \ln x$ $dv = x^2 dx$
 $du = \frac{1}{x} dx$ $v = \frac{1}{3} x^3$
16. False: Use a product identity.
17. False: $\frac{x^2}{x^2 - 1} = 1 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$
18. True: $\frac{x^2 + 2}{x(x^2 - 1)} = -\frac{2}{x} + \frac{3}{2(x+1)} + \frac{3}{2(x-1)}$
19. True: $\frac{x^2 + 2}{x(x^2 + 1)} = \frac{2}{x} + \frac{-x}{x^2 + 1}$
20. False: $\frac{x+2}{x^2(x^2 - 1)}$
 $= -\frac{1}{x} - \frac{2}{x^2} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}$
21. False: To complete the square, add $\left(\frac{b}{2a}\right)^2$.
22. False: Polynomials can be factored into products of linear and quadratic polynomials with real coefficients.
23. True: Polynomials with the same values for all x will have identical coefficients for like degree terms.

Sample Test Problems

1. $\int_0^4 \frac{t}{\sqrt{9+t^2}} dt = \left[\sqrt{9+t^2} \right]_0^4 = 5 - 3 = 2$
2. $\int \cot^2(2\theta) d\theta = \int \frac{\cos^2 2\theta}{\sin^2 2\theta} d\theta$
 $= \int \frac{1 - \sin^2 2\theta}{\sin^2 2\theta} d\theta = \int (\csc^2 2\theta - 1) d\theta$
 $= -\frac{1}{2} \cot 2\theta - \theta + C$
3. $\int_0^{\pi/2} e^{\cos x} \sin x dx = \left[-e^{\cos x} \right]_0^{\pi/2} = e - 1 \approx 1.718$
4. $\int_0^{\pi/4} x \sin 2x dx = \left[\frac{\sin 2x}{4} - \frac{x}{2} \cos 2x \right]_0^{\pi/4} = \frac{1}{4}$
 (Use integration by parts with $u = x$,
 $dv = \sin 2x dx$.)
5. $\int \frac{y^3 + y}{y+1} dy = \int \left(y^2 - y + 2 - \frac{2}{1+y} \right) dy$
 $= \frac{1}{3} y^3 - \frac{1}{2} y^2 + 2y - 2 \ln|1+y| + C$
6. $\int \sin^3(2t) dt = \int [1 - \cos^2(2t)] \sin(2t) dt$
 $= -\frac{1}{2} \cos(2t) + \frac{1}{6} \cos^3(2t) + C$
7. $\int \frac{y-2}{y^2-4y+2} dy = \frac{1}{2} \int \frac{2y-4}{y^2-4y+2} dy$
 $= \frac{1}{2} \ln|y^2-4y+2| + C$
8. $\int_0^{3/2} \frac{dy}{\sqrt{2y+1}} = \left[\sqrt{2y+1} \right]_0^{3/2} = 2 - 1 = 1$
9. $\int \frac{e^{2t}}{e^t - 2} dt = e^t + 2 \ln|e^t - 2| + C$
 (Use the substitution $u = e^t - 2$,
 $du = e^t dt$
 which gives the integral $\int \frac{u+2}{u} du$.)
10. $\int \frac{\sin x + \cos x}{\tan x} dx = \int \left(\cos x + \frac{\cos^2 x}{\sin x} \right) dx$
 $= \int \left(\cos x + \frac{1 - \sin^2 x}{\sin x} \right) dx$

$$\begin{aligned}
&= \int (\cos x + \csc x - \sin x) dx \\
&= \sin x + \ln |\csc x - \cot x| + \cos x + C \\
&\text{(Use Formula 15 for } \int \csc x dx \text{.)}
\end{aligned}$$

$$\begin{aligned}
11. \quad \int \frac{dx}{\sqrt{16+4x-2x^2}} &= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x-1}{3} \right) + C \\
&\text{(Complete the square.)}
\end{aligned}$$

$$\begin{aligned}
12. \quad \int x^2 e^x dx &= e^x (2 - 2x + x^2) + C \\
&\text{Use integration by parts twice.}
\end{aligned}$$

$$\begin{aligned}
13. \quad y &= \sqrt{\frac{2}{3}} \tan t, \quad dy = \sqrt{\frac{2}{3}} \sec^2 t \, dt \\
\int \frac{dy}{\sqrt{2+3y^2}} &= \int \frac{\sqrt{\frac{2}{3}} \sec^2 t}{\sqrt{2} \sec t} dt \\
&= \frac{1}{\sqrt{3}} \int \sec t \, dt = \frac{1}{\sqrt{3}} \ln |\sec t + \tan t| + C_1 \\
&= \frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{y^2 + \frac{2}{3}}}{\sqrt{\frac{2}{3}}} + \frac{y}{\sqrt{\frac{2}{3}}} \right| + C_1 \\
&= \frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{y^2 + \frac{2}{3}} + y}{\sqrt{\frac{2}{3}}} \right| + C_1 \\
&= \frac{1}{\sqrt{3}} \ln \left| \sqrt{y^2 + \frac{2}{3}} + y \right| + C \\
&\text{Note that } \tan t = \frac{y}{\sqrt{\frac{2}{3}}}, \text{ so } \sec t = \frac{\sqrt{y^2 + \frac{2}{3}}}{\sqrt{\frac{2}{3}}}.
\end{aligned}$$

$$\begin{aligned}
14. \quad \int \frac{w^3}{1-w^2} dw &= -\frac{1}{2} w^2 - \frac{1}{2} \ln |1-w^2| + C \\
&\text{Divide the numerator by the denominator.}
\end{aligned}$$

$$\begin{aligned}
15. \quad \int \frac{\tan x}{\ln |\cos x|} dx &= -\ln |\ln |\cos x|| + C \\
&\text{Use the substitution } u = \ln |\cos x|.
\end{aligned}$$

$$\begin{aligned}
16. \quad \int \frac{3dt}{t^3-1} &= \int \frac{1}{t-1} dt - \int \frac{t+2}{t^2+t+1} dt \\
&= \int \frac{1}{t-1} dt - \frac{1}{2} \int \frac{2t+4}{t^2+t+1} dt \\
&= \int \frac{1}{t-1} dt - \frac{1}{2} \int \frac{2t+1+3}{t^2+t+1} dt
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{1}{t-1} dt - \frac{1}{2} \int \frac{2t+1}{t^2+t+1} dt - \frac{3}{2} \int \frac{1}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}} dt \\
&= \ln |t-1| - \frac{1}{2} \ln |t^2+t+1| - \sqrt{3} \tan^{-1} \left(\frac{2t+1}{\sqrt{3}} \right) + C
\end{aligned}$$

$$17. \quad \int \sinh x \, dx = \cosh x + C$$

$$\begin{aligned}
18. \quad u &= \ln y, \quad du = \frac{1}{y} dy \\
\int \frac{(\ln y)^5}{y} dy &= \int u^5 du = \frac{1}{6} (\ln y)^6 + C
\end{aligned}$$

$$\begin{aligned}
19. \quad u &= x & dv &= \cot^2 x \, dx \\
du &= dx & v &= -\cot x - x \\
\int x \cot^2 x \, dx &= -x \cot x - x^2 - \int (-\cot x - x) dx \\
&= -x \cot x - \frac{1}{2} x^2 + \ln |\sin x| + C \\
&\text{Use } \cot^2 x = \csc^2 x - 1 \text{ for } \int \cot^2 x \, dx.
\end{aligned}$$

$$\begin{aligned}
20. \quad u &= \sqrt{x}, \quad du = \frac{1}{2} x^{-1/2} dx \\
\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx &= 2 \int \sin u \, du \\
&= -2 \cos \sqrt{x} + C
\end{aligned}$$

$$\begin{aligned}
21. \quad u &= \ln t^2, \quad du = \frac{2}{t} dt \\
\int \frac{\ln t^2}{t} dt &= \frac{[\ln(t^2)]^2}{4} + C
\end{aligned}$$

$$\begin{aligned}
22. \quad u &= \ln(y^2+9) & dv &= dy \\
du &= \frac{2y}{y^2+9} dy & v &= y \\
\int \ln(y^2+9) dy &= y \ln(y^2+9) - \int \frac{2y^2}{y^2+9} dy \\
&= y \ln(y^2+9) - \int \left(2 - \frac{18}{y^2+9} \right) dy \\
&= y \ln(y^2+9) - 2y + 6 \tan^{-1} \left(\frac{y}{3} \right) + C
\end{aligned}$$

$$\begin{aligned}
23. \quad \int e^{t/3} \sin 3t \, dt &= \frac{-3e^{t/3} (9 \cos 3t - \sin 3t)}{82} + C \\
&\text{Use integration by parts twice.}
\end{aligned}$$

$$\begin{aligned}
 24. \quad \int \frac{t+9}{t^3+9t} dt &= \int \frac{1}{t} dt + \int \frac{-t+1}{t^2+9} dt \\
 &= \int \frac{1}{t} dt - \int \frac{t}{t^2+9} dt + \int \frac{1}{t^2+9} dt \\
 &= \ln|t| - \frac{1}{2} \ln|t^2+9| + \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) + C
 \end{aligned}$$

$$25. \quad \int \sin \frac{3x}{2} \cos \frac{x}{2} dx = -\frac{\cos x}{2} - \frac{\cos 2x}{4} + C$$

Use a product identity.

$$\begin{aligned}
 26. \quad \int \cos^4\left(\frac{x}{2}\right) dx &= \int \left(\frac{1+\cos x}{2}\right)^2 dx \\
 &= \frac{1}{4} \int dx + \frac{1}{4} \int 2 \cos x dx + \frac{1}{4} \int \cos^2 x dx \\
 &= \frac{1}{4} \int dx + \frac{1}{2} \int \cos x dx + \frac{1}{8} \int (1+\cos 2x) dx \\
 &= \frac{3}{8} x + \frac{1}{2} \sin x + \frac{1}{16} \sin 2x + C
 \end{aligned}$$

$$\begin{aligned}
 29. \quad \int \tan^{3/2} x \sec^4 x dx &= \int \tan^{3/2} x (1+\tan^2 x) \sec^2 x dx = \int \tan^{3/2} x \sec^2 x dx + \int \tan^{7/2} x \sec^2 x dx \\
 &= \frac{2}{5} \tan^{5/2} x + \frac{2}{9} \tan^{9/2} x + C
 \end{aligned}$$

$$\begin{aligned}
 30. \quad u &= t^{1/6} + 1, (u-1)^6 = t, 6(u-1)^5 du = dt \\
 \int \frac{dt}{t(t^{1/6}+1)} &= \int \frac{6(u-1)^5 du}{(u-1)^6 u} = \int \frac{6 du}{u(u-1)} = -6 \int \frac{1}{u} du + 6 \int \frac{1}{u-1} du = -6 \ln|t^{1/6}+1| + 6 \ln|t^{1/6}| + C
 \end{aligned}$$

$$\begin{aligned}
 31. \quad u &= 9 - e^{2y}, du = -2e^{2y} dy \\
 \int \frac{e^{2y}}{\sqrt{9-e^{2y}}} dy &= -\frac{1}{2} \int u^{-1/2} du = -\sqrt{u} + C = -\sqrt{9-e^{2y}} + C
 \end{aligned}$$

$$\begin{aligned}
 32. \quad \int \cos^5 x \sqrt{\sin x} dx &= \int (1-\sin^2 x)^2 (\sin^{1/2} x) \cos x dx = \int \sin^{1/2} x \cos x dx - 2 \int \sin^{5/2} x \cos x dx + \int \sin^{9/2} x \cos x dx \\
 &= \frac{2}{3} \sin^{3/2} x - \frac{4}{7} \sin^{7/2} x + \frac{2}{11} \sin^{11/2} x + C
 \end{aligned}$$

$$33. \quad \int e^{\ln(3 \cos x)} dx = \int 3 \cos x dx = 3 \sin x + C$$

$$\begin{aligned}
 34. \quad y &= 3 \sin t, dy = 3 \cos t dt \\
 \int \frac{\sqrt{9-y^2}}{y} dy &= \int \frac{3 \cos t}{3 \sin t} \cdot 3 \cos t dt \\
 &= 3 \int \frac{1-\sin^2 t}{\sin t} dt = 3 \int (\csc t - \sin t) dt \\
 &= 3 [\ln|\csc t - \cot t| + \cos t] + C
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \int \tan^3 2x \sec 2x dx &= \frac{1}{2} \int (\sec^2 2x - 1) d(\sec 2x) \\
 &= \frac{1}{6} \sec^3(2x) - \frac{1}{2} \sec(2x) + C
 \end{aligned}$$

$$\begin{aligned}
 28. \quad u &= \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx \\
 \int \frac{\sqrt{x}}{1+\sqrt{x}} dx &= \int \frac{2x}{1+\sqrt{x}} \left(\frac{1}{2\sqrt{x}} dx \right) = 2 \int \frac{u^2}{1+u} du \\
 &= 2 \int \frac{(u+1)(u-1)+1}{u+1} du = 2 \int \left(u-1 + \frac{1}{u+1} \right) du \\
 &= 2 \left(\frac{u^2}{2} - u + \ln|u+1| \right) + C \\
 &= x - 2\sqrt{x} + 2 \ln|1+\sqrt{x}| + C
 \end{aligned}$$

$$= 3 \ln \left| \frac{3}{y} - \frac{\sqrt{9-y^2}}{y} \right| + \sqrt{9-y^2} + C$$

Note that $\sin t = \frac{y}{3}$, so $\csc t = \frac{3}{y}$ and

$$\cot t = \frac{\sqrt{9-y^2}}{y}.$$

$$35. u = e^{4x}, du = 4e^{4x} dx$$

$$\int \frac{e^{4x}}{1+e^{8x}} dx = \frac{1}{4} \int \frac{du}{1+u^2} = \frac{1}{4} \tan^{-1}(e^{4x}) + C$$

$$36. x = a \tan t, dx = a \sec^2 t dt$$

$$\int \frac{\sqrt{x^2 + a^2}}{x^4} dx = \int \frac{a \sec t}{a^4 \tan^4 t} a \sec^2 t dt$$

$$= \frac{1}{a^2} \int \frac{\sec^3 t}{\tan^4 t} dt = \frac{1}{a^2} \int \frac{\cos t}{\sin^4 t} dt$$

$$= \frac{1}{a^2} \left(-\frac{1}{3} \frac{1}{\sin^3 t} \right) + C = -\frac{1}{3a^2} \csc^3 t + C$$

$$= -\frac{1}{3a^2} \frac{(x^2 + a^2)^{3/2}}{x^3} + C$$

Note that $\tan t = \frac{x}{a}$, so $\csc t = \frac{\sqrt{x^2 + a^2}}{a}$.

$$37. u = \sqrt{w+5}, u^2 = w+5, 2u du = dw$$

$$\int \frac{w}{\sqrt{w+5}} dw = 2 \int (u^2 - 5) du = \frac{2}{3} u^3 - 10u + C$$

$$= \frac{2}{3} (w+5)^{3/2} - 10(w+5)^{1/2} + C$$

$$38. u = 1 + \cos t, du = -\sin t dt$$

$$\int \frac{\sin t dt}{\sqrt{1 + \cos t}} = -\int \frac{du}{\sqrt{u}} = -2\sqrt{1 + \cos t} + C$$

$$42. x = 4 \tan t, dx = 4 \sec^2 t dt$$

$$\int \frac{dx}{(16 + x^2)^{3/2}} = \frac{1}{16} \int \cos t dt = \frac{1}{16} \sin t + C = \frac{1}{16} \left(\frac{x}{\sqrt{x^2 + 16}} \right) + C = \frac{x}{16\sqrt{x^2 + 16}} + C$$

$$43. a. \frac{3 - 4x^2}{(2x+1)^3} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3}$$

$$b. \frac{7x-41}{(x-1)^2(2-x)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2-x} + \frac{D}{(2-x)^2} + \frac{E}{(2-x)^3}$$

$$c. \frac{3x+1}{(x^2+x+10)^2} = \frac{Ax+B}{x^2+x+10} + \frac{Cx+D}{(x^2+x+10)^2}$$

$$d. \frac{(x+1)^2}{(x^2-x+10)^2(1-x^2)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2} + \frac{Ex+F}{x^2-x+10} + \frac{Gx+H}{(x^2-x+10)^2}$$

$$e. \frac{x^5}{(x+3)^4(x^2+2x+10)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} + \frac{Ex+F}{x^2+2x+10} + \frac{Gx+H}{(x^2+2x+10)^2}$$

$$39. u = \cos^2 y, du = -2 \cos y \sin y dy$$

$$\int \frac{\sin y \cos y}{9 + \cos^4 y} dy = -\frac{1}{2} \int \frac{du}{9 + u^2}$$

$$= -\frac{1}{6} \tan^{-1} \left(\frac{\cos^2 y}{3} \right) + C$$

$$40. \int \frac{dx}{\sqrt{1-6x-x^2}} = \int \frac{dx}{\sqrt{10-(x+3)^2}}$$

$$= \sin^{-1} \left(\frac{x+3}{\sqrt{10}} \right) + C$$

$$41. \frac{4x^2 + 3x + 6}{x^2(x^2 + 3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 3}$$

$$A = 1, B = 2, C = -1, D = 2$$

$$\int \frac{4x^2 + 3x + 6}{x^2(x^2 + 3)} dx = \int \frac{1}{x} dx + 2 \int \frac{1}{x^2} dx + \int \frac{-x + 2}{x^2 + 3} dx$$

$$= \int \frac{1}{x} dx + 2 \int \frac{1}{x^2} dx - \frac{1}{2} \int \frac{2x}{x^2 + 3} dx + 2 \int \frac{1}{x^2 + 3} dx$$

$$= \ln|x| - \frac{2}{x} - \frac{1}{2} \ln|x^2 + 3| + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + C$$

$$f. \frac{(3x^2 + 2x - 1)^2}{(2x^2 + x + 10)^3} = \frac{Ax + B}{2x^2 + x + 10} + \frac{Cx + D}{(2x^2 + x + 10)^2} + \frac{Ex + F}{(2x^2 + x + 10)^3}$$

$$44. a. V = \pi \int_1^2 \left[\frac{1}{\sqrt{3x - x^2}} \right]^2 dx = \pi \int_1^2 \frac{1}{3x - x^2} dx$$

$$\frac{1}{3x - x^2} = \frac{A}{x} + \frac{B}{3 - x}$$

$$A = \frac{1}{3}, B = \frac{1}{3}$$

$$V = \pi \int_1^2 \frac{1}{3} \left(\frac{1}{x} + \frac{1}{3 - x} \right) dx = \frac{\pi}{3} [\ln|x| - \ln|3 - x|]_1^2 = \frac{\pi}{3} (\ln 2 + \ln 2) = \frac{2\pi}{3} \ln 2 \approx 1.4517$$

$$b. V = 2\pi \int_1^2 \frac{x}{\sqrt{3x - x^2}} dx = -\pi \int_1^2 \frac{-2x + 3 - 3}{\sqrt{3x - x^2}} dx = -\pi \int_1^2 \frac{3 - 2x}{\sqrt{3x - x^2}} dx + 3\pi \int_1^2 \frac{1}{\sqrt{3x - x^2}} dx$$

$$= -\pi \left[2\sqrt{3x - x^2} \right]_1^2 + 3\pi \int_1^2 \frac{1}{\sqrt{\frac{9}{4} - \left(x - \frac{3}{2}\right)^2}} dx = \left[-2\pi\sqrt{3x - x^2} + 3\pi \sin^{-1} \left(\frac{2x - 3}{3} \right) \right]_1^2$$

$$= -2\pi\sqrt{2} + 3\pi \sin^{-1} \frac{1}{3} + 2\pi\sqrt{2} - 3\pi \sin^{-1} \left(-\frac{1}{3} \right) = 6\pi \sin^{-1} \frac{1}{3} \approx 6.4058$$

$$45. y = \frac{x^2}{16}, y' = \frac{x}{8}$$

$$L = \int_0^4 \sqrt{1 + \left(\frac{x}{8}\right)^2} dx = \int_0^4 \sqrt{1 + \frac{x^2}{64}} dx$$

$$x = 8 \tan t, dx = 8 \sec^2 t$$

$$L = \int_0^{\tan^{-1} \frac{1}{2}} \sec t \cdot 8 \sec^2 t dt = 8 \int_0^{\tan^{-1} \frac{1}{2}} \sec^3 t dt = 4 \left[\sec t \tan t + \ln |\sec t + \tan t| \right]_0^{\tan^{-1} \frac{1}{2}}$$

$$= 4 \left[\left(\frac{\sqrt{5}}{2} \right) \left(\frac{1}{2} \right) + \ln \left| \frac{1}{2} + \frac{\sqrt{5}}{2} \right| \right] = \sqrt{5} + 4 \ln \left(\frac{1 + \sqrt{5}}{2} \right) \approx 4.1609$$

Use Formula 28 for $\int \sec^3 t dt$.

$$46. V = \pi \int_0^3 \frac{1}{(x^2 + 5x + 6)^2} dx = \pi \int_0^3 \frac{1}{(x+3)^2(x+2)^2} dx$$

$$\frac{1}{(x+3)^2(x+2)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$$

$$A = 2, B = 1, C = -2, D = 1$$

$$V = \pi \int_0^3 \left[\frac{2}{x+3} + \frac{1}{(x+3)^2} - \frac{2}{x+2} + \frac{1}{(x+2)^2} \right] dx = \pi \left[2 \ln|x+3| - \frac{1}{x+3} - 2 \ln|x+2| - \frac{1}{x+2} \right]_0^3$$

$$= \pi \left[\left(2 \ln 6 - \frac{1}{6} - 2 \ln 5 - \frac{1}{5} \right) - \left(2 \ln 3 - \frac{1}{3} - 2 \ln 2 - \frac{1}{2} \right) \right] = \pi \left(\frac{7}{15} + 2 \ln \frac{4}{5} \right) \approx 0.06402$$

$$47. V = 2\pi \int_0^3 \frac{x}{x^2 + 5x + 6} dx$$

$$\frac{x}{x^2 + 5x + 6} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$A = -2, B = 3$$

$$V = 2\pi \int_0^3 \left[-\frac{2}{x+2} + \frac{3}{x+3} \right] dx = 2\pi \left[-2 \ln(x+2) + 3 \ln(x+3) \right]_0^3$$

$$= 2\pi [(-2 \ln 5 + 3 \ln 6) - (-2 \ln 2 + 3 \ln 3)] = 2\pi \left(3 \ln 2 + 2 \ln \frac{2}{5} \right) = 2\pi \ln \frac{32}{25} \approx 1.5511$$

$$48. V = 2\pi \int_0^2 4x^2 \sqrt{2-x} dx$$

$$u = 2 - x \quad du = -dx$$

$$x = 2 - u \quad dx = -du$$

$$V = 2\pi \int_2^0 4(2-u)^2 \sqrt{u} (-du) = 8\pi \int_0^2 (4u^{1/2} - 4u^{3/2} + u^{5/2}) du = 8\pi \left[\frac{8}{3} u^{3/2} - \frac{8}{5} u^{5/2} + \frac{2}{7} u^{7/2} \right]_0^2$$

$$= 8\pi \left(\frac{16\sqrt{2}}{3} - \frac{32\sqrt{2}}{5} + \frac{16\sqrt{2}}{7} \right) = 8\pi \left(\frac{128\sqrt{2}}{105} \right) = \frac{1024\sqrt{2}\pi}{105} \approx 43.3287$$

$$49. V = 2\pi \int_0^{\ln 3} 2(e^x - 1)(\ln 3 - x) dx = 4\pi \int_0^{\ln 3} [(\ln 3)e^x - xe^x - \ln 3 + x] dx$$

Note that $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$ by using integration by parts.

$$V = 4\pi \left[(\ln 3)e^x - xe^x + e^x - (\ln 3)x + \frac{1}{2}x^2 \right]_0^{\ln 3} = 4\pi \left[\left(3 \ln 3 - 3 \ln 3 + 3 - (\ln 3)^2 + \frac{1}{2}(\ln 3)^2 \right) - (\ln 3 + 1) \right]$$

$$= 4\pi \left[2 - \ln 3 - \frac{1}{2}(\ln 3)^2 \right] \approx 3.7437$$

$$50. A = \int_{\sqrt{3}}^{3\sqrt{3}} \frac{18}{x^2 \sqrt{x^2 + 9}} dx$$

$$x = 3 \tan t, \quad dx = 3 \sec^2 t \, dt$$

$$A = \int_{\pi/6}^{\pi/3} \frac{18}{27 \tan^2 t \sec t} 3 \sec^2 t \, dt = 2 \int_{\pi/6}^{\pi/3} \frac{\cos t}{\sin^2 t} dt = 2 \left[-\frac{1}{\sin t} \right]_{\pi/6}^{\pi/3} = 2 \left(-\frac{2}{\sqrt{3}} + 2 \right) = 4 \left(1 - \frac{1}{\sqrt{3}} \right) \approx 1.6906$$

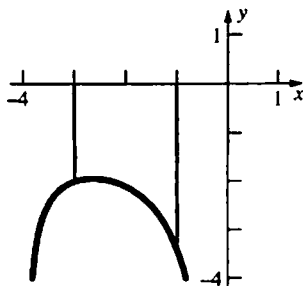
$$51. A = - \int_{-6}^0 \frac{t}{(t-1)^2} dt$$

$$\frac{t}{(t-1)^2} = \frac{A}{(t-1)} + \frac{B}{(t-1)^2}$$

$$A = 1, B = 1$$

$$A = - \int_{-6}^0 \left[\frac{1}{t-1} + \frac{1}{(t-1)^2} \right] dt = - \left[\ln|t-1| - \frac{1}{t-1} \right]_{-6}^0 = - \left[(0+1) - \left(\ln 7 + \frac{1}{7} \right) \right] = \ln 7 - \frac{6}{7} \approx 1.0888$$

52.



$$V = \pi \int_{-3}^1 \left(\frac{6}{x\sqrt{x+4}} \right)^2 dx = \pi \int_{-3}^1 \frac{36}{x^2(x+4)} dx$$

$$\frac{36}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$$

$$A = -\frac{9}{4}, B = 9, C = \frac{9}{4}$$

$$\begin{aligned} V &= \pi \int_{-3}^1 \left[-\frac{9}{4x} + \frac{9}{x^2} + \frac{9}{4(x+4)} \right] dx = \frac{9\pi}{4} \int_{-3}^1 \left(-\frac{1}{x} + \frac{4}{x^2} + \frac{1}{x+4} \right) dx = \frac{9\pi}{4} \left[-\ln|x| - \frac{4}{x} + \ln|x+4| \right]_{-3}^1 \\ &= \frac{9\pi}{4} \left[(4 + \ln 3) - \left(-\ln 3 + \frac{4}{3} \right) \right] = \frac{9\pi}{4} \left(\frac{8}{3} + 2\ln 3 \right) = \frac{3\pi}{2} (4 + 3\ln 3) \approx 34.3808 \end{aligned}$$

53. The length is given by

$$\begin{aligned} \int_{\pi/6}^{\pi/3} \sqrt{1 + [f'(x)]^2} dx &= \int_{\pi/6}^{\pi/3} \sqrt{1 + \frac{\cos^2 x}{\sin^2 x}} dx = \int_{\pi/6}^{\pi/3} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\pi/6}^{\pi/3} \frac{1}{\sin x} dx = \int_{\pi/6}^{\pi/3} \csc x dx \\ &= \left[\ln |\csc x - \cot x| \right]_{\pi/6}^{\pi/3} = \ln \left| \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right| - \ln |2 - \sqrt{3}| = \ln \left(\frac{1}{\sqrt{3}} \right) - \ln(2 - \sqrt{3}) = \ln \left(\frac{2\sqrt{3} + 3}{3} \right) \approx 0.768 \end{aligned}$$

9.1 Concepts Review

- $\lim_{x \rightarrow a} f(x); \lim_{x \rightarrow a} g(x)$
- $\frac{f'(x)}{g'(x)}$
- $\sec^2 x; 1; \lim_{x \rightarrow 0} \cos x \neq 0$
- Cauchy's Mean Value

Problem Set 9.1

- The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{2x - \sin x}{x} = \lim_{x \rightarrow 0} \frac{2 - \cos x}{1} = 1$$
- The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow \pi/2} \frac{\cos x}{\pi/2 - x} = \lim_{x \rightarrow \pi/2} \frac{-\sin x}{-1} = 1$$
- The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{x - \sin 2x}{\tan x} = \lim_{x \rightarrow 0} \frac{1 - 2 \cos 2x}{\sec^2 x} = \frac{1 - 2}{1} = -1$$
- The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{\tan^{-1} 3x}{\sin^{-1} x} = \lim_{x \rightarrow 0} \frac{\frac{3}{1+9x^2}}{\frac{1}{\sqrt{1-x^2}}} = \frac{3}{1} = 3$$
- The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow -2} \frac{x^2 + 6x + 8}{x^2 - 3x - 10} = \lim_{x \rightarrow -2} \frac{2x + 6}{2x - 3} = \frac{2}{-7} = -\frac{2}{7}$$

- The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{x^3 - 3x^2 + x}{x^3 - 2x} = \lim_{x \rightarrow 0} \frac{3x^2 + 6x + 1}{3x^2 - 2} = \frac{1}{-2} = -\frac{1}{2}$$
- The limit is not of the form $\frac{0}{0}$.

As $x \rightarrow 1^-$, $x^2 - 2x + 2 \rightarrow 1$, and $x^2 - 1 \rightarrow 0^-$ so

$$\lim_{x \rightarrow 1^-} \frac{x^2 - 2x + 2}{x^2 - 1} = -\infty$$

- The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 1} \frac{\ln x^2}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{\frac{1}{x^2} 2x}{2x} = \lim_{x \rightarrow 1} \frac{1}{x^2} = 1$$
- The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow \pi/2} \frac{\ln(\sin x)^3}{\pi/2 - x} = \lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sin^3 x} 3 \sin^2 x \cos x}{-1} = \frac{0}{-1} = 0$$
- The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{2 \sin x} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{2 \cos x} = \frac{2}{2} = 1$$
- The limit is of the form $\frac{0}{0}$.

$$\lim_{t \rightarrow 1} \frac{\sqrt{t} - t^2}{\ln t} = \lim_{t \rightarrow 1} \frac{\frac{1}{2\sqrt{t}} - 2t}{\frac{1}{t}} = \frac{-\frac{3}{2}}{1} = -\frac{3}{2}$$

12. The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0^+} \frac{7\sqrt{x} - 1}{2\sqrt{x} - 1} = \lim_{x \rightarrow 0^+} \frac{\frac{7\sqrt{x} \ln 7}{2\sqrt{x}}}{\frac{2\sqrt{x} \ln 2}{2\sqrt{x}}} = \lim_{x \rightarrow 0^+} \frac{7\sqrt{x} \ln 7}{2\sqrt{x} \ln 2}$$

$$= \frac{\ln 7}{\ln 2} \approx 2.81$$

13. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x \rightarrow 0} \frac{\ln \cos 2x}{7x^2} = \lim_{x \rightarrow 0} \frac{\frac{-2 \sin 2x}{\cos 2x}}{14x} = \lim_{x \rightarrow 0} \frac{-2 \sin 2x}{14x \cos 2x}$$

$$= \lim_{x \rightarrow 0} \frac{-4 \cos 2x}{14 \cos 2x - 28x \sin 2x} = \frac{-4}{14 - 0} = -\frac{2}{7}$$

14. The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0^-} \frac{3 \sin x}{\sqrt{-x}} = \lim_{x \rightarrow 0^-} \frac{3 \cos x}{-\frac{1}{2\sqrt{-x}}}$$

$$= \lim_{x \rightarrow 0^-} -6\sqrt{-x} \cos x = 0$$

15. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{\sin 2x - 2x} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{2 \cos 2x - 2}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{-4 \sin 2x} = \lim_{x \rightarrow 0} \frac{2 \sec^4 x + 4 \sec^2 x \tan^2 x}{-8 \cos 2x}$$

$$= \frac{2 + 0}{-8} = -\frac{1}{4}$$

16. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^2 \sin x} = \lim_{x \rightarrow 0} \frac{\cos x - \sec^2 x}{2x \sin x + x^2 \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{-\sin x - 2 \sec^2 x \tan x}{2 \sin x + 4x \cos x - x^2 \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{-\cos x - 2 \sec^4 x - 4 \sec^2 x \tan^2 x}{6 \cos x - x^2 \cos x - 6x \sin x}$$

$$= \frac{-1 - 2 - 0}{6 - 0 - 0} = -\frac{1}{2}$$

17. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x \rightarrow 0^+} \frac{x^2}{\sin x - x} = \lim_{x \rightarrow 0^+} \frac{2x}{\cos x - 1} = \lim_{x \rightarrow 0^+} \frac{2}{-\sin x}$$

This limit is not of the form $\frac{0}{0}$. As $x \rightarrow 0^+$, $2 \rightarrow 2$,

$$\text{and } -\sin x \rightarrow 0^-, \text{ so } \lim_{x \rightarrow 0^+} \frac{2}{-\sin x} = -\infty.$$

18. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x \rightarrow 0} \frac{e^x - \ln(1+x) - 1}{x^2} = \lim_{x \rightarrow 0} \frac{e^x - \frac{1}{1+x}}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{e^x + \frac{1}{(1+x)^2}}{2} = \frac{1+1}{2} = 1$$

19. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x \rightarrow 0} \frac{\tan^{-1} x - x}{8x^3} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x^2} - 1}{24x^2} = \lim_{x \rightarrow 0} \frac{\frac{-2x}{(1+x^2)^2}}{48x}$$

$$= \lim_{x \rightarrow 0} -\frac{1}{24(1+x^2)^2} = -\frac{1}{24}$$

20. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x \rightarrow 0} \frac{\cosh x - 1}{x^2} = \lim_{x \rightarrow 0} \frac{\sinh x}{2x} = \lim_{x \rightarrow 0} \frac{\cosh x}{2} = \frac{1}{2}$$

21. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x \rightarrow 0^+} \frac{1 - \cos x - x \sin x}{2 - 2 \cos x - \sin^2 x} = \lim_{x \rightarrow 0^+} \frac{-x \cos x}{2 \sin x - 2 \cos x \sin x}$$

$$= \lim_{x \rightarrow 0^+} \frac{x \sin x - \cos x}{2 \cos x - 2 \cos^2 x + 2 \sin^2 x}$$

This limit is not of the form $\frac{0}{0}$.

As $x \rightarrow 0^+$, $x \sin x - \cos x \rightarrow -1$ and $2 \cos x - 2 \cos^2 x + 2 \sin^2 x \rightarrow 0^+$, so

$$\lim_{x \rightarrow 0^+} \frac{x \sin x - \cos x}{2 \cos x - 2 \cos^2 x + 2 \sin^2 x} = -\infty$$

22. The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0^-} \frac{\sin x + \tan x}{e^x + e^{-x} - 2} = \lim_{x \rightarrow 0^-} \frac{\cos x + \sec^2 x}{e^x - e^{-x}}$$

This limit is not of the form $\frac{0}{0}$.

As $x \rightarrow 0^-$, $\cos x + \sec^2 x \rightarrow 2$, and

$e^x - e^{-x} \rightarrow 0^-$, so $\lim_{x \rightarrow 0^-} \frac{\cos x + \sec^2 x}{e^x - e^{-x}} = -\infty$.

23. The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{\int_0^x \sqrt{1 + \sin t} dt}{x} = \lim_{x \rightarrow 0} \sqrt{1 + \sin x} = 1$$

24. The limit is of the form $\frac{0}{0}$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\int_0^x \sqrt{t} \cos t dt}{x^2} &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x} \cos x}{2x} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos x}{2\sqrt{x}} = \infty \end{aligned}$$

25. It would not have helped us because we proved

$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ in order to find the derivative of $\sin x$.

26. Note that $\sin(1/0)$ is undefined (not zero), so l'Hôpital's Rule cannot be used.

As $x \rightarrow 0$, $\frac{1}{x} \rightarrow \infty$ and $\sin\left(\frac{1}{x}\right)$ oscillates rapidly between -1 and 1 , so

$$\lim_{x \rightarrow 0} \left| \frac{x^2 \sin\left(\frac{1}{x}\right)}{\tan x} \right| \leq \lim_{x \rightarrow 0} \frac{x^2}{\tan x}.$$

$$\frac{x^2}{\tan x} = \frac{x^2 \cos x}{\sin x}$$

$$\lim_{x \rightarrow 0} \frac{x^2 \cos x}{\sin x} = \lim_{x \rightarrow 0} \left[\left(\frac{x}{\sin x} \right) x \cos x \right] = 0.$$

$$\text{Thus, } \lim_{x \rightarrow 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\tan x} = 0.$$

A table of values or graphing utility confirms this.

27. a. $\overline{OB} = \cos t$, $\overline{BC} = \sin t$ and $\overline{AB} = 1 - \cos t$, so the area of triangle ABC is $\frac{1}{2} \sin t(1 - \cos t)$.

The area of the sector COA is $\frac{1}{2}t$ while the area of triangle COB is $\frac{1}{2} \cos t \sin t$, thus the area of the curved region ABC is $\frac{1}{2}(t - \cos t \sin t)$.

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{\text{area of triangle } ABC}{\text{area of curved region } ABC} &= \lim_{t \rightarrow 0^+} \frac{\frac{1}{2} \sin t(1 - \cos t)}{\frac{1}{2}(t - \cos t \sin t)} \\ &= \lim_{t \rightarrow 0^+} \frac{\sin t(1 - \cos t)}{t - \cos t \sin t} = \lim_{t \rightarrow 0^+} \frac{\cos t - \cos^2 t + \sin^2 t}{1 - \cos^2 t + \sin^2 t} = \lim_{t \rightarrow 0^+} \frac{4 \sin t \cos t - \sin t}{4 \cos t \sin t} = \lim_{t \rightarrow 0^+} \frac{4 \cos t - 1}{4 \cos t} = \frac{3}{4} \end{aligned}$$

(L'Hôpital's Rule was applied twice.)

- b. The area of the sector BOD is $\frac{1}{2}t \cos^2 t$, so the area of the curved region BCD is $\frac{1}{2} \cos t \sin t - \frac{1}{2}t \cos^2 t$.

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{\text{area of curved region } BCD}{\text{area of curved region } ABC} &= \lim_{t \rightarrow 0^+} \frac{\frac{1}{2} \cos t(\sin t - t \cos t)}{\frac{1}{2}(t - \cos t \sin t)} \\ &= \lim_{t \rightarrow 0^+} \frac{\cos t(\sin t - t \cos t)}{t - \sin t \cos t} = \lim_{t \rightarrow 0^+} \frac{\sin t(2t \cos t - \sin t)}{1 - \cos^2 t + \sin^2 t} = \lim_{t \rightarrow 0^+} \frac{2t(\cos^2 t - \sin^2 t)}{4 \cos t \sin t} = \lim_{t \rightarrow 0^+} \frac{t(\cos^2 t - \sin^2 t)}{2 \cos t \sin t} \\ &= \lim_{t \rightarrow 0^+} \frac{\cos^2 t - 4t \cos t \sin t - \sin^2 t}{2 \cos^2 t - 2 \sin^2 t} = \frac{1 - 0 - 0}{2 - 0} = \frac{1}{2} \end{aligned}$$

(L'Hôpital's Rule was applied three times.)

28. a. Note that $\angle DOE$ has measure t radians. Thus the coordinates of E are $(\cos t, \sin t)$.

Also, slope $\overline{BC}$ = slope $\overline{CE}$. Thus,

$$\frac{0-y}{(1-t)-0} = \frac{\sin t-0}{\cos t-(1-t)}$$

$$-y = \frac{(1-t)\sin t}{\cos t+t-1}$$

$$y = \frac{(t-1)\sin t}{\cos t+t-1}$$

$$\lim_{t \rightarrow 0^+} y = \lim_{t \rightarrow 0^+} \frac{(t-1)\sin t}{\cos t+t-1}$$

This limit is of the form $\frac{0}{0}$.

$$\lim_{t \rightarrow 0^+} \frac{(t-1)\sin t}{\cos t+t-1} = \lim_{t \rightarrow 0^+} \frac{\sin t + (t-1)\cos t}{-\sin t + 1} = \frac{0+(-1)(1)}{-0+1} = -1$$

- b. Slope $\overline{AF}$ = slope $\overline{EF}$. Thus,

$$\frac{t}{1-x} = \frac{t-\sin t}{1-\cos t}$$

$$\frac{t(1-\cos t)}{t-\sin t} = 1-x$$

$$x = 1 - \frac{t(1+\cos t)}{t-\sin t}$$

$$x = \frac{t\cos t - \sin t}{t-\sin t}$$

$$\lim_{t \rightarrow 0^+} x = \lim_{t \rightarrow 0^+} \frac{t\cos t - \sin t}{t-\sin t}$$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{t \rightarrow 0^+} \frac{t\cos t - \sin t}{t-\sin t} = \lim_{t \rightarrow 0^+} \frac{-t\sin t}{1-\cos t}$$

$$= \lim_{t \rightarrow 0^+} \frac{-\sin t - t\cos t}{\sin t} = \lim_{t \rightarrow 0^+} \frac{t\sin t - 2\cos t}{\cos t} = \frac{0-2}{1} = -2$$

29. A should approach $4\pi b^2$, the surface area of a sphere of radius b .

$$\lim_{a \rightarrow b^+} \left[2\pi b^2 + \frac{2\pi a^2 b \arcsin \frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}} \right] = 2\pi b^2 + 2\pi b \lim_{a \rightarrow b^+} \frac{a^2 \arcsin \frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}$$

Focusing on the limit, we have

$$\lim_{a \rightarrow b^+} \frac{a^2 \arcsin \frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}} = \lim_{a \rightarrow b^+} \frac{2a \arcsin \frac{\sqrt{a^2-b^2}}{a} + a^2 \left(\frac{b}{a\sqrt{a^2-b^2}} \right)}{\frac{a}{\sqrt{a^2-b^2}}} = \lim_{a \rightarrow b^+} \left(2\sqrt{a^2-b^2} \arcsin \frac{\sqrt{a^2-b^2}}{a} + b \right) = b.$$

Thus, $\lim_{a \rightarrow b^+} A = 2\pi b^2 + 2\pi b(b) = 4\pi b^2$.

30. In order for l'Hôpital's Rule to be of any use, $a(1)^4 + b(1)^3 + 1 = 0$, so $b = -1 - a$.

Using l'Hôpital's Rule,

$$\lim_{x \rightarrow 1} \frac{ax^4 + bx^3 + 1}{(x-1)\sin \pi x} = \lim_{x \rightarrow 1} \frac{4ax^3 + 3bx^2}{\sin \pi x + \pi(x-1)\cos \pi x}$$

To use l'Hôpital's Rule here,

$$4a(1)^3 + 3b(1)^2 = 0, \text{ so } 4a + 3b = 0, \text{ hence } a = 3, b = -4.$$

$$\lim_{x \rightarrow 1} \frac{3x^4 - 4x^3 + 1}{(x-1)\sin \pi x} = \lim_{x \rightarrow 1} \frac{12x^3 - 12x^2}{\sin \pi x + \pi(x-1)\cos \pi x} = \lim_{x \rightarrow 1} \frac{36x^2 - 24x}{2\pi \cos \pi x - \pi^2(x-1)\sin \pi x} = \frac{12}{-2\pi} = -\frac{6}{\pi}$$

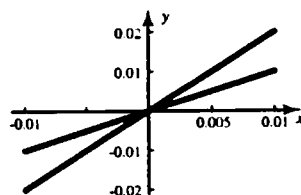
$$a = 3, b = -4, c = -\frac{6}{\pi}$$

31. If $f'(a)$ and $g'(a)$ both exist, then f and g are both continuous at a . Thus, $\lim_{x \rightarrow a} f(x) = 0 = f(a)$

$$\text{and } \lim_{x \rightarrow a} g(x) = 0 = g(a).$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$\lim_{x \rightarrow a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \frac{f'(a)}{g'(a)}$$



The slopes are approximately $0.02/0.01 = 2$ and $0.01/0.01 = 1$. The ratio of the slopes is therefore $2/1 = 2$, indicating that the limit of the ratio should be about 2. An application of l'Hopital's Rule confirms this.

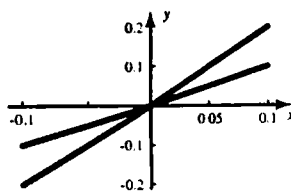
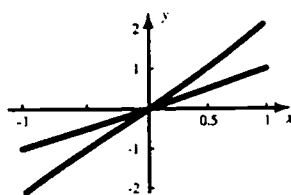
$$32. \lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{1}{24}$$

$$33. \lim_{x \rightarrow 0} \frac{e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6}}{x^4} = \frac{1}{24}$$

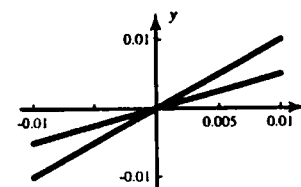
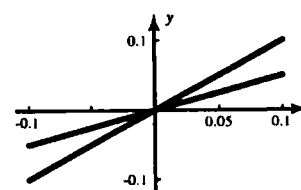
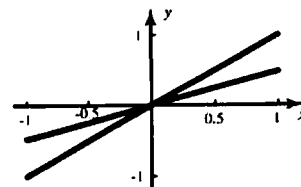
$$34. \lim_{x \rightarrow 0} \frac{1 - \cos(x^2)}{x^3 \sin x} = \frac{1}{2}$$

$$35. \lim_{x \rightarrow 0} \frac{\tan x - x}{\arcsin x - x} = 2$$

36.

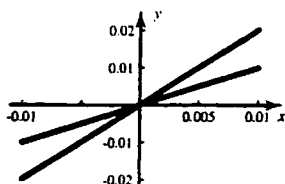
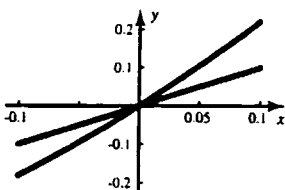
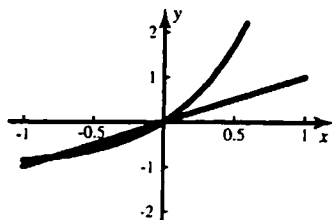


37.



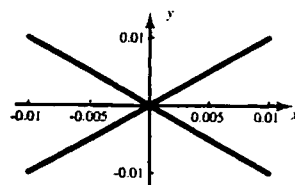
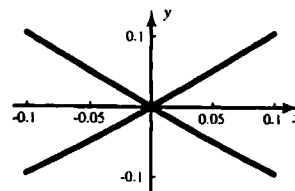
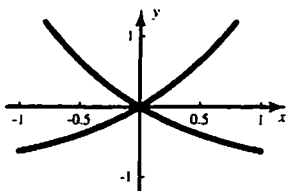
The slopes are approximately $0.005/0.01 = 1/2$ and $0.01/0.01 = 1$. The ratio of the slopes is therefore $1/2$, indicating that the limit of the ratio should be about $1/2$. An application of l'Hopital's Rule confirms this.

38.



The slopes are approximately $0.01/0.01 = 1$ and $0.02/0.01 = 2$. The ratio of the slopes is therefore $1/2$, indicating that the limit of the ratio should be about $1/2$. An application of l'Hopital's Rule confirms this.

39.



The slopes are approximately $0.01/0.01 = 1$ and $-0.01/0.01 = -1$. The ratio of the slopes is therefore $-1/1 = -1$, indicating that the limit of the ratio should be about -1 . An application of l'Hopital's Rule confirms this.

40. If f and g are locally linear at zero, then, since $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = 0$, $f(x) \approx px$ and $g(x) \approx qx$, where $p = f'(0)$ and $q = g'(0)$. Then $f(x)/g(x) \approx px/qx = p/q$ when x is near 0.

9.2 Concepts Review

- $\frac{f'(x)}{g'(x)}$
- $\lim_{x \rightarrow a} \frac{f(x)}{\frac{1}{g(x)}} \text{ or } \lim_{x \rightarrow a} \frac{g(x)}{\frac{1}{f(x)}}$
- $\infty - \infty, 0^\circ, \infty^\circ, 1^\infty$
- $\ln x$

Problem Set 9.2

1. The limit is of the form $\frac{\infty}{\infty}$.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln x^{1000}}{x} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x^{1000}} 1000x^{999}}{1} \\ &= \lim_{x \rightarrow \infty} \frac{1000}{x} = 0 \end{aligned}$$

2. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{2^x} = \lim_{x \rightarrow \infty} \frac{2(\ln x) \frac{1}{x}}{2^x \ln 2}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow \infty} \frac{2 \ln x}{x \cdot 2^x \ln 2} = \lim_{x \rightarrow \infty} \frac{2\left(\frac{1}{x}\right)}{2^x \ln 2(1+x \ln 2)} \\
 &= \lim_{x \rightarrow \infty} \frac{2}{x \cdot 2^x \ln 2(1+x \ln 2)} = 0
 \end{aligned}$$

3. $\lim_{x \rightarrow \infty} \frac{x^{10000}}{e^x} = 0$ (See Example 2).

4. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's

Rule three times.)

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{3x}{\ln(100x + e^x)} &= \lim_{x \rightarrow \infty} \frac{3}{\frac{1}{100x + e^x}(100 + e^x)} \\
 &= \lim_{x \rightarrow \infty} \frac{300x + 3e^x}{100 + e^x} = \lim_{x \rightarrow \infty} \frac{300 + 3e^x}{e^x} \\
 &= \lim_{x \rightarrow \infty} \frac{3e^x}{e^x} = 3
 \end{aligned}$$

5. The limit is of the form $\frac{\infty}{\infty}$.

$$\begin{aligned}
 \lim_{x \rightarrow \frac{\pi}{2}} \frac{3 \sec x + 5}{\tan x} &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{3 \sec x \tan x}{\sec^2 x} \\
 &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{3 \tan x}{\sec x} = \lim_{x \rightarrow \frac{\pi}{2}} 3 \sin x = 3
 \end{aligned}$$

6. The limit is of the form $\frac{-\infty}{-\infty}$.

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} \frac{\ln \sin^2 x}{3 \ln \tan x} &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sin^2 x} 2 \sin x \cos x}{\frac{3}{\tan x} \sec^2 x} \\
 &= \lim_{x \rightarrow 0^+} \frac{2 \cos^2 x}{3} = \frac{2}{3}
 \end{aligned}$$

7. The limit is of the form $\frac{\infty}{\infty}$.

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{\ln(\ln x^{1000})}{\ln x} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln x^{1000}} \left(\frac{1}{x^{1000}} 1000x^{999} \right)}{\frac{1}{x}} \\
 &= \lim_{x \rightarrow \infty} \frac{1000}{x \ln x^{1000}} = 0
 \end{aligned}$$

8. The limit is of the form $\frac{-\infty}{\infty}$. (Apply l'Hôpital's

Rule twice.)

$$\lim_{x \rightarrow \left(\frac{1}{2}\right)^-} \frac{\ln(4-8x)^2}{\tan \pi x} = \lim_{x \rightarrow \left(\frac{1}{2}\right)^-} \frac{\frac{1}{(4-8x)^2} 2(4-8x)(-8)}{\pi \sec^2 \pi x}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow \left(\frac{1}{2}\right)^-} \frac{-16 \cos^2 \pi x}{\pi(4-8x)} = \lim_{x \rightarrow \left(\frac{1}{2}\right)^-} \frac{32 \pi \cos \pi x \sin \pi x}{-8 \pi} \\
 &= \lim_{x \rightarrow \left(\frac{1}{2}\right)^-} -4 \cos \pi x \sin \pi x = 0
 \end{aligned}$$

9. The limit is of the form $\frac{\infty}{\infty}$.

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} \frac{\cot x}{\sqrt{-\ln x}} &= \lim_{x \rightarrow 0^+} \frac{-\csc^2 x}{-\frac{1}{2x\sqrt{-\ln x}}} \\
 &= \lim_{x \rightarrow 0^+} \frac{2x\sqrt{-\ln x}}{\sin^2 x} \\
 &= \lim_{x \rightarrow 0^+} \left[\frac{2x}{\sin x} \csc x \sqrt{-\ln x} \right] = \infty
 \end{aligned}$$

since $\lim_{x \rightarrow 0^+} \frac{x}{\sin x} = 1$ while $\lim_{x \rightarrow 0^+} \csc x = \infty$ and $\lim_{x \rightarrow 0^+} \sqrt{-\ln x} = \infty$.

10. The limit is of the form $\frac{\infty}{\infty}$, but the fraction can be simplified.

$$\lim_{x \rightarrow 0} \frac{2 \csc^2 x}{\cot^2 x} = \lim_{x \rightarrow 0} \frac{2}{\cos^2 x} = \frac{2}{1^2} = 2$$

11. $\lim_{x \rightarrow 0} (x \ln x^{1000}) = \lim_{x \rightarrow 0} \frac{\ln x^{1000}}{\frac{1}{x}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\ln x^{1000}}{\frac{1}{x}} &= \lim_{x \rightarrow 0} \frac{\frac{1}{x^{1000}} 1000x^{999}}{-\frac{1}{x^2}} \\
 &= \lim_{x \rightarrow 0} -1000x = 0
 \end{aligned}$$

12. $\lim_{x \rightarrow 0} 3x^2 \csc^2 x = \lim_{x \rightarrow 0} 3 \left(\frac{x}{\sin x} \right)^2 = 3$ since

$$\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$$

13. $\lim_{x \rightarrow 0} (\csc^2 x - \cot^2 x) = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\sin^2 x}$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin^2 x} = 1$$

$$14. \lim_{x \rightarrow \frac{\pi}{2}} (\tan x - \sec x) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\cos x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\cos x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{-\sin x} = \frac{0}{-1} = 0$$

15. The limit is of the form 0^0 .

Let $y = (3x)^{x^2}$, then $\ln y = x^2 \ln 3x$

$$\lim_{x \rightarrow 0^+} x^2 \ln 3x = \lim_{x \rightarrow 0^+} \frac{\ln 3x}{\frac{1}{x^2}}$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x \rightarrow 0^+} \frac{\ln 3x}{\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{3x} \cdot 3}{-\frac{2}{x^3}} = \lim_{x \rightarrow 0^+} -\frac{x^2}{2} = 0$$

$$\lim_{x \rightarrow 0^+} (3x)^{x^2} = \lim_{x \rightarrow 0^+} e^{\ln y} = 1$$

16. The limit is of the form 1^∞ .

Let $y = (\cos x)^{\csc x}$, then $\ln y = \csc x (\ln(\cos x))$

$$\lim_{x \rightarrow 0} \csc x (\ln(\cos x)) = \lim_{x \rightarrow 0} \frac{\ln(\cos x)}{\sin x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{\ln(\cos x)}{\sin x} = \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x}(-\sin x)}{\cos x}$$

$$= \lim_{x \rightarrow 0} -\frac{\sin x}{\cos^2 x} = -\frac{0}{1} = 0$$

$$\lim_{x \rightarrow 0} (\cos x)^{\csc x} = \lim_{x \rightarrow 0} e^{\ln y} = 1$$

17. The limit is of the form 0^∞ , which is not an

indeterminate form. $\lim_{x \rightarrow (\pi/2)^-} (5 \cos x)^{\tan x} = 0$

$$18. \lim_{x \rightarrow 0} \left(\csc^2 x - \frac{1}{x^2} \right)^2 = \lim_{x \rightarrow 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right)^2 = \lim_{x \rightarrow 0} \left(\frac{x^2 - \sin^2 x}{x^2 \sin^2 x} \right)^2$$

Consider $\lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule four times.)

$$\lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x \rightarrow 0} \frac{2x - 2 \sin x \cos x}{2x \sin^2 x + 2x^2 \sin x \cos x} = \lim_{x \rightarrow 0} \frac{x - \sin x \cos x}{x \sin^2 x + x^2 \sin x \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x + \sin^2 x}{\sin^2 x + 4x \sin x \cos x + x^2 \cos^2 x - x^2 \sin^2 x} = \frac{4 \sin x \cos x}{6x \cos^2 x + 6 \cos x \sin x - 4x^2 \cos x \sin x - 6x \sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{4 \cos^2 x - 4 \sin^2 x}{12 \cos^2 x - 4x^2 \cos^2 x - 32x \cos x \sin x - 12 \sin^2 x + 4x^2 \sin^2 x} = \frac{4}{12} = \frac{1}{3}$$

$$\text{Thus, } \lim_{x \rightarrow 0} \left(\frac{x^2 - \sin^2 x}{x^2 \sin^2 x} \right)^2 = \left(\frac{1}{3} \right)^2 = \frac{1}{9}$$

19. The limit is of the form 1^∞ .

Let $y = (x + e^{x/3})^{3/x}$, then $\ln y = \frac{3}{x} \ln(x + e^{x/3})$.

$$\lim_{x \rightarrow 0} \frac{3}{x} \ln(x + e^{x/3}) = \lim_{x \rightarrow 0} \frac{3 \ln(x + e^{x/3})}{x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{3 \ln(x + e^{x/3})}{x} = \lim_{x \rightarrow 0} \frac{\frac{3}{x + e^{x/3}} \left(1 + \frac{1}{3} e^{x/3} \right)}{1}$$

$$= \lim_{x \rightarrow 0} \frac{3 + e^{x/3}}{x + e^{x/3}} = \frac{4}{1} = 4$$

$$\lim_{x \rightarrow 0} (x + e^{x/3})^{3/x} = \lim_{x \rightarrow 0} e^{\ln y} = e^4$$

20. The limit is of the form $(-1)^0$.

The limit does not exist.

21. The limit is of the form 1^0 , which is not an indeterminate form.

$$\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\cos x} = 1$$

22. The limit is of the form ∞^∞ , which is not an indeterminate form.

$$\lim_{x \rightarrow \infty} x^x = \infty$$

23. The limit is of the form ∞^0 . Let

$$y = x^{1/x}, \text{ then } \ln y = \frac{1}{x} \ln x.$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} \ln x = \lim_{x \rightarrow \infty} \frac{\ln x}{x}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow \infty} x^{1/x} = \lim_{x \rightarrow \infty} e^{\ln y} = 1$$

24. The limit is of the form 1^∞ .

$$\text{Let } y = (\cos x)^{1/x^2}, \text{ then } \ln y = \frac{1}{x^2} \ln(\cos x).$$

$$\lim_{x \rightarrow 0} \frac{1}{x^2} \ln(\cos x) = \lim_{x \rightarrow 0} \frac{\ln(\cos x)}{x^2}$$

The limit is of the form $\frac{0}{0}$.

(Apply l'Hôpital's rule twice.)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln(\cos x)}{x^2} &= \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x} = \lim_{x \rightarrow 0} \frac{-\tan x}{2x} \\ &= \lim_{x \rightarrow 0} \frac{-\sec^2 x}{2} = \frac{-1}{2} = -\frac{1}{2} \end{aligned}$$

$$\lim_{x \rightarrow 0} (\cos x)^{1/x^2} = \lim_{x \rightarrow 0} e^{\ln y} = e^{-1/2} = \frac{1}{\sqrt{e}}$$

25. The limit is of the form 0^∞ , which is not an indeterminate form.

$$\lim_{x \rightarrow 0^+} (\tan x)^{2/x} = 0$$

26. The limit is of the form $\infty + \infty$, which is not an indeterminate form.

$$\lim_{x \rightarrow -\infty} (e^{-x} - x) = \lim_{x \rightarrow \infty} (e^x + x) = \infty$$

27. The limit is of the form 0^0 . Let

$$y = (\sin x)^x, \text{ then } \ln y = x \ln(\sin x).$$

$$\lim_{x \rightarrow 0^+} x \ln(\sin x) = \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{\frac{1}{x}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{\frac{1}{x}} &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sin x} \cos x}{-\frac{1}{x^2}} \\ &= \lim_{x \rightarrow 0^+} \left[\frac{x}{\sin x} (-x \cos x) \right] = 1 \cdot 0 = 0 \end{aligned}$$

$$\lim_{x \rightarrow 0^+} (\sin x)^x = \lim_{x \rightarrow 0^+} e^{\ln y} = 1$$

28. The limit is of the form 1^∞ . Let

$$y = (\cos x - \sin x)^{1/x}, \text{ then } \ln y = \frac{1}{x} \ln(\cos x - \sin x).$$

$$\lim_{x \rightarrow 0} \frac{1}{x} \ln(\cos x - \sin x) = \lim_{x \rightarrow 0} \frac{\ln(\cos x - \sin x)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x - \sin x} (-\sin x - \cos x)}{1}$$

$$= \lim_{x \rightarrow 0} \frac{-\sin x - \cos x}{\cos x - \sin x} = -1$$

$$\lim_{x \rightarrow 0} (\cos x - \sin x)^{1/x} = \lim_{x \rightarrow 0} e^{\ln y} = e^{-1}$$

29. The limit is of the form $\infty - \infty$.

$$\lim_{x \rightarrow 0} \left(\csc x - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x}$$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's

Rule twice.)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0 \end{aligned}$$

30. The limit is of the form 1^∞ .

$$\text{Let } y = \left(1 + \frac{1}{x} \right)^x, \text{ then } \ln y = x \ln \left(1 + \frac{1}{x} \right).$$

$$\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x} \right) = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x} \right)}{\frac{1}{x}}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x} \right)}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{1}{x}} \left(-\frac{1}{x^2} \right)}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x}} = 1$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow \infty} e^{\ln y} = e^1 = e$$

31. The limit is of the form 3^∞ , which is not an indeterminate form.

$$\lim_{x \rightarrow 0^+} (1 + 2e^x)^{1/x} = \infty$$

32. The limit is of the form $\infty - \infty$.

$$\lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{x}{\ln x} \right) = \lim_{x \rightarrow 1} \frac{\ln x - x^2 + x}{(x-1)\ln x}$$

$$\text{The limit is of the form } \frac{0}{0}.$$

Apply l'Hôpital's Rule twice.

$$\lim_{x \rightarrow 1} \frac{\ln x - x^2 + x}{(x-1)\ln x} = \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 2x + 1}{\ln x + \frac{x-1}{x}}$$

$$= \lim_{x \rightarrow 1} \frac{1 - 2x^2 + x}{x \ln x + x - 1} = \lim_{x \rightarrow 1} \frac{-4x + 1}{\ln x + 2} = \frac{-3}{2} = -\frac{3}{2}$$

33. The limit is of the form 1^∞ .

$$\text{Let } y = (\cos x)^{1/x}, \text{ then } \ln y = \frac{1}{x} \ln(\cos x).$$

$$\lim_{x \rightarrow 0} \frac{1}{x} \ln(\cos x) = \lim_{x \rightarrow 0} \frac{\ln(\cos x)}{x}$$

$$\text{The limit is of the form } \frac{0}{0}.$$

$$\lim_{x \rightarrow 0} \frac{\ln(\cos x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x}(-\sin x)}{1} = \lim_{x \rightarrow 0} -\frac{\sin x}{\cos x} = 0$$

$$\lim_{x \rightarrow 0} (\cos x)^{1/x} = \lim_{x \rightarrow 0} e^{\ln y} = 1$$

34. The limit is of the form $0 \cdot -\infty$.

$$\lim_{x \rightarrow 0^+} (x^{1/2} \ln x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}}$$

$$\text{The limit is of the form } \frac{-\infty}{\infty}.$$

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}} = \lim_{x \rightarrow 0^+} -2\sqrt{x} = 0$$

35. Since $\cos x$ oscillates between -1 and 1 as $x \rightarrow \infty$, this limit is not of an indeterminate form previously seen.

$$\text{Let } y = e^{\cos x}, \text{ then } \ln y = (\cos x) \ln e = \cos x$$

$$\lim_{x \rightarrow \infty} \cos x \text{ does not exist, so } \lim_{x \rightarrow \infty} e^{\cos x} \text{ does not exist.}$$

36. The limit is of the form $\infty - \infty$.

$$\lim_{x \rightarrow \infty} [\ln(x+1) - \ln(x-1)] = \lim_{x \rightarrow \infty} \ln \frac{x+1}{x-1}$$

$$\lim_{x \rightarrow \infty} \frac{x+1}{x-1} = \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} = 1, \text{ so } \lim_{x \rightarrow \infty} \ln \frac{x+1}{x-1} = 0$$

37. The limit is of the form $\frac{0}{-\infty}$, which is not an indeterminate form.

$$\lim_{x \rightarrow 0^+} \frac{x}{\ln x} = 0$$

38. The limit is of the form $-\infty \cdot \infty$, which is not an indeterminate form.

$$\lim_{x \rightarrow 0^+} (\ln x \cot x) = -\infty$$

39. $\sqrt{1+e^{-t}} > 1$ for all t , so

$$\int_1^x \sqrt{1+e^{-t}} dt > \int_1^x dt = x-1.$$

$$\text{The limit is of the form } \frac{\infty}{\infty}.$$

$$\lim_{x \rightarrow \infty} \frac{\int_1^x \sqrt{1+e^{-t}} dt}{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{1+e^{-x}}}{1} = 1$$

40. This limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 1^+} \frac{\int_1^x \sin t dt}{x-1} = \lim_{x \rightarrow 1^+} \frac{\sin x}{1} = \sin(1)$$

41. a. Let $y = \sqrt[n]{a}$, then $\ln y = \frac{1}{n} \ln a$.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln a = 0$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a} = \lim_{n \rightarrow \infty} e^{\ln y} = 1$$

- b. The limit is of the form ∞^0 .

$$\text{Let } y = \sqrt[n]{n}, \text{ then } \ln y = \frac{1}{n} \ln n.$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln n = \lim_{n \rightarrow \infty} \frac{\ln n}{n}$$

$$\text{This limit is of the form } \frac{\infty}{\infty}.$$

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} = 0$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = \lim_{n \rightarrow \infty} e^{\ln y} = 1$$

$$c. \lim_{n \rightarrow \infty} n(\sqrt[n]{a} - 1) = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a} - 1}{\frac{1}{n}}$$

This limit is of the form $\frac{0}{0}$,

since $\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$ by part a.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a} - 1}{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \frac{-\frac{1}{n^2} \sqrt[n]{a} \ln a}{-\frac{1}{n^2}} \\ &= \lim_{n \rightarrow \infty} \sqrt[n]{a} \ln a = \ln a \end{aligned}$$

$$d. \lim_{n \rightarrow \infty} n(\sqrt[n]{n} - 1) = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n} - 1}{\frac{1}{n}}$$

This limit is of the form $\frac{0}{0}$,

since $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ by part b.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n} - 1}{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n} \left(\frac{1}{n^2} \right) (1 - \ln n)}{-\frac{1}{n^2}} \\ &= \lim_{n \rightarrow \infty} \sqrt[n]{n} (\ln n - 1) = \infty \end{aligned}$$

42. a. The limit is of the form 0^0 .

Let $y = x^x$, then $\ln y = x \ln x$.

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -x = 0$$

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{\ln y} = 1$$

b. The limit is of the form 1^0 , since

$$\lim_{x \rightarrow 0^+} x^x = 1 \text{ by part a.}$$

Let $y = (x^x)^x$, then $\ln y = x \ln(x^x)$.

$$\lim_{x \rightarrow 0^+} x \ln(x^x) = 0$$

$$\lim_{x \rightarrow 0^+} (x^x)^x = \lim_{x \rightarrow 0^+} e^{\ln y} = 1$$

Note that 1^0 is not an indeterminate form.

c. The limit is of the form 0^1 , since

$$\lim_{x \rightarrow 0^+} x^x = 1 \text{ by part a.}$$

Let $y = x^{(x^x)}$, then $\ln y = x^x \ln x$

$$\lim_{x \rightarrow 0^+} x^x \ln x = -\infty$$

$$\lim_{x \rightarrow 0^+} x^{(x^x)} = \lim_{x \rightarrow 0^+} e^{\ln y} = 0$$

Note that 0^1 is not an indeterminate form.

d. The limit is of the form 1^0 , since

$$\lim_{x \rightarrow 0^+} (x^x)^x = 1 \text{ by part b.}$$

Let $y = ((x^x)^x)^x$, then $\ln y = x \ln((x^x)^x)$.

$$\lim_{x \rightarrow 0^+} x \ln((x^x)^x) = 0$$

$$\lim_{x \rightarrow 0^+} ((x^x)^x)^x = \lim_{x \rightarrow 0^+} e^{\ln y} = 1$$

Note that 1^0 is not an indeterminate form.

e. The limit is of the form 0^0 , since

$$\lim_{x \rightarrow 0^+} (x^{(x^x)}) = 0 \text{ by part c.}$$

Let $y = x^{(x^{(x^x)})}$, then $\ln y = x^{(x^x)} \ln x$.

$$\lim_{x \rightarrow 0^+} x^{(x^x)} \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^{(x^x)}}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^{(x^x)}}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-x^{(x^x)} \left[x^x (\ln x + 1) \ln x + \frac{x^x}{x} \right]}{(x^{(x^x)})^2}}$$

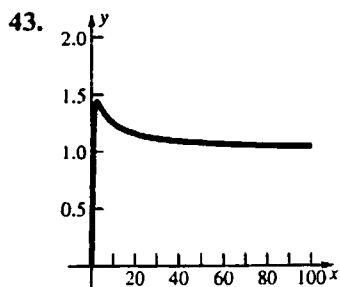
$$= \lim_{x \rightarrow 0^+} \frac{-x^{(x^x)}}{x^x x (\ln x)^2 + x^x x \ln x + x^x}$$

$$= \frac{0}{1 \cdot 0 + 1 \cdot 0 + 1} = 0$$

$$\text{Note: } \lim_{x \rightarrow 0^+} x (\ln x)^2 = \lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow 0^+} \frac{2 \ln x}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -2x \ln x = 0$$

$$\lim_{x \rightarrow 0^+} x^{(x^{(x^x)})} = \lim_{x \rightarrow 0^+} e^{\ln y} = 1$$



$$\ln y = \frac{\ln x}{x}$$

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x} = -\infty, \text{ so } \lim_{x \rightarrow 0^+} x^{1/x} = \lim_{x \rightarrow 0^+} e^{\ln y} = 0$$

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0, \text{ so } \lim_{x \rightarrow \infty} x^{1/x} = \lim_{x \rightarrow \infty} e^{\ln y} = 1$$

$$y = x^{1/x} = e^{\frac{1}{x} \ln x}$$

$$y' = \left(\frac{1}{x^2} - \frac{\ln x}{x^2} \right) e^{\frac{1}{x} \ln x}$$

$$y' = 0 \text{ when } x = e.$$

y is maximum at $x = e$ since $y' > 0$ on $(0, e)$ and

$y' < 0$ on (e, ∞) . When $x = e$, $y = e^{1/e}$.

44. a. The limit is of the form $(1+1)^\infty = 2^\infty$, which is not an indeterminate form.

$$\lim_{x \rightarrow 0^+} (1^x + 2^x)^{1/x} = \infty$$

- b. The limit is of the form $(1+1)^{-\infty} = 2^{-\infty}$, which is not an indeterminate form.

$$\lim_{x \rightarrow 0^-} (1^x + 2^x)^{1/x} = 0$$

- c. The limit is of the form ∞^0 .

Let $y = (1^x + 2^x)^{1/x}$, then

$$\ln y = \frac{1}{x} \ln(1^x + 2^x)$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} \ln(1^x + 2^x) = \lim_{x \rightarrow \infty} \frac{\ln(1^x + 2^x)}{x}$$

The limit is of the form $\frac{\infty}{\infty}$. (Apply

L'Hôpital's Rule twice.)

$$\lim_{x \rightarrow \infty} \frac{\ln(1^x + 2^x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1^x + 2^x} (1^x \ln 1 + 2^x \ln 2)}{1}$$

$$= \lim_{x \rightarrow \infty} \frac{2^x \ln 2}{1^x + 2^x} = \lim_{x \rightarrow \infty} \frac{2^x (\ln 2)^2}{1^x \ln 1 + 2^x \ln 2} = \ln 2$$

$$\lim_{x \rightarrow \infty} (1^x + 2^x)^{1/x} = \lim_{x \rightarrow \infty} e^{\ln y} = e^{\ln 2} = 2$$

- d. The limit is of the form 1^0 , since $1^x = 1$ for all x . This is not an indeterminate form.

$$\lim_{x \rightarrow -\infty} (1^x + 2^x)^{1/x} = 1$$

$$45. \lim_{n \rightarrow \infty} \frac{1^k + 2^k + \dots + n^k}{n^{k+1}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(\frac{1}{n} \right)^k + \left(\frac{2}{n} \right)^k + \dots + \left(\frac{n}{n} \right)^k \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{n^k} + \frac{2^k}{n^k} + \dots + 1 \right] = 0$$

$$46. \text{ Let } y = \left(\sum_{i=1}^n c_i x_i^t \right)^{1/t}, \text{ then } \ln y = \frac{1}{t} \ln \left(\sum_{i=1}^n c_i x_i^t \right).$$

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \ln \left(\sum_{i=1}^n c_i x_i^t \right) = \lim_{t \rightarrow 0^+} \frac{\ln \left(\sum_{i=1}^n c_i x_i^t \right)}{t}$$

The limit is of the form $\frac{0}{0}$, since $\sum_{i=1}^n c_i = 1$.

$$\lim_{t \rightarrow 0^+} \frac{\ln \left(\sum_{i=1}^n c_i x_i^t \right)}{t} = \lim_{t \rightarrow 0^+} \frac{1}{\sum_{i=1}^n c_i x_i^t} \sum_{i=1}^n c_i x_i^t \ln x_i$$

$$= \sum_{i=1}^n c_i \ln x_i = \sum_{i=1}^n \ln x_i^{c_i}$$

$$\lim_{t \rightarrow 0^+} \left(\sum_{i=1}^n c_i x_i^t \right)^{1/t} = \lim_{t \rightarrow 0^+} e^{\ln y}$$

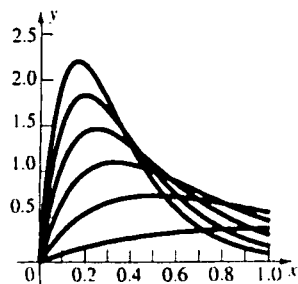
$$= e^{\sum_{i=1}^n \ln x_i^{c_i}} = x_1^{c_1} x_2^{c_2} \dots x_n^{c_n} = \prod_{i=1}^n x_i^{c_i}$$

$$47. \text{ a. } \lim_{t \rightarrow 0^+} \left(\frac{1}{2} 2^t + \frac{1}{2} 5^t \right)^{1/t} = \sqrt{2} \sqrt{5} \approx 3.162$$

$$\text{ b. } \lim_{t \rightarrow 0^+} \left(\frac{1}{5} 2^t + \frac{4}{5} 5^t \right)^{1/t} = \sqrt[5]{2} \cdot \sqrt[5]{5^4} \approx 4.163$$

$$\text{ c. } \lim_{t \rightarrow 0^+} \left(\frac{1}{10} 2^t + \frac{9}{10} 5^t \right)^{1/t} = \sqrt[10]{2} \cdot \sqrt[10]{5^9} \approx 4.562$$

48. a.



b. $n^2 x e^{-nx} = \frac{n^2 x}{e^{nx}}$, so the limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{n \rightarrow \infty} \frac{n^2 x}{e^{nx}} = \lim_{n \rightarrow \infty} \frac{2nx}{x e^{nx}}$$

This limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{n \rightarrow \infty} \frac{2nx}{x e^{nx}} = \lim_{n \rightarrow \infty} \frac{2x}{x^2 e^{nx}} = 0$$

$$\begin{aligned} \text{c. } \int_0^1 x e^{-x} dx &= \left[-x e^{-x} - e^{-x} \right]_0^1 = 1 - \frac{2}{e} \\ \int_0^1 4x e^{-2x} dx &= \left[-2x e^{-2x} - e^{-2x} \right]_0^1 = 1 - \frac{3}{e^2} \\ \int_0^1 9x e^{-3x} dx &= \left[-3x e^{-3x} - e^{-3x} \right]_0^1 = 1 - \frac{4}{e^3} \\ \int_0^1 16x e^{-4x} dx &= \left[-4x e^{-4x} - e^{-4x} \right]_0^1 = 1 - \frac{5}{e^4} \\ \int_0^1 25x e^{-5x} dx &= \left[-5x e^{-5x} - e^{-5x} \right]_0^1 = 1 - \frac{6}{e^5} \\ \int_0^1 36x e^{-6x} dx &= \left[-6x e^{-6x} - e^{-6x} \right]_0^1 = 1 - \frac{7}{e^6} \end{aligned}$$

9.3 Concepts Review

1. converge

$$2. \lim_{b \rightarrow \infty} \int_0^b \cos x dx$$

$$3. \int_{-\infty}^0 f(x) dx; \int_0^{\infty} f(x) dx$$

4. $p > 1$

Problem Set 9.3

In this section and the chapter review, it is understood that $[g(x)]_a^{\infty}$ means $\lim_{b \rightarrow \infty} [g(x)]_a^b$ and likewise for similar expressions.

$$\text{d. Guess: } \lim_{n \rightarrow \infty} \int_0^1 n^2 x e^{-nx} dx = 1$$

$$\int_0^1 n^2 x e^{-nx} dx = \left[-n x e^{-nx} - e^{-nx} \right]_0^1$$

$$= -(n+1)e^{-n} + 1 = 1 - \frac{n+1}{e^n}$$

$$\lim_{n \rightarrow \infty} \int_0^1 n^2 x e^{-nx} dx = \lim_{n \rightarrow \infty} \left(1 - \frac{n+1}{e^n} \right)$$

$$= 1 - \lim_{n \rightarrow \infty} \frac{n+1}{e^n} \text{ if this last limit exists. The}$$

limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{n \rightarrow \infty} \frac{n+1}{e^n} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0, \text{ so}$$

$$\lim_{n \rightarrow \infty} \int_0^1 n^2 x e^{-nx} dx = 1.$$

49. Note $f(x) > 0$ on $[0, \infty)$.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(\frac{x^{25}}{e^x} + \frac{x^3}{e^x} + \left(\frac{2}{e} \right)^x \right) = 0$$

Therefore there is no absolute minimum.

$$f'(x) = (25x^{24} + 3x^2 + 2^x \ln 2) e^{-x}$$

$$- (x^{25} + x^3 + 2^x) e^{-x}$$

$$= (-x^{25} + 25x^{24} - x^3 + 3x^2 - 2^x + 2^x \ln 2) e^{-x}$$

Solve for x when $f'(x) = 0$. Using a numerical method, $x \approx 25$.

A graph using a computer algebra system verifies that an absolute maximum occurs at about $x = 25$.

$$1. \int_{100}^{\infty} e^x dx = \left[e^x \right]_{100}^{\infty} = \infty - e^{100} = \infty$$

The integral diverges.

$$2. \int_{-\infty}^5 \frac{dx}{x^4} = \left[-\frac{1}{3x^3} \right]_{-\infty}^5 = -\frac{1}{3(-125)} - 0 = \frac{1}{375}$$

$$3. \int_1^{\infty} 2x e^{-x^2} dx = \left[-e^{-x^2} \right]_1^{\infty} = 0 - (-e^{-1}) = \frac{1}{e}$$

$$4. \int_{-\infty}^1 e^{4x} dx = \left[\frac{1}{4} e^{4x} \right]_{-\infty}^1 = \frac{1}{4} e^4 - 0 = \frac{1}{4} e^4$$

$$5. \int_9^{\infty} \frac{x dx}{\sqrt{1+x^2}} = \left[\sqrt{1+x^2} \right]_9^{\infty} = \infty - \sqrt{82} = \infty$$

The integral diverges.

$$6. \int_1^{\infty} \frac{dx}{\sqrt{\pi x}} = \left[2\sqrt{\frac{x}{\pi}} \right]_1^{\infty} = \infty - \frac{2}{\sqrt{\pi}} = \infty$$

The integral diverges.

$$7. \int_1^{\infty} \frac{dx}{x^{1.00001}} = \left[-\frac{1}{0.00001x^{0.00001}} \right]_1^{\infty} \\ = 0 - \left(-\frac{1}{0.00001} \right) = \frac{1}{0.00001} = 100,000$$

$$8. \int_0^{\infty} \frac{x}{1+x^2} dx = \frac{1}{2} \left[\ln(1+x^2) \right]_0^{\infty} \\ = \infty - \frac{1}{2} \ln|1| = \infty$$

The integral diverges.

$$9. \int_1^{\infty} \frac{dx}{x^{0.99999}} = \left[\frac{x^{0.00001}}{0.00001} \right]_1^{\infty} = \infty - 100,000 = \infty$$

The integral diverges.

$$10. \int_1^{\infty} \frac{x}{(1+x^2)^2} dx = \left[-\frac{1}{2(1+x^2)} \right]_1^{\infty} \\ = 0 - \left(-\frac{1}{4} \right) = \frac{1}{4}$$

$$11. \int_e^{\infty} \frac{1}{x \ln x} dx = [\ln(\ln x)]_e^{\infty} = \infty - 0 = \infty$$

The integral diverges.

$$12. \int_e^{\infty} \frac{\ln x}{x} dx = \left[\frac{1}{2} (\ln x)^2 \right]_e^{\infty} = \infty - \frac{1}{2} = \infty$$

The integral diverges.

$$17. \int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+9}} dx = \int_{-\infty}^0 \frac{x}{\sqrt{x^2+9}} dx + \int_0^{\infty} \frac{x}{\sqrt{x^2+9}} dx = \left[\sqrt{x^2+9} \right]_{-\infty}^0 + \left[\sqrt{x^2+9} \right]_0^{\infty} = (3 - \infty) + (\infty - 3)$$

The integral diverges since both $\int_{-\infty}^0 \frac{x}{\sqrt{x^2+9}} dx$ and $\int_0^{\infty} \frac{x}{\sqrt{x^2+9}} dx$ diverge.

$$18. \int_{-\infty}^{\infty} \frac{dx}{(x^2+16)^2} = \int_{-\infty}^0 \frac{dx}{(x^2+16)^2} + \int_0^{\infty} \frac{dx}{(x^2+16)^2} \\ \int \frac{dx}{(x^2+16)^2} = \frac{1}{128} \tan^{-1} \frac{x}{4} + \frac{x}{32(x^2+16)} \text{ by using the substitution } x = 4 \tan \theta. \\ \int_{-\infty}^0 \frac{dx}{(x^2+16)^2} = \left[\frac{1}{128} \tan^{-1} \frac{x}{4} + \frac{x}{32(x^2+16)} \right]_{-\infty}^0 = 0 - \left[\frac{1}{128} \left(-\frac{\pi}{2} \right) + 0 \right] = \frac{\pi}{256}$$

$$13. \text{ Let } u = \ln x, du = \frac{1}{x} dx, dv = \frac{1}{x^2} dx, v = -\frac{1}{x}.$$

$$\int_2^{\infty} \frac{\ln x}{x^2} dx = \left[-\frac{\ln x}{x} \right]_2^{\infty} + \int_2^{\infty} \frac{1}{x^2} dx \\ = \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_2^{\infty} = \lim_{b \rightarrow \infty} \left(-\frac{\ln b + 1}{b} \right) + \frac{\ln 2 + 1}{2} \\ = \frac{\ln 2 + 1}{2}$$

$$14. \int_1^{\infty} x e^{-x} dx \\ u = x, du = dx \\ dv = e^{-x} dx, v = -e^{-x} \\ \int_1^{\infty} x e^{-x} dx = \left[-x e^{-x} \right]_1^{\infty} + \int_1^{\infty} e^{-x} dx \\ = \left[-x e^{-x} - e^{-x} \right]_1^{\infty} = 0 - 0 - (-e^{-1} - e^{-1}) = \frac{2}{e}$$

$$15. \int_{-\infty}^1 \frac{dx}{(2x-3)^3} = \left[-\frac{1}{4(2x-3)^2} \right]_{-\infty}^1 \\ = -\frac{1}{4} - (-0) = -\frac{1}{4}$$

$$16. \int_4^{\infty} \frac{dx}{(\pi-x)^{2/3}} = \left[-3(\pi-x)^{1/3} \right]_4^{\infty} \\ = \infty + 3\sqrt[3]{\pi-4} = \infty \\ \text{The integral diverges.}$$

$$\int_0^{\infty} \frac{dx}{(x^2+16)^2} = \left[\frac{1}{128} \tan^{-1} \frac{x}{4} + \frac{x}{32(x^2+16)} \right]_0^{\infty} = \frac{1}{128} \left(\frac{\pi}{2} \right) + 0 - (0) = \frac{\pi}{256}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+16)^2} = \frac{\pi}{256} + \frac{\pi}{256} = \frac{\pi}{128}$$

$$19. \int_{-\infty}^{\infty} \frac{1}{x^2+2x+10} dx = \int_{-\infty}^{\infty} \frac{1}{(x+1)^2+9} dx = \int_{-\infty}^0 \frac{1}{(x+1)^2+9} dx + \int_0^{\infty} \frac{1}{(x+1)^2+9} dx$$

$$\int \frac{1}{(x+1)^2+9} dx = \frac{1}{3} \tan^{-1} \frac{x+1}{3} \text{ by using the substitution } x+1 = 3 \tan \theta.$$

$$\int_{-\infty}^0 \frac{1}{(x+1)^2+9} dx = \left[\frac{1}{3} \tan^{-1} \frac{x+1}{3} \right]_{-\infty}^0 = \frac{1}{3} \tan^{-1} \frac{1}{3} - \frac{1}{3} \left(-\frac{\pi}{2} \right) = \frac{1}{6} \left(\pi + 2 \tan^{-1} \frac{1}{3} \right)$$

$$\int_0^{\infty} \frac{1}{(x+1)^2+9} dx = \left[\frac{1}{3} \tan^{-1} \frac{x+1}{3} \right]_0^{\infty} = \frac{1}{3} \left(\frac{\pi}{2} \right) - \frac{1}{3} \tan^{-1} \frac{1}{3} = \frac{1}{6} \left(\pi - 2 \tan^{-1} \frac{1}{3} \right)$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+10} dx = \frac{1}{6} \left(\pi + 2 \tan^{-1} \frac{1}{3} \right) + \frac{1}{6} \left(\pi - 2 \tan^{-1} \frac{1}{3} \right) = \frac{\pi}{3}$$

$$20. \int_{-\infty}^{\infty} \frac{x}{e^{2|x|}} dx = \int_{-\infty}^0 \frac{x}{e^{-2x}} dx + \int_0^{\infty} \frac{x}{e^{2x}} dx$$

$$\text{For } \int_{-\infty}^0 \frac{x}{e^{-2x}} dx = \int_{-\infty}^0 x e^{2x} dx, \text{ use } u = x, du = dx, dv = e^{2x} dx, v = \frac{1}{2} e^{2x}.$$

$$\int_{-\infty}^0 x e^{2x} dx = \left[\frac{1}{2} x e^{2x} \right]_{-\infty}^0 - \frac{1}{2} \int_{-\infty}^0 e^{2x} dx = \left[\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \right]_{-\infty}^0 = 0 - \frac{1}{4} - (0) = -\frac{1}{4}$$

$$\text{For } \int_0^{\infty} \frac{x}{e^{2x}} dx = \int_0^{\infty} x e^{-2x} dx, \text{ use } u = x, du = dx, dv = e^{-2x} dx, v = -\frac{1}{2} e^{-2x}.$$

$$\int_0^{\infty} x e^{-2x} dx = \left[-\frac{1}{2} x e^{-2x} \right]_0^{\infty} + \frac{1}{2} \int_0^{\infty} e^{-2x} dx = \left[-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \right]_0^{\infty} = 0 - \left(0 - \frac{1}{4} \right) = \frac{1}{4}$$

$$\int_{-\infty}^{\infty} \frac{x}{e^{2|x|}} dx = -\frac{1}{4} + \frac{1}{4} = 0$$

$$21. \int_{-\infty}^{\infty} \operatorname{sech} x dx = \int_{-\infty}^0 \operatorname{sech} x dx + \int_0^{\infty} \operatorname{sech} x dx$$

$$= [\tan^{-1}(\sinh x)]_{-\infty}^0 + [\tan^{-1}(\sinh x)]_0^{\infty}$$

$$= \left[0 - \left(-\frac{\pi}{2} \right) \right] + \left[\frac{\pi}{2} - 0 \right] = \pi$$

$$= 0 - \ln \frac{e-1}{e+1} \approx 0.7719$$

$$\left(\lim_{b \rightarrow \infty} \ln \frac{b-1}{b+1} = 0 \text{ since } \lim_{b \rightarrow \infty} \frac{b-1}{b+1} = 1 \right)$$

$$22. \int_1^{\infty} \operatorname{csch} x dx = \int_1^{\infty} \frac{1}{\sinh x} dx = \int_1^{\infty} \frac{2}{e^x - e^{-x}} dx$$

$$= \int_1^{\infty} \frac{2e^x}{e^{2x} - 1} dx$$

$$\text{Let } u = e^x, du = e^x dx.$$

$$\int_1^{\infty} \frac{2e^x}{e^{2x} - 1} dx = \int_e^{\infty} \frac{2}{u^2 - 1} du = \int_e^{\infty} \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du$$

$$= [\ln(u-1) - \ln(u+1)]_e^{\infty} = \left[\ln \frac{u-1}{u+1} \right]_e^{\infty}$$

$$23. \int_0^{\infty} e^{-x} \cos x dx = \left[\frac{1}{2e^x} (\sin x - \cos x) \right]_0^{\infty}$$

$$= 0 - \frac{1}{2} (0 - 1) = \frac{1}{2}$$

(Use Formula 68 with $a = -1$ and $b = 1$.)

$$24. \int_0^{\infty} e^{-x} \sin x dx = \left[-\frac{1}{2e^x} (\cos x + \sin x) \right]_0^{\infty}$$

$$= 0 + \frac{1}{2} (1 + 0) = \frac{1}{2}$$

(Use Formula 67 with $a = -1$ and $b = 1$.)

25. The area is given by

$$\begin{aligned}\int_1^{\infty} \frac{2}{4x^2 - 1} dx &= \int_1^{\infty} \left(\frac{1}{2x-1} - \frac{1}{2x+1} \right) dx \\ &= \frac{1}{2} [\ln|2x-1| - \ln|2x+1|]_1^{\infty} = \frac{1}{2} \left[\ln \left| \frac{2x-1}{2x+1} \right| \right]_1^{\infty} \\ &= \frac{1}{2} \left(0 - \ln \left(\frac{1}{3} \right) \right) = \frac{1}{2} \ln 3\end{aligned}$$

Note: $\lim_{x \rightarrow \infty} \ln = \frac{2x-1}{2x+1} = 0$ since

$$\lim_{x \rightarrow \infty} \left(\frac{2x-1}{2x+1} \right) = 1.$$

26. The area is

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2 + x} dx &= \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \\ &= [\ln|x| - \ln|x+1|]_1^{\infty} = \left[\ln \left| \frac{x}{x+1} \right| \right]_1^{\infty} = 0 - \ln \frac{1}{2} = \ln 2\end{aligned}$$

30. $FP = \int_0^{\infty} e^{-0.08t} (100,000 + 1000t) dt$

$$= \left[-1,250,000e^{-0.08t} - 12,500te^{-0.08t} - 156,250e^{-0.08t} \right]_0^{\infty} = 1,406,250$$

The present value is \$1,406,250.

31. a. $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a 0 dx + \int_a^b \frac{1}{b-a} dx + \int_b^{\infty} 0 dx$

$$= 0 + \frac{1}{b-a} [x]_a^b + 0$$

$$= \frac{1}{b-a} (b-a)$$

b. $\mu = \int_{-\infty}^{\infty} x f(x) dx$

$$= \int_{-\infty}^a x \cdot 0 dx + \int_a^b x \frac{1}{b-a} dx + \int_b^{\infty} x \cdot 0 dx$$

$$= 0 + \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b + 0$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

$$= \frac{a+b}{2}$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 dx$$

$$= \int_{-\infty}^a (x-\mu)^2 \cdot 0 dx + \int_a^b (x-\mu)^2 \frac{1}{b-a} dx + \int_b^{\infty} (x-\mu)^2 \cdot 0 dx$$

27. The integral would take the form

$$k \int_{3960}^{\infty} \frac{1}{x} dx = [k \ln x]_{3960}^{\infty} = \infty$$

which would make it impossible to send anything out of the earth's gravitational field.

28. At $x = 1080$ mi, $F = 165$, so

$k = 165(1080)^2 \approx 1.925 \times 10^8$. So the work done in mi-lb is

$$\begin{aligned}1.925 \times 10^8 \int_{1080}^{\infty} \frac{1}{x^2} dx &= 1.925 \times 10^8 [-x^{-1}]_{1080}^{\infty} \\ &= \frac{1.925 \times 10^8}{1080} \approx 1.782 \times 10^5 \text{ mi-lb.}\end{aligned}$$

29. $FP = \int_0^{\infty} e^{-rt} f(t) dt = \int_0^{\infty} 100,000e^{-0.08t}$

$$= \left[-\frac{1}{0.08} 100,000e^{-0.08t} \right]_0^{\infty} = 1,250,000$$

The present value is \$1,250,000.

$$\begin{aligned}
&= 0 + \frac{1}{b-a} \left[\frac{(x-\mu)^3}{3} \right]_a^b + 0 \\
&= \frac{1}{b-a} \frac{(b-\mu)^3 - (a-\mu)^3}{3} \\
&= \frac{1}{b-a} \frac{b^3 - 3b^2\mu + 3b\mu^2 - a^3 + 3a^2\mu - 3a\mu^2}{3}
\end{aligned}$$

Next, substitute $\mu = (a+b)/2$ to obtain

$$\begin{aligned}
\sigma^2 &= \frac{1}{3(b-a)} \left[\frac{1}{4}b^3 - \frac{3}{4}b^2a + \frac{3}{4}ba^2 - \frac{1}{4}a^3 \right] \\
&= \frac{1}{12(b-a)} (b-a)^3 \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

c.

$$\begin{aligned}
P(X < 2) &= \int_{-\infty}^2 f(x) dx \\
&= \int_{-\infty}^0 0 dx + \int_0^2 \frac{1}{10-0} dx \\
&= \frac{2}{10} = \frac{1}{5}
\end{aligned}$$

32. a.

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 0 dx + \int_0^{\infty} \frac{\beta}{\theta} \left(\frac{x}{\theta} \right)^{\beta-1} e^{-(x/\theta)^\beta} dx$$

In the second integral, let $u = (x/\theta)^\beta$. Then,

$du = (\beta/\theta)(x/\theta)^{\beta-1} dx$. When $x = 0, u = 0$ and when $x \rightarrow \infty, u \rightarrow \infty$. Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) dx &= \int_0^{\infty} \frac{\beta}{\theta} \left(\frac{x}{\theta} \right)^{\beta-1} e^{-(x/\theta)^\beta} dx \\
&= \int_0^{\infty} e^{-u} du \\
&= \left[-e^{-u} \right]_0^{\infty} \\
&= -0 + e^0 = 1
\end{aligned}$$

b.

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} xf(x) dx = \int_{-\infty}^0 x \cdot 0 dx + \int_0^{\infty} \frac{\beta}{\theta} x \left(\frac{x}{\theta} \right)^{\beta-1} e^{-(x/\theta)^\beta} dx \\
&= \frac{2}{3} \int_0^{\infty} x^2 e^{-(x/3)^2} dx = \frac{3}{2} \sqrt{\pi} \\
\sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_{-\infty}^0 (x-\mu)^2 \cdot 0 dx + \frac{2}{9} \int_0^{\infty} (x-\mu)^2 x e^{-(x^2/9)} dx \\
&= \frac{3}{2} \sqrt{\pi} - \mu = \frac{3}{2} \sqrt{\pi} - \frac{3}{2} \sqrt{\pi} = 0
\end{aligned}$$

c. The probability of being less than 2 is

$$\begin{aligned}
\int_{-\infty}^2 f(x) dx &= \int_{-\infty}^0 0 dx + \int_0^2 \frac{\beta}{\theta} \left(\frac{x}{\theta} \right)^{\beta-1} e^{-(x/\theta)^\beta} dx = 0 + \left[-e^{-(x/\theta)^\beta} \right]_0^2 \\
&= 1 - e^{-(2/\theta)^\beta} = 1 - e^{-(2/3)^2} \approx 0.359
\end{aligned}$$

$$33. \quad \text{a.} \quad \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 0 dx + \int_0^{\infty} \alpha e^{-\alpha x} dx$$

$$= 0 + \left[-e^{-\alpha x} \right]_0^{\infty} = 1$$

$$\text{b.} \quad \mu = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^0 x \cdot 0 dx + \int_0^{\infty} x \alpha e^{-\alpha x} dx$$

$$= \int_0^{\infty} x \alpha e^{-\alpha x} dx$$

Integrate by parts: Let $u = x$, $dv = \alpha e^{-\alpha x}$. Then

$du = dx$, $v = -e^{-\alpha x}$. Thus,

$$\mu = \left[-x e^{-\alpha x} \right]_0^{\infty} - \int_0^{\infty} (-e^{-\alpha x}) dx$$

$$= [0 - 0] + \int_0^{\infty} e^{-\alpha x} dx$$

$$= \left[-\frac{1}{\alpha} e^{-\alpha x} \right]_0^{\infty} = -0 + \frac{1}{\alpha} = \frac{1}{\alpha}$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

$$= \int_{-\infty}^0 (x - \mu)^2 \cdot 0 dx + \int_0^{\infty} (x - \mu)^2 \alpha e^{-\alpha x} dx$$

$$= 0 + \int_0^{\infty} (x^2 - 2x\mu + \mu^2) \alpha e^{-\alpha x} dx$$

$$= \int_0^{\infty} x^2 \alpha e^{-\alpha x} dx - 2\mu \int_0^{\infty} x \alpha e^{-\alpha x} dx$$

$$+ \mu^2 \int_0^{\infty} \alpha e^{-\alpha x} dx$$

$$= \int_0^{\infty} x^2 \alpha e^{-\alpha x} dx - 2 \frac{1}{\alpha} \left(\frac{1}{\alpha} \right) + \left(\frac{1}{\alpha} \right)^2$$

On this last line, we have used the results that

$$\mu = 1/\alpha, \quad \int_0^{\infty} \alpha e^{-\alpha x} dx = 1, \quad \text{and}$$

$$\int_0^{\infty} x \alpha e^{-\alpha x} dx = 1/\alpha. \quad \text{To evaluate the integral,}$$

use integration by parts. Let

$$u = x^2, dv = \alpha e^{-\alpha x}. \quad \text{Then, } du = 2x dx \quad \text{and}$$

$$v = -e^{-\alpha x}. \quad \text{Thus}$$

$$\sigma^2 = \left[x^2 e^{-\alpha x} \right]_0^{\infty} - \int_0^{\infty} (-e^{-\alpha x}) 2x dx$$

$$- 2 \frac{1}{\alpha} \left(\frac{1}{\alpha} \right) + \left(\frac{1}{\alpha} \right)^2$$

$$= 0 + \frac{2}{\alpha} \int_0^{\infty} x \alpha e^{-\alpha x} dx - 2 \frac{1}{\alpha^2} + \frac{1}{\alpha^2}$$

$$= 2 \frac{1}{\alpha^2} - 2 \frac{1}{\alpha^2} + \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$$

$$34. \quad u = Ar \int_a^{\infty} \frac{dx}{(r^2 + x^2)^{3/2}}$$

$$= \frac{A}{r} \left[\frac{x}{\sqrt{r^2 + x^2}} \right]_a^{\infty} = \frac{A}{r} \left(1 - \frac{a}{\sqrt{r^2 + a^2}} \right)$$

Note that $\int \frac{dx}{(r^2 + x^2)^{3/2}} = \frac{x}{r^2 \sqrt{r^2 + x^2}}$ by using the substitution $x = r \tan \theta$.

35. a. $\int_{-\infty}^{\infty} \sin x \, dx = \int_{-\infty}^0 \sin x \, dx + \int_0^{\infty} \sin x \, dx$
 $= \lim_{a \rightarrow -\infty} [-\cos x]_0^a + \lim_{a \rightarrow \infty} [-\cos x]_a^0$
 Both do not converge since $-\cos x$ is oscillating between -1 and 1 , so the integral diverges.

b. $\lim_{a \rightarrow \infty} \int_{-a}^a \sin x \, dx = \lim_{a \rightarrow \infty} [-\cos x]_{-a}^a$
 $= \lim_{a \rightarrow \infty} [-\cos a + \cos(-a)]$
 $= \lim_{a \rightarrow \infty} [-\cos a + \cos a] = \lim_{a \rightarrow \infty} 0 = 0$

36. a. The total mass of the wire is
 $\int_0^{\infty} \frac{1}{1+x^2} \, dx = \frac{\pi}{2}$ from Example 4.

b. $\int_0^{\infty} \frac{x}{1+x^2} \, dx = \left[\frac{1}{2} \ln |1+x^2| \right]_0^{\infty}$ which diverges. Thus, the wire does not have a center of mass.

37. For example, the region under the curve $y = \frac{1}{x}$ to the right of $x = 1$.
 Rotated about the x -axis the volume is
 $\pi \int_1^{\infty} \frac{1}{x^2} \, dx = \pi$. Rotated about the y -axis, the volume is $2\pi \int_1^{\infty} \frac{1}{x} \, dx$ which diverges.

38. a. Suppose $\lim_{x \rightarrow \infty} f(x) = M \neq 0$, so the limit exists but is non-zero. Since $\lim_{x \rightarrow \infty} f(x) = M$, there is some $N > 0$ such that when $x \geq N$,
 $|f(x) - M| \leq \frac{M}{2}$, or
 $M - \frac{M}{2} \leq f(x) \leq M + \frac{M}{2}$
 Since $f(x)$ is nonnegative, $M > 0$, thus
 $\frac{M}{2} > 0$ and
 $\int_0^{\infty} f(x) \, dx = \int_0^N f(x) \, dx + \int_N^{\infty} f(x) \, dx$
 $\geq \int_0^N f(x) \, dx + \int_N^{\infty} \frac{M}{2} \, dx = \int_0^N f(x) \, dx + \left[\frac{Mx}{2} \right]_N^{\infty} = \infty$
 so the integral diverges. Thus, if the limit exists, it must be 0.

b. For example, let $f(x)$ be given by

$$f(x) = \begin{cases} 2n^2x - 2n^3 + 1 & \text{if } n - \frac{1}{2n^2} \leq x \leq n \\ -2n^2x + 2n^3 + 1 & \text{if } n < x \leq n + \frac{1}{2n^2} \\ 0 & \text{otherwise} \end{cases}$$

for every positive integer n .

$$f\left(n - \frac{1}{2n^2}\right) = 2n^2\left(n - \frac{1}{2n^2}\right) - 2n^3 + 1$$

$$= 2n^3 - 1 - 2n^3 + 1 = 0$$

$$f(n) = 2n^2(n) - 2n^3 + 1 = 1$$

$$\lim_{x \rightarrow n^+} f(x) = \lim_{x \rightarrow n^+} (-2n^2x + 2n^3 + 1) = 1 = f(n)$$

$$f\left(n + \frac{1}{2n^2}\right) = -2n^2\left(n + \frac{1}{2n^2}\right) + 2n^3 + 1$$

$$= -2n^3 - 1 + 2n^3 + 1 = 0$$

Thus, f is continuous at

$$n - \frac{1}{2n^2}, n, \text{ and } n + \frac{1}{2n^2}.$$

Note that the intervals

$$\left[n, n + \frac{1}{2n^2}\right] \text{ and } \left[n + 1 - \frac{1}{2(n+1)^2}, n + 1\right]$$

will never overlap since $\frac{1}{2n^2} \leq \frac{1}{2}$ and

$$\frac{1}{2(n+1)^2} \leq \frac{1}{8}.$$

The graph of f consists of a series of isosceles triangles, each of height 1, vertices at

$$\left(n - \frac{1}{2n^2}, 0\right), (n, 1), \text{ and } \left(n + \frac{1}{2n^2}, 0\right),$$

based on the x -axis, and centered over each integer n .

$\lim_{x \rightarrow \infty} f(x)$ does not exist, since $f(x)$ will be 1

at each integer, but 0 between the triangles.

Each triangle has area

$$\frac{1}{2}bh = \frac{1}{2} \left[n + \frac{1}{2n^2} - \left(n - \frac{1}{2n^2} \right) \right] (1)$$

$$= \frac{1}{2} \left(\frac{1}{n^2} \right) = \frac{1}{2n^2}$$

$\int_0^{\infty} f(x) \, dx$ is the area in all of the triangles, thus

$$\int_0^{\infty} f(x) \, dx = \sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n^2} \leq \frac{1}{2} + \frac{1}{2} \int_1^{\infty} \frac{1}{x^2} dx$$

$$= \frac{1}{2} + \frac{1}{2} \left[-\frac{1}{x} \right]_1^{\infty} = \frac{1}{2} + \frac{1}{2} (-0 + 1) = 1$$

(By viewing $\sum_{n=2}^{\infty} \frac{1}{n^2}$ as a lower Riemann sum

for $\frac{1}{x^2}$)

Thus, $\int_0^{\infty} f(x) dx$ converges, although $\lim_{x \rightarrow \infty} f(x)$ does not exist.

$$39. \int_1^{100} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{100} = 0.99$$

$$\int_1^{100} \frac{1}{x^{1.1}} dx = \left[-\frac{1}{0.1x^{0.1}} \right]_1^{100} \approx 3.69$$

$$\int_1^{100} \frac{1}{x^{1.01}} dx = \left[-\frac{1}{0.01x^{0.01}} \right]_1^{100} \approx 4.50$$

$$\int_1^{100} \frac{1}{x} dx = [\ln x]_1^{100} = \ln 100 \approx 4.61$$

$$\int_1^{100} \frac{1}{x^{0.99}} dx = \left[\frac{x^{0.01}}{0.01} \right]_1^{100} \approx 4.71$$

9.4 Concepts Review

1. unbounded

2. 2

$$3. \lim_{x \rightarrow 4^-} \int_0^x \frac{1}{\sqrt{4-x}} dx$$

4. $p < 1$

Problem Set 9.4

$$1. \int_1^3 \frac{dx}{(x-1)^{1/3}} = \lim_{b \rightarrow 1^+} \left[\frac{3(x-1)^{2/3}}{2} \right]_b^3$$

$$= \frac{3}{2} \sqrt[3]{2^2} - \lim_{b \rightarrow 1^+} \frac{3(b-1)^{2/3}}{2} = \frac{3}{\sqrt[3]{2}} - 0 = \frac{3}{\sqrt[3]{2}}$$

$$40. \int_{-10}^{10} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} \left[\tan^{-1} x \right]_{-10}^{10}$$

$$\approx \frac{2.9423}{\pi} \approx 0.937$$

$$\int_{-50}^{50} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} \left[\tan^{-1} x \right]_{-50}^{50}$$

$$\approx \frac{3.1016}{\pi} \approx 0.987$$

$$\int_{-100}^{100} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} \left[\tan^{-1} x \right]_{-100}^{100}$$

$$\approx \frac{3.1216}{\pi} \approx 0.994$$

$$41. \int_{-1}^1 \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2) dx \approx 0.6827$$

$$\int_{-2}^2 \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2) dx \approx 0.9545$$

$$\int_{-3}^3 \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2) dx \approx 0.9973$$

$$\int_{-4}^4 \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2) dx \approx 0.9999$$

$$2. \int_1^3 \frac{dx}{(x-1)^{4/3}} = \lim_{b \rightarrow 1^+} \left[-\frac{3}{(x-1)^{1/3}} \right]_b^3$$

$$= -\frac{3}{\sqrt[3]{2}} + \lim_{b \rightarrow 1^+} \frac{3}{(x-1)^{1/3}} = -\frac{3}{\sqrt[3]{2}} + \infty$$

The integral diverges.

$$3. \int_3^{10} \frac{dx}{\sqrt{x-3}} = \lim_{b \rightarrow 3^+} \left[2\sqrt{x-3} \right]_b^{10}$$

$$= 2\sqrt{7} - \lim_{b \rightarrow 3^+} 2\sqrt{b-3} = 2\sqrt{7}$$

$$4. \int_0^9 \frac{dx}{\sqrt{9-x}} = \lim_{b \rightarrow 9^-} \left[-2\sqrt{9-x} \right]_0^b$$

$$= \lim_{b \rightarrow 9^-} -2\sqrt{9-b} + 2\sqrt{9} = 6$$

$$5. \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b \rightarrow 1^-} \left[\sin^{-1} x \right]_0^b$$

$$= \lim_{b \rightarrow 1^-} \sin^{-1} b - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\begin{aligned}
 6. \quad \int_{100}^{\infty} \frac{x}{\sqrt{1+x^2}} dx &= \lim_{b \rightarrow \infty} \left[\sqrt{1+x^2} \right]_{100}^b \\
 &= \lim_{b \rightarrow \infty} \sqrt{1+b^2} + \sqrt{10,001} = \infty
 \end{aligned}$$

The integral diverges.

$$\begin{aligned}
 &= \left(\lim_{b \rightarrow 0^-} -\frac{1}{2b^2} + \frac{1}{2} \right) + \left(-\frac{1}{18} + \lim_{b \rightarrow 0^+} \frac{1}{2b^2} \right) \\
 &= \left(-\infty + \frac{1}{2} \right) + \left(-\frac{1}{8} + \infty \right)
 \end{aligned}$$

The integral diverges.

$$\begin{aligned}
 7. \quad \int_{-1}^3 \frac{1}{x^3} dx &= \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{1}{x^3} dx + \lim_{b \rightarrow 0^+} \int_b^3 \frac{1}{x^3} dx \\
 &= \lim_{b \rightarrow 0^-} \left[-\frac{1}{2x^2} \right]_{-1}^b + \lim_{b \rightarrow 0^+} \left[-\frac{1}{2x^2} \right]_b^3
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \int_5^{-5} \frac{1}{x^{2/3}} dx &= \lim_{b \rightarrow 0^+} \int_5^b \frac{1}{x^{2/3}} dx + \lim_{b \rightarrow 0^-} \int_b^{-5} \frac{1}{x^{2/3}} dx = \lim_{b \rightarrow 0^+} \left[3x^{1/3} \right]_5^b + \lim_{b \rightarrow 0^-} \left[3x^{1/3} \right]_b^{-5} \\
 &= \lim_{b \rightarrow 0^+} 3b^{1/3} - 3\sqrt[3]{5} + 3\sqrt[3]{-5} - \lim_{b \rightarrow 0^-} 3b^{1/3} = 0 - 3\sqrt[3]{5} + 3\sqrt[3]{-5} - 0 = 3\sqrt[3]{-5} - 3\sqrt[3]{5}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \int_{-1}^{128} x^{-5/7} dx &= \lim_{b \rightarrow 0^-} \int_{-1}^b x^{-5/7} dx + \lim_{b \rightarrow 0^+} \int_b^{128} x^{-5/7} dx = \lim_{b \rightarrow 0^-} \left[\frac{7}{2} x^{2/7} \right]_{-1}^b + \lim_{b \rightarrow 0^+} \left[\frac{7}{2} x^{2/7} \right]_b^{128} \\
 &= \lim_{b \rightarrow 0^-} \frac{7}{2} b^{2/7} - \frac{7}{2} (-1)^{2/7} + \frac{7}{2} (128)^{2/7} - \lim_{b \rightarrow 0^+} \frac{7}{2} b^{2/7} = 0 - \frac{7}{2} + \frac{7}{2} (4) - 0 = \frac{21}{2}
 \end{aligned}$$

$$10. \quad \int_0^1 \frac{x}{\sqrt[3]{1-x^2}} dx = \lim_{b \rightarrow 1^-} \int_0^b \frac{x}{\sqrt[3]{1-x^2}} dx = \lim_{b \rightarrow 1^-} \left[-\frac{3}{4} (1-x^2)^{2/3} \right]_0^b = \lim_{b \rightarrow 1^-} -\frac{3}{4} (1-b^2)^{2/3} + \frac{3}{4} = -0 + \frac{3}{4} = \frac{3}{4}$$

$$\begin{aligned}
 11. \quad \int_0^4 \frac{dx}{(2-3x)^{1/3}} &= \lim_{b \rightarrow \frac{2}{3}^-} \int_0^b \frac{dx}{(2-3x)^{1/3}} + \lim_{b \rightarrow \frac{2}{3}^+} \int_b^4 \frac{dx}{(2-3x)^{1/3}} = \lim_{b \rightarrow \frac{2}{3}^-} \left[-\frac{1}{2} (2-3x)^{2/3} \right]_0^b + \lim_{b \rightarrow \frac{2}{3}^+} \left[-\frac{1}{2} (2-3x)^{2/3} \right]_b^4 \\
 &= \lim_{b \rightarrow \frac{2}{3}^-} -\frac{1}{2} (2-3b)^{2/3} + \frac{1}{2} (2)^{2/3} - \frac{1}{2} (-10)^{2/3} + \lim_{b \rightarrow \frac{2}{3}^+} \frac{1}{2} (2-3b)^{2/3} \\
 &= 0 + \frac{1}{2} 2^{2/3} - \frac{1}{2} 10^{2/3} + 0 = \frac{1}{2} (2^{2/3} - 10^{2/3})
 \end{aligned}$$

$$12. \quad \int_{\sqrt{5}}^{\sqrt{8}} \frac{x}{(16-2x^2)^{2/3}} dx = \lim_{b \rightarrow \sqrt{8}^-} \left[-\frac{3}{4} (16-2x^2)^{1/3} \right]_{\sqrt{5}}^b = \lim_{b \rightarrow \sqrt{8}^-} -\frac{3}{4} (16-2b^2)^{1/3} + \frac{3}{4} \sqrt[3]{6} = \frac{3}{4} \sqrt[3]{6}$$

$$\begin{aligned}
 13. \quad \int_0^4 \frac{x}{16-2x^2} dx &= \lim_{b \rightarrow -\sqrt{8}^+} \int_0^b \frac{x}{16-2x^2} dx + \lim_{b \rightarrow -\sqrt{8}^-} \int_b^4 \frac{x}{16-2x^2} dx \\
 &= \lim_{b \rightarrow -\sqrt{8}^+} \left[-\frac{1}{4} \ln |16-2x^2| \right]_0^b + \lim_{b \rightarrow -\sqrt{8}^-} \left[-\frac{1}{4} \ln |16-2x^2| \right]_b^4 \\
 &= \lim_{b \rightarrow -\sqrt{8}^+} -\frac{1}{4} \ln |16-2b^2| + \frac{1}{4} \ln 16 - \frac{1}{4} \ln 16 + \lim_{b \rightarrow -\sqrt{8}^-} \frac{1}{4} \ln |16-2b^2| \\
 &= \left[-(-\infty) + \frac{1}{4} \ln 16 \right] + \left[-\frac{1}{4} \ln 16 + (-\infty) \right]
 \end{aligned}$$

The integral diverges.

$$14. \int_0^3 \frac{x}{\sqrt{9-x^2}} dx = \lim_{b \rightarrow 3^-} \left[-\sqrt{9-x^2} \right]_0^b = \lim_{b \rightarrow 3^-} -\sqrt{9-b^2} + \sqrt{9} = 3$$

$$15. \int_{-2}^1 \frac{dx}{(x+1)^{4/3}} = \lim_{b \rightarrow -1^-} \left[-\frac{3}{(x+1)^{1/3}} \right]_{-2}^b = \lim_{b \rightarrow -1^-} -\frac{3}{(b+1)^{1/3}} + \frac{3}{(-1)^{1/3}} = -(-\infty) - 3$$

The integral diverges.

$$16. \text{ Note that } \int \frac{dx}{x^2+x-2} = \int \frac{dx}{(x-1)(x+2)} = \int \left[\frac{1}{3(x-1)} - \frac{1}{3(x+2)} \right] dx \text{ by using a partial fraction decomposition.}$$

$$\begin{aligned} \int_0^3 \frac{dx}{x^2+x-2} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{x^2+x-2} + \lim_{b \rightarrow 1^+} \int_b^3 \frac{dx}{x^2+x-2} \\ &= \lim_{b \rightarrow 1^-} \left[\frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|x+2| \right]_0^b + \lim_{b \rightarrow 1^+} \left[\frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|x+2| \right]_b^3 \\ &= \lim_{b \rightarrow 1^-} \left[\frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| \right]_0^b + \lim_{b \rightarrow 1^+} \left[\frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| \right]_b^3 = \lim_{b \rightarrow 1^-} \frac{1}{3} \ln \left| \frac{b-1}{b+2} \right| - \frac{1}{3} \ln \frac{1}{2} + \frac{1}{3} \ln \frac{2}{5} - \lim_{b \rightarrow 1^+} \frac{1}{3} \ln \left| \frac{b-1}{b+2} \right| \\ &= \left(-\infty - \frac{1}{3} \ln \frac{1}{2} \right) + \left(\frac{1}{3} \ln \frac{2}{5} + \infty \right) \end{aligned}$$

The integral diverges.

$$17. \text{ Note that } \frac{1}{x^3-x^2-x+1} = \frac{1}{2(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{4(x+1)}$$

$$\begin{aligned} \int_0^3 \frac{dx}{x^3-x^2-x+1} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{x^3-x^2-x+1} + \lim_{b \rightarrow 1^+} \int_b^3 \frac{dx}{x^3-x^2-x+1} \\ &= \lim_{b \rightarrow 1^-} \left[-\frac{1}{2(x-1)} - \frac{1}{4} \ln|x-1| + \frac{1}{4} \ln|x+1| \right]_0^b + \lim_{b \rightarrow 1^+} \left[-\frac{1}{2(x-1)} - \frac{1}{4} \ln|x-1| + \frac{1}{4} \ln|x+1| \right]_b^3 \\ &= \lim_{b \rightarrow 1^-} \left\{ \left(-\frac{1}{2(b-1)} + \frac{1}{4} \ln \left| \frac{b+1}{b-1} \right| \right) + \left(-\frac{1}{2} + 0 \right) + \left[-\frac{1}{4} + \frac{1}{4} \ln 2 - \left(-\frac{1}{2(b-1)} + \frac{1}{4} \ln \left| \frac{b+1}{b-1} \right| \right) \right] \right\} \\ &= \left(\infty + \infty - \frac{1}{2} \right) + \left(-\frac{1}{4} + \frac{1}{4} \ln 2 + \infty - \infty \right) \end{aligned}$$

The integral diverges.

$$18. \text{ Note that } \frac{x^{1/3}}{x^{2/3}-9} = \frac{1}{x^{1/3}} - \frac{9}{x^{1/3}(x^{2/3}-9)}.$$

$$\begin{aligned} \int_0^{27} \frac{x^{1/3}}{x^{2/3}-9} dx &= \lim_{b \rightarrow 27^-} \left[\frac{3}{2} x^{2/3} + \frac{27}{2} \ln|x^{2/3}-9| \right]_0^b = \lim_{b \rightarrow 27^-} \left(\frac{3}{2} b^{2/3} + \frac{27}{2} \ln|b^{2/3}-9| \right) - \left(0 + \frac{27}{2} \ln 9 \right) \\ &= \frac{27}{2} - \infty - \frac{27}{2} \ln 9 \end{aligned}$$

The integral diverges.

$$\begin{aligned} 19. \int_0^{\pi/4} \tan 2x dx &= \lim_{b \rightarrow \pi/4^-} \left[-\frac{1}{2} \ln|\cos 2x| \right]_0^b \\ &= \lim_{b \rightarrow \pi/4^-} -\frac{1}{2} \ln|\cos 2b| + \frac{1}{2} \ln 1 = -(-\infty) + 0 \end{aligned}$$

The integral diverges.

$$\begin{aligned} 20. \int_0^{\pi/2} \csc x dx &= \lim_{b \rightarrow 0^+} \left[\ln|\csc x - \cot x| \right]_b^{\pi/2} \\ &= \ln|1-0| - \lim_{b \rightarrow 0^+} \ln|\csc b - \cot b| \\ &= 0 - \lim_{b \rightarrow 0^+} \ln \left| \frac{1-\cos b}{\sin b} \right| \\ &= \lim_{b \rightarrow 0^+} \frac{1-\cos b}{\sin b} \text{ is of the form } \frac{0}{0}. \end{aligned}$$

$$\lim_{b \rightarrow 0^+} \frac{1 - \cos b}{\sin b} = \lim_{b \rightarrow 0^+} \frac{\sin b}{\cos b} = \frac{0}{1} = 0$$

Thus, $\lim_{b \rightarrow 0^+} \ln \left| \frac{1 - \cos b}{\sin b} \right| = -\infty$ and the integral diverges.

$$\begin{aligned} 21. \int_0^{\pi/2} \frac{\sin x}{1 - \cos x} dx &= \lim_{b \rightarrow 0^+} [\ln |1 - \cos x|]_b^{\pi/2} \\ &= \ln 1 - \lim_{b \rightarrow 0^+} \ln |1 - \cos b| = 0 - (-\infty) \end{aligned}$$

The integral diverges.

$$\begin{aligned} 22. \int_0^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin x}} dx &= \lim_{b \rightarrow 0^+} \left[\frac{3}{2} \sin^{2/3} x \right]_b^{\pi/2} \\ &= \frac{3}{2} (1)^{2/3} - \frac{3}{2} (0)^{2/3} = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} 23. \int_0^{\pi/2} \tan^2 x \sec^2 x dx &= \lim_{b \rightarrow \frac{\pi}{2}^-} \left[\frac{1}{3} \tan^3 x \right]_0^b \\ &= \lim_{b \rightarrow \frac{\pi}{2}^-} \frac{1}{3} \tan^3 b - \frac{1}{3} (0)^3 = \infty \end{aligned}$$

The integral diverges.

$$24. \int_0^{\pi/4} \frac{\sec^2 x}{(\tan x - 1)^2} dx = \lim_{b \rightarrow \frac{\pi}{4}^-} \left[-\frac{1}{\tan x - 1} \right]_0^b$$

$$28. \text{ Note that } \sqrt{4x - x^2} = \sqrt{4 - (x^2 - 4x + 4)} = \sqrt{2^2 - (x - 2)^2}.$$

$$\int_2^4 \frac{dx}{\sqrt{4x - x^2}} = \lim_{b \rightarrow 4^-} \int_2^b \frac{dx}{\sqrt{4x - x^2}} = \lim_{b \rightarrow 4^-} \left[\sin^{-1} \frac{x-2}{2} \right]_2^b = \lim_{b \rightarrow 4^-} \sin^{-1} \frac{b-2}{2} - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$29. \int_1^e \frac{dx}{x \ln x} = \lim_{b \rightarrow 1^+} [\ln(\ln x)]_b^e = \ln(\ln e) - \lim_{b \rightarrow 1^+} \ln(\ln b) = \ln 1 - \ln 0 = 0 + \infty$$

The integral diverges.

$$30. \int_1^{10} \frac{dx}{x \ln^{100} x} = \lim_{b \rightarrow 1^+} \left[-\frac{1}{99 \ln^{99} x} \right]_b^{10} = -\frac{1}{99 \ln^{99} 10} + \lim_{b \rightarrow 1^+} \frac{1}{99 \ln^{99} b} = -\frac{1}{99 \ln^{99} 10} + \infty$$

The integral diverges.

$$\begin{aligned} 31. \int_{2c}^{4c} \frac{dx}{\sqrt{x^2 - 4c^2}} &= \lim_{b \rightarrow 2c^+} \left[\ln \left| x + \sqrt{x^2 - 4c^2} \right| \right]_b^{4c} = \ln \left[(4 + 2\sqrt{3})c \right] - \lim_{b \rightarrow 2c^+} \ln \left| b + \sqrt{b^2 - 4c^2} \right| \\ &= \ln \left[(4 + 2\sqrt{3})c \right] - \ln 2c = \ln(2 + \sqrt{3}) \end{aligned}$$

$$= \lim_{b \rightarrow \frac{\pi}{4}^-} -\frac{1}{\tan b - 1} + \frac{1}{0 - 1} = -(-\infty) - 1$$

The integral diverges.

$$25. \text{ Since } \frac{1 - \cos x}{2} = \sin^2 \frac{x}{2},$$

$$\frac{1}{\cos x - 1} = -\frac{1}{2} \csc^2 \frac{x}{2}.$$

$$\begin{aligned} \int_0^{\pi} \frac{dx}{\cos x - 1} &= \lim_{b \rightarrow 0^+} \left[\cot \frac{x}{2} \right]_b^{\pi} \\ &= \cot \frac{\pi}{2} - \lim_{b \rightarrow 0^+} \cot \frac{b}{2} = 0 - \infty \end{aligned}$$

The integral diverges.

$$\begin{aligned} 26. \int_{-3}^{-1} \frac{dx}{x \sqrt{\ln(-x)}} &= \lim_{b \rightarrow -1^-} \left[2\sqrt{\ln(-x)} \right]_{-3}^b \\ &= \lim_{b \rightarrow -1^-} 2\sqrt{\ln(-b)} - 2\sqrt{\ln 3} = 0 - 2\sqrt{\ln 3} \\ &= -2\sqrt{\ln 3} \end{aligned}$$

$$\begin{aligned} 27. \int_0^{\ln 3} \frac{e^x dx}{\sqrt{e^x - 1}} &= \lim_{b \rightarrow 0^+} \left[2\sqrt{e^x - 1} \right]_b^{\ln 3} \\ &= 2\sqrt{3 - 1} - \lim_{b \rightarrow 0^+} 2\sqrt{e^b - 1} = 2\sqrt{2} - 0 = 2\sqrt{2} \end{aligned}$$

$$\begin{aligned}
32. \quad \int_c^{2c} \frac{x \, dx}{\sqrt{x^2 + xc - 2c^2}} &= \int_c^{2c} \frac{x \, dx}{\sqrt{\left(x + \frac{c}{2}\right)^2 - \frac{9}{4}c^2}} = \int_c^{2c} \frac{\left(x + \frac{c}{2}\right) dx}{\sqrt{\left(x + \frac{c}{2}\right)^2 - \frac{9}{4}c^2}} - \frac{c}{2} \int_0^{2c} \frac{dx}{\sqrt{\left(x + \frac{c}{2}\right)^2 - \frac{9}{4}c^2}} \\
&= \lim_{b \rightarrow c^+} \left[\sqrt{x^2 + xc - 2c^2} - \frac{c}{2} \ln \left| x + \frac{c}{2} + \sqrt{x^2 + xc - 2c^2} \right| \right]_b^{2c} \\
&= \sqrt{4c^2} - \frac{c}{2} \ln \left| \frac{5c}{2} + \sqrt{4c^2} \right| - \lim_{b \rightarrow c^+} \left[\sqrt{b^2 + bc - 2c^2} - \frac{c}{2} \ln \left| b + \frac{c}{2} + \sqrt{b^2 + bc - 2c^2} \right| \right] \\
&= 2c - \frac{c}{2} \ln \frac{9c}{2} - \left(0 - \frac{c}{2} \ln \left| \frac{3c}{2} + 0 \right| \right) = 2c - \frac{c}{2} \ln \frac{9c}{2} + \frac{c}{2} \ln \frac{3c}{2} = 2c - \frac{c}{2} \ln 3
\end{aligned}$$

33. For $0 < c < 1$, $\frac{1}{\sqrt{x(1+x)}}$ is continuous. Let $u = \frac{1}{1+x}$, $du = -\frac{1}{(1+x)^2} dx$.

$$dv = \frac{1}{\sqrt{x}} dx, v = 2\sqrt{x}.$$

$$\int_c^1 \frac{1}{\sqrt{x(1+x)}} dx = \left[\frac{2\sqrt{x}}{1+x} \right]_c^1 + 2 \int_c^1 \frac{\sqrt{x} dx}{(1+x)^2} = \frac{2}{2} - \frac{2\sqrt{c}}{1+c} + 2 \int_c^1 \frac{\sqrt{x} dx}{(1+x)^2} = 1 - \frac{2\sqrt{c}}{1+c} + 2 \int_c^1 \frac{\sqrt{x} dx}{(1+x)^2}$$

$$\text{Thus, } \lim_{c \rightarrow 0} \int_c^1 \frac{1}{\sqrt{x(1+x)}} dx = \lim_{c \rightarrow 0} \left[1 - \frac{2\sqrt{c}}{1+c} + 2 \int_c^1 \frac{\sqrt{x} dx}{(1+x)^2} \right] = 1 - 0 + 2 \int_0^1 \frac{\sqrt{x} dx}{(1+x)^2}$$

This last integral is a proper integral.

34. Let $u = \frac{1}{\sqrt{1+x}}$, $du = -\frac{1}{2(1+x)^{3/2}} dx$

$$dv = \frac{1}{\sqrt{x}} dx, v = 2\sqrt{x}.$$

$$\text{For } 0 < c < 1, \int_c^1 \frac{dx}{\sqrt{x(1+x)}} = \left[\frac{2\sqrt{x}}{\sqrt{1+x}} \right]_c^1 + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}} dx = \frac{2\sqrt{1}}{\sqrt{2}} - \frac{2\sqrt{c}}{\sqrt{1+c}} + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}} dx$$

$$\text{Thus, } \int_0^1 \frac{dx}{\sqrt{x(1+x)}} = \lim_{c \rightarrow 0} \int_c^1 \frac{dx}{\sqrt{x(1+x)}} = \lim_{c \rightarrow 0} \left[\frac{2\sqrt{x}}{\sqrt{1+x}} - \frac{2\sqrt{c}}{\sqrt{1+c}} + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}} dx \right] = \sqrt{2} - 0 + \int_0^1 \frac{\sqrt{x}}{(1+x)^{3/2}} dx$$

This is a proper integral.

$$\begin{aligned}
35. \quad \int_{-3}^3 \frac{x}{\sqrt{9-x^2}} dx &= \int_{-3}^0 \frac{x}{\sqrt{9-x^2}} dx + \int_0^3 \frac{x}{\sqrt{9-x^2}} dx = \lim_{b \rightarrow -3^+} \left[-\sqrt{9-x^2} \right]_b^0 + \lim_{b \rightarrow 3^-} \left[-\sqrt{9-x^2} \right]_0^b \\
&= -\sqrt{9} + \lim_{b \rightarrow -3^+} \sqrt{9-b^2} - \lim_{b \rightarrow 3^-} \sqrt{9-b^2} + \sqrt{9} = -3 + 0 - 0 + 3 = 0
\end{aligned}$$

$$\begin{aligned}
36. \quad \int_{-3}^3 \frac{x}{9-x^2} dx &= \int_{-3}^0 \frac{x}{9-x^2} dx + \int_0^3 \frac{x}{9-x^2} dx = \lim_{b \rightarrow -3^+} \left[-\frac{1}{2} \ln |9-x^2| \right]_b^0 + \lim_{b \rightarrow 3^-} \left[-\frac{1}{2} \ln |9-x^2| \right]_0^b \\
&= -\ln 3 + \lim_{b \rightarrow -3^+} \frac{1}{2} \ln |9-b^2| - \lim_{b \rightarrow 3^-} \frac{1}{2} \ln |9-b^2| + \ln 3 = (-\ln 3 - \infty) + (\infty + \ln 3)
\end{aligned}$$

The integral diverges.

$$37. \quad \int_{-4}^4 \frac{1}{16-x^2} dx = \int_{-4}^0 \frac{1}{16-x^2} dx + \int_0^4 \frac{1}{16-x^2} dx = \lim_{b \rightarrow -4^+} \left[\frac{1}{8} \ln \left| \frac{x+4}{x-4} \right| \right]_b^0 + \lim_{b \rightarrow 4^-} \left[\frac{1}{8} \ln \left| \frac{x+4}{x-4} \right| \right]_0^b$$

$$= \frac{1}{8} \ln 1 - \lim_{b \rightarrow -4^+} \frac{1}{8} \ln \left| \frac{b+4}{b-4} \right| + \lim_{b \rightarrow -4^-} \frac{1}{8} \ln \left| \frac{b+4}{b-4} \right| - \frac{1}{8} \ln 1 = (0 + \infty) + (\infty - 0)$$

The integral diverges.

$$\begin{aligned} 38. \int_{-1}^1 \frac{1}{x\sqrt{-\ln|x|}} dx &= \int_{-1}^{-1/2} \frac{1}{x\sqrt{-\ln|x|}} dx + \int_{-1/2}^0 \frac{1}{x\sqrt{-\ln|x|}} dx + \int_0^{1/2} \frac{1}{x\sqrt{-\ln|x|}} dx + \int_{1/2}^1 \frac{1}{x\sqrt{-\ln|x|}} dx \\ &= \lim_{b \rightarrow -1^+} \left[-2\sqrt{-\ln|x|} \right]_b^{-1/2} + \lim_{b \rightarrow 0^-} \left[-2\sqrt{-\ln|x|} \right]_{-1/2}^b + \lim_{b \rightarrow 0^+} \left[-2\sqrt{-\ln|x|} \right]_b^{1/2} + \lim_{b \rightarrow 1^-} \left[-2\sqrt{-\ln|x|} \right]_{1/2}^b \\ &= (-2\sqrt{\ln 2} + 0) + (-\infty + 2\sqrt{\ln 2}) + (-2\sqrt{\ln 2} + \infty) + (0 + 2\sqrt{\ln 2}) \end{aligned}$$

The integral diverges.

$$39. \int_0^\infty \frac{1}{x^p} dx = \int_0^1 \frac{1}{x^p} dx + \int_1^\infty \frac{1}{x^p} dx$$

$$\text{If } p > 1, \int_0^1 \frac{1}{x^p} dx = \left[\frac{1}{-p+1} x^{-p+1} \right]_0^1 \text{ diverges}$$

$$\text{since } \lim_{x \rightarrow 0^+} x^{-p+1} = \infty.$$

$$\text{If } p < 1 \text{ and } p \neq 0, \int_1^\infty \frac{1}{x^p} dx = \left[\frac{1}{-p+1} x^{-p+1} \right]_1^\infty$$

$$\text{diverges since } \lim_{x \rightarrow \infty} x^{-p+1} = \infty.$$

$$\text{If } p = 0, \int_0^\infty dx = \infty.$$

$$\text{If } p = 1, \text{ both } \int_0^1 \frac{1}{x} dx \text{ and } \int_1^\infty \frac{1}{x} dx \text{ diverge.}$$

$$\begin{aligned} 40. \int_0^\infty f(x) dx &= \lim_{b \rightarrow 1^-} \int_0^b f(x) dx + \lim_{b \rightarrow 1^+} \int_b^c f(x) dx + \lim_{b \rightarrow \infty} \int_c^b f(x) dx \\ &\text{where } 1 < c < \infty. \end{aligned}$$

$$\begin{aligned} 41. \int_0^8 (x-8)^{-2/3} dx &= \lim_{b \rightarrow 8^-} \left[3(x-8)^{1/3} \right]_0^b \\ &= 3(0) - 3(-2) = 6 \end{aligned}$$

$$\begin{aligned} 42. \int_0^1 \left(\frac{1}{x} - \frac{1}{x^3 + x} \right) dx &= \lim_{b \rightarrow 0^-} \int_b^1 \frac{x}{x^2 + 1} dx = \lim_{b \rightarrow 0^-} \left[\frac{1}{2} \ln|x^2 + 1| \right]_b^1 \\ &= \frac{1}{2} \ln 2 - \lim_{b \rightarrow 0^-} \frac{1}{2} \ln|b^2 + 1| = \frac{1}{2} \ln 2 \end{aligned}$$

$$43. \text{ a. } \int_0^1 x^{-2/3} dx = \lim_{b \rightarrow 0^+} \left[3x^{1/3} \right]_b^1 = 3$$

$$\begin{aligned} \text{b. } V &= \pi \int_0^1 x^{-4/3} dx = \lim_{b \rightarrow 0^+} \pi \left[-3x^{-1/3} \right]_b^1 \\ &= -3\pi + 3\pi \lim_{b \rightarrow 0^+} b^{-1/3} \end{aligned}$$

The limit tends to infinity as $b \rightarrow 0$, so the volume is infinite.

$$\begin{aligned} 44. \text{ Since } \ln x < 0 \text{ for } 0 < x < 1, b > 1 \\ \int_0^b \ln x dx &= \lim_{c \rightarrow 0^-} \int_c^1 \ln x dx + \int_1^b \ln x dx \\ &= \lim_{c \rightarrow 0^+} [x \ln x - x]_c^1 + [x \ln x - x]_1^b \\ &= -1 - \lim_{c \rightarrow 0^+} (c \ln c - c) + b \ln b - b + 1 \\ &= b \ln b - b \end{aligned}$$

Thus, $b \ln b - b = 0$ when $b = e$.

$$\begin{aligned} 45. \int_0^1 \frac{\sin x}{x} dx &\text{ is not an improper integral since } \\ \frac{\sin x}{x} &\text{ is bounded in the interval } 0 \leq x \leq 1. \end{aligned}$$

$$46. \text{ For } x \geq 1, \frac{1}{1+x^4} < 1 \text{ so } \frac{1}{x^4(1+x^4)} < \frac{1}{x^4}.$$

$$\begin{aligned} \int_1^\infty \frac{1}{x^4} dx &= \lim_{b \rightarrow \infty} \left[-\frac{1}{3x^3} \right]_1^b = -\lim_{b \rightarrow \infty} \frac{1}{3b^3} + \frac{1}{3} \\ &= -0 + \frac{1}{3} = \frac{1}{3} \end{aligned}$$

Thus, by the Comparison Test $\int_1^\infty \frac{1}{x^4(1+x^4)} dx$ converges.

$$\begin{aligned} 47. \text{ For } x \geq 1, x^2 \geq x \text{ so } -x^2 \leq -x, \text{ thus } \\ e^{-x^2} &\leq e^{-x}. \end{aligned}$$

$$\int_1^\infty e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}]_1^b = -\lim_{b \rightarrow \infty} \frac{1}{e^b} + e^{-1}$$

$$= -0 + \frac{1}{e} = \frac{1}{e}$$

Thus, by the Comparison Test, $\int_1^{\infty} e^{-x^2} dx$ converges.

48. Since $\sqrt{x+2} - 1 \leq \sqrt{x+2}$ we know that $\frac{1}{\sqrt{x+2}-1} \geq \frac{1}{\sqrt{x+2}}$. Consider $\int_0^{\infty} \frac{1}{\sqrt{x+2}} dx$
- $$\int_2^{\infty} \frac{1}{\sqrt{x+2}} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{\sqrt{x+2}} dx$$
- $$= \lim_{b \rightarrow \infty} \left[2\sqrt{x+2} \right]_2^{\infty} = \lim_{b \rightarrow \infty} 2(\sqrt{b+2} - 2) = \infty$$

Thus, by the Comparison Test of Problem 46, we conclude that $\int_0^{\infty} \frac{1}{\sqrt{x+2}} dx$ diverges.

49. Since $x^2 \ln(x+1) \geq x^2$, we know that $\frac{1}{x^2 \ln(x+1)} \leq \frac{1}{x^2}$. Since $\int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{\infty} = 1$ we can apply the Comparison Test of Problem 46 to conclude that $\int_1^{\infty} \frac{1}{x^2 \ln(x+1)} dx$ converges.

50. If $0 \leq f(x) \leq g(x)$ on $[a, b]$ and either $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \infty$ or $\lim_{x \rightarrow b} f(x) = \lim_{x \rightarrow b} g(x) = \infty$, then the convergence of $\int_a^b g(x) dx$ implies the convergence of $\int_a^b f(x) dx$ and the divergence of $\int_a^b f(x) dx$ implies the divergence of $\int_a^b g(x) dx$.

51. a. From Example 2 of Section 9.2, $\lim_{x \rightarrow \infty} \frac{x^a}{e^x} = 0$ for any positive real number. Thus $\lim_{x \rightarrow \infty} \frac{x^{n+1}}{e^x} = 0$ for any positive real number n , hence there is a number M such that $0 < \frac{x^{n+1}}{e^x} \leq 1$ for $x \geq M$. Divide the inequality by x^2 to get that $0 < \frac{x^{n-1}}{e^x} \leq \frac{1}{x^2}$ for $x \geq M$.

- b. $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = -\lim_{b \rightarrow \infty} \frac{1}{b} + \frac{1}{1} = -0 + 1 = 1$
- $$\int_1^{\infty} x^{n-1} e^{-x} dx = \int_1^M x^{n-1} e^{-x} dx + \int_M^{\infty} x^{n-1} e^{-x} dx$$
- $$\leq \int_1^M x^{n-1} e^{-x} dx + \int_1^{\infty} \frac{1}{x^2} dx$$
- $$= 1 + \int_1^M x^{n-1} e^{-x} dx$$
- by part a and Problem 46. The remaining integral is finite, so $\int_1^{\infty} x^{n-1} e^{-x} dx$ converges.

52. $\int_0^1 e^{-x} dx = \left[-e^{-x} \right]_0^1 = -e^{-1} + 1 = 1 - \frac{1}{e}$, so the integral converges when $n = 1$. For $0 \leq x \leq 1$, $0 \leq x^{n-1} \leq 1$ for $n > 1$. Thus, $\frac{x^{n-1}}{e^x} = x^{n-1} e^{-x} \leq e^{-x}$. By the comparison test from Problem 50, $\int_0^1 x^{n-1} e^{-x} dx$ converges.

53. a. $\Gamma(1) = \int_0^{\infty} x^0 e^{-x} dx = \left[-e^{-x} \right]_0^{\infty} = 1$

- b. $\Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx$
- Let $u = x^n$, $dv = e^{-x} dx$,
 $du = nx^{n-1} dx$, $v = -e^{-x}$.
- $$\Gamma(n+1) = \left[-x^n e^{-x} \right]_0^{\infty} + \int_0^{\infty} nx^{n-1} e^{-x} dx$$
- $$= 0 + n \int_0^{\infty} x^{n-1} e^{-x} dx = n\Gamma(n)$$

- c. From parts a and b,
 $\Gamma(1) = 1$, $\Gamma(2) = 1 \cdot \Gamma(1) = 1$,
 $\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2!$.
 Suppose $\Gamma(n) = (n-1)!$, then by part b,
 $\Gamma(n+1) = n\Gamma(n) = n[(n-1)!] = n!$.

54. $n = 1$, $\int_0^{\infty} e^{-x} dx = 1 = 0! = (1-1)!$
 $n = 2$, $\int_0^{\infty} xe^{-x} dx = 1 = 1! = (2-1)!$
 $n = 3$, $\int_0^{\infty} x^2 e^{-x} dx = 2 = 2! = (3-1)!$
 $n = 4$, $\int_0^{\infty} x^3 e^{-x} dx = 6 = 3! = (4-1)!$
 $n = 5$, $\int_0^{\infty} x^4 e^{-x} dx = 24 = 4! = (5-1)!$

55. a. The integral is the area between the curve $y^2 = \frac{1-x}{x}$ and the x -axis from $x = 0$ to $x = 1$.

$$y^2 = \frac{1-x}{x}; xy^2 = 1-x; x(y^2+1) = 1$$

$$x = \frac{1}{y^2+1}$$

As $x \rightarrow 0$, $y = \sqrt{\frac{1-x}{x}} \rightarrow \infty$, while

when $x = 1$, $y = \sqrt{\frac{1-1}{1}} = 0$, thus the area is

$$\int_0^\infty \frac{1}{y^2+1} dy = \lim_{b \rightarrow \infty} [\tan^{-1} y]_0^b$$

$$= \lim_{b \rightarrow \infty} \tan^{-1} b - \tan^{-1} 0 = \frac{\pi}{2}$$

- b. The integral is the area between the curve

$$y^2 = \frac{1+x}{1-x} \text{ and the } x\text{-axis from } x = -1 \text{ to}$$

$$x = 1.$$

$$y^2 = \frac{1+x}{1-x}; y^2 - xy^2 = 1+x; y^2 - 1 = x(y^2+1);$$

$$x = \frac{y^2-1}{y^2+1}$$

When $x = -1$, $y = \sqrt{\frac{1+(-1)}{1-(-1)}} = \sqrt{\frac{0}{2}} = 0$, while

as $x \rightarrow 1$, $y = \sqrt{\frac{1+x}{1-x}} \rightarrow \infty$.

The area in question is the area to the right of

the curve $y = \sqrt{\frac{1+x}{1-x}}$ and to the left of the line $x = 1$. Thus, the area is

$$\int_0^\infty \left(1 - \frac{y^2-1}{y^2+1}\right) dy = \int_0^\infty \frac{2}{y^2+1} dy$$

$$= \lim_{b \rightarrow \infty} [2 \tan^{-1} y]_0^b$$

$$\lim_{b \rightarrow \infty} 2 \tan^{-1} b - 2 \tan^{-1} 0 = 2 \left(\frac{\pi}{2}\right) = \pi$$

56. For $0 < x < 1$, $x^p > x^q$ so $2x^p > x^p + x^q$ and

$$\frac{1}{x^p + x^q} > \frac{1}{2x^p}. \text{ For } 1 < x, x^q > x^p \text{ so}$$

$$2x^q > x^p + x^q \text{ and } \frac{1}{x^p + x^q} > \frac{1}{2x^q}.$$

$$\int_0^\infty \frac{1}{x^p + x^q} dx = \int_0^1 \frac{1}{x^p + x^q} dx + \int_1^\infty \frac{1}{x^p + x^q} dx$$

Both of these integrals must converge.

$$\int_0^1 \frac{1}{x^p + x^q} dx > \int_0^1 \frac{1}{2x^p} dx = \frac{1}{2} \int_0^1 \frac{1}{x^p} dx \text{ which}$$

converges if and only if $p < 1$.

$$\int_1^\infty \frac{1}{x^p + x^q} dx > \int_1^\infty \frac{1}{2x^q} dx = \frac{1}{2} \int_1^\infty \frac{1}{x^q} dx \text{ which}$$

converges if and only if $q > 1$. Thus, $0 < p < 1$ and $1 < q$.

9.5 Chapter Review

Concepts Test

1. True: See Example 2 of Section 9.2.

2. True: Use l'Hôpital's Rule.

3. False: $\lim_{x \rightarrow \infty} \frac{1000x^4 + 1000}{0.001x^4 + 1} = \frac{1000}{0.001} = 10^6$

4. False: $\lim_{x \rightarrow \infty} xe^{-1/x} = \infty$ since $e^{-1/x} \rightarrow 1$ and $x \rightarrow \infty$ as $x \rightarrow \infty$.

5. False: For example, if $f(x) = x$ and $g(x) = e^x$,

$$\lim_{x \rightarrow \infty} \frac{x}{e^x} = 0.$$

6. False: See Example 7 of Section 9.2.

7. True: Take the inner limit first.

8. True: Raising a small number to a large exponent results in an even smaller number.

9. True: Since $\lim_{x \rightarrow a} f(x) = -1 \neq 0$, it serves only to affect the sign of the limit of the product.

10. False: Consider $f(x) = (x-a)^2$ and

$$g(x) = \frac{1}{(x-a)^2}, \text{ then } \lim_{x \rightarrow a} f(x) = 0$$

and $\lim_{x \rightarrow a} g(x) = \infty$, while
 $\lim_{x \rightarrow a} [f(x)g(x)] = 1$.

11. False: Consider $f(x) = 3x^2$ and $g(x) = x^2 + 1$, then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{3x^2}{x^2 + 1} = \lim_{x \rightarrow \infty} \frac{3}{1 + \frac{1}{x^2}} = 3, \text{ but}$$

$$\begin{aligned} \lim_{x \rightarrow \infty} [f(x) - 3g(x)] &= \lim_{x \rightarrow \infty} [3x^2 - 3(x^2 + 1)] \\ &= \lim_{x \rightarrow \infty} [-3] = -3 \end{aligned}$$

12. True: As $x \rightarrow a$, $f(x) \rightarrow 2$ while

$$\frac{1}{|g(x)|} \rightarrow \infty.$$

13. True: See Example 7 of Section 9.2.

14. True: Let $y = [1 + f(x)]^{1/f(x)}$, then

$$\ln y = \frac{1}{f(x)} \ln[1 + f(x)].$$

$$\lim_{x \rightarrow a} \frac{1}{f(x)} \ln[1 + f(x)] = \lim_{x \rightarrow a} \frac{\ln[1 + f(x)]}{f(x)}$$

This limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow a} \frac{\ln[1 + f(x)]}{f(x)} = \lim_{x \rightarrow a} \frac{\frac{1}{1+f(x)} f'(x)}{f'(x)}$$

$$= \lim_{x \rightarrow a} \frac{1}{1 + f(x)} = 1$$

$$\lim_{x \rightarrow a} [1 + f(x)]^{1/f(x)} = \lim_{x \rightarrow a} e^{\ln y} = e^1 = e$$

15. True: Use repeated applications of l'Hôpital's Rule.

16. True: $e^0 = 1$ and $p(0)$ is the constant term.

17. False: Consider $f(x) = 3x^2 + x + 1$ and $g(x) = 4x^3 + 2x + 3$; $f'(x) = 6x + 1$ $g'(x) = 12x^2 + 2$, and so
 $\lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 0} \frac{6x + 1}{12x^2 + 2} = \frac{1}{2}$ while

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{3x^2 + x + 1}{4x^3 + 2x + 3} = \frac{1}{3}$$

18. False: $p > 1$. See Example 4 of Section 9.4.

19. True: $\int_0^{\infty} \frac{1}{x^p} dx = \int_0^1 \frac{1}{x^p} dx + \int_1^{\infty} \frac{1}{x^p} dx$;
 $\int_0^1 \frac{1}{x^p} dx$ diverges for $p \geq 1$ and
 $\int_1^{\infty} \frac{1}{x^p} dx$ diverges for $p \leq 1$.

20. False: Consider $\int_0^{\infty} \frac{1}{x+1} dx$.

21. True: $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$
 If f is an even function, then $f(-x) = f(x)$ so
 $\int_{-\infty}^0 f(x) dx = \int_0^{\infty} f(x) dx$.
 Thus, both integrals making up $\int_{-\infty}^{\infty} f(x) dx$ converge so their sum converges.

22. False: See Problem 33 of Section 9.3.

23. True: $\int_0^{\infty} f'(x) dx = \lim_{b \rightarrow \infty} \int_0^b f'(x) dx$
 $= \lim_{b \rightarrow \infty} [f(x)]_0^b = \lim_{b \rightarrow \infty} f(b) - f(0)$
 $= 0 - f(0) = -f(0)$.
 $f(0)$ must exist and be finite since $f'(x)$ is continuous on $[0, \infty)$.

24. True: $\int_0^{\infty} f(x) dx \leq \int_0^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}]_0^b$
 $= \lim_{b \rightarrow \infty} -e^{-b} + 1 = 1$, so $\int_0^{\infty} f(x) dx$ must converge.

25. False: The integrand is bounded on the interval $\left[0, \frac{\pi}{4}\right]$.

Sample Test Problems

1. The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{4x}{\tan x} = \lim_{x \rightarrow 0} \frac{4}{\sec^2 x} = 4$$

2. The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{\tan 2x}{\sin 3x} = \lim_{x \rightarrow 0} \frac{2 \sec^2 2x}{3 \cos 3x} = \frac{2}{3}$$

3. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's

Rule twice.)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x - \tan x}{\frac{1}{3}x^2} &= \lim_{x \rightarrow 0} \frac{\cos x - \sec^2 x}{\frac{2}{3}x} \\ &= \lim_{x \rightarrow 0} \frac{-\sin x - 2 \sec x (\sec x \tan x)}{\frac{2}{3}} = 0 \end{aligned}$$

4. $\lim_{x \rightarrow 0} \frac{\cos x}{x^2} = \infty$ (L'Hôpital's Rule does not apply since $\cos(0) = 1$.)

5. $\lim_{x \rightarrow 0} 2x \cot x = \lim_{x \rightarrow 0} \frac{2x \cos x}{\sin x}$

The limit is of the form $\frac{0}{0}$.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2x \cos x}{\sin x} &= \lim_{x \rightarrow 0} \frac{2 \cos x - 2x \sin x}{\cos x} \\ &= \frac{2 - 0}{1} = 2 \end{aligned}$$

6. The limit is of the form $\frac{\infty}{\infty}$.

$$\begin{aligned} \lim_{x \rightarrow 1^-} \frac{\ln(1-x)}{\cot \pi x} &= \lim_{x \rightarrow 1^-} \frac{-\frac{1}{1-x}}{-\pi \csc^2 \pi x} \\ &= \lim_{x \rightarrow 1^-} \frac{\sin^2 \pi x}{\pi(1-x)} \end{aligned}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 1^-} \frac{\sin^2 \pi x}{\pi(1-x)} = \lim_{x \rightarrow 1^-} \frac{2\pi \sin \pi x \cos \pi x}{-\pi} = 0$$

7. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{t \rightarrow \infty} \frac{\ln t}{t^2} = \lim_{t \rightarrow \infty} \frac{\frac{1}{t}}{2t} = \lim_{t \rightarrow \infty} \frac{1}{2t^2} = 0$$

8. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x \rightarrow \infty} \frac{2x^3}{\ln x} = \lim_{x \rightarrow \infty} \frac{6x^2}{\frac{1}{x}} = \lim_{x \rightarrow \infty} 6x^3 = \infty$$

9. As $x \rightarrow 0$, $\sin x \rightarrow 0$, and $\frac{1}{x} \rightarrow \infty$. A number less than 1, raised to a large power, is a very small number $\left(\left(\frac{1}{2}\right)^{32} = 2.328 \times 10^{-10}\right)$ so

$$\lim_{x \rightarrow 0^+} (\sin x)^{1/x} = 0.$$

10. $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -x = 0$$

11. The limit is of the form 0^0 .

Let $y = x^x$, then $\ln y = x \ln x$.

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -x = 0$$

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{\ln y} = 1$$

12. The limit is of the form 1^∞ .

Let $y = (1 + \sin x)^{2/x}$, then $\ln y = \frac{2}{x} \ln(1 + \sin x)$.

$$\lim_{x \rightarrow 0} \frac{2}{x} \ln(1 + \sin x) = \lim_{x \rightarrow 0} \frac{2 \ln(1 + \sin x)}{x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{2 \ln(1 + \sin x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{2}{1 + \sin x} \cos x}{1}$$

$$= \lim_{x \rightarrow 0} \frac{2 \cos x}{1 + \sin x} = \frac{2}{1} = 2$$

$$\lim_{x \rightarrow 0} (1 + \sin x)^{2/x} = \lim_{x \rightarrow 0} e^{\ln y} = e^2$$

13. $\lim_{x \rightarrow 0^+} \sqrt{x} \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}} = \lim_{x \rightarrow 0^+} -2\sqrt{x} = 0$$

14. The limit is of the form ∞^0 .

Let $y = t^{1/t}$, then $\ln y = \frac{1}{t} \ln t$.

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln t = \lim_{t \rightarrow \infty} \frac{\ln t}{t}$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{t \rightarrow \infty} \frac{\ln t}{t} = \lim_{t \rightarrow \infty} \frac{1}{1} = \lim_{t \rightarrow \infty} \frac{1}{t} = 0$$

$$\lim_{t \rightarrow \infty} t^{1/t} = \lim_{t \rightarrow \infty} e^{\ln y} = 1$$

15. $\lim_{x \rightarrow 0^+} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0^+} \frac{x - \sin x}{x \sin x}$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's

Rule twice.)

$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x \sin x} = \lim_{x \rightarrow 0^+} \frac{1 - \cos x}{\sin x + x \cos x}$$

$$= \lim_{x \rightarrow 0^+} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0$$

16. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's

Rule three times.)

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 3x}{\tan x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{3 \sec^2 3x}{\sec^2 x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{3 \cos^2 x}{\cos^2 3x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x \sin x}{\cos 3x \sin 3x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos^2 x - \sin^2 x}{3(\cos^2 3x - \sin^2 3x)} = -\frac{1}{3(0-1)} = \frac{1}{3}$$

17. The limit is of the form 1^∞ .

Let $y = (\sin x)^{\tan x}$, then $\ln y = \tan x \ln(\sin x)$.

$$\lim_{x \rightarrow \frac{\pi}{2}} \tan x \ln(\sin x) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x \ln(\sin x)}{\cos x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x \ln(\sin x)}{\cos x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x \ln(\sin x) + \frac{\sin x}{\sin x} \cos x}{\sin x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x(1 + \ln(\sin x))}{\sin x} = \frac{0}{1} = 0$$

$$\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x} = \lim_{x \rightarrow \frac{\pi}{2}} e^{\ln y} = 1$$

18. $\lim_{x \rightarrow \frac{\pi}{2}} \left(x \tan x - \frac{\pi}{2} \sec x \right) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{x \sin x - \frac{\pi}{2}}{\cos x}$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{x \sin x - \frac{\pi}{2}}{\cos x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x + x \cos x}{-\sin x} = \frac{1}{-1} = -1$$

19. $\int_0^\infty \frac{dx}{(x+1)^2} = \left[-\frac{1}{x+1} \right]_0^\infty = 0 + 1 = 1$

20. $\int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1} x \right]_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

21. $\int_{-\infty}^1 e^{2x} dx = \left[\frac{1}{2} e^{2x} \right]_{-\infty}^1 = \frac{1}{2} e^2 - 0 = \frac{1}{2} e^2$

22. $\int_{-1}^1 \frac{dx}{1-x} = \lim_{b \rightarrow 1} [-\ln(1-x)]_{-1}^b$
 $= -\lim_{b \rightarrow 1} \ln(1-b) + \ln 2 = \infty$

The integral diverges.

23. $\int_0^\infty \frac{dx}{x+1} = [\ln(x+1)]_0^\infty = \infty - 0 = \infty$

The integral diverges.

24. $\int_{\frac{1}{2}}^2 \frac{dx}{x(\ln x)^{1/5}} = \lim_{b \rightarrow 1^-} \int_{\frac{1}{2}}^b \frac{dx}{x(\ln x)^{1/5}} + \lim_{b \rightarrow 1^+} \int_b^2 \frac{dx}{x(\ln x)^{1/5}} = \lim_{b \rightarrow 1^-} \left[\frac{5}{4} (\ln x)^{4/5} \right]_{\frac{1}{2}}^b + \lim_{b \rightarrow 1^+} \left[\frac{5}{4} (\ln x)^{4/5} \right]_b^2$
 $= \left(\frac{5}{4}(0) - \frac{5}{4} \left(\ln \frac{1}{2} \right)^{4/5} \right) + \left(\frac{5}{4} (\ln 2)^{4/5} - \frac{5}{4}(0) \right) = \frac{5}{4} (\ln 2)^{4/5} - \frac{5}{4} \left(\ln \frac{1}{2} \right)^{4/5} = \frac{5}{4} [(\ln 2)^{4/5} - (-\ln 2)^{4/5}]$
 $= \frac{5}{4} [(\ln 2)^{4/5} - (\ln 2)^{4/5}] = 0$

$$25. \int_1^{\infty} \frac{dx}{x^2 + x^4} = \int_1^{\infty} \left(\frac{1}{x^2} - \frac{1}{1+x^2} \right) dx = \left[-\frac{1}{x} - \tan^{-1} x \right]_1^{\infty} = 0 - \frac{\pi}{2} + 1 + \tan^{-1} 1 = 1 + \frac{\pi}{4} - \frac{\pi}{2} = 1 - \frac{\pi}{4}$$

$$26. \int_{-\infty}^1 \frac{dx}{(2-x)^2} = \left[\frac{1}{2-x} \right]_{-\infty}^1 = \frac{1}{1} - 0 = 1$$

$$27. \int_{-2}^0 \frac{dx}{2x+3} = \lim_{b \rightarrow -\frac{3}{2}^-} \int_{-2}^b \frac{dx}{2x+3} + \lim_{b \rightarrow -\frac{3}{2}^+} \int_b^0 \frac{dx}{2x+3} = \lim_{b \rightarrow -\frac{3}{2}^-} \left[\frac{1}{2} \ln |2x+3| \right]_{-2}^b + \lim_{b \rightarrow -\frac{3}{2}^+} \left[\frac{1}{2} \ln |2x+3| \right]_b^0$$

$$= \left(\lim_{b \rightarrow -\frac{3}{2}^-} \frac{1}{2} \ln |2b+3| - \frac{1}{2} \ln(1) \right) + \left(\frac{1}{2} \ln 3 - \lim_{b \rightarrow -\frac{3}{2}^+} \frac{1}{2} \ln |2b+3| \right) = (-\infty) + \left(\frac{1}{2} \ln 3 + \infty \right)$$

The integral diverges.

$$28. \int_1^4 \frac{dx}{\sqrt{x-1}} = \lim_{b \rightarrow 1^+} [2\sqrt{x-1}]_b^4 = 2\sqrt{3} - \lim_{b \rightarrow 1^+} 2\sqrt{x-1} = 2\sqrt{3} - 0 = 2\sqrt{3}$$

$$29. \int_2^{\infty} \frac{dx}{x(\ln x)^2} = \left[-\frac{1}{\ln x} \right]_2^{\infty} = -0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

$$30. \int_0^{\infty} \frac{dx}{e^{x/2}} = \left[-\frac{2}{e^{x/2}} \right]_0^{\infty} = -0 + \frac{2}{1} = 2$$

$$31. \int_3^5 \frac{dx}{(4-x)^{2/3}} = \lim_{b \rightarrow 4^-} \int_3^b \frac{dx}{(4-x)^{2/3}} + \lim_{b \rightarrow 4^+} \int_b^5 \frac{dx}{(4-x)^{2/3}} = \lim_{b \rightarrow 4^-} \left[-3(4-x)^{1/3} \right]_3^b + \lim_{b \rightarrow 4^+} \left[-3(4-x)^{1/3} \right]_b^5$$

$$= \lim_{b \rightarrow 4^-} -3(4-b)^{1/3} + 3(1)^{1/3} - 3(-1)^{1/3} + \lim_{b \rightarrow 4^+} 3(4-b)^{1/3} = 0 + 3 + 3 + 0 = 6$$

$$32. \int_2^{\infty} x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_2^{\infty} = 0 + \frac{1}{2} e^{-4} = \frac{1}{2} e^{-4}$$

$$35. \frac{e^x}{e^{2x} + 1} = \frac{e^x}{(e^x)^2 + 1}$$

Let $u = e^x$, $du = e^x dx$

$$33. \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} dx = \int_{-\infty}^0 \frac{x}{x^2 + 1} dx + \int_0^{\infty} \frac{x}{x^2 + 1} dx$$

$$= \frac{1}{2} \left[\ln(x^2 + 1) \right]_{-\infty}^0 + \frac{1}{2} \left[\ln(x^2 + 1) \right]_0^{\infty} = (0 + \infty) + (\infty - 0)$$

The integral diverges.

$$\int_0^{\infty} \frac{e^x}{e^{2x} + 1} dx = \int_1^{\infty} \frac{1}{u^2 + 1} du = \left[\tan^{-1} u \right]_1^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$34. \int_{-\infty}^{\infty} \frac{x}{1+x^4} dx = \int_{-\infty}^0 \frac{x}{1+x^4} dx + \int_0^{\infty} \frac{x}{1+x^4} dx$$

$$= \left[\frac{1}{2} \tan^{-1} x^2 \right]_{-\infty}^0 + \left[\frac{1}{2} \tan^{-1} x^2 \right]_0^{\infty}$$

$$= \frac{1}{2} \tan^{-1} 0 - \frac{1}{2} \left(\frac{\pi}{2} \right) + \frac{1}{2} \left(\frac{\pi}{2} \right) - \frac{1}{2} \tan^{-1} 0$$

$$= 0 - \frac{\pi}{4} + \frac{\pi}{4} - 0 = 0$$

$$36. \text{ Let } u = x^3, du = 3x^2 dx$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^3} dx = \int_{-\infty}^{\infty} \frac{1}{3} e^{-u} du$$

$$= \frac{1}{3} \int_{-\infty}^0 e^{-u} du + \frac{1}{3} \int_0^{\infty} e^{-u} du$$

$$= \frac{1}{3} \left[-e^{-u} \right]_{-\infty}^0 + \frac{1}{3} \left[-e^{-u} \right]_0^{\infty}$$

$$= \frac{1}{3} (-1 + \infty) + \frac{1}{3} (-0 + 1)$$

The integral diverges.

37. $\int_3^3 \frac{x}{\sqrt{9-x^2}} dx = 0$

See Problem 35 in Section 9.4.

38. let $u = \ln(\cos x)$, then

$$du = \frac{1}{\cos x} \cdot -\sin x dx = -\tan x dx$$

$$\begin{aligned} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\tan x}{(\ln \cos x)^2} dx &= \int_{\ln \frac{1}{2}}^{-\infty} \frac{1}{u^2} du = \int_{-\infty}^{\ln \frac{1}{2}} \frac{1}{u^2} du \\ &= \left[-\frac{1}{u} \right]_{-\infty}^{\ln \frac{1}{2}} = -\frac{1}{\ln \frac{1}{2}} + 0 = \frac{1}{\ln 2} \end{aligned}$$

39. For $p \neq 1, p \neq 0$, $\int_1^{\infty} \frac{1}{x^p} dx = \left[-\frac{1}{(p-1)x^{p-1}} \right]_1^{\infty}$

$$= \lim_{b \rightarrow \infty} \frac{1}{(1-p)b^{p-1}} + \frac{1}{p-1}$$

$$\lim_{b \rightarrow \infty} \frac{1}{b^{p-1}} = 0 \text{ when } p-1 > 0 \text{ or } p > 1,$$

$$\text{and } \lim_{b \rightarrow \infty} \frac{1}{b^{p-1}} = \infty \text{ when } p < 1, p \neq 0.$$

$$\text{When } p = 1, \int_1^{\infty} \frac{1}{x} dx = [\ln x]_1^{\infty} = \infty - 0$$

The integral diverges.

When $p = 0$, $\int_1^{\infty} 1 dx = [x]_1^{\infty} = \infty - 1$. The integral diverges.

$\int_1^{\infty} \frac{1}{x^p} dx$ converges when $p > 1$ and diverges when $p \leq 1$.

40. For $p \neq 1, p \neq 0$, $\int_0^1 \frac{1}{x^p} dx = \left[-\frac{1}{(p-1)x^{p-1}} \right]_0^1$

$$= \frac{1}{1-p} + \lim_{b \rightarrow 0} \frac{1}{(p-1)b^{p-1}}$$

$$\lim_{b \rightarrow 0} \frac{1}{b^{p-1}} \text{ converges when } p-1 < 0 \text{ or } p < 1.$$

$$\text{When } p = 1, \int_0^1 \frac{1}{x} dx = [\ln x]_0^1 = 0 - \lim_{b \rightarrow 0^+} \ln b = \infty$$

The integral diverges.

$$\text{When } p = 0, \int_0^1 1 dx = [x]_0^1 = 1 - 0 = 1$$

$\int_0^1 \frac{1}{x^p} dx$ converges when $p < 1$ and diverges when $1 \leq p$.

41. For $x \geq 1$, $x^6 + x > x^6$, so $\sqrt{x^6 + x} > \sqrt{x^6} = x^3$

$$\text{and } \frac{1}{\sqrt{x^6 + x}} < \frac{1}{x^3}. \text{ Hence,}$$

$$\int_1^{\infty} \frac{1}{\sqrt{x^6 + x}} dx < \int_1^{\infty} \frac{1}{x^3} dx \text{ which converges}$$

since $3 > 1$ (see Problem 39). Thus

$$\int_1^{\infty} \frac{1}{\sqrt{x^6 + x}} dx \text{ converges.}$$

42. For $x > 1$, $\ln x < e^x$, so $\frac{\ln x}{e^x} < 1$ and

$$\frac{\ln x}{e^{2x}} = \frac{\ln x}{(e^x)^2} < \frac{1}{e^x}.$$

$$\text{Hence, } \int_1^{\infty} \frac{\ln x}{e^{2x}} dx < \int_1^{\infty} e^{-x} dx = [-e^{-x}]_1^{\infty}$$

$$= -0 + e^{-1} = \frac{1}{e}.$$

$$\text{Thus, } \int_1^{\infty} \frac{\ln x}{e^{2x}} dx \text{ converges.}$$

43. For $x > 3$, $\ln x > 1$, so $\frac{\ln x}{x} > \frac{1}{x}$. Hence,

$$\int_3^{\infty} \frac{\ln x}{x} dx > \int_3^{\infty} \frac{1}{x} dx = [\ln x]_3^{\infty} = \infty - \ln 3.$$

The integral diverges, thus $\int_3^{\infty} \frac{\ln x}{x} dx$ also diverges.

44. For $x \geq 1$, $\ln x < x$, so $\frac{\ln x}{x} < 1$ and $\frac{\ln x}{x^3} < \frac{1}{x^2}$.

Hence,

$$\int_1^{\infty} \frac{\ln x}{x^3} dx < \int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{\infty} = -0 + 1 = 1.$$

$$\text{Thus, } \int_1^{\infty} \frac{\ln x}{x^3} dx \text{ converges.}$$

9.6 Additional Problem Set

1. a. For $1 < x < c$, $\frac{1}{x}$ is a differentiable function.

Let $u = \frac{1}{x}$, so $x = \frac{1}{u}$; $du = -\frac{1}{x^2} dx$ so

$$dx = -x^2 du = -\frac{1}{u^2} du.$$

When $x = 1$, $u = 1$, while when $x = c$, $u = \frac{1}{c}$.

$$\int_1^c \frac{1}{1+x^2} dx = \int_1^{\frac{1}{c}} \frac{1}{1+\frac{1}{u^2}} \left(-\frac{1}{u^2} du\right)$$

$$= -\int_1^{\frac{1}{c}} \frac{1}{u^2+1} du = \int_{\frac{1}{c}}^1 \frac{1}{u^2+1} du$$

- b. $\int_1^{\infty} \frac{1}{1+x^2} dx = \lim_{c \rightarrow \infty} \int_1^c \frac{1}{1+x^2} dx$
 $= \lim_{c \rightarrow \infty} \int_{\frac{1}{c}}^1 \frac{1}{u^2+1} du = \int_0^1 \frac{1}{u^2+1} du$ by using
 part a and $\lim_{c \rightarrow \infty} \frac{1}{c} = 0$.

$$\int_0^1 \frac{1}{u^2+1} du = [\tan^{-1} u]_0^1 = \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

2. For $1 < x < c$, $\frac{1}{x}$ is a differentiable function.

Let $u = \frac{1}{x}$, so $x = \frac{1}{u}$; $du = -\frac{1}{x^2} dx$, so

$$dx = -x^2 du = -\frac{1}{u^2} du.$$

When $x = 1$, $u = 1$, while when $x = c$, $u = \frac{1}{c}$.

$$\int_1^c \frac{x}{x^3+1} dx = \int_1^{\frac{1}{c}} \frac{\frac{1}{u}}{\frac{1}{u^3}+1} \left(-\frac{1}{u^2} du\right)$$

$$= -\int_1^{\frac{1}{c}} \frac{1}{1+u^3} du = \int_{\frac{1}{c}}^1 \frac{1}{u^3+1} du$$

$$\text{Thus, } \int_1^{\infty} \frac{x}{x^3+1} dx = \lim_{c \rightarrow \infty} \int_1^c \frac{x}{x^3+1} dx$$

$$= \lim_{c \rightarrow \infty} \int_{\frac{1}{c}}^1 \frac{1}{u^3+1} du = \int_0^1 \frac{1}{u^3+1} du$$

3. a. $f(x)$ is an even function, so $\int_{-\infty}^0 f(x) dx = \int_0^{\infty} f(x) dx$, thus $\int_{-\infty}^{\infty} f(x) dx = 2 \int_0^{\infty} f(x) dx$.

$$2 \int_0^{\infty} C|x|e^{-kx^2} dx = \left[-\frac{C}{k} e^{-kx^2} \right]_0^{\infty} = -0 + \frac{C}{k}$$

$$\frac{C}{k} = 1 \text{ when } C = k$$

- b. $\int_{-\infty}^{\infty} kx|x|e^{-kx^2} dx = \int_{-\infty}^0 -kx^2 e^{-kx^2} dx + \int_0^{\infty} kx^2 e^{-kx^2} dx$

In both integrals, use $u = x$, $du = dx$, $dv = kxe^{-kx^2} dx$, $v = -\frac{1}{2}e^{-kx^2}$

$$\int_{-\infty}^0 -kx^2 e^{-kx^2} dx + \int_0^{\infty} kx^2 e^{-kx^2} dx = -\left(\left[-\frac{1}{2}xe^{-kx^2} \right]_{-\infty}^0 + \int_{-\infty}^0 \frac{1}{2}e^{-kx^2} dx \right) + \left(\left[-\frac{1}{2}xe^{-kx^2} \right]_0^{\infty} + \int_0^{\infty} \frac{1}{2}e^{-kx^2} dx \right)$$

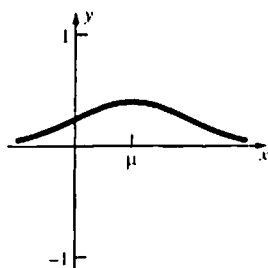
$$= -(0+0) - \frac{1}{2} \int_{-\infty}^0 e^{-kx^2} dx + (0-0) + \frac{1}{2} \int_0^{\infty} e^{-kx^2} dx = -\frac{1}{2} \int_{-\infty}^0 e^{-kx^2} dx + \frac{1}{2} \int_0^{\infty} e^{-kx^2} dx$$

Now let $u = \sqrt{2k}x$, so that $kx^2 = \frac{u^2}{2}$ and $du = \sqrt{2k}dx$. Then the integrals become

$$-\frac{1}{2\sqrt{2k}} \int_{-\infty}^0 e^{-u^2/2} du + \frac{1}{2\sqrt{2k}} \int_0^{\infty} e^{-u^2/2} du = -\frac{1}{2\sqrt{2k}} \left(\frac{1}{2}\sqrt{2\pi} \right) + \frac{1}{2\sqrt{2k}} \left(\frac{1}{2}\sqrt{2\pi} \right) = 0.$$

$$\int_0^{\infty} e^{-u^2/2} du = \int_{-\infty}^0 e^{-u^2/2} du = \frac{1}{2}\sqrt{2\pi} \text{ from Example 5 in Section 9.3.}$$

4.



$$f'(x) = -\frac{x-\mu}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\begin{aligned} f''(x) &= -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} + \frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \\ &= \left(\frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}} - \frac{1}{\sigma^3\sqrt{2\pi}} \right) e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{\sigma^5\sqrt{2\pi}} [(x-\mu)^2 - \sigma^2] e^{-(x-\mu)^2/2\sigma^2} \end{aligned}$$

$f''(x) = 0$ when $(x-\mu)^2 = \sigma^2$ so $x = \mu \pm \sigma$ and the distance from μ to each inflection point is σ .

5. a. $\int_{-\infty}^{\infty} f(x) dx = \int_M^{\infty} \frac{CM^k}{x^{k+1}} dx = CM^k \left[-\frac{1}{kx^k} \right]_M^{\infty} = CM^k \left(0 + \frac{1}{kM^k} \right) = \frac{C}{k}$. Thus, $\frac{C}{k} = 1$ when $C = k$.

b. $\mu = \int_{-\infty}^{\infty} xf(x) dx = \int_M^{\infty} x \frac{kM^k}{x^{k+1}} dx = kM^k \int_M^{\infty} \frac{1}{x^k} dx = kM^k \left(\lim_{b \rightarrow \infty} \int_M^b \frac{1}{x^k} dx \right)$

This integral converges when $k > 1$.

When $k > 1$, $\mu = kM^k \left(\lim_{b \rightarrow \infty} \left[-\frac{1}{(k-1)x^{k-1}} \right]_M^b \right) = kM^k \left(-0 + \frac{1}{(k-1)M^{k-1}} \right) = \frac{kM}{k-1}$

The mean is finite only when $k > 1$.

c. Since the mean is finite only when $k > 1$, the variance is only defined when $k > 1$.

$$\begin{aligned} \sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_M^{\infty} \left(x - \frac{kM}{k-1} \right)^2 \frac{kM^k}{x^{k+1}} dx = kM^k \int_M^{\infty} \left(x^2 - \frac{2kM}{k-1}x + \frac{k^2M^2}{(k-1)^2} \right) \frac{1}{x^{k+1}} dx \\ &= kM^k \int_M^{\infty} \frac{1}{x^{k-1}} dx - \frac{2k^2M^{k+1}}{k-1} \int_M^{\infty} \frac{1}{x^k} dx + \frac{k^3M^{k+2}}{(k-1)^2} \int_M^{\infty} \frac{1}{x^{k+1}} dx \end{aligned}$$

The first integral converges only when $k-1 > 1$ or $k > 2$. The second integral converges only when $k > 1$, which is taken care of by requiring $k > 2$.

$$\begin{aligned} \sigma^2 &= kM^k \left[-\frac{1}{(k-2)x^{k-2}} \right]_M^{\infty} - \frac{2k^2M^{k+1}}{k-1} \left[-\frac{1}{(k-1)x^{k-1}} \right]_M^{\infty} + \frac{k^3M^{k+2}}{(k-1)^2} \left[-\frac{1}{kx^k} \right]_M^{\infty} \\ &= kM^k \left(-0 + \frac{1}{(k-2)M^{k-2}} \right) - \frac{2k^2M^{k+1}}{k-1} \left(-0 + \frac{1}{(k-1)M^{k-1}} \right) + \frac{k^3M^{k+2}}{(k-1)^2} \left(-0 + \frac{1}{kM^k} \right) \\ &= \frac{kM^2}{k-2} - \frac{2k^2M^2}{(k-1)^2} + \frac{k^2M^2}{(k-1)^2} \\ &= kM^2 \left(\frac{1}{k-2} - \frac{k}{(k-1)^2} \right) = kM^2 \left(\frac{k^2 - 2k + 1 - k^2 + 2k}{(k-2)(k-1)^2} \right) = \frac{kM^2}{(k-2)(k-1)^2} \end{aligned}$$

6. a. $\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} Cx^{\alpha-1} e^{-\beta x} dx$

Let $y = \beta x$, so $x = \frac{y}{\beta}$ and $dx = \frac{1}{\beta} dy$.

$$\int_0^{\infty} Cx^{\alpha-1} e^{-\beta x} dx = \int_0^{\infty} C \left(\frac{y}{\beta} \right)^{\alpha-1} e^{-y} \frac{1}{\beta} dy = \frac{C}{\beta^{\alpha}} \int_0^{\infty} y^{\alpha-1} e^{-y} dy = C \beta^{-\alpha} \Gamma(\alpha)$$

$$C \beta^{-\alpha} \Gamma(\alpha) = 1 \text{ when } C = \frac{1}{\beta^{-\alpha} \Gamma(\alpha)} = \frac{\beta^{\alpha}}{\Gamma(\alpha)}.$$

b. $\mu = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha} e^{-\beta x} dx$

Let $y = \beta x$, so $x = \frac{y}{\beta}$ and $dx = \frac{1}{\beta} dy$.

$$\mu = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \left(\frac{y}{\beta} \right)^{\alpha} e^{-y} \frac{1}{\beta} dy = \frac{1}{\beta \Gamma(\alpha)} \int_0^{\infty} y^{\alpha} e^{-y} dy = \frac{1}{\beta \Gamma(\alpha)} \Gamma(\alpha+1) = \frac{1}{\beta \Gamma(\alpha)} \alpha \Gamma(\alpha) = \frac{\alpha}{\beta}$$

(Recall that $\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$ for $\alpha > 0$.)

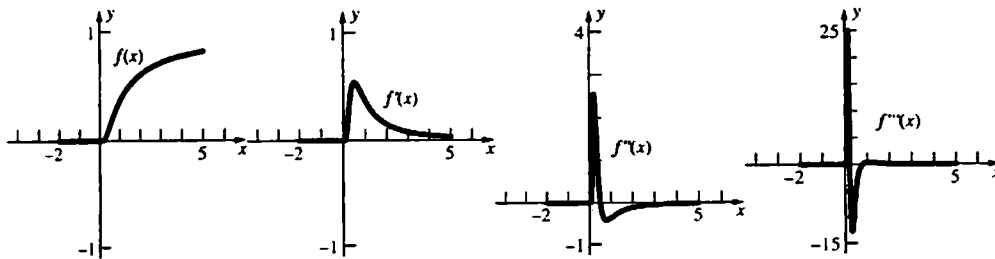
c. $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_0^{\infty} \left(x - \frac{\alpha}{\beta} \right)^2 \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \left(x^2 - \frac{2\alpha}{\beta} x + \frac{\alpha^2}{\beta^2} \right) x^{\alpha-1} e^{-\beta x} dx$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha+1} e^{-\beta x} dx - \frac{2\alpha\beta^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha} e^{-\beta x} dx + \frac{\alpha^2\beta^{\alpha-2}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-\beta x} dx$$

In all three integrals, let $y = \beta x$, so $x = \frac{y}{\beta}$ and $dx = \frac{1}{\beta} dy$.

$$\begin{aligned} \sigma^2 &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \left(\frac{y}{\beta} \right)^{\alpha+1} e^{-y} \frac{1}{\beta} dy - \frac{2\alpha\beta^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\infty} \left(\frac{y}{\beta} \right)^{\alpha} e^{-y} \frac{1}{\beta} dy + \frac{\alpha^2\beta^{\alpha-2}}{\Gamma(\alpha)} \int_0^{\infty} \left(\frac{y}{\beta} \right)^{\alpha-1} e^{-y} \frac{1}{\beta} dy \\ &= \frac{1}{\beta^2 \Gamma(\alpha)} \int_0^{\infty} y^{\alpha+1} e^{-y} dy - \frac{2\alpha}{\beta^2 \Gamma(\alpha)} \int_0^{\infty} y^{\alpha} e^{-y} dy + \frac{\alpha^2}{\beta^2 \Gamma(\alpha)} \int_0^{\infty} y^{\alpha-1} e^{-y} dy \\ &= \frac{1}{\beta^2 \Gamma(\alpha)} \Gamma(\alpha+2) - \frac{2\alpha}{\beta^2 \Gamma(\alpha)} \Gamma(\alpha+1) + \frac{\alpha^2}{\beta^2 \Gamma(\alpha)} \Gamma(\alpha) = \frac{1}{\beta^2 \Gamma(\alpha)} (\alpha+1)\alpha \Gamma(\alpha) - \frac{2\alpha}{\beta^2 \Gamma(\alpha)} \alpha \Gamma(\alpha) + \frac{\alpha^2}{\beta^2} \\ &= \frac{\alpha^2 + \alpha}{\beta^2} - \frac{2\alpha^2}{\beta^2} + \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2} \end{aligned}$$

7. a.



b. $f^{(1)}(x) = \begin{cases} \frac{1}{x^2} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

$$f^{(2)}(x) = \begin{cases} \left(\frac{1}{x^4} - \frac{2}{x^3} \right) e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$f^{(3)}(x) = \begin{cases} \left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4} \right) e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} f^{(0)}(x) = \lim_{x \rightarrow 0^+} e^{-1/x} = \lim_{x \rightarrow 0^+} \frac{1}{e^{1/x}}$$

Let $v = \frac{1}{x}$, then as $x \rightarrow 0^+$, $v \rightarrow \infty$ and

$$\lim_{x \rightarrow 0^+} \frac{1}{e^{1/x}} = \lim_{v \rightarrow \infty} \frac{1}{e^v} = 0.$$

Use the same change of variables.

$$\lim_{x \rightarrow 0^+} f^{(1)}(x) = \lim_{x \rightarrow 0^+} \frac{1}{x^2} e^{-1/x} = \lim_{v \rightarrow \infty} v^2 e^{-v} = \lim_{v \rightarrow \infty} \frac{v^2}{e^v} = 0$$

$$\lim_{x \rightarrow 0^+} f^{(2)}(x) = \lim_{x \rightarrow 0^+} \left[\left(\frac{1}{x^4} - \frac{2}{x^3} \right) e^{-1/x} \right] = \lim_{v \rightarrow \infty} [(v^4 - 2v^3) e^{-v}] = \lim_{v \rightarrow \infty} \left(\frac{v^4}{e^v} - \frac{2v^3}{e^v} \right) = 0$$

$$\lim_{x \rightarrow 0^+} f^{(3)}(x) = \lim_{x \rightarrow 0^+} \left[\left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4} \right) e^{-1/x} \right] = \lim_{v \rightarrow \infty} [(v^6 - 6v^5 + 6v^4) e^{-v}] = \lim_{v \rightarrow \infty} \left(\frac{v^6}{e^v} - \frac{6v^5}{e^v} + \frac{6v^4}{e^v} \right) = 0$$

(Recall from Example 2 of Section 9.2 that $\lim_{x \rightarrow \infty} \frac{x^a}{e^x} = 0$ when a is any positive real number.)

For $x > 0$ and $x < 0$, $f^{(i)}(x)$ is continuous and since $\lim_{x \rightarrow 0^+} f^{(i)}(x) = 0 = f^{(i)}(0)$ for $i = 0, 1, 2, 3$, these are continuous for all x .

c. For $x \leq 0$, $f^{(n)}(x) = 0$ for all n , while for $x > 0$, $f^{(n)}(x)$ will be a sum of terms of the form $\frac{b}{x^a} e^{-1/x}$. Using

the same change of variables as in part b, this is $\frac{bv^a}{e^v}$ which approaches 0 as $v \rightarrow \infty$. Thus

$$\lim_{x \rightarrow 0^+} f^{(n)}(x) = 0 = f^{(n)}(0) \text{ so } f^{(n)}(x) \text{ will be continuous.}$$

8. a. $L\{t^\alpha\}(s) = \int_0^\infty t^\alpha e^{-st} dt$

Let $t = \frac{x}{s}$, so $dt = \frac{1}{s} dx$, then

$$\int_0^\infty t^\alpha e^{-st} dt = \int_0^\infty \left(\frac{x}{s} \right)^\alpha e^{-x} \frac{1}{s} dx = \int_0^\infty \frac{1}{s^{\alpha+1}} x^\alpha e^{-x} dx = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

If $s \leq 0$, $t^\alpha e^{-st} \rightarrow \infty$ as $t \rightarrow \infty$, so the integral does not converge. Thus, the transform is defined only when $s > 0$.

b. $L\{e^{at}\}(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{(a-s)t} dt = \left[\frac{1}{\alpha-s} e^{(\alpha-s)t} \right]_0^\infty = \frac{1}{\alpha-s} \left[\lim_{b \rightarrow \infty} e^{(\alpha-s)b} - 1 \right]$

$$\lim_{b \rightarrow \infty} e^{(\alpha-s)b} = \begin{cases} \infty & \text{if } \alpha > s \\ 0 & \text{if } s > \alpha \end{cases}$$

Thus, $L\{e^{at}\}(s) = \frac{-1}{\alpha-s} = \frac{1}{s-\alpha}$ when $s > \alpha$. (When $s \leq \alpha$, the integral does not converge.)

c. $L\{\sin(\alpha t)\}(s) = \int_0^{\infty} \sin(\alpha t)e^{-st} dt$

Let $I = \int_0^{\infty} \sin(\alpha t)e^{-st} dt$ and use integration by parts with $u = \sin(\alpha t)$, $du = \alpha \cos(\alpha t)dt$,

$$dv = e^{-st} dt, \text{ and } v = -\frac{1}{s}e^{-st}.$$

$$\text{Then } I = \left[-\frac{1}{s}\sin(\alpha t)e^{-st} \right]_0^{\infty} + \frac{\alpha}{s} \int_0^{\infty} \cos(\alpha t)e^{-st} dt$$

Use integration by parts on this integral with

$$u = \cos(\alpha t), du = -\alpha \sin(\alpha t)dt, dv = e^{-st} dt, \text{ and } v = -\frac{1}{s}e^{-st}.$$

$$I = \left[-\frac{1}{s}\sin(\alpha t)e^{-st} \right]_0^{\infty} + \frac{\alpha}{s} \left(\left[-\frac{1}{s}\cos(\alpha t)e^{-st} \right]_0^{\infty} - \frac{\alpha}{s} \int_0^{\infty} \sin(\alpha t)e^{-st} dt \right)$$

$$= -\frac{1}{s} \left[e^{-st} \left(\sin(\alpha t) + \frac{\alpha}{s} \cos(\alpha t) \right) \right]_0^{\infty} - \frac{\alpha^2}{s^2} I$$

Thus,

$$I \left(1 + \frac{\alpha^2}{s^2} \right) = -\frac{1}{s} \left[e^{-st} \left(\sin(\alpha t) + \frac{\alpha}{s} \cos(\alpha t) \right) \right]_0^{\infty}$$

$$I = -\frac{1}{s \left(1 + \frac{\alpha^2}{s^2} \right)} \left[e^{-st} \left(\sin(\alpha t) + \frac{\alpha}{s} \cos(\alpha t) \right) \right]_0^{\infty} = -\frac{s}{s^2 + \alpha^2} \left[\lim_{b \rightarrow \infty} e^{-sb} \left(\sin(\alpha b) + \frac{\alpha}{s} \cos(\alpha b) \right) - \frac{\alpha}{s} \right]$$

$$\lim_{b \rightarrow \infty} e^{-sb} \left(\sin(\alpha b) + \frac{\alpha}{s} \cos(\alpha b) \right) = \begin{cases} 0 & \text{if } s > 0 \\ \infty & \text{if } s \leq 0 \end{cases}$$

$$\text{Thus, } I = \frac{\alpha}{s^2 + \alpha^2} \text{ when } s > 0.$$

d. $L\{\cos(\alpha t)\}(s) = \int_0^{\infty} \cos(\alpha t)e^{-st} dt$

Let $I = \int_0^{\infty} \cos(\alpha t)e^{-st} dt$ and use integration by parts with $u = \cos(\alpha t)$, $du = -\alpha \sin(\alpha t)dt$,

$$dv = e^{-st} dt, v = -\frac{1}{s}e^{-st}. \text{ Then,}$$

$$I = \left[-\frac{1}{s}\cos(\alpha t)e^{-st} \right]_0^{\infty} - \frac{\alpha}{s} \int_0^{\infty} \sin(\alpha t)e^{-st} dt$$

Note that this last integral is the same as in part c. Thus,

$$I = \left[-\frac{1}{s}\cos(\alpha t)e^{-st} \right]_0^{\infty} - \frac{\alpha}{s} \left(\frac{\alpha}{s^2 + \alpha^2} \right) = -\frac{1}{s} \lim_{b \rightarrow \infty} \cos(\alpha t)e^{-sb} + \frac{1}{s} - \frac{\alpha^2}{s(s^2 + \alpha^2)}$$

The integral from part c converges only when $s > 0$ and when this is the case $\lim_{b \rightarrow \infty} \cos(\alpha t)e^{-sb} = 0$.

Thus, when $s > 0$,

$$I = \frac{1}{s} - \frac{\alpha^2}{s(s^2 + \alpha^2)} = \frac{s^2 + \alpha^2 - \alpha^2}{s(s^2 + \alpha^2)} = \frac{s}{s^2 + \alpha^2}$$

10.1 Concepts Review

1. a sequence
2. $\lim_{n \rightarrow \infty} a_n$ exists (finite sense)
3. bounded above
4. $-1; 1$

Problem Set 10.1

$$1. a_1 = \frac{1}{2}, a_2 = \frac{2}{5}, a_3 = \frac{3}{8}, a_4 = \frac{4}{11}, a_5 = \frac{5}{14}$$

$$\lim_{n \rightarrow \infty} \frac{n}{3n-1} = \lim_{n \rightarrow \infty} \frac{1}{3-\frac{1}{n}} = \frac{1}{3}; \text{ converges}$$

$$2. a_1 = \frac{5}{2}, a_2 = \frac{8}{3}, a_3 = \frac{11}{4}, a_4 = \frac{14}{5}, a_5 = \frac{17}{6}$$

$$\lim_{n \rightarrow \infty} \frac{3n+2}{n+1} = \lim_{n \rightarrow \infty} \frac{3+\frac{2}{n}}{1+\frac{1}{n}} = 3; \text{ converges}$$

$$3. a_1 = \frac{6}{3} = 2, a_2 = \frac{18}{9} = 2, a_3 = \frac{38}{17},$$

$$a_4 = \frac{66}{27} = \frac{22}{9}, a_5 = \frac{102}{39} = \frac{34}{13}$$

$$\lim_{n \rightarrow \infty} \frac{4n^2+2}{n^2+3n-1} = \lim_{n \rightarrow \infty} \frac{4+\frac{2}{n^2}}{1+\frac{3}{n}-\frac{1}{n^2}} = 4; \text{ converges}$$

$$4. a_1 = 5, a_2 = \frac{14}{3}, a_3 = \frac{29}{5}, a_4 = \frac{50}{7}, a_5 = \frac{77}{9}$$

$$\lim_{n \rightarrow \infty} \frac{3n^2+2}{2n-1} = \lim_{n \rightarrow \infty} \frac{3n+\frac{2}{n}}{2-\frac{1}{n}} = \infty; \text{ diverges}$$

$$5. a_1 = \frac{7}{8}, a_2 = \frac{26}{27}, a_3 = \frac{63}{64}, a_4 = \frac{124}{125}, a_5 = \frac{215}{216}$$

$$\lim_{n \rightarrow \infty} \frac{n^3+3n^2+3n}{(n+1)^3} = \lim_{n \rightarrow \infty} \frac{n^3+3n^2+3n}{n^3+3n^2+3n+1}$$

$$= \lim_{n \rightarrow \infty} \frac{1+\frac{3}{n}+\frac{3}{n^2}}{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}} = 1$$

$$6. a_1 = \frac{\sqrt{5}}{3}, a_2 = \frac{\sqrt{14}}{5}, a_3 = \frac{\sqrt{29}}{7},$$

$$a_4 = \frac{\sqrt{50}}{9} = \frac{5\sqrt{2}}{9}, a_5 = \frac{\sqrt{77}}{11}$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{3n^2+2}}{2n+1} = \lim_{n \rightarrow \infty} \frac{\sqrt{3+\frac{2}{n^2}}}{2+\frac{1}{n}} = \frac{\sqrt{3}}{2}; \text{ converges}$$

$$7. a_1 = -\frac{1}{3}, a_2 = \frac{2}{4} = \frac{1}{2}, a_3 = -\frac{3}{5}, a_4 = \frac{4}{6} = \frac{2}{3},$$

$$a_5 = -\frac{5}{7}$$

$$\lim_{n \rightarrow \infty} \frac{n}{n+2} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{2}{n}} = 1, \text{ but since it alternates}$$

between positive and negative, the sequence diverges.

$$8. a_1 = -1, a_2 = \frac{2}{3}, a_3 = -\frac{3}{5}, a_4 = \frac{4}{7}, a_5 = -\frac{5}{9}$$

$$\cos(n\pi) = \begin{cases} -1 & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases}$$

$$\lim_{n \rightarrow \infty} \frac{n}{2n-1} = \lim_{n \rightarrow \infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}, \text{ but since } \cos(n\pi)$$

alternates between 1 and -1 , the sequence diverges.

$$9. a_1 = -1, a_2 = \frac{1}{2}, a_3 = -\frac{1}{3}, a_4 = \frac{1}{4}, a_5 = -\frac{1}{5}$$

$$-1 \leq \cos(n\pi) \leq 1 \text{ for all } n, \text{ so}$$

$$-\frac{1}{n} \leq \frac{\cos(n\pi)}{n} \leq \frac{1}{n}.$$

$$\lim_{n \rightarrow \infty} -\frac{1}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \text{ so by the Squeeze}$$

Theorem, the sequence converges to 0.

10. $a_1 = e^{-1} \sin 1 \approx 0.3096, a_2 = e^{-2} \sin 2 \approx 0.1231,$
 $a_3 = e^{-3} \sin 3 \approx 0.0070, a_4 = e^{-4} \sin 4 \approx -0.0139,$
 $a_5 = e^{-5} \sin 5 \approx -0.0065$
 $-1 \leq \sin n \leq 1$ for all n , so
 $-e^{-n} \leq e^{-n} \sin n \leq e^{-n}.$
 $\lim_{n \rightarrow \infty} -e^{-n} = \lim_{n \rightarrow \infty} e^{-n} = 0$, so by the Squeeze
 Theorem, the sequence converges to 0.

11. $a_1 = \frac{e^2}{3} \approx 2.4630, a_2 = \frac{e^4}{9} \approx 6.0665,$
 $a_3 = \frac{e^6}{17} \approx 23.7311, a_4 = \frac{e^8}{27} \approx 110.4059,$
 $a_5 = \frac{e^{10}}{39} \approx 564.7812$

Consider

$$\lim_{x \rightarrow \infty} \frac{e^{2x}}{x^2 + 3x - 1} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2x + 3} = \lim_{x \rightarrow \infty} \frac{4e^{2x}}{2} = \infty$$

by using l'Hôpital's Rule twice. The sequence diverges.

12. $a_1 = \frac{e^2}{4} \approx 1.8473, a_2 = \frac{e^4}{16} \approx 3.4124,$
 $a_3 = \frac{e^6}{64} \approx 6.3036, a_4 = \frac{e^8}{256} \approx 11.6444,$
 $a_5 = \frac{e^{10}}{1024} \approx 21.510$
 $\frac{e^{2n}}{4^n} = \left(\frac{e^2}{4}\right)^n, \frac{e^2}{4} > 1$ so the sequence diverges.

13. $a_1 = -\frac{\pi}{5} \approx -0.6283, a_2 = \frac{\pi^2}{25} \approx 0.3948,$
 $a_3 = -\frac{\pi^3}{125} \approx -0.2481, a_4 = \frac{\pi^4}{625} \approx 0.1559,$
 $a_5 = -\frac{\pi^5}{3125} \approx -0.0979$
 $\frac{(-\pi)^n}{5^n} = \left(-\frac{\pi}{5}\right)^n, -1 < -\frac{\pi}{5} < 1$, thus the sequence
 converges to 0.

14. $a_1 = \frac{1}{4} + \sqrt{3} \approx 1.9821, a_2 = \frac{1}{16} + 3 = 3.0625,$
 $a_3 = \frac{1}{64} + 3\sqrt{3} \approx 5.2118, a_4 = \frac{1}{256} + 9 \approx 9.0039,$

$$a_5 = \frac{1}{1024} + 9\sqrt{3} \approx 15.589$$

$$\left(\frac{1}{4}\right)^n \text{ converges to 0 since } -1 < \frac{1}{4} < 1.$$

$$3^{n/2} = (\sqrt{3})^n \text{ diverges since } \sqrt{3} \approx 1.732 > 1.$$

Thus, the sum diverges.

15. $a_1 = 2.99, a_2 = 2.9801, a_3 \approx 2.9703,$
 $a_4 \approx 2.9606, a_5 \approx 2.9510$
 $(0.99)^n$ converges to 0 since $-1 < 0.99 < 1$, thus
 $2 + (0.99)^n$ converges to 2.

16. $a_1 = \frac{1}{e} \approx 0.3679, a_2 = \frac{2^{100}}{e^2} \approx 1.72 \times 10^{29},$
 $a_3 = \frac{3^{100}}{e^3} \approx 2.57 \times 10^{46}, a_4 = \frac{4^{100}}{e^4} \approx 2.94 \times 10^{58},$
 $a_5 = \frac{5^{100}}{e^5} \approx 5.32 \times 10^{67}$

Consider $\lim_{x \rightarrow \infty} \frac{x^{100}}{e^x}$. By Example 2 of

Section 9.2, $\lim_{x \rightarrow \infty} \frac{x^{100}}{e^x} = 0$. Thus, $\lim_{n \rightarrow \infty} \frac{n^{100}}{e^n} = 0$;
 converges

17. $a_1 = \frac{\ln 1}{\sqrt{1}} = 0, a_2 = \frac{\ln 2}{\sqrt{2}} \approx 0.4901,$
 $a_3 = \frac{\ln 3}{\sqrt{3}} \approx 0.6343, a_4 = \frac{\ln 4}{2} \approx 0.6931,$
 $a_5 = \frac{\ln 5}{\sqrt{5}} \approx 0.7198$

Consider $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$ by

using l'Hôpital's Rule. Thus, $\lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = 0$;
 converges.

18. $a_1 = \frac{\ln 1}{\sqrt{2}} = 0, a_2 = \frac{\ln \frac{1}{2}}{2} \approx -0.3466,$
 $a_3 = \frac{\ln \frac{1}{3}}{\sqrt{6}} \approx -0.4485, a_4 = \frac{\ln \frac{1}{4}}{2\sqrt{2}} \approx -0.4901,$
 $a_5 = \frac{\ln \frac{1}{5}}{\sqrt{10}} \approx -0.5089$

Consider $\lim_{x \rightarrow \infty} \frac{\ln \frac{1}{x}}{\sqrt{2x}} = \lim_{x \rightarrow \infty} \frac{-\ln x}{\sqrt{2x}} = \lim_{x \rightarrow \infty} \frac{-\frac{1}{x}}{\frac{1}{\sqrt{2x}}}$

$$= \lim_{x \rightarrow \infty} -\frac{\sqrt{2}}{\sqrt{x}} = 0 \text{ by using l'Hôpital's Rule. Thus,}$$

$$\lim_{n \rightarrow \infty} \frac{\ln \frac{1}{n}}{\sqrt{2n}} = 0; \text{ converges}$$

$$19. a_1 = \left(1 + \frac{2}{1}\right)^{1/2} = \sqrt{3} \approx 1.7321,$$

$$a_2 = \left(1 + \frac{2}{2}\right)^{2/2} = 2,$$

$$a_3 = \left(1 + \frac{2}{3}\right)^{3/2} = \left(\frac{5}{3}\right)^{3/2} \approx 2.1517,$$

$$a_4 = \left(1 + \frac{2}{4}\right)^{4/2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4},$$

$$a_5 = \left(1 + \frac{2}{5}\right)^{5/2} = \left(\frac{7}{5}\right)^{5/2} \approx 2.3191$$

$$\text{Let } \frac{2}{n} = h, \text{ then as } n \rightarrow \infty, h \rightarrow 0 \text{ and}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^{n/2} = \lim_{h \rightarrow 0} (1+h)^{1/h} = e \text{ by}$$

Theorem 7.5A; converges

$$20. a_1 = 2^{1/2} \approx 1.4142, a_2 = 4^{1/4} = 2^{1/2} \approx 1.4142,$$

$$a_3 = 6^{1/6} \approx 1.3480, a_4 = 8^{1/8} = 2^{3/8} \approx 1.2968,$$

$$a_5 = 10^{1/10} \approx 1.2589$$

Consider $\lim_{x \rightarrow \infty} (2x)^{1/2x}$. This limit is of the form

$$\infty^0. \text{ Let } y = (2x)^{1/2x}, \text{ then } \ln y = \frac{1}{2x} \ln 2x.$$

$$\lim_{x \rightarrow \infty} \frac{1}{2x} \ln 2x = \lim_{x \rightarrow \infty} \frac{\ln 2x}{2x}$$

This limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x \rightarrow \infty} \frac{\ln 2x}{2x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2x}}{2} = \lim_{x \rightarrow \infty} \frac{1}{4x} = 0$$

$$\lim_{x \rightarrow \infty} (2x)^{1/2x} = \lim_{x \rightarrow \infty} e^{\ln y} = 1$$

Thus $\lim_{n \rightarrow \infty} (2n)^{1/2n} = 1$; converges

$$21. a_n = \frac{n}{n+1} \text{ or } a_n = 1 - \frac{1}{n+1};$$

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1 - \lim_{n \rightarrow \infty} \frac{1}{n+1} = 1; \text{ converges}$$

$$22. a_n = \frac{n}{2^{n+1}}$$

$$\text{Consider } \frac{x}{2^x}. \text{ Now, } \lim_{x \rightarrow \infty} \frac{x}{2^x} = \lim_{x \rightarrow \infty} \frac{1}{2^x \ln 2} = 0$$

$$\text{by l'Hôpital's Rule. Thus, } \lim_{n \rightarrow \infty} \frac{n}{2^{n+1}} = 0;$$

converges

$$23. a_n = (-1)^n \frac{n}{2n-1}; \lim_{n \rightarrow \infty} \frac{n}{2n-1}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}, \text{ but due to } (-1)^n, \text{ the terms of}$$

the sequence alternate between positive and negative, so the sequence diverges.

$$24. a_n = \frac{1}{1 - \frac{n-1}{n}} = n;$$

$$\lim_{n \rightarrow \infty} n = \infty; \text{ diverges}$$

$$25. a_n = \frac{n}{n^2 - (n-1)^2} = \frac{n}{n^2 - (n^2 - 2n + 1)} = \frac{n}{2n-1};$$

$$\lim_{n \rightarrow \infty} \frac{n}{2n-1} = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}; \text{ converges}$$

$$26. a_n = \frac{n}{(n+1) - \frac{1}{n+1}} = \frac{n(n+1)}{(n+1)^2 - 1} = \frac{n^2 + n}{n^2 + 2n};$$

$$\lim_{n \rightarrow \infty} \frac{n^2 + n}{n^2 + 2n} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} = 1; \text{ converges}$$

$$27. a_n = n \sin \frac{1}{n}; \lim_{n \rightarrow \infty} n \sin \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1 \text{ since}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1; \text{ converges}$$

$$28. a_n = (-1)^n \frac{n^2}{3^n};$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{3^n} = \lim_{n \rightarrow \infty} \frac{2n}{3^n \ln 3} = \lim_{n \rightarrow \infty} \frac{2}{3^n (\ln 3)^2} = 0$$

by using l'Hôpital's Rule twice; converges

$$29. a_n = \frac{2^n}{n^2};$$

$$\lim_{n \rightarrow \infty} \frac{2^n}{n^2} = \lim_{n \rightarrow \infty} \frac{2^n \ln 2}{2n} = \lim_{n \rightarrow \infty} \frac{2^n (\ln 2)^2}{2} = \infty;$$

diverges

$$30. a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)};$$

$$\lim_{n \rightarrow \infty} \frac{1}{n(n+1)} = 0; \text{ converges}$$

$$31. a_1 = \frac{1}{2}, a_2 = \frac{5}{4}, a_3 = \frac{9}{8}, a_4 = \frac{13}{16}$$

a_n is positive for all n , and $a_{n+1} < a_n$ for all

$$n \geq 2 \text{ since } a_{n+1} - a_n = -\frac{4n-7}{2^{n+1}}, \text{ so } \{a_n\}$$

converges to a limit $L \geq 0$.

$$32. a_1 = \frac{1}{2}; a_2 = \frac{7}{6}; a_3 = \frac{17}{12}; a_4 = \frac{31}{20}$$

$$a_n = \frac{2n^2-1}{n^2+n} < 2 \text{ for all } n, \text{ and } a_n < a_{n+1} \text{ for all}$$

n since $a_{n+1} - a_n = \frac{2}{n^2+2n}$, so $\{a_n\}$ converges to a limit $L \leq 2$.

$$33. a_2 = \frac{3}{4}; a_3 = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right) = \frac{2}{3};$$

$$a_4 = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{15}{16}\right) = \frac{5}{8};$$

$$a_5 = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{15}{16}\right)\left(\frac{24}{25}\right) = \frac{3}{5}$$

$a_n > 0$ for all n and $a_{n+1} < a_n$ since

$$a_{n+1} = a_n \left(1 - \frac{1}{(n+1)^2}\right) \text{ and } 1 - \frac{1}{(n+1)^2} < 1, \text{ so}$$

$\{a_n\}$ converges to a limit $L \geq 0$.

$$34. a_1 = 1; a_2 = \frac{3}{2}; a_3 = \frac{5}{3}; a_4 = \frac{41}{24}$$

$a_n < 2$ for all n since

$$1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \leq \frac{1}{2^0} + \frac{1}{2^1} + \cdots + \frac{1}{2^{n+1}} < \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$$

the sum never reaches 2. $a_n < a_{n+1}$ since each term is the previous term plus a positive quantity, so $\{a_n\}$ converges to a limit $L \leq 2$.

$$35. a_1 = 1, a_2 = 1 + \frac{1}{2}(1) = \frac{3}{2}, a_3 = 1 + \frac{1}{2}\left(\frac{3}{2}\right) = \frac{7}{4},$$

$$a_4 = 1 + \frac{1}{2}\left(\frac{7}{4}\right) = \frac{15}{8}$$

Suppose that $1 < a_n < 2$, then $\frac{1}{2} < \frac{1}{2}a_n < 1$, so

$$\frac{3}{2} < 1 + \frac{1}{2}a_n < 2, \text{ or } \frac{3}{2} < a_{n+1} < 2. \text{ Thus, since}$$

$1 < a_2 < 2$, every subsequent term is between $\frac{3}{2}$ and 2.

$$a_n < 2 \text{ thus } \frac{1}{2}a_n < 1, \text{ so } a_n < 1 + \frac{1}{2}a_n = a_{n+1}$$

and the sequence is nondecreasing, so $\{a_n\}$ converges to a limit $L \leq 2$.

$$36. a_1 = 2, a_2 = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{3}{2},$$

$$a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{17}{12}, a_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{24}{17}\right) = \frac{577}{408}$$

Suppose $a_n > \sqrt{2}$, and consider

$$a_{n+1} \left(a_n + \frac{2}{a_n}\right) > \sqrt{2}.$$

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) > \sqrt{2} \Leftrightarrow a_n + \frac{2}{a_n} > 2\sqrt{2} \Leftrightarrow$$

$$a_n^2 + 2 > 2\sqrt{2}a_n \Leftrightarrow a_n^2 - 2\sqrt{2}a_n + 2 > 0 \Leftrightarrow$$

$$(a_n - \sqrt{2})^2 > 0, \text{ which is always true. Hence,}$$

$a_n > \sqrt{2}$ for all n . Thus, $\{a_n\}$ converges to a limit $L \geq \sqrt{2}$.

37.

n	u_n
1	1.73205
2	2.17533
3	2.27493
4	2.29672
5	2.30146
6	2.30249
7	2.30271
8	2.30276
9	2.30277
10	2.30278
11	2.30278

$$\lim_{n \rightarrow \infty} u_n \approx 2.3028$$

38. Suppose that $0 < u_n < \frac{1}{2}(1 + \sqrt{13})$, then

$$3 < 3 + u_n < \frac{1}{2}(7 + \sqrt{13}) \text{ and}$$

$$\sqrt{3} < \sqrt{3 + u_n} = u_{n+1} < \sqrt{\frac{1}{2}(7 + \sqrt{13})} = \frac{1}{2}(1 + \sqrt{13})$$

$$\left(\sqrt{\frac{1}{2}(7 + \sqrt{13})}\right) = \frac{1}{2}(1 + \sqrt{13}) \text{ can be seen by}$$

squaring both sides of the equality and noting

that both sides are positive.) Hence, since

$$0 < u_1 = \sqrt{3} \approx 1.73 < \frac{1}{2}(1 + \sqrt{13}) \approx 2.3028,$$

$\sqrt{3} < u_n < \frac{1}{2}(1 + \sqrt{13})$ for all n ; $\{u_n\}$ is bounded above.

$$u_{n+1} = \sqrt{3 + u_n} > u_n \text{ if } 3 + u_n > u_n^2 \text{ or}$$

$$u_n^2 - u_n - 3 < 0. \quad u_n^2 - u_n - 3 = 0 \text{ when}$$

$$u_n = \frac{1}{2}(1 \pm \sqrt{13}), \text{ thus } u_{n+1} > u_n \text{ if}$$

$$\frac{1}{2}(1 - \sqrt{13}) < u_n < \frac{1}{2}(1 + \sqrt{13}), \quad \frac{1}{2}(1 - \sqrt{13}) < 0$$

and $0 < u_n < \frac{1}{2}(1 + \sqrt{13})$ for all n , as shown

above, so $\{u_n\}$ is increasing. Hence, by Theorem D, $\{u_n\}$ converges.

39. If $u = \lim_{n \rightarrow \infty} u_n$, then $u = \sqrt{3 + u}$ or $u^2 = 3 + u$;

$$u^2 - u - 3 = 0 \text{ when } u = \frac{1}{2}(1 \pm \sqrt{13}) \text{ so}$$

$$u = \frac{1}{2}(1 + \sqrt{13}) \approx 2.3028 \text{ since } u > 0 \text{ and}$$

$$\frac{1}{2}(1 - \sqrt{13}) < 0.$$

40. If $a = \lim_{n \rightarrow \infty} a_n$ where $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, then

$$a = \frac{1}{2}\left(a + \frac{2}{a}\right) \text{ or } 2a^2 = a^2 + 2; \quad a^2 = 2 \text{ when}$$

$$a = \pm\sqrt{2}, \text{ so } a = \sqrt{2}, \text{ since } a > 0.$$

41.

n	u_n
1	0
2	1
3	1.1
4	1.11053
5	1.11165
6	1.11177
7	1.11178
8	1.11178

$$\lim_{n \rightarrow \infty} u_n \approx 1.1118$$

42. Since $1.1 > 1$, $1.1^a > 1.1^b$ if $a > b$. Thus, since

$$u_3 = 1.1 > 1 = u_2, \quad u_4 = 1.1^{1.1} > 1.1^1 = u_3.$$

Suppose that $u_n < u_{n+1}$ for all $n \leq N$. Then

$$u_{N+1} = 1.1^{u_N} > 1.1^{u_{N-1}} = u_N, \text{ since } u_N > u_{N-1}$$

by the induction hypothesis. Thus, u_n is increasing.

$1.1^{u_n} < 2$ if and only if $u_n \ln 1.1 < \ln 2$;

$$u_n < \frac{\ln 2}{\ln 1.1} \approx 7.3. \text{ Thus, unless } u_n > 7.3,$$

$u_{n+1} = 1.1^{u_n} < 2$. This means that $\{u_n\}$ is bounded above by 2, since $u_1 = 0$.

43. As $n \rightarrow \infty$, $\frac{k}{n} \rightarrow 0$; using $\Delta x = \frac{1}{n}$, an equivalent definite integral is

$$\int_0^1 \sin x \, dx = [-\cos x]_0^1 = -\cos 1 + \cos 0 = 1 - \cos 1 \approx 0.4597$$

44. As $n \rightarrow \infty$, $\frac{k}{n} \rightarrow 0$; using $\Delta x = \frac{1}{n}$, an equivalent definite integral is

$$\int_0^1 \frac{1}{1+x^2} \, dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

$$45. \left| \frac{n}{n+1} - 1 \right| = \left| \frac{n - (n+1)}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1};$$

$$\frac{1}{n+1} < \varepsilon \text{ is the same as } \frac{1}{\varepsilon} < n+1. \text{ For whatever}$$

$$\varepsilon \text{ is given, choose } N > \frac{1}{\varepsilon} - 1 \text{ then}$$

$$n \geq N \Rightarrow \left| \frac{n}{n+1} - 1 \right| < \varepsilon.$$

$$46. \text{ For } n > 0, \left| \frac{n}{n^2+1} \right| = \frac{n}{n^2+1}. \quad \frac{n}{n^2+1} < \varepsilon \text{ is the}$$

$$\text{same as } \frac{n^2+1}{n} = n + \frac{1}{n} > \frac{1}{\varepsilon}. \text{ For } n > 1, \frac{1}{n} < 1 \text{ so}$$

$$n+1 > n + \frac{1}{n}; \quad n + \frac{1}{n} > \frac{1}{\varepsilon} \Leftrightarrow n > \frac{1}{\varepsilon}.$$

For whatever ε is chosen, choose $N > \frac{1}{\varepsilon}$, then

$$n \geq N \Rightarrow \left| \frac{n}{n^2+1} \right| < \varepsilon.$$

47. Recall from Section 1.2 that every rational number can be written as either a terminating or a repeating decimal.

Thus if the sequence $1, 1.4, 1.41, 1.414, \dots$ has a limit within the rational numbers, the terms of the sequence would eventually either repeat or terminate, which they do not since they are the decimal approximations to $\sqrt{2}$, which is irrational. Within the real numbers, the least upper bound is $\sqrt{2}$.

48. Suppose that $\{a_n\}$ is a nondecreasing sequence, and U is an upper bound for $\{a_n\}$, so $S = \{a_n : n \in \mathbb{N}\}$ is bounded above. By the completeness property, S has a least upper bound, which we call A . Then $A \leq U$ by definition and $a_n \leq A$ for all n . Suppose that $\lim_{n \rightarrow \infty} a_n \neq A$, i.e., that $\{a_n\}$ either does not converge, or does not converge to A . Then there is some $\varepsilon > 0$ such that $A - a_n > \varepsilon$ for all n , since if $A - a_N \leq \varepsilon$, $A - a_n \leq \varepsilon$ for $n \geq N$ since $\{a_n\}$ is nondecreasing and $a_n \leq A$ for all n . However, if $A - a_n > \varepsilon$ for all n , $a_n < A - \frac{\varepsilon}{2} < A$ for all n , which contradicts A being the least upper bound for the set S . For the second part of Theorem D, suppose that $\{a_n\}$ is a nonincreasing sequence, and L is a lower bound for $\{a_n\}$. Then $\{-a_n\}$ is a nondecreasing sequence and $-L$ is an upper bound for $\{-a_n\}$. By what was just proven, $\{-a_n\}$ converges to a limit $A \leq -L$, so $\{a_n\}$ converges to a limit $B = -A \geq L$.

49. If $\{b_n\}$ is bounded, there are numbers N and M with $N \leq |b_n| \leq M$ for all n . Then $|a_n N| \leq |a_n b_n| \leq |a_n M|$. $\lim_{n \rightarrow \infty} |a_n N| = |N| \lim_{n \rightarrow \infty} |a_n| = 0$ and $\lim_{n \rightarrow \infty} |a_n M| = |M| \lim_{n \rightarrow \infty} |a_n| = 0$, so $\lim_{n \rightarrow \infty} |a_n b_n| = 0$ by the Squeeze Theorem, and by Theorem C, $\lim_{n \rightarrow \infty} a_n b_n = 0$.

50. Suppose $\{a_n + b_n\}$ converges. Then, by Theorem A $\lim_{n \rightarrow \infty} [(a_n + b_n) - a_n] = \lim_{n \rightarrow \infty} (a_n + b_n) - \lim_{n \rightarrow \infty} a_n$. But since $(a_n + b_n) - a_n = b_n$, this would mean that $\{b_n\}$ converges. Thus $\{a_n + b_n\}$ diverges.
51. No. Consider $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Both $\{a_n\}$ and $\{b_n\}$ diverge, but $a_n + b_n = (-1)^n + (-1)^{n+1} = (-1)^n(1 + (-1)) = 0$ so $\{a_n + b_n\}$ converges.

52. a. $f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8,$
 $f_7 = 13, f_8 = 21, f_9 = 34, f_{10} = 55$

b. Using the formula,

$$f_1 = \frac{1}{\sqrt{5}} \left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \right] = \frac{1}{\sqrt{5}} \left[\frac{2\sqrt{5}}{2} \right] = 1$$

$$\begin{aligned} f_2 &= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^2 - \left(\frac{1-\sqrt{5}}{2} \right)^2 \right] \\ &= \frac{1}{\sqrt{5}} \left[\frac{1+2\sqrt{5}+5 - (1-2\sqrt{5}+5)}{4} \right] \\ &= \frac{1}{\sqrt{5}} \left[\frac{4\sqrt{5}}{4} \right] = 1. \\ \lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} &= \lim_{n \rightarrow \infty} \frac{\phi^{n+1} - (-1)^{n+1} \phi^{-n-1}}{\phi^n - (-1)^n \phi^{-n}} \\ &= \lim_{n \rightarrow \infty} \frac{\phi^{n+1} - \frac{(-1)^{n+1}}{\phi^{n+1}}}{\phi^n - \frac{(-1)^n}{\phi^n}} = \lim_{n \rightarrow \infty} \frac{\phi - \frac{(-1)^{n+1}}{\phi^{2n+1}}}{1 - \frac{(-1)^n}{\phi^{2n}}} = \phi \end{aligned}$$

c. $\phi^2 - \phi - 1 = \left[\frac{1}{2}(1+\sqrt{5}) \right]^2 - \frac{1}{2}(1+\sqrt{5}) - 1$
 $= \left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right) - \left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right) - 1 = 0$

Therefore ϕ satisfies $x^2 - x - 1 = 0$.

Using the Quadratic Formula on

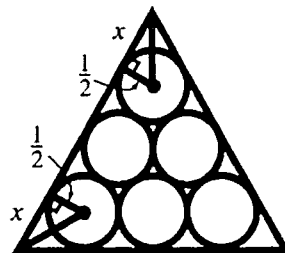
$x^2 - x - 1 = 0$ yields

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

$$\phi = \frac{1+\sqrt{5}}{2};$$

$$-\frac{1}{\phi} = -\frac{2}{1+\sqrt{5}} = -\frac{2(1-\sqrt{5})}{1-5} = \frac{1-\sqrt{5}}{2}$$

53.



From the figure shown, the sides of the triangle have length $n-1+2x$. The small right triangles marked are 30-60-90 right triangles, so $x = \frac{\sqrt{3}}{2}$;

thus the sides of the large triangle have lengths $n-1+\sqrt{3}$ and $B_n = \frac{\sqrt{3}}{4}(n-1+\sqrt{3})^2$

$$= \frac{\sqrt{3}}{4}(n^2 + 2\sqrt{3}n - 2n - 2\sqrt{3} + 4) \text{ while}$$

$$A_n = \frac{n(n+1)}{2} \pi \left(\frac{1}{2} \right)^2 = \frac{\pi}{8}(n^2 + n)$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{A_n}{B_n} &= \lim_{n \rightarrow \infty} \frac{\frac{\pi}{8}(n^2 + n)}{\frac{\sqrt{3}}{4}(n^2 + 2\sqrt{3}n - 2n - 2\sqrt{3} + 4)} \\ &= \lim_{n \rightarrow \infty} \frac{\pi(1 + \frac{1}{n})}{2\sqrt{3}(1 + \frac{2\sqrt{3}}{n} - \frac{2}{n} - \frac{2\sqrt{3}}{n^2} + \frac{4}{n^2})} = \frac{\pi}{2\sqrt{3}}\end{aligned}$$

54. Let $f(x) = \left(1 + \frac{1}{x}\right)^x$.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow 0^+} (1+x)^{1/x} = e, \text{ so}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

55. Let $f(x) = \left(1 + \frac{1}{2x}\right)^x$.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{2x}\right)^x = \lim_{x \rightarrow 0^+} \left(1 + \frac{x}{2}\right)^{1/x}$$

$$= \lim_{x \rightarrow 0^+} \left[\left(1 + \frac{x}{2}\right)^{2/x}\right]^{1/2} = e^{1/2}, \text{ so}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n}\right)^n = e^{1/2}.$$

56. Let $f(x) = \left(1 + \frac{1}{x^2}\right)^x$.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x^2}\right)^x = \lim_{x \rightarrow 0^+} (1+x^2)^{1/x^2} = 1, \text{ so}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2}\right)^n = 1.$$

57. Let $f(x) = \left(\frac{x-1}{x+1}\right)^x$.

$$\lim_{x \rightarrow \infty} \left(\frac{x-1}{x+1}\right)^x = \lim_{x \rightarrow 0^+} \left(\frac{\frac{1-x}{x}}{\frac{1}{x}+1}\right)^{1/x} = \lim_{x \rightarrow 0^+} \left(\frac{1-x}{1+x}\right)^{1/x}$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{1-x}{1+x}\right)^{1/x} = e^{-2}, \text{ so}$$

$$\lim_{n \rightarrow \infty} \left(\frac{n-1}{n+1}\right)^n = e^{-2}.$$

58. Let $f(x) = \left(\frac{2+x^2}{3+x^2}\right)^x$.

$$\lim_{x \rightarrow \infty} \left(\frac{2+x^2}{3+x^2}\right)^x = \lim_{x \rightarrow 0^+} \left(\frac{2+\frac{1}{x^2}}{3+\frac{1}{x^2}}\right)^{1/x}$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{\frac{2x^2+1}{x^2}}{\frac{3x^2+1}{x^2}}\right)^{1/x} = \lim_{x \rightarrow 0^+} \left(\frac{2x^2+1}{3x^2+1}\right)^{1/x} = 1, \text{ so}$$

$$\lim_{n \rightarrow \infty} \left(\frac{2+n^2}{3+n^2}\right)^n = 1.$$

59. Let $f(x) = \left(\frac{2+x^2}{3+x^2}\right)^{x^2}$.

$$\lim_{x \rightarrow \infty} \left(\frac{2+x^2}{3+x^2}\right)^{x^2} = \lim_{x \rightarrow 0^+} \left(\frac{2+\frac{1}{x^2}}{3+\frac{1}{x^2}}\right)^{1/x^2}$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{\frac{2x^2+1}{x^2}}{\frac{3x^2+1}{x^2}}\right)^{1/x^2} = \lim_{x \rightarrow 0^+} \left(\frac{2x^2+1}{3x^2+1}\right)^{1/x^2} = e^{-1},$$

$$\text{so } \lim_{n \rightarrow \infty} \left(\frac{2+n^2}{3+n^2}\right)^{n^2} = e^{-1}.$$

10.2 Concepts Review

1. an infinite series
2. $a_1 + a_2 + \dots + a_n$
3. $|r| < 1; \frac{a}{1-r}$
4. diverges

Problem Set 10.2

1. $\sum_{k=1}^{\infty} \left(\frac{1}{7}\right)^k = \frac{1}{7} + \frac{1}{7} \cdot \frac{1}{7} + \frac{1}{7} \left(\frac{1}{7}\right)^2 + \dots$; a geometric series with $a = \frac{1}{7}, r = \frac{1}{7}; S = \frac{\frac{1}{7}}{1 - \frac{1}{7}} = \frac{\frac{1}{7}}{\frac{6}{7}} = \frac{1}{6}$

$$2. \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^{-k-2} = \left(-\frac{1}{4}\right)^{-3} + \left(-\frac{1}{4}\right)^{-4} + \left(-\frac{1}{4}\right)^{-5} + \dots \\ = (-4)^3 + (-4)^4 + (-4)^5 + \dots; \text{ a geometric series with} \\ a = (-4)^3, r = -4; |r| = 4 > 1 \text{ so the series diverges.}$$

$$3. \sum_{k=0}^{\infty} 2\left(\frac{1}{4}\right)^k = 2 + 2 \cdot \frac{1}{4} + 2\left(\frac{1}{4}\right)^2 + \dots; \text{ a geometric series with } a = 2, r = \frac{1}{4}; S = \frac{2}{1 - \frac{1}{4}} = \frac{2}{\frac{3}{4}} = \frac{8}{3}.$$

$$\sum_{k=0}^{\infty} 3\left(-\frac{1}{5}\right)^k = 3 - 3 \cdot \frac{1}{5} + 3\left(\frac{1}{5}\right)^2 - \dots; \text{ a geometric series with } a = 3, r = -\frac{1}{5};$$

$$S = \frac{3}{1 - \left(-\frac{1}{5}\right)} = \frac{3}{\frac{6}{5}} = \frac{5}{2}$$

Thus, by Theorem B.

$$\sum_{k=0}^{\infty} \left[2\left(\frac{1}{4}\right)^k + 3\left(-\frac{1}{5}\right)^k \right] = \frac{8}{3} + \frac{5}{2} = \frac{31}{6}$$

$$4. \sum_{k=1}^{\infty} 5\left(\frac{1}{2}\right)^k = \frac{5}{2} + \frac{5}{2} \cdot \frac{1}{2} + \frac{5}{2}\left(\frac{1}{2}\right)^2 + \dots; \text{ a geometric series with } a = \frac{5}{2}, r = \frac{1}{2}; S = \frac{\frac{5}{2}}{1 - \frac{1}{2}} = \frac{\frac{5}{2}}{\frac{1}{2}} = 5.$$

$$\sum_{k=1}^{\infty} 3\left(\frac{1}{7}\right)^{k+1} = \frac{3}{49} + \frac{3}{49} \cdot \frac{1}{7} + \frac{3}{49}\left(\frac{1}{7}\right)^2 + \dots; \text{ a}$$

$$\text{geometric series with } a = \frac{3}{49}, r = \frac{1}{7};$$

$$S = \frac{\frac{3}{49}}{1 - \frac{1}{7}} = \frac{\frac{3}{49}}{\frac{6}{7}} = \frac{1}{14}$$

Thus, by Theorem B,

$$\sum_{k=1}^{\infty} \left[2\left(\frac{1}{4}\right)^k - 3\left(\frac{1}{7}\right)^{k+1} \right] = 5 - \frac{1}{14} = \frac{69}{14}.$$

$$5. \sum_{k=1}^{\infty} \frac{k-5}{k+2} = -\frac{4}{3} - \frac{3}{4} - \frac{2}{5} - \frac{1}{6} + 0 + \frac{1}{8} + \frac{2}{9} + \dots;$$

$$\lim_{k \rightarrow \infty} \frac{k-5}{k+2} = \lim_{k \rightarrow \infty} \frac{1 - \frac{5}{k}}{1 + \frac{2}{k}} = 1 \neq 0; \text{ the series}$$

diverges.

$$6. \sum_{k=1}^{\infty} \left(\frac{9}{8}\right)^k = \frac{9}{8} + \frac{9}{8} \cdot \frac{9}{8} + \frac{9}{8}\left(\frac{9}{8}\right)^2 + \dots; \text{ a geometric}$$

$$\text{series with } a = \frac{9}{8}, r = \frac{9}{8}; \left|\frac{9}{8}\right| > 1, \text{ so the series diverges.}$$

$$7. \sum_{k=2}^{\infty} \left(\frac{1}{k} - \frac{1}{k-1}\right) = \left(\frac{1}{2} - \frac{1}{1}\right) + \left(\frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{4} - \frac{1}{3}\right) + \dots;$$

$$S_n = \left(\frac{1}{2} - 1\right) + \left(\frac{1}{3} - \frac{1}{2}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n-2}\right) + \left(\frac{1}{n} - \frac{1}{n-1}\right) = -1 + \frac{1}{n};$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} -1 + \frac{1}{n} = -1, \text{ so } \sum_{k=2}^{\infty} \left(\frac{1}{k} - \frac{1}{k-1}\right) = -1$$

$$8. \sum_{k=1}^{\infty} \frac{3}{k} = 3 \sum_{k=1}^{\infty} \frac{1}{k} \text{ which diverges since } \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges.}$$

$$9. \sum_{k=1}^{\infty} \frac{k!}{100^k} = \frac{1}{100} + \frac{2}{10,000} + \frac{6}{1,000,000} + \dots$$

Consider $\{a_n\}$, where $a_{n+1} = \frac{n+1}{100} a_n, a_1 = \frac{1}{100}$. $a_n > 0$ for all n , and for $n > 99$, $a_{n+1} > a_n$, so the

sequence is eventually an increasing sequence, hence $\lim_{n \rightarrow \infty} a_n \neq 0$. The sequence can also be described by

$$a_n = \frac{n!}{100^n}, \text{ hence } \sum_{k=1}^{\infty} \frac{k!}{100^k} \text{ diverges.}$$

$$\begin{aligned} 10. \quad \sum_{k=1}^{\infty} \frac{2}{(k+2)k} &= \frac{2}{3} + \frac{2}{8} + \frac{2}{15} + \dots = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2} \right) = \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \dots \\ S_n &= \left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \left(\frac{1}{n} - \frac{1}{n+2} \right) \\ &= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)} = \frac{3}{2} - \frac{2n+3}{n^2+3n+2} \\ \lim_{n \rightarrow \infty} S_n &= \frac{3}{2} - \lim_{n \rightarrow \infty} \frac{2n+3}{n^2+3n+2} = \frac{3}{2} - \lim_{n \rightarrow \infty} \frac{\frac{2}{n} + \frac{3}{n^2}}{1 + \frac{3}{n} + \frac{2}{n^2}} = \frac{3}{2}, \text{ so } \sum_{k=1}^{\infty} \frac{2}{(k+2)k} = \frac{3}{2}. \end{aligned}$$

$$\begin{aligned} 11. \quad \sum_{k=1}^{\infty} \left(\frac{e}{\pi} \right)^{k+1} &= \left(\frac{e}{\pi} \right)^2 + \left(\frac{e}{\pi} \right)^2 \cdot \frac{e}{\pi} + \left(\frac{e}{\pi} \right)^2 \left(\frac{e}{\pi} \right)^2 + \dots; \text{ a geometric series with } a = \left(\frac{e}{\pi} \right)^2, r = \frac{e}{\pi} < 1; \\ S &= \frac{\left(\frac{e}{\pi} \right)^2}{1 - \frac{e}{\pi}} = \frac{\left(\frac{e}{\pi} \right)^2}{\frac{\pi-e}{\pi}} = \frac{e^2}{\pi(\pi-e)} \approx 5.5562 \end{aligned}$$

$$12. \quad \sum_{k=1}^{\infty} \frac{4^{k+1}}{7^{k-1}} = \frac{16}{1} + 16 \cdot \frac{4}{7} + 16 \left(\frac{4}{7} \right)^2 + \dots; \text{ a geometric series with } a = 16, r = \frac{4}{7} < 1; S = \frac{16}{1 - \frac{4}{7}} = \frac{16}{\frac{3}{7}} = \frac{112}{3}$$

$$\begin{aligned} 13. \quad \sum_{k=2}^{\infty} \left(\frac{3}{(k-1)^2} - \frac{3}{k^2} \right) &= \left(\frac{3}{1} - \frac{3}{4} \right) + \left(\frac{3}{4} - \frac{3}{9} \right) + \left(\frac{3}{9} - \frac{3}{16} \right) + \dots; \\ S_n &= \left(3 - \frac{3}{4} \right) + \left(\frac{3}{4} - \frac{3}{9} \right) + \left(\frac{3}{9} - \frac{3}{16} \right) + \dots + \left(\frac{3}{(n-2)^2} - \frac{3}{(n-1)^2} \right) + \left(\frac{3}{(n-1)^2} - \frac{3}{n^2} \right) \\ &= 3 - \frac{3}{n^2}; \lim_{n \rightarrow \infty} S_n = 3 - \lim_{n \rightarrow \infty} \frac{3}{n^2} = 3, \text{ so} \\ \sum_{k=2}^{\infty} \left(\frac{3}{(k-1)^2} - \frac{3}{k^2} \right) &= 3. \end{aligned}$$

$$\begin{aligned} 14. \quad \sum_{k=6}^{\infty} \frac{2}{k-5} &= \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \dots \\ &= 2 \sum_{k=1}^{\infty} \frac{1}{k} \text{ which diverges since } \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges.} \end{aligned}$$

$$\begin{aligned} 16. \quad 0.21212121\dots &= \sum_{k=1}^{\infty} \frac{21}{100} \left(\frac{1}{100} \right)^{k-1} \\ &= \frac{\frac{21}{100}}{1 - \frac{1}{100}} = \frac{21}{99} = \frac{7}{33} \end{aligned}$$

$$\begin{aligned} 15. \quad 0.22222\dots &= \sum_{k=1}^{\infty} \frac{2}{10} \left(\frac{1}{10} \right)^{k-1} \\ &= \frac{\frac{2}{10}}{1 - \frac{1}{10}} = \frac{2}{9} \end{aligned}$$

$$\begin{aligned} 17. \quad 0.013013013\dots &= \sum_{k=1}^{\infty} \frac{13}{1000} \left(\frac{1}{1000} \right)^{k-1} \\ &= \frac{\frac{13}{1000}}{1 - \frac{1}{1000}} = \frac{13}{999} \end{aligned}$$

$$18. 0.125125125... = \sum_{k=1}^{\infty} \frac{125}{1000} \left(\frac{1}{1000} \right)^{k-1}$$

$$= \frac{\frac{125}{1000}}{1 - \frac{1}{1000}} = \frac{125}{999}$$

$$19. 0.4999... = \frac{4}{10} + \sum_{k=1}^{\infty} \frac{9}{100} \left(\frac{1}{10} \right)^{k-1}$$

$$= \frac{4}{10} + \frac{\frac{9}{100}}{1 - \frac{1}{10}} = \frac{1}{2}$$

$$20. 0.36717171... = \frac{36}{100} + \sum_{k=1}^{\infty} \frac{71}{10,000} \left(\frac{1}{100} \right)^{k-1}$$

$$= \frac{36}{100} + \frac{\frac{71}{10,000}}{1 - \frac{1}{100}} = \frac{727}{1980}$$

21. Let $s = 1 - r$, so $r = 1 - s$. Since $0 < r < 2$,
 $-1 < 1 - r < 1$, so

23. $\ln \frac{k}{k+1} = \ln k - \ln(k+1)$

$$S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \dots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) = \ln 1 - \ln(n+1) = -\ln(n+1)$$

$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} -\ln(n+1) = -\infty$, thus $\sum_{k=1}^{\infty} \ln \frac{k}{k+1}$ diverges.

24. $\ln \left(1 - \frac{1}{k^2} \right) = \ln \frac{k^2 - 1}{k^2} = \ln(k^2 - 1) - \ln k^2 = \ln[(k+1)(k-1)] - \ln k^2 = \ln(k+1) + \ln(k-1) - 2 \ln k$

$$S_n = (\ln 3 + \ln 1 - 2 \ln 2) + (\ln 4 + \ln 2 - 2 \ln 3) + (\ln 5 + \ln 3 - 2 \ln 4) + \dots$$

$$+ (\ln n + \ln(n-2) - 2 \ln(n-1)) + (\ln(n+1) + \ln(n-1) - 2 \ln n)$$

$$= -\ln 2 + \ln(n+1) - \ln n = -\ln 2 + \ln \frac{n+1}{n}$$

$\lim_{n \rightarrow \infty} S_n = -\ln 2 + \lim_{n \rightarrow \infty} \ln \frac{n+1}{n} = -\ln 2 + \ln \left(\lim_{n \rightarrow \infty} \frac{n+1}{n} \right) = -\ln 2 + \ln 1 = -\ln 2$

25. The ball drops 100 feet, rebounds up $100 \left(\frac{2}{3} \right)$ feet, drops $100 \left(\frac{2}{3} \right)$ feet, rebounds up $100 \left(\frac{2}{3} \right)^2$ feet, drops $100 \left(\frac{2}{3} \right)^2$, etc. The total distance it travels is

$$100 + 200 \left(\frac{2}{3} \right) + 200 \left(\frac{2}{3} \right)^2 + 200 \left(\frac{2}{3} \right)^3 + \dots = -100 + 200 + 200 \left(\frac{2}{3} \right) + 200 \left(\frac{2}{3} \right)^2 + 200 \left(\frac{2}{3} \right)^3 + \dots$$

$$= -100 + \sum_{k=1}^{\infty} 200 \left(\frac{2}{3} \right)^{k-1} = -100 + \frac{200}{1 - \frac{2}{3}} = 500 \text{ feet}$$

$|s| < 1$, and $\sum_{k=0}^{\infty} r(1-r)^k = \sum_{k=0}^{\infty} (1-s)s^k$

$$= \sum_{k=1}^{\infty} (1-s)s^{k-1} = \frac{1-s}{1-s} = 1$$

22. $\sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=1}^{\infty} (-x)^{k-1}$;
 if $-1 < x < 1$ then
 $-1 < -x < 1$ so $|x| < 1$;

$$\sum_{k=1}^{\infty} (-x)^{k-1} = \frac{1}{1 - (-x)} = \frac{1}{1+x}$$

$$26. \text{ Each gets } \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \left(\frac{1}{4} \cdot \frac{1}{4} \right) + \dots = \sum_{k=1}^{\infty} \frac{1}{4} \left(\frac{1}{4} \right)^{k-1} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}$$

(This can be seen intuitively, since the size of the leftover piece is approaching 0, and each person gets the same amount.)

$$27. \$1 \text{ billion} + 75\% \text{ of } \$1 \text{ billion} + 75\% \text{ of } 75\% \text{ of } \$1 \text{ billion} + \dots = \sum_{k=1}^{\infty} (\$1 \text{ billion}) 0.75^{k-1} = \frac{\$1 \text{ billion}}{1 - 0.75} = \$4 \text{ billion}$$

$$28. \sum_{k=1}^{\infty} \$1 \text{ billion } (0.90)^{k-1} = \frac{\$1 \text{ billion}}{1 - 0.90} = \$10 \text{ billion}$$

$$29. \text{ As the midpoints of the sides of a square are connected, a new square is formed. The new square has sides } \frac{1}{\sqrt{2}}$$

times the sides of the old square. Thus, the new square has area $\frac{1}{2}$ the area of the old square. Then in the next step,

$\frac{1}{8}$ of each new square is shaded.

$$\text{Area} = \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{4} + \dots = \sum_{k=1}^{\infty} \frac{1}{8} \left(\frac{1}{2} \right)^{k-1} = \frac{\frac{1}{8}}{1 - \frac{1}{2}} = \frac{1}{4}$$

The area will be $\frac{1}{4}$.

$$30. \frac{1}{9} + \frac{1}{9} \left(\frac{8}{9} \right) + \frac{1}{9} \left(\frac{8}{9} \cdot \frac{8}{9} \right) + \dots = \sum_{k=1}^{\infty} \frac{1}{9} \left(\frac{8}{9} \right)^{k-1} = \frac{\frac{1}{9}}{1 - \frac{8}{9}} = 1; \text{ the whole square will be pointed.}$$

$$31. \frac{3}{4} + \frac{3}{4} \left(\frac{1}{4} \cdot \frac{1}{4} \right) + \frac{3}{4} \left(\frac{1}{4} \cdot \frac{1}{4} \right) \left(\frac{1}{4} \cdot \frac{1}{4} \right) + \dots = \sum_{k=1}^{\infty} \frac{3}{4} \left(\frac{1}{16} \right)^{k-1} = \frac{\frac{3}{4}}{1 - \frac{1}{16}} = \frac{4}{5}$$

The original does not need to be equilateral since each smaller triangle will have $\frac{1}{4}$ area of the previous larger triangle.

$$32. \text{ Ratio of inscribed circle to triangle is } \frac{\pi}{3\sqrt{3}}, \text{ so}$$

$$\sum_{k=1}^{\infty} \frac{\pi}{3\sqrt{3}} \cdot \frac{3}{4} \left(\frac{1}{4} \right)^{k-1} = \frac{\left(\frac{\pi}{4\sqrt{3}} \right)}{1 - \frac{1}{4}} = \frac{\pi}{3\sqrt{3}}$$

(This can be seen intuitively, since *every* small triangle has a circle inscribed in it.)

$$33. \text{ Both Achilles and the tortoise will have moved.}$$

$$100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \dots = \sum_{k=1}^{\infty} 100 \left(\frac{1}{10} \right)^{k-1} = \frac{100}{1 - \frac{1}{10}} = 111 \frac{1}{9} \text{ yards}$$

Also, one can see this by the following reasoning.

In the time it takes the tortoise to run $\frac{d}{10}$ yards,

Achilles will run d yards. Solve

$$d = 100 + \frac{d}{10}, d = \frac{1000}{9} = 111 \frac{1}{9} \text{ yards}$$

34. a. Say Trot and Tom start from the left. Joel from the right. Trot and Joel run towards each other at 30 mph. Since they are 60 miles apart they will meet in 2 hours. Trot will have run 40 miles and Tom will have run 20 miles, so they will be 20 miles apart. Trot and Tom will now be approaching each other at 30 mph, so they will meet after $\frac{2}{3}$ hour. Trot will have run another $\frac{40}{3}$ miles and will be $\frac{80}{3}$ miles from the left. Joel will have run another $\frac{20}{3}$ miles and will be at $\frac{100}{3}$ miles from the left, so

they will be $20/3$ miles apart. They will meet after $2/9$ hour, during which Trot will have run $40/9$ miles, etc. So Trot runs

$$40 + \frac{40}{3} + \frac{40}{9} + \cdots = \sum_{k=1}^{\infty} 40 \left(\frac{1}{3} \right)^{k-1} = \frac{40}{1 - \frac{1}{3}} = 60 \text{ miles.}$$

- b. Tom and Joel are approaching each other at 20 mph. They are 60 miles apart, so they will meet in 3 hours. Trot is running at 20 mph during that entire time, so he runs 60 miles.

35. (Proof by contradiction) Assume $\sum_{k=1}^{\infty} ca_k$ converges, and $c \neq 0$. Then $\frac{1}{c}$ is defined, so

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{c} ca_k = \frac{1}{c} \sum_{k=1}^{\infty} ca_k \text{ would also converge, by Theorem B(i).}$$

36. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{1}{k} \right) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k}$ diverges since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

37. a. The top block is supported *exactly* at its center of mass. The location of the center of mass of the top n blocks is the average of the locations of their individual centers of mass, so the n th block moves the center of mass left by $\frac{1}{n}$ of the location of its center of

mass, that is, $\frac{1}{n} \cdot \frac{1}{2}$ or $\frac{1}{2n}$ to the left. But this is exactly how far the $(n+1)$ st block underneath it is offset.

- b. Since $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k}$, which diverges, there is no limit to how far the top block can protrude.

38. $N = 31$; $S_{31} \approx 4.0272$ and $S_{30} \approx 3.9950$.

39. (Proof by contradiction) Assume $\sum_{k=1}^{\infty} (a_k + b_k)$

converges. Since $\sum_{k=1}^{\infty} b_k$ converges, so would

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (a_k + b_k) + (-1) \sum_{k=1}^{\infty} b_k, \text{ by}$$

Theorem B(ii).

40. (Answers may vary). $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1) \frac{1}{n} \text{ both diverge, but}$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n} \right) \text{ converges to 0.}$$

41. Taking vertical strips, the area is

$$1 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} + \cdots = \sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^{k-1}.$$

Taking horizontal strips, the area is

$$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{16} \cdot 4 + \cdots = \sum_{k=1}^{\infty} \frac{k}{2^k}.$$

$$\text{a. } \sum_{k=1}^{\infty} \frac{k}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^{k-1} = \frac{1}{1 - \frac{1}{2}} = 2$$

- b. The moment about $x = 0$ is

$$\sum_{k=0}^{\infty} \left(\frac{1}{2} \right)^k \cdot (1)k = \sum_{k=0}^{\infty} \frac{k}{2^k} = \sum_{k=1}^{\infty} \frac{k}{2^k} = 2.$$

$$\bar{x} = \frac{\text{moment}}{\text{area}} = \frac{2}{2} = 1$$

42. If $\sum_{k=1}^{\infty} kr^k$ converges, so will $r \sum_{k=1}^{\infty} kr^k$, by

Theorem B.

$$rS = r \sum_{k=1}^{\infty} kr^k = \sum_{k=1}^{\infty} kr^{k+1} = \sum_{k=2}^{\infty} (k-1)r^k \text{ while}$$

$$S = \sum_{k=1}^{\infty} kr^k = r + \sum_{k=2}^{\infty} kr^k \text{ so}$$

$$S - rS = r + \sum_{k=2}^{\infty} kr^k - \sum_{k=2}^{\infty} (k-1)r^k$$

$$= r + \sum_{k=2}^{\infty} [k - (k-1)]r^k = r + \sum_{k=2}^{\infty} r^k = \sum_{k=1}^{\infty} r^k$$

$$\text{Since } |r| < 1, \sum_{k=1}^{\infty} r^k = \frac{r}{1-r}, \text{ thus}$$

$$S = \frac{1}{1-r} \sum_{k=1}^{\infty} r^k = \frac{r}{(1-r)^2}.$$

43. a. $A = \sum_{n=0}^{\infty} Ce^{-nkt} = \sum_{n=1}^{\infty} C \left(\frac{1}{e^{kt}} \right)^{n-1}$

$$= \frac{C}{1 - \frac{1}{e^{kt}}} = \frac{Ce^{kt}}{e^{kt} - 1}$$

$$b. \quad \frac{1}{2} = e^{-kt} = e^{-6k} \Rightarrow k = \frac{\ln 2}{6} \Rightarrow A = \frac{4}{3}C;$$

$$\text{if } C = 2 \text{ mg, then } A = \frac{8}{3} \text{ mg.}$$

$$44. \text{ Using partial fractions, } \frac{2^k}{(2^{k+1}-1)(2^k-1)} = \frac{1}{2^k-1} - \frac{1}{2^{k+1}-1}$$

$$S_n = \left(\frac{1}{2^1-1} - \frac{1}{2^2-1} \right) + \left(\frac{1}{2^2-1} - \frac{1}{2^3-1} \right) + \dots + \left(\frac{1}{2^{n-1}-1} - \frac{1}{2^n-1} \right) + \left(\frac{1}{2^n-1} - \frac{1}{2^{n+1}-1} \right)$$

$$= \frac{1}{2-1} - \frac{1}{2^{n+1}-1} = 1 - \frac{1}{2^{n+1}-1}$$

$$\lim_{n \rightarrow \infty} S_n = 1 - \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}-1} = 1 - 0 = 1$$

$$45. \quad \frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1} f_{k+2}} = \frac{f_{k+2} - f_k}{f_k f_{k+1} f_{k+2}} = \frac{1}{f_k f_{k+2}}$$

since $f_{k+2} = f_{k+1} + f_k$. Thus,

$$\sum_{k=1}^{\infty} \frac{1}{f_k f_{k+2}} = \sum_{k=1}^{\infty} \left(\frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1} f_{k+2}} \right) \text{ and}$$

$$S_n = \left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3} \right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4} \right) + \dots + \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} \right) + \left(\frac{1}{f_n f_{n+1}} - \frac{1}{f_{n+1} f_{n+2}} \right)$$

$$= \frac{1}{f_1 f_2} - \frac{1}{f_{n+1} f_{n+2}} = \frac{1}{1 \cdot 1} - \frac{1}{f_{n+1} f_{n+2}} = 1 - \frac{1}{f_{n+1} f_{n+2}}$$

The terms of the Fibonacci sequence increase without bound, so

$$\lim_{n \rightarrow \infty} S_n = 1 - \lim_{n \rightarrow \infty} \frac{1}{f_{n+1} f_{n+2}} = 1 - 0 = 1$$

10.3 Concepts Review

1. bounded above
2. $f(k)$; continuous; positive; nonincreasing
3. convergence or divergence
4. $p > 1$

Problem Set 10.3

1. $\frac{1}{x+3}$ is continuous, positive, and nonincreasing on $[0, \infty)$.

$$\int_0^{\infty} \frac{1}{x+3} dx = [\ln|x+3|]_0^{\infty} = \infty - \ln 3 = \infty$$

The series diverges.

2. $\frac{3}{2x-3}$ is continuous, positive, and nonincreasing on $[2, \infty)$.

$$\int_2^{\infty} \frac{3}{2x-3} dx = \left[\frac{3}{2} \ln|2x-3| \right]_2^{\infty} = \infty - \frac{3}{2} \ln 1 = \infty$$

This series diverges.

3. $\frac{x}{x^2+3}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{x}{x^2+3} dx = \left[\frac{1}{2} \ln|x^2+3| \right]_1^{\infty} = \infty - \frac{1}{2} \ln 4 = \infty$$

The series diverges.

4. $\frac{3}{2x^2+1}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{3}{2x^2+1} dx = \left[\frac{3}{\sqrt{2}} \tan^{-1} \sqrt{2}x \right]_1^{\infty}$$

$$= \frac{3}{\sqrt{2}} \left(\frac{\pi}{2} - \tan^{-1} 1 \right) < \infty$$

The series converges.

5. $\frac{2}{\sqrt{x+2}}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{2}{\sqrt{x+2}} dx = \left[4\sqrt{x+2} \right]_1^{\infty} = \infty - 4\sqrt{3} = \infty$$

Thus $\sum_{k=1}^{\infty} \frac{2}{\sqrt{k+2}}$ diverges, hence

$$\sum_{k=1}^{\infty} \frac{-2}{\sqrt{k+2}} = -\sum_{k=1}^{\infty} \frac{2}{\sqrt{k+2}} \text{ also diverges.}$$

6. $\frac{3}{(x+2)^2}$ is continuous, positive, and nonincreasing on $[100, \infty)$.

$$\int_{100}^{\infty} \frac{3}{(x+2)^2} dx = \left[-\frac{3}{x+2} \right]_{100}^{\infty} = 0 + \frac{3}{102} = \frac{3}{102} < \infty$$

The series converges.

7. $\frac{7}{4x+2}$ is continuous, positive, and nonincreasing on $[2, \infty)$.

$$\int_2^{\infty} \frac{7}{4x+2} dx = \left[\frac{7}{4} \ln|4x+2| \right]_2^{\infty} = \infty - \frac{7}{4} \ln 10 = \infty$$

The series diverges.

8. $\frac{x^2}{e^x}$ is continuous, positive, and nonincreasing on $[2, \infty)$. Using integration by parts twice, with

$$u = x^i, \quad i = 1, 2 \text{ and } dv = e^{-x} dx,$$

$$\begin{aligned} \int_2^{\infty} x^2 e^{-x} dx &= [-x^2 e^{-x}]_2^{\infty} + 2 \int_2^{\infty} x e^{-x} dx \\ &= [-x^2 e^{-x}]_2^{\infty} + 2 \left([-x e^{-x}]_2^{\infty} + \int_2^{\infty} e^{-x} dx \right) \end{aligned}$$

$$\begin{aligned} &= [-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}]_2^{\infty} \\ &= 0 + 4e^{-2} + 4e^{-2} + 2e^{-2} = 10e^{-2} < \infty \end{aligned}$$

The series converges.

9. $\frac{3}{(4+3x)^{7/6}}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{3}{(4+3x)^{7/6}} dx = \left[-\frac{6}{(4+3x)^{1/6}} \right]_1^{\infty}$$

$$= 0 + \frac{6}{7^{1/6}} = 6 \cdot 7^{-1/6} < \infty$$

The series converges.

10. $\frac{1000x^2}{1+x^3}$ is continuous, positive, and nonincreasing on $[2, \infty)$.

$$\int_2^{\infty} \frac{1000x^2}{1+x^3} dx = \left[\frac{1000}{3} \ln|1+x^3| \right]_2^{\infty}$$

$$= \infty - \frac{1000}{3} \ln 9 = \infty$$

The series diverges.

11. xe^{-3x^2} is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} xe^{-3x^2} dx = \left[-\frac{1}{6} e^{-3x^2} \right]_1^{\infty} = 0 + \frac{1}{6} e^{-3}$$

$$= \frac{1}{6e^3} < \infty$$

The series converges.

12. $\frac{1000}{x(\ln x)^2}$ is continuous, positive, and nonincreasing on $[5, \infty)$.

$$\int_5^{\infty} \frac{1000}{x(\ln x)^2} dx = \left[-\frac{1000}{\ln x} \right]_5^{\infty} = 0 + \frac{1000}{\ln 5}$$

$$= \frac{1000}{\ln 5} < \infty$$

The series converges.

13. $\lim_{k \rightarrow \infty} \frac{k^2+1}{k^2+5} = \lim_{k \rightarrow \infty} \frac{1+\frac{1}{k^2}}{1+\frac{5}{k^2}} = 1 \neq 0$, so the series diverges.

14. $\sum_{k=1}^{\infty} \left(\frac{3}{\pi}\right)^k = \sum_{k=1}^{\infty} \frac{3}{\pi} \left(\frac{3}{\pi}\right)^{k-1}$; a geometric series with $a = \frac{3}{\pi}, r = \frac{3}{\pi}; \left|\frac{3}{\pi}\right| < 1$ so the series converges.

15. $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ is a geometric series with $r = \frac{1}{2}; \left|\frac{1}{2}\right| < 1$ so the series converges.

$$\text{In } \sum_{k=1}^{\infty} \frac{k-1}{2k+1}, \lim_{k \rightarrow \infty} \frac{k-1}{2k+1} = \lim_{k \rightarrow \infty} \frac{1-\frac{1}{k}}{2+\frac{1}{k}} = \frac{1}{2} \neq 0, \text{ so}$$

the series diverges. Thus, the sum of the series diverges.

16. $\frac{1}{x^2}$ is continuous, positive, and nonincreasing on

$$[1, \infty). \int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{\infty} = 0 + 1 = 1 < \infty, \text{ so}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges.}$$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^k; \text{ a geometric series with}$$

$$r = \frac{1}{2}; \left| \frac{1}{2} \right| < 1, \text{ so the series converges. Thus, the}$$

sum of the series converges.

$$17. \sin\left(\frac{k\pi}{2}\right) = \begin{cases} 1 & k = 4j + 1 \\ -1 & k = 4j + 3, \\ 0 & k \text{ is even} \end{cases}$$

where j is any nonnegative integer.

$$20. S_n = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n-1} \right) = 1 - \frac{1}{n-1}$$

$$\lim_{n \rightarrow \infty} S_n = 1 - \lim_{n \rightarrow \infty} \frac{1}{n-1} = 1 - 0 = 1$$

The series converges to 1.

$$21. \frac{\tan^{-1} x}{1+x^2} \text{ is continuous, positive, and}$$

nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2} dx = \left[\frac{1}{2} (\tan^{-1} x)^2 \right]_1^{\infty}$$

$$= \frac{1}{2} \left(\frac{\pi}{2} \right)^2 - \frac{1}{2} \left(\frac{\pi}{4} \right)^2 = \frac{3\pi^2}{32} < \infty, \text{ so the series}$$

converges.

$$22. \frac{1}{1+4x^2} \text{ is continuous, positive, and}$$

nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{1}{1+4x^2} dx = \left[\frac{1}{2} \tan^{-1}(2x) \right]_1^{\infty}$$

$$= \frac{1}{2} \left(\frac{\pi}{2} \right) - \frac{1}{2} \tan^{-1} 2 < \infty,$$

so the series converges.

$$23. \frac{x}{e^x} \text{ is continuous, positive, and nonincreasing on}$$

$[5, \infty)$.

Thus $\lim_{k \rightarrow \infty} \left| \sin\left(\frac{k\pi}{2}\right) \right|$ does not exist, hence

$\lim_{k \rightarrow \infty} \left| \sin\left(\frac{k\pi}{2}\right) \right| \neq 0$ and the series diverges.

$$18. \text{ As } k \rightarrow \infty, \frac{1}{k} \rightarrow 0. \text{ Let } y = \frac{1}{k}, \text{ then}$$

$$\lim_{k \rightarrow \infty} k \sin \frac{1}{k} = \lim_{y \rightarrow 0} \frac{1}{y} \sin y = \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \neq 0, \text{ so}$$

the series diverges.

$$19. x^2 e^{-x^3} \text{ is continuous, positive, and nonincreasing on } [1, \infty).$$

$$\int_1^{\infty} x^2 e^{-x^3} dx = \left[-\frac{1}{3} e^{-x^3} \right]_1^{\infty} = 0 + \frac{1}{3} e^{-1} < \infty, \text{ so}$$

the series converges.

$$E = \sum_{k=6}^{\infty} \frac{k}{e^k} \leq \int_5^{\infty} \frac{x}{e^x} dx = [-xe^{-x}]_5^{\infty} + \int_5^{\infty} e^{-x} dx$$

$$= [-xe^{-x} - e^{-x}]_5^{\infty} = 0 + 5e^{-5} + e^{-5} = 6e^{-5}$$

$$\approx 0.0404$$

$$24. \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}} \text{ is continuous, positive, and nonincreasing on } [5, \infty).$$

$$E = \sum_{k=6}^{\infty} \frac{1}{k\sqrt{k}} \leq \int_5^{\infty} \frac{1}{x^{3/2}} dx = \left[-\frac{2}{\sqrt{x}} \right]_5^{\infty} = 0 + \frac{2}{\sqrt{5}}$$

$$\approx 0.8944$$

$$25. \frac{1}{1+x^2} \text{ is continuous, positive, and nonincreasing on } [5, \infty).$$

$$E = \sum_{k=6}^{\infty} \frac{1}{1+k^2} \leq \int_5^{\infty} \frac{1}{1+x^2} dx = [\tan^{-1} x]_5^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1} 5 \approx 0.1974$$

26. $\frac{1}{x(x+1)}$ is continuous, positive, and nonincreasing on $[5, \infty)$.

$$\begin{aligned} E &= \sum_{k=6}^{\infty} \frac{1}{k(k+1)} \leq \int_5^{\infty} \frac{1}{x(x+1)} dx = \int_5^{\infty} \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \\ &= \left[\ln|x| - \ln|x+1| \right]_5^{\infty} = \left[\ln \left| \frac{x}{x+1} \right| \right]_5^{\infty} = 0 - \ln \frac{5}{6} \\ &= \ln \frac{6}{5} \approx 0.1823 \end{aligned}$$

27. Consider $\int_2^{\infty} \frac{1}{x(\ln x)^p} dx$. Let $u = \ln x$,

$$du = \frac{1}{x} dx.$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_{\ln 2}^{\infty} \frac{1}{u^p} du \text{ which converges for } p > 1.$$

28. $\frac{1}{x \ln x \ln(\ln x)}$ is continuous, positive, and nonincreasing on $[3, \infty)$.

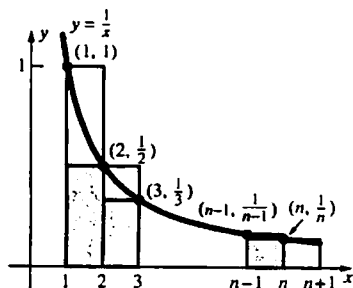
$$\int_3^{\infty} \frac{1}{x \ln x \ln(\ln x)} dx$$

$$\text{Let } u = \ln(\ln x), \quad du = \frac{1}{x \ln x} dx.$$

$$\int_3^{\infty} \frac{1}{x \ln x \ln(\ln x)} dx = \int_{\ln(\ln 3)}^{\infty} \frac{1}{u} du = [\ln u]_{\ln(\ln 3)}^{\infty} = \infty - \ln(\ln 3) = \infty$$

The series diverges.

29.



The upper rectangles, which extend to $n+1$ on the right, have area $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. These rectangles are above the curve $y = \frac{1}{x}$ from $x = 1$ to $x = n+1$. Thus,

$$\begin{aligned} \int_1^{n+1} \frac{1}{x} dx &= [\ln x]_1^{n+1} = \ln(n+1) - \ln 1 = \ln(n+1) \\ &< 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}. \end{aligned}$$

The lower (shaded) rectangles have area $\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. These rectangles lie below the curve $y = \frac{1}{x}$ from $x = 1$ to $x = n$. Thus

$$\begin{aligned} \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} &< \int_1^n \frac{1}{x} dx = \ln n, \text{ so} \\ 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} &< 1 + \ln n. \end{aligned}$$

30. From Problem 29, B_n is the area of the region within the upper rectangles but above the curve $y = \frac{1}{x}$. Each time n is incremented by 1, the added area is a positive amount, thus B_n is increasing. From the inequalities in Problem 29, $0 < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln(n+1) < 1 + \ln n - \ln(n+1) = 1 + \ln \frac{n}{n+1}$. Since $\frac{n}{n+1} < 1$, $\ln \frac{n}{n+1} < 0$, thus $B_n < 1$ for all n , and B_n is bounded by 1.

31. $\{B_n\}$ is a nondecreasing sequence that is bounded above, thus by the Monotone Sequence Theorem (Theorem D of Section 10.1), $\lim_{n \rightarrow \infty} B_n$ exists. The rationality of γ is a famous unsolved problem.

32. From Problem 29, $\ln(n+1) < \sum_{k=1}^n \frac{1}{k} < 1 + \ln n$, thus

$$\begin{aligned} \ln(10,000,001) &\approx 16.1181 < \sum_{k=1}^{10,000,000} \frac{1}{k} \\ &< 1 + \ln(10,000,000) \approx 17.1181 \end{aligned}$$

33. $\gamma + \ln(n+1) > 20 \Rightarrow \ln(n+1) > 20 - \gamma \approx 19.4228$
 $\Rightarrow n+1 > e^{19.4228} \approx 272,404,867$
 $\Rightarrow n > 272,404,866$

34. a. Each time n is incremented by 1, a positive amount of area is added.
 b. The leftmost rectangle has area $1 \cdot f(1) = f(1)$. If each shaded region to the right of $x = 2$ is shifted until it is in the leftmost rectangle, there will be no overlap of the shaded area, since the top of each rectangle is at the bottom of the shaded

region to the left. Thus, the total shaded area is less than or equal to the area of the leftmost rectangle, or $B_n \leq f(1)$.

- c. By parts a and b, $\{B_n\}$ is a nondecreasing sequence that is bounded above, so $\lim_{n \rightarrow \infty} B_n$ exists.

- d. Let $f(x) = \frac{1}{x}$, then

$$\int_1^{n+1} f(x) dx = \int_1^{n+1} \frac{1}{x} dx = \ln(n+1) \text{ and}$$

$$\lim_{n \rightarrow \infty} B_n = \gamma \text{ as defined in Problem 31.}$$

35. Every time n is incremented by 1, a positive amount of area is added, thus $\{A_n\}$ is an increasing sequence.
Each curved region has horizontal width 1, and

can be moved into the heavily outlined triangle without any overlap. This can be done by shifting the n th shaded region, which goes from $(n, f(n))$ to $(n+1, f(n+1))$, as follows:

shift $(n+1, f(n+1))$ to $(2, f(2))$ and $(n, f(n))$ to $(1, f(2) - [f(n+1) - f(n)])$.

The slope of the line forming the bottom of the shaded region between $x = n$ and $x = n+1$ is

$$\frac{f(n+1) - f(n)}{(n+1) - n} = f(n+1) - f(n) > 0$$

since f is increasing. By the Mean Value

Theorem, $f(n+1) - f(n) = f'(c)$ for some c in $(n, n+1)$. Since f is concave down, $n < c < n+1$ means that $f'(c) < f'(b)$ for all b in $[1, n]$. Thus, the n th shaded region will not overlap any other shaded region when shifted into the heavily outlined triangle. Thus, the area of all of the shaded regions is less than or equal to the area of the heavily outlined triangle, so $\lim_{n \rightarrow \infty} A_n$ exists.

36. $\ln x$ is continuous, increasing, and concave down on $[1, \infty)$, so the conditions of Problem 35 are met.

- a. See the figure in the text for Problem 35. The area under the curve from $x = 1$ to $x = n$ is $\int_1^n \ln x dx$ and the area of the n th trapezoid is $\frac{\ln n + \ln(n+1)}{2}$, thus $A_n = \int_1^n \ln x dx - \left[\frac{\ln 1 + \ln 2}{2} + \dots + \frac{\ln(n-1) + \ln(n)}{2} \right]$.

Using integration by parts with $u = \ln x$, $du = \frac{1}{x} dx$, $dv = dx$, $v = x$

$$\int_1^n \ln x dx = [x \ln x]_1^n - \int_1^n dx = [x \ln x - x]_1^n = n \ln n - n - (\ln 1 - 1) = n \ln n - n + 1$$

The sum of the areas of the n trapezoids is

$$\frac{\ln 1 + \ln 2}{2} + \frac{\ln 2 + \ln 3}{2} + \dots + \frac{\ln(n-2) + \ln(n-1)}{2} + \frac{\ln(n-1) + \ln(n)}{2} = \frac{2 \ln 2 + 2 \ln 3 + \dots + 2 \ln(n-1) + \ln n}{2}$$

$$= \ln 2 + \ln 3 + \dots + \ln n - \frac{\ln n}{2} = \ln(2 \cdot 3 \cdot \dots \cdot n) - \frac{\ln n}{2} = \ln n! - \ln \sqrt{n}$$

$$\text{Thus, } A_n = n \ln n - n + 1 - (\ln n! - \ln \sqrt{n}) = n \ln n - n + 1 - \ln n! + \ln \sqrt{n} = \ln n^n - \ln e^n + 1 - \ln n! + \ln \sqrt{n}$$

$$= \ln \left(\frac{n}{e} \right)^n + 1 + \ln \frac{\sqrt{n}}{n!} = 1 + \ln \left[\left(\frac{n}{e} \right)^n \frac{\sqrt{n}}{n!} \right]$$

- b. By Problem 35, $\lim_{n \rightarrow \infty} A_n$ exists, hence part a says that $\lim_{n \rightarrow \infty} \left[1 + \ln \left[\left(\frac{n}{e} \right)^n \frac{\sqrt{n}}{n!} \right] \right]$ exists.

$$\lim_{n \rightarrow \infty} \left[1 + \ln \left[\left(\frac{n}{e} \right)^n \frac{\sqrt{n}}{n!} \right] \right] = 1 + \lim_{n \rightarrow \infty} \ln \left[\left(\frac{n}{e} \right)^n \frac{\sqrt{n}}{n!} \right] = 1 + \ln \left[\lim_{n \rightarrow \infty} \left(\frac{n}{e} \right)^n \frac{\sqrt{n}}{n!} \right]$$

Since the limit exists, $\lim_{n \rightarrow \infty} \left(\frac{n}{e} \right)^n \frac{\sqrt{n}}{n!} = m$. m cannot be 0 since $\lim_{x \rightarrow 0^+} \ln x = -\infty$.

Thus, $\lim_{n \rightarrow \infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\frac{\left(\frac{n}{e}\right)^n \sqrt{n}}{n!}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{\left(\frac{n}{e}\right)^n \sqrt{n}}{n!}} = \frac{1}{m}$, i.e., the limit exists.

c. From part b, $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, thus, $15! \approx \sqrt{30\pi} \left(\frac{15}{e}\right)^{15} \approx 1.3004 \times 10^{12}$

The exact value is $15! = 1,307,674,368,000$.

10.4 Concepts Review

- $0 \leq a_k \leq b_k$
- $\lim_{k \rightarrow \infty} \frac{a_k}{b_k}$
- $\rho < 1$; $\rho > 1$; $\rho = 1$
- Ratio; Limit Comparison

Problem Set 10.4

$$1. a_n = \frac{n}{n^2 + 2n + 3}; b_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 2n + 3} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n} + \frac{3}{n^2}} = 1;$$

$$0 < 1 < \infty$$

$$\sum_{n=1}^{\infty} b_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

$$2. a_n = \frac{3n+1}{n^3-4}; b_n = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{3n^3 + n^2}{n^3 - 4} = \lim_{n \rightarrow \infty} \frac{3 + \frac{1}{n}}{1 - \frac{4}{n^3}} = 3;$$

$$0 < 3 < \infty$$

$$\sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$3. a_n = \frac{1}{n\sqrt{n+1}} = \frac{1}{\sqrt{n^3+n^2}}; b_n = \frac{1}{n^{3/2}}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{\sqrt{n^3+n^2}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n^3}{n^3+n^2}}$$

$$= \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1; 0 < 1 < \infty$$

$$\sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$4. a_n = \frac{\sqrt{2n+1}}{n^2}; b_n = \frac{1}{n^{3/2}}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{3/2} \sqrt{2n+1}}{n^2} = \lim_{n \rightarrow \infty} \sqrt{\frac{2n^4+n^3}{n^4}}$$

$$= \lim_{n \rightarrow \infty} \sqrt{\frac{2+\frac{1}{n}}{1}} = \sqrt{2}; 0 < \sqrt{2} < \infty$$

$$\sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$5. \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{8^{n+1} n!}{(n+1)! 8^n} = \lim_{n \rightarrow \infty} \frac{8}{n+1} = 0 < 1$$

The series converges.

$$6. \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{5^{n+1} n^5}{(n+1)^5 5^n} = \lim_{n \rightarrow \infty} \frac{5n^5}{(n+1)^5}$$

$$= \lim_{n \rightarrow \infty} \frac{5n^5}{n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1}$$

$$= \lim_{n \rightarrow \infty} \frac{5}{1 + \frac{5}{n} + \frac{10}{n^2} + \frac{10}{n^3} + \frac{5}{n^4} + \frac{1}{n^5}} = 5 > 1$$

The series diverges.

$$7. \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)! n^{100}}{(n+1)^{100} n!} = \lim_{n \rightarrow \infty} \frac{n^{100}}{(n+1)^{99}}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{\left(\frac{n+1}{n}\right)^{99}} = \infty \text{ since } \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^{99} = 1$$

The series diverges.

$$8. \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1) \left(\frac{1}{3}\right)^{n+1}}{n \left(\frac{1}{3}\right)^n} = \lim_{n \rightarrow \infty} \frac{n+1}{3n}$$

$$= \lim_{n \rightarrow \infty} \frac{1+\frac{1}{n}}{3} = \frac{1}{3} < 1$$

The series converges.

$$\begin{aligned}
 9. \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^3 (2n)!}{(2n+2)! n^3} \\
 &= \lim_{n \rightarrow \infty} \frac{(n+1)^3}{(2n+2)(2n+1)n^3} = \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 3n + 1}{4n^5 + 6n^4 + 2n^3} \\
 &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} + \frac{3}{n^3} + \frac{3}{n^4} + \frac{1}{n^5}}{4 + \frac{6}{n} + \frac{2}{n^2}} = 0 < 1
 \end{aligned}$$

The series converges.

$$\begin{aligned}
 10. \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(3^{n+1} + n + 1)n!}{(n+1)!(3^n + n)} \\
 &= \lim_{n \rightarrow \infty} \frac{3^{n+1} + n + 1}{(3^n + n)(n+1)} = \lim_{n \rightarrow \infty} \frac{3^{n+1} + n + 1}{n3^n + 3^n + n^2 + n} \\
 &= \lim_{n \rightarrow \infty} \frac{3 + \frac{n}{3^n} + \frac{1}{3^n}}{n + 1 + \frac{n^2}{3^n} + \frac{n}{3^n}} = 0 < \infty \quad \text{since} \quad \lim_{n \rightarrow \infty} \frac{n}{3^n} = 0
 \end{aligned}$$

and $\lim_{n \rightarrow \infty} \frac{n^2}{3^n} = 0$ which can be seen by using

L'Hôpital's Rule. The series converges.

$$11. \quad \lim_{n \rightarrow \infty} \frac{n}{n+200} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{200}{n}} = 1 \neq 0$$

The series diverges; n th-Term Test

$$\begin{aligned}
 12. \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)!(5+n)}{(6+n)n!} \\
 &= \lim_{n \rightarrow \infty} \frac{(n+1)(5+n)}{6+n} = \lim_{n \rightarrow \infty} \frac{n^2 + 6n + 5}{6+n} \\
 &= \lim_{n \rightarrow \infty} \frac{n + 6 + \frac{5}{n}}{\frac{6}{n} + 1} = \infty > 1
 \end{aligned}$$

The series diverges; Ratio Test

$$\begin{aligned}
 13. \quad a_n &= \frac{n+3}{n^2 \sqrt{n}}; b_n = \frac{1}{n^{3/2}} \\
 \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{n^{5/2} + 3n^{3/2}}{n^{5/2}} = \lim_{n \rightarrow \infty} \frac{1 + \frac{3}{n}}{1} = 1;
 \end{aligned}$$

$$0 < 1 < \infty. \quad \sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n$$

converges: Limit Comparison Test

$$\begin{aligned}
 14. \quad a_n &= \frac{\sqrt{n+1}}{n^2 + 1}; b_n = \frac{1}{n^{3/2}} \\
 \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{n^{3/2} \sqrt{n+1}}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^4 + n^3}}{n^2 + 1} \\
 &= \lim_{n \rightarrow \infty} \frac{\sqrt{1 + \frac{1}{n}}}{1 + \frac{1}{n^2}} = 1; 0 < 1 < \infty.
 \end{aligned}$$

$$\sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges; Limit}$$

Comparison Test

$$\begin{aligned}
 15. \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^2 n!}{(n+1)! n^2} = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{(n+1)n^2} \\
 &= \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{n^3 + n^2} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + \frac{2}{n^2} + \frac{1}{n^3}}{1 + \frac{1}{n}} = 0 < 1
 \end{aligned}$$

The series converges; Ratio Test

$$16. \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\ln(n+1)2^n}{2^{n+1} \ln n} = \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{2 \ln n}$$

Using L'Hôpital's Rule,

$$\lim_{n \rightarrow \infty} \frac{\ln(n+1)}{2 \ln n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{2(n+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2 + \frac{2}{n}} = \frac{1}{2} < 1.$$

The series converges; Ratio Test

$$17. \quad a_n = \frac{4n^3 + 3n}{n^5 - 4n^2 + 1}; b_n = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{4n^5 + 3n^3}{n^5 - 4n^2 + 1} = \lim_{n \rightarrow \infty} \frac{4 + \frac{3}{n^2}}{1 - \frac{4}{n^3} + \frac{1}{n^5}} = 4;$$

$$0 < 4 < \infty$$

$$\sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges; Limit}$$

Comparison Test

$$\begin{aligned}
 18. \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{[(n+1)^2 + 1]3^n}{3^{n+1}(n^2 + 1)} \\
 &= \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 2}{3n^2 + 3} = \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{n} + \frac{2}{n^2}}{3 + \frac{3}{n^2}} = \frac{1}{3} < 1
 \end{aligned}$$

The series converges; Ratio Test

$$19. \quad a_n = \frac{1}{n(n+1)} = \frac{1}{n^2 + n}; b_n = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + n} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1;$$

$$0 < 1 < \infty$$

$$\sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges;}$$

Limit Comparison Test

$$20. a_n = \frac{n}{(n+1)^2} = \frac{n}{n^2 + 2n + 1}; b_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 2n + 1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}} = 1;$$

$$0 < 1 < \infty$$

$$\sum_{n=1}^{\infty} b_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges};$$

Limit Comparison Test

$$21. a_n = \frac{n+1}{n(n+2)(n+3)} = \frac{n+1}{n^3 + 5n^2 + 6n}; b_n = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^3 + n^2}{n^3 + 5n^2 + 6n} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{1 + \frac{5}{n} + \frac{6}{n^2}} = 1;$$

$$0 < 1 < \infty$$

$$\sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges};$$

Limit Comparison Test

$$22. a_n = \frac{n}{n^2 + 1}; b_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n^2}} = 1; 0 < 1 < \infty$$

$$\sum_{n=1}^{\infty} b_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges};$$

Limit Comparison Test

$$23. a_n = \frac{n}{3^n}; \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)3^n}{3^{n+1}n}$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{3n} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{3} = \frac{1}{3} < 1$$

The series converges; Ratio Test

$$24. a_n = \frac{3^n}{n!};$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{3^{n+1}n!}{(n+1)!3^n} = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0 < 1$$

The series converges; Ratio Test

$$25. a_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}; \frac{1}{x^{3/2}} \text{ is continuous, positive, and nonincreasing on } [1, \infty).$$

$$\int_1^{\infty} \frac{1}{x^{3/2}} dx = \left[-\frac{2}{\sqrt{x}} \right]_1^{\infty} = 0 + 2 = 2 < \infty$$

The series converges; Integral Test

$$26. a_n = \frac{\ln n}{n^2}; \frac{\ln x}{x^2} \text{ is continuous, positive, and nonincreasing on } [2, \infty). \text{ Use integration by parts with } u = \ln x \text{ and } dv = \frac{1}{x^2} dx \text{ for}$$

$$\int_2^{\infty} \frac{\ln x}{x^2} dx = \left[-\frac{\ln x}{x} \right]_2^{\infty} + \int_2^{\infty} \frac{1}{x^2} dx$$

$$= \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_2^{\infty} = 0 + \frac{\ln 2}{2} + \frac{1}{2} < \infty$$

$$\left(\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0 \text{ by l'Hôpital's Rule.} \right)$$

The series converges; Integral Test

$$27. 0 \leq \sin^2 n \leq 1 \text{ for all } n, \text{ so}$$

$$2 \leq 2 + \sin^2 n \leq 3 \Rightarrow \frac{1}{2} \geq \frac{1}{2 + \sin^2 n} \geq \frac{1}{3} \text{ for all } n.$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \frac{1}{2 + \sin^2 n} \neq 0 \text{ and the series diverges;}$$

n th-Term Test

$$28. \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{5(3^{n+1} + 1)}{(3^{n+1} + 1)5} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{3^n}}{3 + \frac{1}{3^n}}$$

$$= \frac{1}{3} < 1$$

The series converges; Ratio Test

$$29. -1 \leq \cos n \leq 1 \text{ for all } n, \text{ so}$$

$$3 \leq 4 + \cos n \leq 5 \Rightarrow \frac{3}{n^3} \leq \frac{4 + \cos n}{n^3} \leq \frac{5}{n^3} \text{ for all } n.$$

$$\sum_{n=1}^{\infty} \frac{5}{n^3} \text{ converges} \Rightarrow \sum_{n=1}^{\infty} \frac{4 + \cos n}{n^3} \text{ converges;}$$

Comparison Test

$$30. \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{5^{2n+2}n!}{(n+1)!5^{2n}} = \lim_{n \rightarrow \infty} \frac{25}{n+1} = 0 < 1$$

The series converges; Ratio Test

$$31. \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}(2n)!}{(2n+2)!n^n}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(2n+2)(2n+1)n^n} = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{2(n+1)(2n+1)n^n}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^n}{2(2n+1)n^n} = \lim_{n \rightarrow \infty} \left[\frac{1}{4n+2} \left(\frac{n+1}{n} \right)^n \right]$$

$$= \left[\lim_{n \rightarrow \infty} \frac{1}{4n+2} \right] \left[\lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n \right] = 0 \cdot e = 0 < 1$$

(The limits can be separated since both limits exist.) The series converges; Ratio Test

32. Let $y = \left(1 - \frac{1}{x}\right)^x$; $\ln y = x \ln \left(1 - \frac{1}{x}\right)$

$$\lim_{x \rightarrow \infty} x \ln \left(1 - \frac{1}{x}\right) = \lim_{x \rightarrow \infty} \frac{\ln \left(1 - \frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1/x^2}{(1-1/x)}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} -\frac{1}{\left(1 - \frac{1}{x}\right)} = -1$$

Thus $\lim_{x \rightarrow \infty} y = e^{-1}$, so $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$.

The series diverges; n th-Term Test

33. $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(4^{n+1} + n + 1)n!}{(n+1)!(4^n + n)}$

$$= \lim_{n \rightarrow \infty} \frac{4^{n+1} + n + 1}{(n+1)(4^n + n)} = \lim_{n \rightarrow \infty} \frac{4 + \frac{n}{4^n} + \frac{1}{4^n}}{(n+1)\left(1 + \frac{n}{4^n}\right)}$$

$$= \lim_{n \rightarrow \infty} \frac{4 + \frac{n}{4^n} + \frac{1}{4^n}}{1 + n + \frac{n}{4^n} + \frac{n^2}{4^n}} = 0$$

since $\lim_{n \rightarrow \infty} \frac{n^2}{4^n} = 0$, $\lim_{n \rightarrow \infty} \frac{n}{4^n} = 0$, and

$$\lim_{n \rightarrow \infty} \frac{1}{4^n} = 0. \text{ The series converges; Ratio Test}$$

34. $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)(2 + n5^n)}{[2 + (n+1)5^{n+1}]n}$

$$= \lim_{n \rightarrow \infty} \frac{2n + n^2 5^n + 2 + n5^n}{2n + n^2 5^{n+1} + n5^{n+1}}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{2}{n5^n} + 1 + \frac{2}{n^2 5^n} + \frac{1}{n}}{\frac{2}{n5^n} + 5 + \frac{5}{n}} = \frac{1}{5} < 1$$

The series converge; Ratio Test

35. Since $\sum a_n$ converges, $\lim_{n \rightarrow \infty} a_n = 0$. Thus, there

is some positive integer N such that $0 < a_n < 1$

for all $n \geq N$. $a_n < 1 \Rightarrow a_n^2 < a_n$, thus

$$\sum_{n=N}^{\infty} a_n^2 < \sum_{n=N}^{\infty} a_n. \text{ Hence } \sum_{n=N}^{\infty} a_n^2 \text{ converges,}$$

and $\sum a_n^2$ also converges, since adding a finite number of terms does not affect the convergence or divergence of a series.

36. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by Example 7, thus

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0 \text{ by the } n\text{th-Term Test.}$$

37. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ then there is some positive integer

N such that $0 \leq \frac{a_n}{b_n} < 1$ for all $n \geq N$. Thus, for

$n \geq N$, $a_n < b_n$. By the Comparison Test, since

$$\sum_{n=N}^{\infty} b_n \text{ converges, } \sum_{n=N}^{\infty} a_n \text{ also converges. Thus,}$$

$\sum a_n$ converges since adding a finite number of terms will not affect the convergence or divergence of a series.

38. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ then there is some positive

integer N such that $\frac{a_n}{b_n} > 1$ for all $n \geq N$. Thus,

for $n \geq N$, $a_n > b_n$ and by the Comparison

Test, since $\sum_{n=N}^{\infty} b_n$ diverges, $\sum_{n=N}^{\infty} a_n$ also

diverges. Thus, $\sum a_n$ diverges since adding a finite number of terms will not affect the convergence or divergence of a series.

39. If $\lim_{n \rightarrow \infty} na_n = 1$ then there is some positive

integer N such that $a_n \geq 0$ for all $n \geq N$. Let

$$b_n = \frac{1}{n}, \text{ so } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} na_n = 1 < \infty.$$

Since $\sum_{n=N}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=N}^{\infty} a_n$ diverges by the

Limit Comparison Test.

Thus $\sum a_n$ diverges since adding a finite number of terms will not affect the convergence or divergence of a series.

40. Consider $f(x) = x - \ln(1+x)$, then

$$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \text{ on } (0, \infty).$$

$f(0) = 0 - \ln 1 = 0$, so since $f(x)$ is increasing, $f(x) > 0$ on $(0, \infty)$, i.e., $x > \ln(1+x)$ for $x > 0$.

Thus, since a_n is a series of positive terms,

$$\sum \ln(1+a_n) < \sum a_n, \text{ hence if } \sum a_n \text{ converges,}$$

$$\sum \ln(1+a_n) \text{ also converges.}$$

41. Suppose that $\lim_{n \rightarrow \infty} (a_n)^{1/n} = R$ where $a_n > 0$.

If $R < 1$, there is some number r with $R < r < 1$ and some positive integer N such that

$$|(a_n)^{1/n} - R| < r - R \text{ for all } n \geq N. \text{ Thus,}$$

$$R - r < (a_n)^{1/n} - R < r - R \text{ or}$$

$$-r < (a_n)^{1/n} < r < 1. \text{ Since } a_n > 0,$$

$$0 < (a_n)^{1/n} < r \text{ and } 0 < a_n < r^n \text{ for all } n \geq N$$

Thus, $\sum_{n=N}^{\infty} a_n < \sum_{n=N}^{\infty} r^n$, which converges since

$$|r| < 1. \text{ Thus, } \sum_{n=N}^{\infty} a_n \text{ converges so } \sum a_n \text{ also}$$

converges.

If $R > 1$, there is some number r with $1 < r < R$ and some positive integer N such that

$$|(a_n)^{1/n} - R| < R - r \text{ for all } n \geq N. \text{ Thus,}$$

$$r - R < (a_n)^{1/n} - R < R - r \text{ or}$$

$$r < (a_n)^{1/n} < 2R - r \text{ for all } n \geq N. \text{ Hence}$$

$$r^n < a_n \text{ for all } n \geq N, \text{ so } \sum_{n=N}^{\infty} r^n < \sum_{n=N}^{\infty} a_n, \text{ and}$$

$$\text{since } \sum_{n=N}^{\infty} r^n \text{ diverges } (r > 1), \sum_{n=N}^{\infty} a_n \text{ also}$$

diverges, so $\sum a_n$ diverges.

42. a. $\lim_{n \rightarrow \infty} (a_n)^{1/n} = \lim_{n \rightarrow \infty} \left[\left(\frac{1}{\ln n} \right)^n \right]^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{\ln n}$

$$= 0 < 1$$

The series converges.

b. $\lim_{n \rightarrow \infty} (a_n)^{1/n} = \lim_{n \rightarrow \infty} \left[\left(\frac{n}{3n+2} \right)^n \right]^{1/n}$

$$= \lim_{n \rightarrow \infty} \frac{n}{3n+2} = \lim_{n \rightarrow \infty} \frac{1}{3 + \frac{2}{n}} = \frac{1}{3} < 1$$

The series converges.

c. $\lim_{n \rightarrow \infty} (a_n)^{1/n} = \lim_{n \rightarrow \infty} \left[\left(\frac{1}{2} + \frac{1}{n} \right)^n \right]^{1/n}$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{n} \right) = \frac{1}{2} < 1$$

The series converges.

43. a. $\ln \left(1 + \frac{1}{n} \right) = \ln \left(\frac{n+1}{n} \right) = \ln(n+1) - \ln n$

$$S_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \dots$$

$$+ (\ln n - \ln(n-1)) + (\ln(n+1) - \ln n)$$

$$= -\ln 1 + \ln(n+1) = \ln(n+1)$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \ln(n+1) = \infty$$

Since the partial sums are unbounded, the series diverges.

b. $\ln \frac{(n+1)^2}{n(n+2)} = 2 \ln(n+1) - \ln n - \ln(n+2)$

$$S_n = (2 \ln 2 - \ln 1 - \ln 3) + (2 \ln 3 - \ln 2 - \ln 4)$$

$$+ (2 \ln 4 - \ln 3 - \ln 5) + \dots$$

$$+ (2 \ln n - \ln(n-1) - \ln(n+1))$$

$$+ (2 \ln(n+1) - \ln n - \ln(n+2))$$

$$= \ln 2 - \ln 1 + \ln(n+1) - \ln(n+2)$$

$$= \ln 2 + \ln \frac{n+1}{n+2}$$

$$\lim_{n \rightarrow \infty} S_n = \ln 2 + \lim_{n \rightarrow \infty} \ln \frac{n+1}{n+2} = \ln 2$$

Since the partial sums converge, the series converges.

c. $\left(\frac{1}{\ln x} \right)^{\ln x}$ is continuous, positive, and

$$\text{nonincreasing on } [2, \infty), \text{ thus } \sum_{n=2}^{\infty} \left(\frac{1}{\ln n} \right)^{\ln n}$$

$$\text{converges if and only if } \int_2^{\infty} \left(\frac{1}{\ln x} \right)^{\ln x} dx$$

converges.

Let $u = \ln x$, so $x = e^u$ and $dx = e^u du$.

$$\int_2^{\infty} \left(\frac{1}{\ln x} \right)^{\ln x} dx = \int_{\ln 2}^{\infty} \left(\frac{1}{u} \right)^u e^u du = \int_{\ln 2}^{\infty} \left(\frac{e}{u} \right)^u du$$

This integral converges if and only if the

associated series, $\sum_{n=1}^{\infty} \left(\frac{e}{n} \right)^n$ converges. With

$$a_n = \left(\frac{e}{n} \right)^n, \text{ the Root Test (Problem 41)}$$

$$\text{gives } \lim_{n \rightarrow \infty} (a_n)^{1/n} = \lim_{n \rightarrow \infty} \left[\left(\frac{e}{n} \right)^n \right]^{1/n}$$

$$= \lim_{n \rightarrow \infty} \frac{e}{n} = 0 < 1$$

$$\text{Thus, } \sum_{n=1}^{\infty} \left(\frac{e}{n} \right)^n \text{ converges, so } \int_{\ln 2}^{\infty} \left(\frac{e}{u} \right)^u du$$

converges, whereby $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converges.

d. $\left(\frac{1}{\ln(\ln x)}\right)^{\ln x}$ is continuous, positive, and nonincreasing on $[3, \infty)$, thus $\sum_{n=3}^{\infty} \left(\frac{1}{\ln(\ln n)}\right)^{\ln n}$ converges if and only if $\int_3^{\infty} \left(\frac{1}{\ln(\ln x)}\right)^{\ln x} dx$ converges.

Let $u = \ln x$, so $x = e^u$ and $dx = e^u du$.

$$\begin{aligned} \int_3^{\infty} \left(\frac{1}{\ln(\ln x)}\right)^{\ln x} dx &= \int_{\ln 3}^{\infty} \left(\frac{1}{\ln u}\right)^u e^u du = \int_{\ln 3}^{\infty} \left(\frac{e}{\ln u}\right)^u du. \end{aligned}$$

This integral converges if and only if the associated series, $\sum_{n=2}^{\infty} \left(\frac{e}{\ln n}\right)^n$ converges.

With $a_n = \left(\frac{e}{\ln n}\right)^n$, the Root Test (Problem 41) gives

$$\begin{aligned} \lim_{n \rightarrow \infty} (a_n)^{1/n} &= \lim_{n \rightarrow \infty} \left[\left(\frac{e}{\ln n}\right)^n \right]^{1/n} \\ &= \lim_{n \rightarrow \infty} \frac{e}{\ln n} = 0 < 1 \end{aligned}$$

Thus, $\sum_{n=2}^{\infty} \left(\frac{e}{\ln n}\right)^n$ converges, so

$$\begin{aligned} \int_{\ln 3}^{\infty} \left(\frac{e}{\ln u}\right)^u du &\text{ converges, whereby} \\ \sum_{n=3}^{\infty} \frac{1}{(\ln(\ln n))^{\ln n}} &\text{ converges.} \end{aligned}$$

e. $a_n = 1/n$; $b_n = 1/(\ln n)^4$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{1/n}{1/(\ln n)^4} = \lim_{n \rightarrow \infty} \frac{(\ln n)^4}{n} \\ &= \lim_{n \rightarrow \infty} \frac{4(\ln n)^3 (1/n)}{1} = \lim_{n \rightarrow \infty} \frac{4(\ln n)^3}{n} \\ &= \lim_{n \rightarrow \infty} \frac{12(\ln n)^2 (1/n)}{1} = \lim_{n \rightarrow \infty} \frac{12(\ln n)^2}{n} \\ &= \lim_{n \rightarrow \infty} \frac{24(\ln n)}{n} = \lim_{n \rightarrow \infty} \frac{24(1/n)}{1} \\ &= \lim_{n \rightarrow \infty} \frac{24}{n} = 0 \\ \sum_{n=2}^{\infty} \frac{1}{n} &\text{ diverges} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{(\ln n)^4} \text{ diverges} \end{aligned}$$

f. $\left(\frac{\ln x}{x}\right)^2$ is continuous, positive, and nonincreasing on $[3, \infty)$. Using integration by parts twice,

$$\begin{aligned} \int_3^{\infty} \left(\frac{\ln x}{x}\right)^2 dx &= \left[-\frac{(\ln x)^2}{x} \right]_3^{\infty} + \int_3^{\infty} \frac{2 \ln x}{x^2} dx \\ &= \left[-\frac{(\ln x)^2}{x} \right]_3^{\infty} + \left[-\frac{2 \ln x}{x} \right]_3^{\infty} + \int_3^{\infty} \frac{2}{x^2} dx \\ &= \left[-\frac{(\ln x)^2}{x} - \frac{2 \ln x}{x} - \frac{2}{x} \right]_3^{\infty} \approx 1.8 < \infty \end{aligned}$$

Thus, $\sum_{n=3}^{\infty} \left(\frac{\ln x}{x}\right)^2$ converges.

44. The degree of $p(n)$ must be at least 2 less than the degree of $q(n)$. If $p(n)$ and $q(n)$ have the same degree, r , then

$$p(n) = c_r n^r + c_{r-1} n^{r-1} + \dots + c_1 n + c_0 \text{ and}$$

$$q(n) = d_r n^r + d_{r-1} n^{r-1} + \dots + d_1 n + d_0 \text{ where } c_r, d_r \neq 0 \text{ and}$$

$$\lim_{n \rightarrow \infty} \frac{p(n)}{q(n)} = \lim_{n \rightarrow \infty} \frac{c_r n^r + c_{r-1} n^{r-1} + \dots + c_1 n + c_0}{d_r n^r + d_{r-1} n^{r-1} + \dots + d_1 n + d_0} = \lim_{n \rightarrow \infty} \frac{c_r + \frac{c_{r-1}}{n} + \dots + \frac{c_1}{n^{r-1}} + \frac{c_0}{n^r}}{d_r + \frac{d_{r-1}}{n} + \dots + \frac{d_1}{n^{r-1}} + \frac{d_0}{n^r}} = \frac{c_r}{d_r} \neq 0.$$

Thus, the series diverges by the n th-Term Test. If the degree of $p(n)$ is r and the degree of $q(n)$ is s , then the Limit

Comparison Test with $a_n = \frac{p(n)}{q(n)}$, $b_n = \frac{1}{n^{s-r}}$ will give $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$, since $\frac{a_n}{b_n} = \frac{n^{s-r} p(n)}{q(n)}$ and

the degrees of $n^{s-r} p(n)$ and $q(n)$ are the same, similar to the previous case. Since $0 < L < \infty$, a_n and b_n either both converge or both diverge.

If $s \geq r + 2$, then $s - r \geq 2$ so $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{s-r}} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$. Thus $\sum_{n=1}^{\infty} b_n$ converges, and hence $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{p(n)}{q(n)}$ converges.

If $s < r + 2$, then $s - r \leq 1$ so $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{s-r}} \geq \sum_{n=1}^{\infty} \frac{1}{n}$. Thus $\sum_{n=1}^{\infty} b_n$ diverges, and hence $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{p(n)}{q(n)}$ diverges.

45. Let $a_n = \frac{1}{n^p} \left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} \right)$ and

$b_n = \frac{1}{n^p}$. Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} \right) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

which converges if $p > 1$. Thus, by the Limit

Comparison Test, if $\sum_{n=1}^{\infty} b_n$ converges for $p > 1$,

so does $\sum_{n=1}^{\infty} a_n$. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

for $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p} \left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} \right)$ also

converges. For $p \leq 1$, since $1 + \frac{1}{2^p} + \dots + \frac{1}{n^p} > 1$,

$\frac{1}{n^p} \left(1 + \frac{1}{2^p} + \dots + \frac{1}{n^p} \right) > \frac{1}{n^p}$. Hence, since

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$,

$\sum_{n=1}^{\infty} \frac{1}{n^p} \left(1 + \frac{1}{2^p} + \dots + \frac{1}{n^p} \right)$ also diverges. The

series converges for $p > 1$ and diverges for $p \leq 1$.

46. a. Let $a_n = \sin^2 \left(\frac{1}{n} \right)$ and $b_n = \frac{1}{n^2}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} n^2 \sin^2 \left(\frac{1}{n} \right) = \lim_{u \rightarrow 0^+} \left(\frac{1}{u} \right)^2 \sin^2 u = \lim_{u \rightarrow 0^+} \left(\frac{\sin u}{u} \right)^2 = 1$

using the substitution $u = \frac{1}{n}$. Since $0 < 1 < \infty$, both $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin^2 \left(\frac{1}{n} \right)$ converge.

b. Let $a_n = \tan \left(\frac{1}{n} \right)$ and $b_n = \frac{1}{n}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} n \tan \left(\frac{1}{n} \right) = \lim_{n \rightarrow \infty} \frac{n \sin \left(\frac{1}{n} \right)}{\cos \left(\frac{1}{n} \right)}$

$= \lim_{u \rightarrow 0} \frac{\left(\frac{1}{u} \right) \sin u}{\cos u} = \lim_{u \rightarrow 0} \left(\frac{\sin u}{u} \cos u \right) = 1$ using the substitution $u = \frac{1}{n}$. Since $0 < 1 < \infty$, both $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ and

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \tan \left(\frac{1}{n} \right)$ diverge.

c. $\sum_{n=1}^{\infty} \sqrt{n} \left(1 - \cos \frac{1}{n} \right) = \sum_{n=1}^{\infty} \sqrt{n} \left(1 - \cos \frac{1}{n} \right) \left(\frac{1 + \cos \frac{1}{n}}{1 + \cos \frac{1}{n}} \right) = \sum_{n=1}^{\infty} \frac{\sqrt{n} (1 - \cos \frac{1}{n})}{1 + \cos \frac{1}{n}} = \sum_{n=1}^{\infty} \frac{\sqrt{n} \sin^2 \frac{1}{n}}{1 + \cos \frac{1}{n}} < \sum_{n=1}^{\infty} \sqrt{n} \sin^2 \frac{1}{n}$

Let $a_n = \sqrt{n} \sin^2 \frac{1}{n}$ and $b_n = \frac{1}{n^{3/2}}$.

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} n^2 \sin^2 \frac{1}{n} = \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}} \right)^2 = 1$, since $\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{u \rightarrow 0^+} \frac{\sin u}{u} = 1$ with $u = \frac{1}{n}$.

Thus, by the Limit Comparison Test, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty} \sqrt{n} \sin^2 \frac{1}{n}$ converges, and hence,

$\sum_{n=1}^{\infty} \sqrt{n} \left(1 - \cos \frac{1}{n}\right)$ converges by the Comparison Test.

10.5 Concepts Review

1. $\lim_{n \rightarrow \infty} a_n = 0$
2. absolutely; conditionally
3. the alternating harmonic series
4. rearranged

Problem Set 10.5

1. $a_n = \frac{2}{3n+1}; \frac{2}{3n+1} > \frac{2}{3n+4}$, so $a_n > a_{n+1}$;
 $\lim_{n \rightarrow \infty} \frac{2}{3n+1} = 0$. $S_9 \approx 0.363$. The error made by using S_9 is not more than $a_{10} \approx 0.065$.
2. $a_n = \frac{1}{\sqrt{n}}; \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$, so $a_n > a_{n+1}$;
 $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$. $S_9 \approx 0.76695$. The error made by using S_9 is not more than $a_{10} \approx 0.31623$.
3. $a_n = \frac{1}{\ln(n+1)}; \frac{1}{\ln(n+1)} > \frac{1}{\ln(n+2)}$, so $a_n > a_{n+1}$;
 $\lim_{n \rightarrow \infty} \frac{1}{\ln(n+1)} = 0$. $S_9 \approx 1.137$. The error made by using S_9 is not more than $a_{10} \approx 0.417$.
4. $a_n = \frac{n}{n^2+1}; \frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1}$, so $a_n > a_{n+1}$;
 $\lim_{n \rightarrow \infty} \frac{n}{n^2+1} = 0$. $S_9 \approx 0.32153$. The error made by using S_9 is not more than $a_{10} \approx 0.09901$.

5. $a_n = \frac{\ln n}{n}; \frac{\ln n}{n} > \frac{\ln(n+1)}{n+1}$ is equivalent to $\ln \frac{n^{n+1}}{(n+1)^n} > 0$ or $\frac{n^{n+1}}{(n+1)^n} > 1$ which is true for $n > 2$. $S_9 \approx -0.041$. The error made by using S_9 is not more than $a_{10} \approx 0.230$.
6. $a_n = \frac{\ln n}{\sqrt{n}}; \frac{\ln n}{\sqrt{n}} > \frac{\ln(n+1)}{\sqrt{n+1}}$ for $n \geq 7$, so $a_n > a_{n+1}$ for $n \geq 7$. $S_9 \approx 0.17199$. The error made by using S_9 is not more than $a_{10} \approx 0.72814$.
7. $\frac{|u_{n+1}|}{|u_n|} = \frac{\left|(-\frac{3}{4})^{n+1}\right|}{\left|(-\frac{3}{4})^n\right|} = \frac{3}{4} < 1$, so the series converges absolutely.
8. $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which converges since $\frac{3}{2} > 1$, thus $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n\sqrt{n}}$ converges absolutely.
9. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2n}$; $\lim_{n \rightarrow \infty} \frac{n+1}{2n} = \frac{1}{2} < 1$, so the series converges absolutely.
10. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \frac{(n+1)^2}{en^2}$;
 $\lim_{n \rightarrow \infty} \frac{(n+1)^2}{en^2} = \frac{1}{e} \approx 0.36788 < 1$, so the series converges absolutely.
11. $n(n+1) = n^2 + n > n^2$ for all $n > 0$, thus $\frac{1}{n(n+1)} < \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

which converges since $2 > 1$, thus

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$ converges absolutely.

$$12. \frac{|u_{n+1}|}{|u_n|} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1}; \quad \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 < 1, \text{ so the}$$

series converges absolutely.

$$13. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ which converges}$$

since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. The series is

conditionally convergent since $\frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$14. \sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{5n^{1.1}} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n^{1.1}} \text{ converges since}$$

$1.1 > 1$. The series is absolutely convergent.

$$15. \lim_{n \rightarrow \infty} \frac{n}{10n+1} = \frac{1}{10} \neq 0. \text{ Thus the sequence of}$$

partial sums does not converge; the series diverges.

$$16. \frac{n}{10n^{1.1}+1} > \frac{n+1}{10(n+1)^{1.1}+1}, \text{ so } a_n > a_{n+1};$$

$$\lim_{n \rightarrow \infty} \frac{n}{10n^{1.1}+1} = \lim_{n \rightarrow \infty} \frac{1}{10n^{0.1} + \frac{1}{n}} = 0. \text{ The}$$

alternating series converges.

$$\text{Let } a_n = \frac{n}{10n^{1.1}+1} \text{ and } b_n = \frac{1}{n^{0.1}}. \text{ Then}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{1.1}}{10n^{1.1}+1} = \frac{1}{10}; \quad 0 < \frac{1}{10} < \infty; \text{ so}$$

both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, since $\sum_{n=1}^{\infty} \frac{1}{n^{0.1}}$ diverges. The series is conditionally convergent.

$$17. \lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0; \frac{1}{n \ln n} > \frac{1}{(n+1) \ln(n+1)} \text{ is}$$

equivalent to $(n+1)^{n+1} > n^n$ which is true for all $n > 0$ so $a_n > a_{n+1}$. The alternating series converges.

$$\sum_{n=2}^{\infty} |u_n| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}; \quad \frac{1}{x \ln x} \text{ is continuous,}$$

positive, and nonincreasing on $[2, \infty)$.

$$\text{Using } u = \ln x, \quad du = \frac{1}{x} dx,$$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_{\ln 2}^{\infty} \frac{1}{u} du = [\ln |u|]_{\ln 2}^{\infty} = \infty. \text{ Thus,}$$

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges and $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ is conditionally convergent.

$$18. \sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n(1+\sqrt{n})} \leq \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

which converges since $\frac{3}{2} > 1$. The series is absolutely convergent.

$$19. \frac{|u_{n+1}|}{|u_n|} = \frac{\frac{(n+1)^4}{2^{n+1}}}{\frac{n^4}{2^n}} = \frac{n^4}{2(n+1)^4};$$

$$\lim_{n \rightarrow \infty} \frac{n^4}{2(n+1)^4} = \frac{1}{2} < 1.$$

The series is absolutely convergent.

$$20. a_n = \frac{1}{\sqrt{n^2-1}}; \frac{1}{\sqrt{n^2-1}} > \frac{1}{\sqrt{n^2+2n}}, \text{ so}$$

$$a_n > a_{n+1}; \quad \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^2-1}} = 0, \text{ hence the}$$

alternating series converges.

$$\text{Let } b_n = \frac{1}{n}, \text{ then}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}} = 1;$$

$$0 < 1 < \infty.$$

Thus, since $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n}$ diverges,

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}} \text{ also diverges. The series}$$

converges conditionally.

$$21. a_n = \frac{n}{n^2+1}; \frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1} \text{ is equivalent to}$$

$$n^2+n-1 > 0, \text{ which is true for } n > 1, \text{ so}$$

$$a_n > a_{n+1}; \quad \lim_{n \rightarrow \infty} \frac{n}{n^2+1} = 0, \text{ hence the alternating}$$

series converges. Let $b_n = \frac{1}{n}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1; \quad 0 < 1 < \infty. \text{ Thus, since}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1} \text{ also}$$

diverges. The series is conditionally convergent.

22. $a_n = \frac{n-1}{n}$; $\lim_{n \rightarrow \infty} \frac{n-1}{n} = 1 \neq 0$

The series is divergent.

23. $\cos n\pi = (-1)^n = \frac{1}{(-1)}(-1)^{n+1}$ so the series is

$$-1 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}, \quad -1 \text{ times the alternating}$$

harmonic series. The series is conditionally convergent.

24. $\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2} = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \dots$, since

$$\sin \frac{n\pi}{2} = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

Thus, $\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)^2}$.

$$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$(2n-1)^2 > n^2$ for $n > 1$, thus

$$\sum_{n=2}^{\infty} \frac{1}{(2n-1)^2} < \sum_{n=2}^{\infty} \frac{1}{n^2}, \text{ which converges since}$$

$2 > 1$. The series is absolutely convergent.

25. $|\sin n| \leq 1$ for all n , so

$$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ which converges}$$

since $\frac{3}{2} > 1$. Thus the series is absolutely convergent.

26. $n \sin\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$. As $n \rightarrow \infty$, $\frac{1}{n} \rightarrow 0$ and

$$\lim_{k \rightarrow 0} \frac{\sin k}{k} = 1, \text{ so } \lim_{n \rightarrow \infty} n \sin\left(\frac{1}{n}\right) = 1. \text{ The series}$$

diverges.

27. $a_n = \frac{1}{\sqrt{n(n+1)}}$; $\frac{1}{\sqrt{n(n+1)}} > \frac{1}{\sqrt{(n+1)(n+2)}}$ and

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n(n+1)}} = 0 \text{ so the alternating series}$$

converges.

Let $b_n = \frac{1}{n}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n}}} = 1;$$

$$0 < 1 < \infty.$$

Thus, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} \text{ also diverges.}$$

The series is conditionally convergent.

28. $a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$;

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} > \frac{1}{\sqrt{n+2} + \sqrt{n+1}}, \text{ so } a_n > a_{n+1};$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0. \text{ The alternating series}$$

converges.

Let $b_n = \frac{1}{\sqrt{n}}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{2};$$

$$0 < \frac{1}{2} < \infty. \text{ Thus, since } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

diverges $\left(\frac{1}{2} < 1\right)$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$ also diverges. The series is conditionally convergent.

29. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n^2}$; $\lim_{n \rightarrow \infty} \frac{3^{n+1}}{n^2} \neq 0$, so the series diverges.

30. $a_n = \sin \frac{\pi}{n}$; for $n \geq 2$, $\sin \frac{\pi}{n} > 0$ and

$$\sin \frac{\pi}{n} > \sin \frac{\pi}{n+1}, \text{ so } a_n > a_{n+1}; \quad \lim_{n \rightarrow \infty} \sin \frac{\pi}{n} = 0.$$

The alternating series converges.

For $n > 2$, $\frac{1}{n} < \sin \frac{\pi}{n}$ which can be seen from

$$\text{noting that } n < \sin n\pi \text{ for } n < \frac{1}{2}. \text{ Thus}$$

$$\sum_{n=3}^{\infty} |u_n| = \sum_{n=3}^{\infty} \sin \frac{\pi}{n} > \sum_{n=3}^{\infty} \frac{1}{n}, \text{ which diverges.}$$

The series is conditionally convergent.

31. Suppose $\sum |a_n|$ converges. Thus, $\sum 2|a_n|$ converges, so $\sum (|a_n| + a_n)$ converges since $0 \leq |a_n| + a_n \leq 2|a_n|$. By the linearity of convergent series, $\sum a_n = \sum (|a_n| + a_n) - \sum |a_n|$ converges, which is a contradiction.

32. Let $\sum a_n = \sum (-1)^{n+1} \frac{1}{\sqrt{n}} = \sum b_n$ and $\sum b_n$ both converge, but $\sum a_n b_n = \sum \frac{1}{n}$ diverges.

33. The positive-term series is

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots = \sum_{n=1}^{\infty} \frac{1}{2n-1}.$$

$\sum_{n=1}^{\infty} \frac{1}{2n-1} > \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges since the harmonic series diverges.

Thus, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges.

The negative-term series is

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \dots = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \text{ which diverges.}$$

since the harmonic series diverges.

36. $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} - \frac{1}{6} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} - \frac{1}{8} + \frac{1}{25} + \frac{1}{27} + \frac{1}{29} + \frac{1}{31}$

$$S_{20} \approx 1.3265$$

37. Written response

38. Possible answer: take several positive terms, add one negative term, then add positive terms whose sum is at least one greater than the negative term previously added. Add another negative term, then add positive terms whose sum is at least one greater than the negative term just added. Continue in this manner and the resulting series will diverge.

39. Consider $1 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{9} + \dots$

It is clear that $\lim_{n \rightarrow \infty} a_n = 0$. Pairing successive

terms, we obtain $\frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2} > 0$ for $n > 1$.

Let $a_n = \frac{n-1}{n^2}$ and $b_n = \frac{1}{n}$. Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2 - n}{n^2} = 1; \quad 0 < 1 < \infty.$$

34. If the positive terms and negative terms both formed convergent series then the series would be absolutely convergent. If one series was convergent and the other was divergent, the sum, which is the original series, would be divergent

35. a. $1 + \frac{1}{3} \approx 1.33$

b. $1 + \frac{1}{3} - \frac{1}{2} \approx 0.833$

c. $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \approx 1.38$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} \approx 1.13$$

Thus, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2} \right) \text{ also diverges.}$$

40. $\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n+1}} = \frac{2}{n-1}$, so

$$\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1} + \dots = \sum_{n=2}^{\infty} \frac{2}{n-1} = 2 \sum_{n=1}^{\infty} \frac{1}{n} \text{ which diverges.}$$

41. Note that $(a_k + b_k)^2 \geq 0$ and $(a_k - b_k)^2 \geq 0$ for all k . Thus, $a_k^2 \pm 2a_k b_k + b_k^2 \geq 0$, or

$$a_k^2 + b_k^2 \geq \pm 2a_k b_k \text{ for all } k, \text{ and}$$

$$a_k^2 + b_k^2 \geq 2|a_k b_k|. \text{ Since } \sum_{k=1}^{\infty} a_k^2 \text{ and } \sum_{k=1}^{\infty} b_k^2 \text{ both}$$

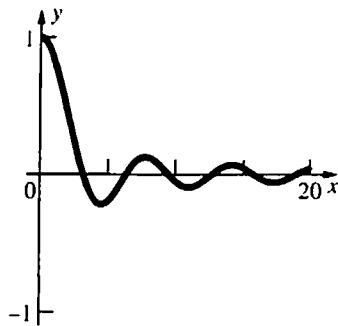
converge, $\sum_{k=1}^{\infty} (a_k^2 + b_k^2)$ also converges, and by

the Comparison Test, $\sum_{k=1}^{\infty} 2|a_k b_k|$ converges.

$\sum_{k=1}^{\infty} a_k b_k$ converges absolutely.

Hence, $\sum_{k=1}^{\infty} |a_k b_k| = \frac{1}{2} \sum_{k=1}^{\infty} 2|a_k b_k|$ converges, i.e.,

42.



$\int_0^{\infty} \frac{\sin x}{x} dx$ gives the area of the region above the x -axis minus the area of the region below.

Note that

$$\begin{aligned} \int_{2k\pi}^{(2k+1)\pi} \left(\frac{\sin x}{x} + \frac{\sin(x+\pi)}{x+\pi} \right) dx &= \int_{2k\pi}^{(2k+1)\pi} \frac{\sin x}{x} dx + \int_{2k\pi}^{(2k+1)\pi} \frac{\sin(x+\pi)}{x+\pi} dx \\ &= \int_{2k\pi}^{(2k+1)\pi} \frac{\sin x}{x} dx + \int_{(2k+1)\pi}^{(2k+2)\pi} \frac{\sin u}{u} du = \int_{2k\pi}^{(2k+2)\pi} \frac{\sin x}{x} dx \end{aligned}$$

by using the substitution $u = x + \pi$, then changing the variable of integration back to x .

$$\begin{aligned} \text{Thus, } \int_0^{\infty} \frac{\sin x}{x} dx &= \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \left(\frac{\sin x}{x} + \frac{\sin(x+\pi)}{x+\pi} \right) dx = \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \frac{(x+\pi) \sin x + x \sin(x+\pi)}{x(x+\pi)} dx \\ &= \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \frac{x \sin x + \pi \sin x - x \sin x}{x(x+\pi)} dx = \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \frac{\pi \sin x}{x(x+\pi)} dx. \end{aligned}$$

For $k > 0$, on $[2k\pi, (2k+1)\pi]$ $0 \leq \sin x \leq 1$ while $0 < \frac{\pi}{x(x+\pi)} \leq \frac{\pi}{2k\pi(2k\pi+\pi)} = \frac{1}{(4k^2+2k)\pi}$.

$$\text{Thus, } 0 \leq \int_{2k\pi}^{(2k+1)\pi} \frac{\pi \sin x}{x(x+\pi)} dx \leq \frac{1}{(4k^2+2k)\pi} \int_{2k\pi}^{(2k+1)\pi} dx = \frac{1}{4k^2+2k}.$$

Hence, $\int_{2\pi}^{\infty} \frac{\sin x}{x} dx \leq \sum_{k=1}^{\infty} \frac{1}{4k^2+2k} \leq \sum_{k=1}^{\infty} \frac{1}{4k^2}$ which converges.

Adding $\int_0^{2\pi} \frac{\sin x}{x} dx$ will not affect the convergence, so $\int_0^{\infty} \frac{\sin x}{x} dx$ converges.

43. Consider the graph of $\frac{|\sin x|}{x}$ on the interval $[k\pi, (k+1)\pi]$.

Note that for $k\pi + \frac{\pi}{6} \leq x \leq k\pi + \frac{5\pi}{6}$, $\frac{1}{2} \leq |\sin x|$ while $\frac{1}{\left(k + \frac{5}{6}\right)\pi} \leq \frac{1}{x}$. Thus on $\left[\left(k + \frac{1}{6}\right)\pi, \left(k + \frac{5}{6}\right)\pi\right]$

$$\frac{1}{2\left(k + \frac{5}{6}\right)\pi} = \frac{1}{\left(2k + \frac{5}{3}\right)\pi} \leq \frac{|\sin x|}{x}, \text{ so } \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \geq \int_{(k+1/6)\pi}^{(k+5/6)\pi} \frac{|\sin x|}{x} dx \geq \frac{1}{\left(2k + \frac{5}{3}\right)\pi} \int_{(k+1/6)\pi}^{(k+5/6)\pi} dx = \frac{1}{3k + \frac{5}{2}}.$$

Hence, $\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx \geq \sum_{k=1}^{\infty} \frac{1}{3k + \frac{5}{2}}$. Let $a_k = \frac{1}{3k + \frac{5}{2}}$ and $b_k = \frac{1}{k}$.

$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{k}{3k + \frac{5}{2}} = \lim_{k \rightarrow \infty} \frac{1}{3 + \frac{5}{2k}} = \frac{1}{3}$; $0 < \frac{1}{3} < \infty$. Thus, since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges, $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{3k + \frac{5}{2}}$ also diverges. Hence, $\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx$ also diverges and adding $\int_0^{\pi} \frac{|\sin x|}{x} dx$ will not affect its divergence.

44. Recall that a straight line is the shortest distance between two points. Note that $\sin \frac{\pi}{x} = 1$ when $x = \frac{2}{5}, \frac{2}{9}, \frac{2}{13}, \dots$

and $\sin \frac{\pi}{x} = -1$ when $x = \frac{2}{3}, \frac{2}{7}, \frac{2}{11}, \dots$. Thus, for $n \geq 1$, the curve $y = x \sin \frac{\pi}{x}$ goes from $\left(\frac{2}{4n+1}, \frac{2}{4n+1}\right)$ to

$\left(\frac{2}{4n+3}, -\frac{2}{4n+3}\right)$. The distance between these two points is

$$\begin{aligned} & \sqrt{\left(\frac{2}{4n+1} - \frac{2}{4n+3}\right)^2 + \left(\frac{2}{4n+1} + \frac{2}{4n+3}\right)^2} = \sqrt{2\left(\frac{2}{4n+1}\right)^2 + 2\left(\frac{2}{4n+3}\right)^2} \\ &= \frac{2\sqrt{2(4n+3)^2 + 2(4n+1)^2}}{(4n+1)(4n+3)} = \frac{2\sqrt{64n^2 + 64n + 20}}{16n^2 + 16n + 3} = \frac{4\sqrt{16n^2 + 16n + 5}}{16n^2 + 16n + 3} \end{aligned}$$

The length of $x \sin \frac{\pi}{x}$ on $(0, 1]$ is greater than $\sum_{n=1}^{\infty} \frac{4\sqrt{16n^2 + 16n + 5}}{16n^2 + 16n + 3}$ because this sum does not even take into

account the distances from $\left(\frac{2}{4n+3}, -\frac{2}{4n+3}\right)$ to $\left(\frac{2}{4(n+1)+1}, \frac{2}{4(n+1)+1}\right)$ which are still shorter than the lengths along the curve.

Let $a_n = \frac{4\sqrt{16n^2 + 16n + 5}}{16n^2 + 16n + 3}$ and $b_n = \frac{1}{n}$.

$$\begin{aligned} \text{Then } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{4n\sqrt{16n^2 + 16n + 5}}{16n^2 + 16n + 3} = \lim_{n \rightarrow \infty} \frac{4\sqrt{16n^4 + 16n^3 + 5n^2}}{16n^2 + 16n + 3} = \lim_{n \rightarrow \infty} \frac{4\sqrt{16 + \frac{16}{n} + \frac{5}{n^2}}}{16 + \frac{16}{n} + \frac{3}{n^2}} \\ &= \frac{16}{16} = 1; 0 < 1 < \infty. \end{aligned}$$

Thus, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{4\sqrt{16n^2 + 16n + 5}}{16n^2 + 16n + 3}$ also diverges.

Since the length of the graph is less than $\sum_{n=1}^{\infty} a_n$, the length of the graph is infinite.

$$45. \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} = \left[\frac{1}{1 + \frac{1}{n}} + \frac{1}{1 + \frac{2}{n}} + \dots + \frac{1}{1 + \frac{n}{n}} \right] \left(\frac{1}{n} \right)$$

This is a Riemann sum for the function $f(x) = \frac{1}{x}$ from $x = 1$ to 2 where $\Delta x = \frac{1}{n}$.

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\frac{1}{1 + \frac{k}{n}} \left(\frac{1}{n} \right) \right] = \int_1^2 \frac{1}{x} dx = \ln 2$$

10.6 Concepts Review

1. power series
2. where it converges
3. interval; endpoints
4. $(-1, 1)$

Problem Set 10.6

$$1. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n(n+1)}; \rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)(n+2)} \div \frac{x^n}{n(n+1)} \right| = \lim_{n \rightarrow \infty} |x| \left| \frac{n}{n+2} \right| = |x|$$

When $x = 1$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$ which converges absolutely by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

When $x = -1$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{2n+1} \frac{1}{n(n+1)}$
 $= \sum_{n=1}^{\infty} (-1) \frac{1}{n(n+1)} = (-1) \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ which converges since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

The series converges on $-1 \leq x \leq 1$.

$$2. \sum_{n=0}^{\infty} \frac{x^n}{n!}; \rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!} \right| = \lim_{n \rightarrow \infty} |x| \left| \frac{1}{n+1} \right| = 0$$

The series converges for all x .

$$3. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}; \rho = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+1}}{(2n+1)!} \div \frac{x^{2n-1}}{(2n-1)!} \right| = \lim_{n \rightarrow \infty} |x^2| \left| \frac{1}{2n(2n+1)} \right| = 0$$

The series converges for all x .

$$4. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}; \rho = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \div \frac{x^{2n}}{(2n)!} \right|$$

$$= \lim_{n \rightarrow \infty} |x^2| \left| \frac{1}{(2n+2)(2n+1)} \right| = 0$$

The series converges for all x .

$$5. \sum_{n=1}^{\infty} nx^n; \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| \frac{n+1}{n} \right| = |x|$$

When $x = 1$, the series is $\sum_{n=1}^{\infty} n$ which clearly diverges.

When $x = -1$, the series is $\sum_{n=1}^{\infty} n(-1)^n$; $a_n = n$;

$\lim_{n \rightarrow \infty} a_n \neq 0$, thus the series diverges.

The series converges on $-1 < x < 1$.

$$6. \sum_{n=1}^{\infty} n^2 x^n; \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 x^{n+1}}{n^2 x^n} \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| \frac{(n+1)^2}{n^2} \right| = |x|$$

When $x = 1$, the series is $\sum_{n=1}^{\infty} n^2$ which clearly diverges.

When $x = -1$, the series is $\sum_{n=1}^{\infty} n^2 (-1)^n$;

$a_n = n^2$; $\lim_{n \rightarrow \infty} a_n \neq 0$, thus the series diverges.

The series converges on $-1 < x < 1$.

$$7. \quad 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}; \rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \div \frac{x^n}{n} \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| \frac{n}{n+1} \right| = |x|$$

When $x = 1$, the series is $1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, which is

1 added to the alternating harmonic series multiplied by -1 , which converges.

When $x = -1$, the series is

$$1 + \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n}{n} = 1 + \sum_{n=1}^{\infty} \frac{1}{n}, \text{ which diverges.}$$

The series converges on $-1 < x \leq 1$.

$$8. \quad 1 + \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}; \rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \div \frac{x^n}{\sqrt{n}} \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| \sqrt{\frac{n}{n+1}} \right| = |x|$$

When $x = 1$, the series is $1 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which

diverges since $\frac{1}{2} < 1$.

When $x = -1$, the series is $1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$;

$$a_n = \frac{1}{\sqrt{n}}; \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}, \text{ so } a_n > a_{n+1} \text{ and}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0, \text{ so the series converges.}$$

The series converges on $-1 \leq x < 1$.

$$9. \quad 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(n+2)};$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)(n+3)} \div \frac{x^n}{n(n+2)} \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| \frac{n^2 + 2n}{n^2 + 4n + 3} \right| = |x|$$

When $x = 1$ the series is $1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n(n+2)}$

which converges absolutely by comparison with

$$\text{the series } \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

When $x = -1$, the series is

$$1 + \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n}{n(n+2)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \text{ which}$$

converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

The series converges on $-1 \leq x \leq 1$.

$$10. \quad \sum_{n=1}^{\infty} \frac{x^n}{(n+1)^2 - 1};$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+2)^2 - 1} \div \frac{x^n}{(n+1)^2 - 1} \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| \frac{n^2 + 2n}{n^2 + 4n + 3} \right| = |x|$$

When $x = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{n^2 + 2n} = \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \text{ which}$$

converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

When $x = -1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2 - 1}$ which

converges absolutely by comparison with

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The series converges on $-1 \leq x \leq 1$.

$$11. \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}; \rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{2^{n+1}} \div \frac{x^n}{2^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{2} \right|$$

$$= \left| \frac{x}{2} \right| < 1 \text{ when } -2 < x < 2.$$

When $x = 2$, the series is $\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$

which diverges.

When $x = -2$, the series is

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n (-1)^n = \sum_{n=0}^{\infty} 1 \text{ which}$$

diverges. The series converges on $-2 < x < 2$.

$$12. \quad \sum_{n=0}^{\infty} 2^n x^n; \rho = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} x^{n+1}}{2^n x^n} \right| = \lim_{n \rightarrow \infty} |2x| = |2x|;$$

$$|2x| < 1 \text{ when } -\frac{1}{2} < x < \frac{1}{2}.$$

When $x = \frac{1}{2}$, the series is $\sum_{n=0}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} 1$

which diverges.

When $x = -\frac{1}{2}$, the series is

$$\sum_{n=0}^{\infty} 2^n \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \text{ which diverges.}$$

The series converges on $-\frac{1}{2} < x < \frac{1}{2}$.

$$13. \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}; \rho = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} x^{n+1}}{(n+1)!} \div \frac{2^n x^n}{n!} \right|$$

$$= \lim_{n \rightarrow \infty} |2x| \left| \frac{1}{n+1} \right| = 0.$$

The series converges for all x .

$$14. \sum_{n=1}^{\infty} \frac{nx^n}{n+1}; \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{n+2} \div \frac{nx^n}{n+1} \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| \frac{n^2 + 2n + 1}{n^2 + 2n} \right| = |x|$$

When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{n}{n+1}$ which

diverges since $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$.

When $x = -1$, the series is $\sum_{n=1}^{\infty} \frac{n(-1)^n}{n+1}$ which

diverges since $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$.

The series converges on $-1 < x < 1$.

$$15. \sum_{n=1}^{\infty} \frac{(x-1)^n}{n}; \rho = \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{n+1} \div \frac{(x-1)^n}{n} \right|$$

$$= \lim_{n \rightarrow \infty} |x-1| \left| \frac{n}{n+1} \right| = |x-1|; |x-1| < 1 \text{ when}$$

$$0 < x < 2.$$

When $x = 0$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges.

When $x = 2$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

The series converges on $0 \leq x < 2$.

$$16. \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}; \rho = \lim_{n \rightarrow \infty} \left| \frac{(x+2)^{n+1}}{(n+1)!} \div \frac{(x+2)^n}{n!} \right|$$

$$= \lim_{n \rightarrow \infty} |x+2| \left| \frac{1}{n+1} \right| = 0$$

The series converges for all x .

$$17. \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n}; \rho = \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{2^{n+1}} \div \frac{(x+1)^n}{2^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x+1}{2} \right| = \left| \frac{x+1}{2} \right|; \left| \frac{x+1}{2} \right| < 1 \text{ when}$$

$$-3 < x < 1.$$

When $x = -3$, the series is $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$

which diverges.

When $x = 1$, the series is $\sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$ which

diverges.

The series converges on $-3 < x < 1$.

$$18. \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}; \rho = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2} \div \frac{(x-2)^n}{n^2} \right|$$

$$= \lim_{n \rightarrow \infty} |x-2| \left| \frac{n^2}{(n+1)^2} \right| = |x-2|; |x-2| < 1 \text{ when}$$

$$1 < x < 3.$$

When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which

converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

When $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which

converges. The series converges on $1 \leq x \leq 3$.

$$19. \sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)}; \rho = \lim_{n \rightarrow \infty} \left| \frac{(x+5)^{n+1}}{(n+1)(n+2)} \div \frac{(x+5)^n}{n(n+1)} \right|$$

$$= \lim_{n \rightarrow \infty} |x+5| \left| \frac{n}{n+2} \right| = |x+5|; |x+5| < 1 \text{ when}$$

$$-6 < x < -4.$$

When $x = -4$, the series is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ which

converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

When $x = -6$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ which

converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

converges.

The series converges on $-6 \leq x \leq -4$.

$$20. \sum_{n=1}^{\infty} (-1)^{n+1} n(x+3)^n; \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x+3)^{n+1}}{n(x+3)^n} \right|$$

$$= \lim_{n \rightarrow \infty} |x+3| \left| \frac{n+1}{n} \right| = |x+3|; |x+3| < 1 \text{ when } -4 < x < -2.$$

When $x = -2$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} n$ which diverges since $\lim_{n \rightarrow \infty} n \neq 0$.

When $x = -4$, the series is

$$\sum_{n=1}^{\infty} (-1)^{n+1} n(-1)^n = \sum_{n=1}^{\infty} -n, \text{ which diverges.}$$

The series converges on $-4 < x < -2$.

21. If for some x_0 , $\lim_{n \rightarrow \infty} \frac{x_0^n}{n!} \neq 0$, then $\sum \frac{x_0^n}{n!}$ could not converge.

22. For any number k , since

$$k - n < k - n + 1 < \dots < k - 2 < k - 1 < k,$$

$$|(k-1)(k-2)\dots(k-n)| < k^n, \text{ thus}$$

$$\lim_{n \rightarrow \infty} \left| \frac{k(k-1)(k-2)\dots(k-n)}{n!} x^n \right| < \lim_{n \rightarrow \infty} \left| \frac{k^{n+1}}{n!} x^n \right|$$

$$= |k| \lim_{n \rightarrow \infty} \left| \frac{k^n}{n!} x^n \right|. \text{ Since } -1 < x < 1, \lim_{n \rightarrow \infty} x^n = 0,$$

$$\text{and by Problem 21, } \lim_{n \rightarrow \infty} \left| \frac{k^n}{n!} \right| = 0, \text{ hence}$$

$$\lim_{n \rightarrow \infty} \frac{k(k-1)(k-2)\dots(k-n)}{n!} x^n = 0.$$

23. The Absolute Ratio Test gives

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{2n+3}}{1 \cdot 3 \cdot 5 \dots (2n+1)} \div \frac{n! x^{2n+1}}{1 \cdot 3 \cdot 5 \dots (2n-1)} \right|$$

$$27. \text{ a. } \rho = \lim_{n \rightarrow \infty} \left| \frac{(3x+1)^{n+1}}{(n+1) \cdot 2^{n+1}} \div \frac{(3x+1)^n}{n \cdot 2^n} \right| = \lim_{n \rightarrow \infty} |3x+1| \left| \frac{n}{2n+2} \right| = \frac{1}{2} |3x+1|; \frac{1}{2} |3x+1| < 1 \text{ when } -1 < x < \frac{1}{3}.$$

$$\text{When } x = -1, \text{ the series is } \sum_{n=1}^{\infty} \frac{(-2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}, \text{ which converges.}$$

$$\text{When } x = \frac{1}{3}, \text{ the series is } \sum_{n=1}^{\infty} \frac{2^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}, \text{ which diverges. The series converges on } -1 < x < \frac{1}{3}.$$

$$= \lim_{n \rightarrow \infty} |x^2| \left| \frac{n+1}{2n+1} \right| = \left| \frac{x^2}{2} \right|; \left| \frac{x^2}{2} \right| < 1 \text{ when } |x| < \sqrt{2}.$$

The radius of convergence is $\sqrt{2}$.

24. Using the Absolute Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{(pn+p)!}{((n+1)!)^p} x^{n+1} \div \frac{(pn)!}{(n!)^p} x^n \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| \frac{(pn+p)(pn+p-1)\dots(pn+p-(p-1))}{(n+1)^p} \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| p \left(p - \frac{1}{n+1} \right) \left(p - \frac{2}{n+1} \right) \dots \left(p - \frac{p-1}{n+1} \right) \right|$$

$$= |x| p^p$$

The radius of convergence is p^{-p} .

25. This is a geometric series, so it converges for $|x-3| < 1$, $2 < x < 4$. For these values of x , the

$$\text{series converges to } \frac{1}{1-(x-3)} = \frac{1}{4-x}.$$

26. $\sum_{n=0}^{\infty} a_n(x-3)^n$ converges on an interval of the

form $(3-a, 3+a)$, where $a \geq 0$. If the series converges at $x = -1$, then $3-a \leq -1$, or $a \geq 4$, since $x = -1$ could be an endpoint where the series converges. If $a \geq 4$, then $3+a \geq 7$ so the series will converge at $x = 6$. The series may not converge at $x = 7$, since $x = 7$ may be an endpoint of the convergence intervals, where the series might or might not converge.

$$\text{b. } \rho = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}(2x-3)^{n+1}}{4^{n+1}\sqrt{n+1}} \div \frac{(-1)^n(2x-3)^n}{4^n\sqrt{n}} \right| = \lim_{n \rightarrow \infty} |2x-3| \left| \frac{\sqrt{n}}{4\sqrt{n+1}} \right| = \frac{1}{4}|2x-3|;$$

$$\frac{1}{4}|2x-3| < 1 \text{ when } -\frac{1}{2} < x < \frac{7}{2}.$$

When $x = -\frac{1}{2}$, the series is $\sum_{n=1}^{\infty} (-1)^n \frac{(-4)^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges since $\frac{1}{2} < 1$.

When $x = \frac{7}{2}$, the series is $\sum_{n=1}^{\infty} (-1)^n \frac{4^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}};$

$$a_n = \frac{1}{\sqrt{n}}; \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}, \text{ so } a_n > a_{n+1};$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0, \text{ so } \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \text{ converges.}$$

The series converges on $-\frac{1}{2} < x \leq \frac{7}{2}$.

28. From Problem 52 of Section 10.1,

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] = \frac{1}{2^n\sqrt{5}} \left[(1+\sqrt{5})^n - (1-\sqrt{5})^n \right]$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{2^{n+1}\sqrt{5}} \left[(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1} \right] \div \frac{x^n}{2^n\sqrt{5}} \left[(1+\sqrt{5})^n - (1-\sqrt{5})^n \right] \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x}{2} \left| \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n} \right| \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{2} \left| \frac{1+\sqrt{5} - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}} \right)^n (1-\sqrt{5})}{1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}} \right)^n} \right| \right|$$

$$= \left| \frac{1+\sqrt{5}}{2} x \right|; \left| \frac{1+\sqrt{5}}{2} x \right| < 1 \text{ when } -\frac{2}{1+\sqrt{5}} < x < \frac{2}{1+\sqrt{5}}.$$

$$\left(\text{Note that } \lim_{n \rightarrow \infty} \left(\frac{1-\sqrt{5}}{1+\sqrt{5}} \right)^n = 0 \text{ since } \left| \frac{1-\sqrt{5}}{1+\sqrt{5}} \right| < 1. \right)$$

$$R = \frac{2}{1+\sqrt{5}} \approx 0.618$$

29. If $a_{n+3} = a_n$, then $a_0 = a_3 = a_6 = a_{3n}$, $a_1 = a_4 = a_7 = a_{3n+1}$, and $a_2 = a_5 = a_8 = a_{3n+2}$. Thus,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_0 x^3 + a_1 x^4 + a_2 x^5 + \cdots = (a_0 + a_1 x + a_2 x^2)(1 + x^3 + x^6 + \cdots)$$

$$= (a_0 + a_1 x + a_2 x^2) \sum_{n=0}^{\infty} x^{3n} = (a_0 + a_1 x + a_2 x^2) \sum_{n=0}^{\infty} (x^3)^n.$$

$a_0 + a_1 x + a_2 x^2$ is a polynomial, which will converge for all x .

$\sum_{n=0}^{\infty} (x^3)^n$ is a geometric series which converges for $|x^3| < 1$, or, equivalently, $|x| < 1$.

$$\text{Since } \sum_{n=0}^{\infty} (x^3)^n = \frac{1}{1-x^3} \text{ for } |x| < 1, S(x) = \frac{a_0 + a_1 x + a_2 x^2}{1-x^3} \text{ for } |x| < 1.$$

30. If $a_n = a_{n+p}$, then $a_0 = a_p = a_{2p} = a_{np}$, $a_1 = a_{p+1} = a_{2p+1} = a_{np+1}$, etc. Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x + \cdots + a_{p-1} x^{p-1} + a_0 x^p + a_1 x^{p+1} + \cdots + a_{p-1} x^{2p-1} + \cdots \\ &= (a_0 + a_1 x + \cdots + a_{p-1} x^{p-1})(1 + x^p + x^{2p} + \cdots) = (a_0 + a_1 x + \cdots + a_{p-1} x^{p-1}) \sum_{n=0}^{\infty} x^{np}\end{aligned}$$

$a_0 + a_1 x + \cdots + a_{p-1} x^{p-1}$ is a polynomial, which will converge for all x .

$\sum_{n=0}^{\infty} x^{np} = \sum_{n=0}^{\infty} (x^p)^n$ is a geometric series which converges for $|x^p| < 1$, or, equivalently, $|x| < 1$.

Since $\sum_{n=0}^{\infty} (x^p)^n = \frac{1}{1-x^p}$ for $|x| < 1$, $S(x) = (a_0 + a_1 x + \cdots + a_{p-1} x^{p-1}) \left(\frac{1}{1-x^p} \right)$ for $|x| < 1$.

10.7 Concepts Review

1. integrated; interior

2. $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}$

3. $1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6}$

4. $1 + x + \frac{3x^2}{2} + \frac{x^3}{3} + \frac{3x^4}{4}$

Problem Set 10.7

1. From the geometric series for $\frac{1}{1-x}$ with x

replaced by $-x$, we get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

radius of convergence 1.

2. $\frac{d}{dx} \left(\frac{1}{1+x} \right) = -\frac{1}{(1+x)^2}$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \text{ so}$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$

radius of convergence 1.

3. $\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2}$; $\frac{d}{dx} \left(\frac{1}{(1-x)^2} \right) = \frac{2}{(1-x)^3}$,

so $\frac{1}{(1-x)^3}$ is $\frac{1}{2}$ of the second derivative of

$$\frac{1}{1-x}. \text{ Thus,}$$

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$$

radius of convergence 1.

4. Using the result of Problem 2,

$$\frac{x}{(1+x)^2} = x - 2x^2 + 3x^3 - 4x^4 + 5x^5 - \cdots$$

radius of convergence 1.

5. From the geometric series for $\frac{1}{1-x}$ with x

replaced by $\frac{3}{2}x$, we get

$$\frac{1}{2-3x} = \frac{1}{2} + \frac{3x}{4} + \frac{9x^2}{8} + \frac{27x^3}{16} + \cdots$$

radius of convergence $\frac{2}{3}$.

6. $\frac{1}{3+2x} = \frac{1}{3} \left(\frac{1}{1+\frac{2}{3}x} \right)$. Since

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

$$\frac{1}{3} \left(\frac{1}{1+\frac{2}{3}x} \right) = \frac{1}{3} - \frac{2x}{9} + \frac{4x^2}{27} - \frac{8x^3}{81} + \frac{16x^4}{243} - \cdots$$

radius of convergence $\frac{3}{2}$.

7. From the geometric series for $\frac{1}{1-x}$ with x

replaced by x^4 , we get

$$\frac{x^2}{1-x^4} = x^2 + x^6 + x^{10} + x^{14} + \dots;$$

radius of convergence 1.

$$8. \frac{x^3}{2-x^3} = \frac{x^3}{2} \left(\frac{1}{1-\frac{x^3}{2}} \right) = \frac{x^3}{2} + \frac{x^6}{4} + \frac{x^9}{8} + \frac{x^{12}}{16} + \dots$$

$$\text{for } \left| \frac{x^3}{2} \right| < 1 \text{ or } -\sqrt[3]{2} < x < \sqrt[3]{2}.$$

9. From the geometric series for $\ln(1+x)$ with x replaced by t , we get

$$\int_0^x \ln(1+t) dt = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \dots;$$

radius of convergence 1.

$$10. \int_0^x \tan^{-1} t dt = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \dots;$$

radius of convergence 1.

$$11. \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, -1 < x \leq 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots, -1 \leq x < 1$$

$$\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$$

$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots; \text{ radius of convergence 1.}$$

12. If $M = \frac{1+x}{1-x}$, then $M - Mx = 1 + x$;

$$M - 1 = (M+1)x; \quad x = \frac{M-1}{M+1}.$$

$$\left| \frac{M-1}{M+1} \right| < 1 \text{ is equivalent to } -M-1 < M-1 < M+1 \text{ or } 0 < 2M < 2M+2 \text{ which is true for } M > 0. \text{ Thus, the natural}$$

logarithm of any positive number can be found by using the series from Problem 11. For $M = 8$, $x = \frac{7}{9}$, so

$$\ln 8 = 2 \left(\frac{7}{9} \right) + \frac{2}{3} \left(\frac{7}{9} \right)^3 + \frac{2}{5} \left(\frac{7}{9} \right)^5 + \frac{2}{7} \left(\frac{7}{9} \right)^7 + \frac{2}{9} \left(\frac{7}{9} \right)^9 + \frac{2}{11} \left(\frac{7}{9} \right)^{11} + \dots$$

$$\approx 1.55556 + 0.31367 + 0.11385 + 0.04919 + 0.02315 + 0.01146 + 0.00586 + 0.00307 + 0.00164 + 0.00089$$

$$+ 0.00049 + 0.00027 + 0.00015 + 0.00008 \approx 2.079$$

13. Substitute $-x$ for x in the series for e^x to get:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots.$$

$$14. xe^{x^2} = x \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots \right) = x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \frac{x^9}{4!} + \dots$$

15. Add the result of Problem 13 to the series for e^x to get:

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} + \dots.$$

$$16. e^{2x} - 1 - 2x = -1 - 2x + \left(1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots\right) = \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \frac{32x^5}{5!} + \dots$$

$$17. e^{-x} \cdot \frac{1}{1-x} = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)(1 + x + x^2 + \dots) = 1 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^4}{8} + \frac{11x^5}{30} + \dots$$

$$18. e^x \tan^{-1} x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) = x + x^2 + \frac{x^3}{6} - \frac{x^4}{6} + \frac{3x^5}{40} + \dots$$

$$19. \frac{\tan^{-1} x}{e^x} = e^{-x} \tan^{-1} x = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) = x - x^2 + \frac{x^3}{6} + \frac{x^4}{6} + \frac{3x^5}{40} + \dots$$

$$20. \frac{e^x}{1 + \ln(1+x)} = \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}{1 + x - \frac{x^2}{2} + \frac{x^3}{3} - \dots} = 1 + x^2 - \frac{7x^3}{6} + \frac{47x^4}{24} - \frac{46x^5}{15} + \dots$$

$$21. (\tan^{-1} x)(1 + x^2 + x^4) = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)(1 + x^2 + x^4) = x + \frac{2x^3}{3} + \frac{13x^5}{15} - \frac{29x^7}{105} + \dots$$

$$22. \frac{\tan^{-1} x}{1 + x^2 + x^4} = \frac{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}{1 + x^2 + x^4} = x - \frac{4x^3}{3} + \frac{8x^5}{15} + \frac{23x^7}{35} - \dots$$

$$23. \text{The series representation of } \frac{e^x}{1+x} \text{ is } 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \frac{11x^5}{30} + \dots, \text{ so } \int_0^x \frac{e^t}{1+t} dt = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5 - \dots$$

$$24. \text{The series representation of } \frac{\tan^{-1} x}{x} \text{ is } 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \dots, \text{ so } \int_0^x \frac{\tan^{-1} t}{t} dt = x - \frac{x^3}{9} + \frac{x^5}{25} - \frac{x^7}{49} + \dots$$

$$25. \text{a. } \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots, \text{ so } \frac{x}{1+x} = x - x^2 + x^3 - x^4 + x^5 - \dots$$

$$\text{b. } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots, \text{ so } \frac{e^x - (1+x)}{x^2} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \dots$$

$$\text{c. } -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \dots, \text{ so } -\ln(1-2x) = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \dots$$

$$26. \text{a. Since } \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots, \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \dots$$

$$\text{b. Again using } \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots, \frac{1}{1-\cos x} - 1 = \cos x + \cos^2 x + \cos^3 x + \dots$$

$$\text{c. } \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots, \text{ so } \ln(1-x^2) = -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \dots, \text{ and}$$

$$-\frac{1}{2} \ln(1-x^2) = \ln \frac{1}{\sqrt{1-x^2}} = \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \dots$$

27. Differentiating the series for $\frac{1}{1-x}$ yields $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$ multiplying this series by x gives

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \dots, \text{ hence } \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \text{ for } -1 < x < 1.$$

28. Differentiating the series for $\frac{1}{x-1}$ twice yields $\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + 20x^3 + \dots$. Multiplying this series by x

$$\text{gives } \frac{2x}{(1-x)^3} = 2x + 3 \cdot 2x^2 + 4 \cdot 3x^3 + 5 \cdot 4x^4 + \dots, \text{ hence } \sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3} \text{ for } -1 < x < 1.$$

$$\begin{aligned} 29. \text{ a. } \tan^{-1}(e^x - 1) &= (e^x - 1) - \frac{(e^x - 1)^3}{3} + \frac{(e^x - 1)^5}{5} - \dots = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - \frac{1}{3} \left(x + \frac{x^2}{2!} + \dots\right)^3 + \dots \\ &= x + \frac{x^2}{2} - \frac{x^3}{6} - \dots \end{aligned}$$

$$\begin{aligned} \text{b. } e^{e^x - 1} &= 1 + (e^x - 1) + \frac{(e^x - 1)^2}{2!} + \frac{(e^x - 1)^3}{3!} + \dots \\ &= 1 + \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + \frac{1}{2!} \left(x + \frac{x^2}{2!} + \dots\right)^2 + \frac{1}{3!} \left(x + \frac{x^2}{2!} + \dots\right)^3 \\ &= 1 + \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + \frac{1}{2!} \left(x^2 + 2 \frac{x^3}{2!} + \dots\right) + \frac{1}{3!} \left(x^3 + 3 \frac{x^4}{2!} + \dots\right) = 1 + x + x^2 + \frac{5x^3}{6} + \dots \end{aligned}$$

$$30. f(x) = a_0 + a_1x + a_2x^2 + \dots = b_0 + b_1x + b_2x^2 + \dots;$$

$$f(0) = a_0 = b_0, \text{ so } a_0 = b_0.$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \dots = b_1 + 2b_2x + 3b_3x^2 + \dots;$$

$$f'(0) = a_1 = b_1, \text{ so } a_1 = b_1.$$

The n th derivative of $f(x)$ is

$$f^{(n)}(x) = n!a_n + (n+1)!a_{n+1}x + \frac{(n+2)!}{2}a_{n+2}x^2 + \dots = n!b_n + (n+1)!b_{n+1}x + \frac{(n+2)!}{2}b_{n+2}x^2 + \dots;$$

$$f^{(n)}(0) = n!a_n = n!b_n, \text{ so } a_n = b_n.$$

$$\begin{aligned} 31. \frac{x}{x^2 - 3x + 2} &= \frac{x}{(x-2)(x-1)} = \frac{2}{x-2} - \frac{1}{x-1} = -\frac{1}{1-\frac{x}{2}} + \frac{1}{1-x} = -\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \dots\right) + (1 + x + x^2 + x^3 + \dots) \\ &= \frac{x}{2} + \frac{3x^2}{4} + \frac{7x^3}{8} + \dots = \sum_{n=1}^{\infty} \frac{(2^n - 1)x^n}{2^n} \end{aligned}$$

32. $y'' = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots = -y$, so $y'' + y = 0$. It is clear that $y(0) = 0$ and $y'(0) = 1$. Both the sine and cosine functions satisfy $y'' + y = 0$, however, only the sine function satisfies the given initial conditions. Thus, $y = \sin x$.

$$\begin{aligned} 33. F(x) - xF(x) - x^2F(x) &= (f_0 + f_1x + f_2x^2 + f_3x^3 + \dots) - (f_0x + f_1x^2 + f_2x^3 + \dots) - (f_0x^2 + f_1x^3 + f_2x^4 + \dots) \\ &= f_0 + (f_1 - f_0)x + (f_2 - f_1 - f_0)x^2 + (f_3 - f_2 - f_1)x^3 + \dots \\ &= f_0 + (f_1 - f_0)x + \sum_{n=2}^{\infty} (f_n - f_{n-1} - f_{n-2})x^n = 0 + x + \sum_{n=0}^{\infty} (f_{n+2} - f_{n+1} - f_n)x^{n+2} \end{aligned}$$

Since $f_{n+2} = f_{n+1} + f_n$, $f_{n+2} - f_{n+1} - f_n = 0$. Thus $F(x) - xF(x) - x^2F(x) = x$.

$$F(x) = \frac{x}{1-x-x^2}$$

$$34. \quad y(x) = \frac{f_0}{0!} + \frac{f_1}{1!}x + \frac{f_2}{2!}x^2 + \frac{f_3}{3!}x^3 + \frac{f_4}{4!}x^4 + \dots;$$

$$y'(x) = \frac{f_1}{0!} + \frac{f_2}{1!}x + \frac{f_3}{2!}x^2 + \frac{f_4}{3!}x^3 + \dots;$$

$$y''(x) = \frac{f_2}{0!} + \frac{f_3}{1!}x + \frac{f_4}{2!}x^2 + \dots$$

(Recall that $0! = 1$.)

$$y''(x) - y'(x) - y(x) = \left(\frac{f_2}{0!} + \frac{f_3}{1!}x + \frac{f_4}{2!}x^2 + \dots \right) - \left(\frac{f_1}{0!} + \frac{f_2}{1!}x + \frac{f_3}{2!}x^2 + \dots \right) - \left(\frac{f_0}{0!} + \frac{f_1}{1!}x + \frac{f_2}{2!}x^2 + \dots \right)$$

$$= \frac{1}{0!}(f_2 - f_1 - f_0) + \frac{1}{1!}(f_3 - f_2 - f_1)x + \frac{1}{2!}(f_4 - f_3 - f_2)x^2 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(f_{n+2} - f_{n+1} - f_n)x^n = 0 \quad \text{since } f_{n+2} = f_{n+1} + f_n \text{ for all } n \geq 0.$$

$$35. \quad \pi \approx 16 \left(\frac{1}{5} - \frac{1}{375} + \frac{1}{15,625} - \frac{1}{546,875} + \frac{1}{17,578,125} \right) - 4 \left(\frac{1}{239} \right) \approx 3.14159$$

36. For any positive integer $k \leq n$, both $\frac{n!}{k}$ and $\frac{n!}{k!}$ are positive integers. Thus, since $q < n$, $n!e = \frac{n!p}{q}$ is a positive

integer and $M = n!e - n! - \frac{n!}{2!} - \frac{n!}{3!} - \dots - \frac{n!}{n!}$ is also an integer. M is positive since

$$e - 1 - 1 - \frac{1}{2!} - \dots - \frac{1}{n!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots$$

$M < \frac{1}{n}$ contradicts that M is a positive integer since for $n \geq 1$, $\frac{1}{n} \leq 1$ and there are no positive integers less than 1.

10.8 Concepts Review

$$1. \quad \frac{f^{(k)}(0)}{k!}$$

$$3. \quad -\infty; \infty$$

$$2. \quad \lim_{n \rightarrow \infty} R_n(x) = 0$$

$$4. \quad 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$$

Problem Set 10.8

$$1. \quad \tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

$$2. \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots} = x - \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

3. $e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \dots$
4. $e^{-x} \cos x = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) = 1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{30} + \dots$
5. $\cos x \ln(1+x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) = x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} - \dots$
6. $(\sin x)\sqrt{1+x} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots\right) = x + \frac{x^2}{2} - \frac{7x^3}{24} - \frac{x^4}{48} - \frac{19x^5}{1920} + \dots, -1 < x < 1$
7. $e^x + x + \sin x = x + \left(1 + x + \frac{x^2}{2!} + \dots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) = 1 + 3x + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{2x^5}{5!} + \dots$
8. $\cos x - 1 + \frac{x^2}{2} = \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$, so $\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{1}{4!} - \frac{x^2}{6!} + \frac{x^4}{8!} - \dots$
9. $\frac{1}{1-x} \cosh x = (1 + x + x^2 + x^3 + \dots) \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) = 1 + x + \frac{3x^2}{2} + \frac{3x^3}{2} + \frac{37x^4}{24} + \frac{37x^5}{24} + \dots, -1 < x < 1$
10. $\frac{-\ln(1+x)}{1+x} = \frac{-\ln(1+x)}{1-(-x)} = \left(-x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \dots\right) (1 - x + x^2 - x^3 + x^4 - \dots)$
 $= -x + \frac{3x^2}{2} - \frac{11x^3}{6} + \frac{25x^4}{12} - \frac{137x^5}{60} + \dots, -1 < x < 1$
11. $\frac{1}{1+x+x^2} = \frac{1}{1-(-x^2-x)} = 1 + (-x^2-x) + (-x^2-x)^2 + (-x^2-x)^3 + \dots$
 $= 1 - x + x^3 - x^4 + \dots, -1 < -x^2 - x < 1 \text{ or } -1 < x^2 + x < 1$
12. $\frac{1}{1-\sin x} = 1 + \sin x + (\sin x)^2 + (\sin x)^3 + \dots$
 $= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) + \left(x - \frac{x^3}{3!} + \dots\right)^2 + \left(x - \frac{x^3}{3!} + \dots\right)^3 + \left(x - \frac{x^3}{3!} + \dots\right)^4 + \left(x - \frac{x^3}{3!} + \dots\right)^5 + \dots$
 $= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) + \left(x^2 - 2\frac{x^4}{3!} + \dots\right) + \left(x^3 - 3\frac{x^5}{3!} + \dots\right) + (x^4 - \dots) + (x^5 - \dots)$
 $= 1 + x + x^2 + \frac{5x^3}{6} + \frac{2x^4}{3} + \frac{61x^5}{120} + \dots, -1 < \sin x < 1 \text{ or } x \neq \frac{(2k+1)\pi}{2} \text{ where } k \text{ is any integer.}$
13. $\sin^3 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) = \left(x^2 - 2\frac{x^4}{3!} + \dots\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) = x^3 - \frac{x^5}{2} + \dots$
14. $x(\sin 2x + \sin 3x) = x \left[\left(2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \dots\right) + \left(3x - \frac{27x^3}{3!} + \frac{243x^5}{5!} - \dots\right) \right] = x \left(5x - \frac{35x^3}{3!} + \dots \right) = 5x^2 - \frac{35x^4}{3!} + \dots$

$$15. \quad x \sec(x^2) + \sin x = \frac{x}{\cos(x^2)} + \sin x = \frac{x}{1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \dots} + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)$$

$$= \left(x + \frac{x^5}{2} + \dots \right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) = 2x - \frac{x^3}{3!} + \frac{61x^5}{120} + \dots$$

$$16. \quad \frac{\cos x}{\sqrt{1+x}} = (\cos x)(1+x)^{-1/2} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) \left(1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \dots \right)$$

$$= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \frac{49x^4}{384} - \frac{85x^5}{768} + \dots, \quad -1 < x < 1$$

$$17. \quad (1+x)^{3/2} = 1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} + \dots, \quad -1 < x < 1$$

$$18. \quad (1-x^2)^{2/3} = [1+(-x^2)]^{2/3} = 1 + \frac{2}{3}(-x^2) - \frac{1}{9}(-x^2)^2 + \frac{4}{81}(-x^2)^3 + \dots = 1 - \frac{2x^2}{3} - \frac{x^4}{9} - \dots,$$

$$-1 < -x^2 < 1 \text{ or } -1 < x < 1$$

$$19. \quad f^{(n)}(x) = e^x \text{ for all } n. \quad f(1) = f'(1) = f''(1) = f'''(1) = e$$

$$e^x \approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3$$

$$20. \quad f\left(\frac{\pi}{6}\right) = \frac{1}{2}; f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}; f''\left(\frac{\pi}{6}\right) = -\frac{1}{2}; f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}; \sin x \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$$

$$21. \quad f\left(\frac{\pi}{3}\right) = \frac{1}{2}; f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}; f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}; f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}; \cos x \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3$$

$$22. \quad f\left(\frac{\pi}{4}\right) = 1; f'\left(\frac{\pi}{4}\right) = 2; f''\left(\frac{\pi}{4}\right) = 4; f'''\left(\frac{\pi}{4}\right) = 16$$

$$\tan x \approx 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

$$23. \quad f(1) = 3; f'(1) = 2 + 3 = 5;$$

$$f''(1) = 2 + 6 = 8; f'''(1) = 6$$

$$1 + x^2 + x^3 = 3 + 5(x-1) + 4(x-1)^2 + (x-1)^3$$

This is exact since $f^{(n)}(x) = 0$ for $n \geq 4$.

$$24. \quad f(-1) = 2 + 1 + 3 + 1 = 7;$$

$$f'(-1) = -1 - 6 - 3 = -10;$$

$$f''(-1) = 6 + 6 = 12; f'''(1) = -6$$

$$2 - x + 3x^2 - x^3 = 7 - 10(x+1) + 6(x+1)^2 - (x+1)^3$$

This is exact since $f^{(n)}(x) = 0$ for $n \geq 4$.

$$25. \quad \text{The derivative of an even function is an odd function and the derivative of an odd function is}$$

an even function. (Problem 50 of Section 3.2).

Since $f(x) = \sum a_n x^n$ is an even function, $f'(x)$ is an odd function, so $f''(x)$ is an even function, hence $f'''(x)$ is an odd function, etc.

Thus $f^{(n)}(x)$ is an even function when n is even and an odd function when n is odd.

By the Uniqueness Theorem, if $f(x) = \sum a_n x^n$, then $a_n = \frac{f^{(n)}(0)}{n!}$. If $g(x)$ is an odd function,

$g(0) = 0$, thence $a_n = 0$ for all odd n since $f^{(n)}(x)$ is an odd function for odd n .

$$26. \quad \text{Let } f(x) = \sum a_n x^n \text{ be an odd function}$$

$(f(-x) = -f(x))$ for x in $(-R, R)$. Then $a_n = 0$ if n is even.

The derivative of an even function is an odd function and the derivative of an odd function is an even function (Problem 50 of Section 3.2).

Since $f(x) = \sum a_n x^n$ is an odd function, $f'(x)$ is an even function, so $f''(x)$ is an odd function, hence $f'''(x)$ is an even function, etc. Thus,

$f^{(n)}(x)$ is an even function when n is odd and an odd function when n is even.

By the Uniqueness Theorem, if $f(x) = \sum a_n x^n$,

then $a_n = \frac{f^{(n)}(0)}{n!}$. If $g(x)$ is an odd function,

$g(0) = 0$, hence $a_n = 0$ for all even n since

$f^{(n)}(x)$ is an odd function for all even n .

$$\begin{aligned} 27. \quad \frac{1}{\sqrt{1-t^2}} &= [1+(-t^2)]^{-1/2} \\ &= 1 - \frac{1}{2}(-t^2) + \frac{3}{8}(-t^2)^2 - \frac{5}{16}(-t^2)^3 + \dots \\ &= 1 + \frac{t^2}{2} + \frac{3t^4}{8} + \frac{5t^6}{16} + \dots \end{aligned}$$

$$\begin{aligned} \text{Thus, } \sin^{-1} x &= \int_0^x \frac{1}{\sqrt{1-t^2}} dt \\ &= \int_0^x \left(1 + \frac{t^2}{2} + \frac{3t^4}{8} + \frac{5t^6}{16} + \dots \right) dt \\ &= \left[t + \frac{t^3}{6} + \frac{3t^5}{40} + \frac{5t^7}{112} + \dots \right]_0^x \\ &= x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \dots \end{aligned}$$

$$\begin{aligned} 30. \quad \sin \sqrt{x} &= \sqrt{x} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} + \frac{x^{9/2}}{9!} - \dots \\ \int_0^{0.5} \sin \sqrt{x} dx &= \int_0^{0.5} \left(\sqrt{x} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} + \frac{x^{9/2}}{9!} - \dots \right) dx \\ &= \left[\frac{2}{3} x^{3/2} - \frac{2}{5} \frac{x^{5/2}}{3!} + \frac{2}{7} \frac{x^{7/2}}{5!} - \frac{2}{9} \frac{x^{9/2}}{7!} + \frac{2}{11} \frac{x^{11/2}}{9!} - \dots \right]_0^{0.5} \\ &= \frac{2}{3} (0.5)^{3/2} - \frac{1}{15} (0.5)^{5/2} + \frac{1}{420} (0.5)^{7/2} - \frac{1}{22,680} (0.5)^{9/2} + \frac{1}{1,995,840} (0.5)^{11/2} - \dots \approx 0.22413 \end{aligned}$$

$$\begin{aligned} 31. \quad \frac{1}{x-1-(1-x)} &= 1 + (1-x) + (1-x)^2 + (1-x)^3 + \dots = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots \\ &\text{for } -1 < 1-x < 1, \text{ or } 0 < x < 2. \end{aligned}$$

$$\begin{aligned} 32. \quad (1+x)^{1/2} &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \dots \\ (1-x)^{1/2} &= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5 - \dots \end{aligned}$$

$$\begin{aligned} 28. \quad \frac{1}{\sqrt{1+t^2}} &= (1+t^2)^{-1/2} \\ &= 1 - \frac{1}{2}t^2 + \frac{3}{8}(t^2)^2 - \frac{5}{16}(t^2)^3 + \dots \\ &= 1 - \frac{t^2}{2} + \frac{3t^4}{8} - \frac{5t^6}{16} + \dots \end{aligned}$$

$$\begin{aligned} \text{Thus, } \sinh^{-1}(x) &= \int_0^x \frac{1}{\sqrt{1+t^2}} dt \\ &= \int_0^x \left(1 - \frac{t^2}{2} + \frac{3t^4}{8} - \frac{5t^6}{16} + \dots \right) dt \\ &= \left[t - \frac{t^3}{6} + \frac{3t^5}{40} - \frac{5t^7}{112} + \dots \right]_0^x \\ &= x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112} + \dots \end{aligned}$$

$$\begin{aligned} 29. \quad \cos(x^2) &= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \dots \\ \int_0^1 \cos(x^2) dx &= \int_0^1 \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \dots \right) dx \\ &= \left[x - \frac{x^5}{10} + \frac{x^9}{216} - \frac{x^{13}}{9360} + \frac{x^{17}}{685,440} - \dots \right]_0^1 \\ &= 1 - \frac{1}{10} + \frac{1}{216} - \frac{1}{9360} + \frac{1}{685,440} - \dots \approx 0.90452 \end{aligned}$$

$$\text{so } f(x) = 2 - \frac{1}{4}x^2 - \frac{5}{64}x^4 - \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Note that $f^{(n)}(0) = 0$ when n is odd.

$$\text{Thus, } \frac{f^{(4)}(0)}{4!} = -\frac{5}{64} \text{ and } \frac{f^{(5)}(0)}{5!} = 0, \text{ so } f^{(4)}(0) = -\frac{5}{64} 4! = -\frac{15}{8} \text{ and } f^{(5)}(0) = 0.$$

$$\begin{aligned} 33. \text{ a. } f(x) &= 1 + (x+x^2) + \frac{(x+x^2)^2}{2!} + \frac{(x+x^2)^3}{3!} + \frac{(x+x^2)^4}{4!} + \dots \\ &= 1 + (x+x^2) + \frac{1}{2}(x^2 + 2x^3 + x^4) + \frac{1}{6}(x^3 + 3x^4 + 3x^5 + x^6) + \frac{1}{24}(x^4 + 4x^5 + 6x^6 + 4x^7 + x^8) + \dots \\ &= 1 + x + \frac{3x^2}{2} + \frac{7x^3}{6} + \frac{25x^4}{24} + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \end{aligned}$$

$$\text{Thus } \frac{f^{(4)}(0)}{4!} = \frac{25}{24} \text{ so } f^{(4)}(0) = \frac{25}{24} 4! = 25.$$

$$\text{b. } f(x) = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \frac{\sin^4 x}{4!} + \dots$$

$$\begin{aligned} &= 1 + \left(x - \frac{x^3}{3!} + \dots\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + \dots\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{3!} + \dots\right)^3 + \frac{1}{24}\left(x - \frac{x^3}{3!} + \dots\right)^4 \\ &= 1 + \left(x - \frac{x^3}{3!} + \dots\right) + \frac{1}{2}\left(x^2 - 2\frac{x^4}{3!} + \dots\right) + \frac{1}{6}(x^3 - \dots) + \frac{1}{24}(x^4 - \dots) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \end{aligned}$$

$$\text{Thus, } \frac{f^{(4)}(0)}{4!} = -\frac{1}{8} \text{ so } f^{(4)}(0) = -\frac{1}{8} 4! = -3.$$

$$\text{c. } e^{t^2} - 1 = -1 + \left(1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \frac{t^8}{4!} + \dots\right) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \frac{t^8}{4!} + \dots$$

$$\text{so } \frac{e^{t^2} - 1}{t^2} = 1 + \frac{t^2}{2} + \frac{t^4}{6} + \frac{t^6}{24} + \dots$$

$$f(x) = \int_0^x \left(1 + \frac{t^2}{2} + \frac{t^4}{6} + \frac{t^6}{24} + \dots\right) dt = \left[t + \frac{t^3}{6} + \frac{t^5}{30} + \dots\right]_0^x = x + \frac{x^3}{6} + \frac{x^5}{30} + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\text{Thus, } \frac{f^{(4)}(0)}{4!} = 0 \text{ so } f^{(4)}(0) = 0.$$

$$\text{d. } e^{\cos x - 1} = 1 + (\cos x - 1) + \frac{(\cos x - 1)^2}{2!} + \frac{(\cos x - 1)^3}{3!} + \dots$$

$$\begin{aligned} &= 1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)^2 + \frac{1}{6}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)^3 + \dots \\ &= 1 + \left(-\frac{x^2}{2} + \frac{x^4}{24} - \dots\right) + \frac{1}{2}\left(\frac{x^4}{4} - \dots\right) + \frac{1}{6}\left(-\frac{x^6}{8} + \dots\right) = 1 - \frac{x^2}{2} + \frac{x^4}{6} - \dots \end{aligned}$$

$$\text{Hence } f(x) = e - \frac{e}{2}x^2 + \frac{e}{6}x^4 - \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\text{Thus, } \frac{f^{(4)}(0)}{4!} = \frac{e}{6} \text{ so } f^{(4)}(0) = \frac{e}{6} 4! = 4e.$$

e. Observe that $\ln(\cos^2 x) = \ln(1 - \sin^2 x)$.

$$\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)^2 = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \dots$$

$$\ln(1 - \sin^2 x) = -\sin^2 x - \frac{\sin^4 x}{2} - \frac{\sin^6 x}{3} - \dots$$

$$= -\left(x^2 - \frac{x^4}{3} + \dots \right) - \frac{1}{2} \left(x^2 - \frac{x^4}{3} + \dots \right)^2 - \frac{1}{3} \left(x^2 - \frac{x^4}{3} + \dots \right)^3$$

$$= -\left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \dots \right) - \frac{1}{2} \left(x^4 - \frac{2x^6}{3} + \dots \right) - \frac{1}{3} (x^6 - \dots) = -x^2 - \frac{x^4}{6} - \frac{2x^6}{45} - \dots$$

$$\text{Hence } f(x) = -x^2 - \frac{1}{6}x^4 - \frac{2}{45}x^6 - \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

$$\text{Thus, } \frac{f^{(4)}(0)}{4!} = -\frac{1}{6} \text{ so } f^{(4)}(0) = -\frac{1}{6} 4! = -4.$$

34. $\sec x = \frac{1}{\cos x} = a_0 + a_1 x + a_2 x^2 + \dots$ so

$$1 = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots \right)$$

$$= a_0 + a_1 x + \left(a_2 - \frac{a_0}{2} \right) x^2 + \left(a_3 - \frac{a_1}{2} \right) x^3 + \left(a_4 - \frac{a_2}{2} + \frac{a_0}{24} \right) x^4 + \dots$$

$$\text{Thus } a_0 = 1, a_1 = 0, a_2 - \frac{a_0}{2} = 0, a_3 - \frac{a_1}{2} = 0, a_4 - \frac{a_2}{2} + \frac{a_0}{24} = 0, \text{ so}$$

$$a_0 = 1, a_1 = 0, a_2 = \frac{1}{2}, a_3 = 0, a_4 = \frac{5}{24}$$

$$\text{and therefore } \sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots$$

35. $\tanh x = \frac{\sinh x}{\cosh x} = a_0 + a_1 x + a_2 x^2 + \dots$

$$\text{so } \sinh x = \cosh x (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$\text{or } x + \frac{x^3}{6} + \frac{x^5}{120} + \dots = \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \dots \right) (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= a_0 + a_1 x + \left(a_2 + \frac{a_0}{2} \right) x^2 + \left(a_3 + \frac{a_1}{2} \right) x^3 + \left(a_4 + \frac{a_2}{2} + \frac{a_0}{24} \right) x^4 + \left(a_5 + \frac{a_3}{2} + \frac{a_1}{24} \right) x^5 + \dots$$

$$\text{Thus } a_0 = 0, a_1 = 1, a_2 + \frac{a_0}{2} = 0, a_3 + \frac{a_1}{2} = \frac{1}{6},$$

$$a_4 + \frac{a_2}{2} + \frac{a_0}{24} = 0, a_5 + \frac{a_3}{2} + \frac{a_1}{24} = \frac{1}{120}, \text{ so}$$

$$a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -\frac{1}{3}, a_4 = 0, a_5 = \frac{2}{15} \text{ and therefore}$$

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \dots$$

$$36. \operatorname{sech} x = \frac{1}{\cosh x} = a_0 + a_1 x + a_2 x^2 + \dots$$

$$\text{so } 1 = \cosh x(a_0 + a_1 x + a_2 x^2 + \dots)$$

$$\text{or } 1 = \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \dots\right)(a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= a_0 + a_1 x + \left(a_2 + \frac{a_0}{2}\right)x^2 + \left(a_3 + \frac{a_1}{2}\right)x^3 + \left(a_4 + \frac{a_2}{2} + \frac{a_0}{24}\right)x^4 + \left(a_5 + \frac{a_3}{2} + \frac{a_1}{24}\right)x^5 + \dots$$

$$\text{Thus, } a_0 = 1, a_1 = 0, \left(a_2 + \frac{a_0}{2}\right) = 0, \left(a_3 + \frac{a_1}{2}\right) = 0, \left(a_4 + \frac{a_2}{2} + \frac{a_0}{24}\right) = 0, \left(a_5 + \frac{a_3}{2} + \frac{a_1}{24}\right) = 0, \text{ so}$$

$$a_0 = 1, a_1 = 0, a_2 = -\frac{1}{2}, a_3 = 0, a_4 = \frac{5}{24}, a_5 = 0 \text{ and therefore}$$

$$\operatorname{sech} x = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \dots$$

37. a. First define $R_3(x)$ by

$$R_3(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \frac{f'''(a)}{3!}(x-a)^3$$

For any t in the interval $[a, x]$ we define

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - R_3(x) \frac{(x-t)^4}{(x-a)^4}$$

Next we differentiate with respect to t using the Product and Power Rules:

$$\begin{aligned} g'(t) &= 0 - f'(t) - [-f''(t) + f''(t)(x-t)] - \frac{1}{2!}[-2f'''(t)(x-t) + f'''(t)(x-t)^2] \\ &\quad - \frac{1}{3!}[-3f^{(4)}(t)(x-t)^2 + f^{(4)}(t)(x-t)^3] + R_3(x) \frac{4(x-t)^3}{(x-a)^4} \\ &= -\frac{f^{(4)}(t)(x-t)^3}{3!} + 4R_3(x) \frac{(x-t)^3}{(x-a)^4} \end{aligned}$$

Since $g(x) = 0$, $g(a) = R_3(x) - R_3(x) = 0$, and $g(t)$ is continuous on $[a, x]$, we can apply the Mean Value Theorem for Derivatives. There exists, therefore, a number c between a and x such that $g'(c) = 0$. Thus,

$$0 = g'(c) = -\frac{f^{(4)}(c)(x-c)^3}{3!} + 4R_3(x) \frac{(x-c)^3}{(x-a)^4}$$

which leads to:

$$R_3(x) = \frac{f^{(4)}(c)}{4!}(x-a)^4$$

b. Like the previous part, first define $R_n(x)$ by

$$R_n(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \dots - \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For any t in the interval $[a, x]$ we define

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \dots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x) \frac{(x-t)^{n+1}}{(x-a)^{n+1}}$$

Next we differentiate with respect to t using the Product and Power Rules:

$$\begin{aligned}
g'(t) &= 0 - f'(t) - [-f'(t) + f''(t)(x-t)] - \frac{1}{2!}[-2f''(t)(x-t) + f'''(t)(x-t)^2] - \dots \\
&\quad - \frac{1}{n!}[-nf^{(n)}(t)(x-t)^{n-1} + f^{(n+1)}(t)(x-t)^n] + R_n(x) \frac{(n+1)(x-t)^n}{(x-a)^{n+1}} \\
&= -\frac{f^{(n+1)}(t)(x-t)^n}{n!} + (n+1)R_n(x) \frac{(x-t)^n}{(x-a)^{n+1}}
\end{aligned}$$

Since $g(x) = 0$, $g(a) = R_n(x) - R_n(x) = 0$, and $g(t)$ is continuous on $[a, x]$, we can apply the Mean Value Theorem for Derivatives. There exists, therefore, a number c between a and x such that $g'(c) = 0$. Thus,

$$0 = g'(c) = -\frac{f^{(n+1)}(c)(x-c)^n}{n!} + (n+1)R_n(x) \frac{(x-c)^n}{(x-a)^{n+1}}$$

which leads to:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

38. a. For $\sum_{n=1}^{\infty} \left(\frac{p}{n}\right) x^n$, $\rho = \lim_{n \rightarrow \infty} \left| \frac{\left(\frac{p}{n+1}\right) x^{n+1}}{\left(\frac{p}{n}\right) x^n} \right| = \lim_{n \rightarrow \infty} |x| \left| \frac{p(p-1)\dots(p-n+1)(p-n)}{(n+1)!} \div \frac{p(p-1)\dots(p-n+1)}{n!} \right|$

$$= \lim_{n \rightarrow \infty} |x| \left| \frac{p-n}{n+1} \right| = |x|$$

Thus $f(x) = 1 + \sum_{n=1}^{\infty} \left(\frac{p}{n}\right) x^n$ converges for $|x| < 1$.

b. It is clear that $f(0) = 1$.

Since $f(x) = 1 + \sum_{n=1}^{\infty} \left(\frac{p}{n}\right) x^n$, $f'(x) = \sum_{n=1}^{\infty} n \left(\frac{p}{n}\right) x^{n-1}$ and

$$\begin{aligned}
(x+1)f'(x) &= \sum_{n=1}^{\infty} n(x+1) \left(\frac{p}{n}\right) x^{n-1} = \sum_{n=1}^{\infty} \left[nx^n \left(\frac{p}{n}\right) + n \left(\frac{p}{n}\right) x^{n-1} \right] = 1 \cdot \left(\frac{p}{1}\right) x^0 + \sum_{n=1}^{\infty} \left[n \left(\frac{p}{n}\right) + (n+1) \left(\frac{p}{n+1}\right) \right] x^n \\
n \left(\frac{p}{n}\right) + (n+1) \left(\frac{p}{n+1}\right) &= n \frac{p(p-1)\dots(p-n+1)}{n!} + (n+1) \frac{p(p-1)\dots(p-n+1)(p-n)}{(n+1)!} \\
&= \frac{1}{n!} [np(p-1)\dots(p-n+1) + p(p-1)\dots(p-n+1)(p-n)] = \frac{p(p-1)\dots(p-n+1)}{n!} [n+p-n] = \left(\frac{p}{n}\right) p
\end{aligned}$$

and since $\left(\frac{p}{1}\right) = p$, $(1+x)f'(x) = p + \sum_{n=1}^{\infty} p \left(\frac{p}{n}\right) x^n = pf(x)$.

c. Let $y = f(x)$, then the differential equation is $(1+x)y' = py$ or $\frac{y'}{y} = \frac{p}{1+x}$.

$$\int \frac{dy}{y} = \int \frac{p}{1+x} dx \Rightarrow \ln|y| = p \ln|1+x| + C_1 \text{ or } y = C(1+x)^p \text{ so } f(x) = C(1+x)^p.$$

Since $f(0) = C(1)^p = C$ and $f(0) = 1$, $C = 1$ and $f(x) = (1+x)^p$.

39. $f'(t) = \begin{cases} 0 & \text{if } t < 0 \\ 4t^3 & \text{if } t \geq 0 \end{cases}$

$$f''(t) = \begin{cases} 0 & \text{if } t < 0 \\ 12t^2 & \text{if } t \geq 0 \end{cases}$$

$$f''(t) = \begin{cases} 0 & \text{if } t < 0 \\ 24t & \text{if } t \geq 0 \end{cases}$$

$$f^{(4)}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 24 & \text{if } t \geq 0 \end{cases}$$

$\lim_{t \rightarrow 0^+} f^{(4)}(t) = 24$ while $\lim_{t \rightarrow 0^-} f^{(4)}(t) = 0$, thus

$f^{(4)}(0)$ does not exist, and $f(t)$ cannot be

represented by a Maclaurin series.

Suppose that $g(t)$ as described in the text is represented by a Maclaurin series, so

$$g(t) = a_0 + a_1 t + a_2 t^2 + \dots = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n \text{ for all}$$

t in $(-R, R)$ for some $R > 0$. It is clear that, for $t \leq 0$, $g(t)$ is represented by

$g(t) = 0 + 0t + 0t^2 + \dots$. However, this will not represent $g(t)$ for any $t > 0$ since the car is moving for $t > 0$. Similarly, any series that represents $g(t)$ for $t > 0$ cannot be 0 everywhere, so it will not represent $g(t)$ for $t < 0$. Thus, $g(t)$ cannot be represented by a Maclaurin series.

$$\begin{aligned} 40. \text{ a. } f'(0) &= \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{h}}{e^{1/h^2}} \\ &= \lim_{h \rightarrow 0} \frac{h e^{-1/h^2}}{2} = 0 \text{ (by l'Hôpital's Rule)} \end{aligned}$$

$$\text{b. } f'(x) = \begin{cases} 2x^{-3} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

so

$$f''(0) = \lim_{h \rightarrow 0} \frac{2e^{-1/h^2}}{h^4} = \lim_{h \rightarrow 0} \frac{\frac{2}{h^4}}{e^{1/h^2}} = \lim_{h \rightarrow 0} \frac{\frac{4}{h^5}}{e^{1/h^2}}$$

$$\begin{aligned} 44. \exp(x^2) &= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \dots \\ \exp(x^2) &= 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{6} + \dots = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \dots \end{aligned}$$

$$45. \sin(\exp x - 1) = x + \frac{x^2}{2} - \frac{5x^4}{24} - \frac{23x^5}{120} - \dots$$

$$\exp x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

$$\begin{aligned} \sin(\exp x - 1) &= \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots \right) - \frac{1}{6} \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \right)^3 + \frac{1}{120} \left(x + \frac{x^2}{2} + \dots \right)^5 - \dots \\ &= \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots \right) - \frac{1}{6} \left(x^3 + \frac{3x^4}{2} + \frac{5x^5}{4} + \dots \right) + \frac{1}{120} (x^5 + \dots) - \dots \\ &= \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots \right) - \left(\frac{x^3}{6} + \frac{x^4}{4} + \frac{5x^5}{24} + \dots \right) + \left(\frac{x^5}{120} + \dots \right) - \dots = x + \frac{x^2}{2} - \frac{5x^4}{24} - \frac{23x^5}{120} - \dots \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{4}{e^{1/h^2}} = 0 \text{ (by using l'Hôpital's Rule twice)}$$

c. If $f^{(n)}(0) = 0$ for all n , then the Maclaurin series for $f(x)$ is 0.

d. No, $f(x) \neq 0$ for $x \neq 0$. It only represents $f(x)$ at $x = 0$.

e. Note that for any n and $x \neq 0$,

$$R(x) = e^{-1/x^2}.$$

$$41. \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots$$

$$42. \exp x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$43. 3 \sin x - 2 \exp x = -2 + x - x^2 - \frac{5x^3}{6} - \dots$$

$$3 \sin x = 3x - \frac{x^3}{2} + \frac{x^5}{40} - \frac{x^7}{1680} + \dots$$

$$-2 \exp x = -2 - 2x - x^2 - \frac{x^3}{3} - \dots$$

$$\text{Thus, } 3 \sin x - 2 \exp x = -2 + x - x^2 - \frac{5x^3}{6} - \dots$$

$$46. \exp(\sin x) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \dots$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$$

$$\begin{aligned} \exp(\sin x) &= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) + \frac{1}{2} \left(x - \frac{x^3}{6} + \dots\right)^2 + \frac{1}{6} \left(x - \frac{x^3}{6} + \dots\right)^3 + \frac{1}{24} \left(x - \frac{x^3}{6} + \dots\right)^4 + \dots \\ &= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) + \frac{1}{2} \left(x^2 - \frac{x^4}{3} + \dots\right) + \frac{1}{6} \left(x^3 - \frac{x^5}{2} + \dots\right) + \frac{1}{24} \left(x^4 - \frac{2x^6}{3} + \dots\right) + \dots \\ &= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) + \left(\frac{x^2}{2} - \frac{x^4}{6} + \dots\right) + \left(\frac{x^3}{6} - \frac{x^5}{12} + \dots\right) + \left(\frac{x^4}{24} - \frac{x^6}{36} + \dots\right) + \dots = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \dots \end{aligned}$$

$$47. (\sin x)(\exp x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \dots$$

$$\begin{aligned} (\sin x)(\exp x) &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right) \\ &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \dots\right) + \left(x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \dots\right) + \left(\frac{x^3}{2} - \frac{x^5}{12} + \dots\right) + \left(\frac{x^4}{6} - \frac{x^6}{36} + \dots\right) + \left(\frac{x^5}{24} - \frac{x^7}{144} + \dots\right) + \dots \\ &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \dots \end{aligned}$$

$$48. \frac{\sin x}{\exp x} = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \dots$$

$$\frac{\sin x}{\exp x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots}{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots} = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \dots$$

10.9 Chapter Review

Concepts Test

1. False: If $b_n = 100$ and $a_n = 50 + (-1)^n$ then

$$\text{since } a_n = \begin{cases} 51 & \text{if } n \text{ is even} \\ 49 & \text{if } n \text{ is odd} \end{cases},$$

$$0 \leq a_n \leq b_n \text{ for all } n \text{ and}$$

$\lim_{n \rightarrow \infty} b_n = 100$ while $\lim_{n \rightarrow \infty} a_n$ does not exist.

2. True: It is clear that $n! \leq n^n$. The inequality $n! \leq n^n \leq (2n-1)!$ is equivalent to

$$1 \leq \frac{n^n}{n!} \leq \frac{(2n-1)!}{n!}.$$

Expanding the terms gives

$$\begin{aligned} \frac{n^n}{n!} &= \frac{n}{1} \cdot \frac{n}{2} \cdot \frac{n}{3} \cdots \frac{n}{n-1} \cdot \frac{n}{n} \\ &\leq (n+1)(n+2) \cdots (n+n-1) \text{ or} \\ n \cdot \frac{n}{2} \cdot \frac{n}{3} \cdots \frac{n}{n-1} &\leq (n+1)(n+2) \cdots (n+n-1) \end{aligned}$$

The left-hand side consists of $n-1$ terms, each of which is less than or equal to n , while the right-hand side consists of $n-1$ terms, each of which is greater than n . Thus, the inequality is true so $n! \leq n^n \leq (2n-1)!$

3. True: If $\lim_{n \rightarrow \infty} a_n = L$ then for any $\varepsilon > 0$

there is a number $M > 0$ such that $|a_n - L| < \varepsilon$ for all $n \geq M$. Thus, for the same ε , $|a_{3n+4} - L| < \varepsilon$ for

- $3n + 4 \geq M$ or $n \geq \frac{M-4}{3}$. Since ε was arbitrary, $\lim_{n \rightarrow \infty} a_{3n+4} = L$.
4. False: Suppose $a_n = 1$ if $n = 2k$ or $n = 3k$ where k is any positive integer and $a_n = 0$ if n is not a multiple of 2 or 3. Then $\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} a_{3n} = 1$ but $\lim_{n \rightarrow \infty} a_n$ does not exist.
5. False: Let a_n be given by

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{if } n \text{ is composite} \end{cases}$$
 Then $a_{mn} = 0$ for all mn since $m \geq 2$, hence $\lim_{n \rightarrow \infty} a_{mn} = 0$ for $m \geq 2$. $\lim_{n \rightarrow \infty} a_n$ does not exist since for any $M > 0$ there will be a_n 's with $a_n = 1$ since there are infinitely many prime numbers.
6. True: Given $\varepsilon > 0$ there are numbers M_1 and M_2 such that $|a_{2n} - L| < \varepsilon$ when $n \geq M_1$ and $|a_{2n+1} - L| < \varepsilon$ when $n \geq M_2$. Let $M = \max\{M_1, M_2\}$, then when $n \geq M$ we have $|a_{2n} - L| < \varepsilon$ and $|a_{2n+1} - L| < \varepsilon$ so $|a_n - L| < \varepsilon$ for all $n \geq 2M + 1$ since every $k \geq 2M + 1$ is either even ($k = 2n$) or odd ($k = 2n + 1$).
7. False: Let $a_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$. Then

$$a_n - a_{n+1} = -\frac{1}{n+1} \text{ so } \lim_{n \rightarrow \infty} (a_n - a_{n+1}) = 0 \text{ but } \lim_{n \rightarrow \infty} a_n \text{ is not finite since } \lim_{n \rightarrow \infty} a_n = \sum_{k=1}^{\infty} \frac{1}{k},$$
 which diverges.
8. False: $\{(-1)^n\}$ and $\{(-1)^{n+1}\}$ both diverge but

$$\{(-1)^n + (-1)^{n+1}\} = \{(-1)^n(1-1)\} = \{0\}$$
 converges.
9. True: If $\{a_n\}$ converges, then for some N , there are numbers m and M with $m \leq a_n \leq M$ for all $n \geq N$. Thus

$$\frac{m}{n} \leq \frac{a_n}{n} \leq \frac{M}{n} \text{ for all } n \geq N.$$
 Since $\left\{\frac{m}{n}\right\}$ and $\left\{\frac{M}{n}\right\}$ both converge to 0, $\left\{\frac{a_n}{n}\right\}$ must also converge to 0.
10. False: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges.

$$a_n = (-1)^n \frac{1}{\sqrt{n}} \text{ so } a_n^2 = \frac{1}{n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$
11. True: The series converges by the Alternating Series Test.

$$S_1 = a_1, S_2 = a_1 - a_2, S_3 = a_1 - a_2 + a_3, S_4 = a_1 - a_2 + a_3 - a_4, \text{ etc.}$$

$$0 < a_2 < a_1 \Rightarrow 0 < a_1 - a_2 = S_2 < a_1;$$

$$0 < a_3 < a_2 \Rightarrow -a_2 < -a_2 + a_3 < 0 \text{ so } 0 < a_1 - a_2 < a_1 - a_2 + a_3 = S_3 < a_1;$$

$$0 < a_4 < a_3 \Rightarrow 0 < a_3 - a_4 < a_3, \text{ so } -a_2 < -a_2 + a_3 - a_4 < -a_2 + a_3 < 0,$$
 hence

$$0 < a_1 - a_2 < a_1 - a_2 + a_3 - a_4 = S_4 < a_1 - a_2 + a_3 < a_1; \text{ etc.}$$
 For each even n , $0 < S_{n-1} - a_n$ while for each odd n , $n > 1$, $S_{n-1} + a_n < a_1$.
12. True: For $n \geq 2$, $\frac{1}{n} \leq \frac{1}{2}$ so

$$\sum_{n=2}^{\infty} \left(\frac{1}{n}\right)^n \leq \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n \text{ which converges since it is a geometric series with } r = \frac{1}{2}.$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n = 1 + \frac{1}{4} + \frac{1}{27} + \dots > 1 \text{ since all terms are positive.}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n = 1 + \sum_{n=2}^{\infty} \left(\frac{1}{n}\right)^n \leq 1 + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{4} + \frac{1}{8} + \dots = -\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = -\frac{1}{2} + \frac{1}{1-\frac{1}{2}}$$

$$= -\frac{1}{2} + 2 = \frac{3}{2}$$

$$\text{Thus, } \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n = S \text{ with}$$

$$1 < S \leq \frac{3}{2} < 2.$$

13. False: $\sum_{n=1}^{\infty} (-1)^n$ diverges but the partial sums are bounded ($S_n = -1$ for odd n and $S_n = 0$ for even n .)

14. False: $0 < \frac{1}{n^2} \leq \frac{1}{n}$ for all n in $\mathbb{N}$ but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges while $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

15. True: $\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$. Ratio Test is inconclusive. (See the discussion before Example 5 in Section 10.4.)

16. False: $\frac{1}{n^2} > 0$ for all n in $\mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = 1$.

17. False: $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} \neq 0$ so the series cannot converge.

18. False: Since $\lim_{n \rightarrow \infty} \frac{n^4 + 1}{e^n} = 0$, there is some number M such that $e^n > n^4 + 1$ for all $n \geq M$, thus $n > \ln(n^4 + 1)$ and $\frac{1}{n} < \frac{1}{\ln(n^4 + 1)}$ for $n \geq M$. Hence, $\sum_{n=M}^{\infty} \frac{1}{n} < \sum_{n=M}^{\infty} \frac{1}{\ln(n^4 + 1)}$ and so $\sum_{n=1}^{\infty} \frac{1}{\ln(n^4 + 1)}$ diverges by the Comparison Test.

19. True: $\sum_{n=2}^{\infty} \frac{n+1}{(n \ln n)^2}$

$$= \sum_{n=2}^{\infty} \left[\frac{n}{(n \ln n)^2} + \frac{1}{(n \ln n)^2} \right]$$

$$= \sum_{n=2}^{\infty} \left[\frac{1}{n(\ln n)^2} + \frac{1}{n^2(\ln n)^2} \right]$$
 $\frac{1}{x(\ln x)^2}$ is continuous, positive, and nonincreasing on $[2, \infty)$. Using $u = \ln x$, $du = \frac{1}{x} dx$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \int_{\ln 2}^{\infty} \frac{1}{u^2} du$$

$$= \left[-\frac{1}{u} \right]_{\ln 2}^{\infty} = 0 + \frac{1}{\ln 2} < \infty$$
 so $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.
 For $n \geq 3$, $\ln n > 1$, so $(\ln n)^2 > 1$ and $\frac{1}{n^2(\ln n)^2} < \frac{1}{n^2}$. Thus $\sum_{n=3}^{\infty} \frac{1}{n^2(\ln n)^2} < \sum_{n=3}^{\infty} \frac{1}{n^2}$ so $\sum_{n=3}^{\infty} \frac{1}{n^2(\ln n)^2}$ converges by the Comparison Test. Since both series converge, so does their sum.

20. False: This series is $\sum_{n=1}^{\infty} \frac{1}{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \dots$ which diverges.

21. True: If $0 \leq a_{n+100} \leq b_n$ for all n in $\mathbb{N}$, then $\sum_{n=101}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$ so $\sum_{n=1}^{\infty} a_n$ also converges, since adding a finite number of terms does not affect the convergence or divergence of a series.

22. True: If $ca_n \geq \frac{1}{n}$ for all n in $\mathbb{N}$ with $c > 0$, then $a_n \geq \frac{1}{cn}$ for all n in $\mathbb{N}$ so $\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} \frac{1}{cn} = \frac{1}{c} \sum_{n=1}^{\infty} \frac{1}{n}$ which

diverges. Thus, $\sum_{n=1}^{\infty} a_n$ diverges by the Comparison Test.

23. True: $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \dots = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$
 $= \sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{1}{3}\right)^{n-1} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$, so the sum of the first thousand terms is less than $\frac{1}{2}$.

24. False: Consider the series with $a_n = \frac{(-1)^{n+1}}{n}$. Then $(-1)^n a_n = \frac{(-1)^{2n+1}}{n} = \frac{-1}{n}$ so $\sum_{n=1}^{\infty} (-1)^n a_n = -1 - \frac{1}{2} - \frac{1}{3} - \dots$ which diverges.

25. True: If $b_n \leq a_n \leq 0$ for all n in $\mathbb{N}$ then $0 \leq -a_n \leq -b_n$ for all n in $\mathbb{N}$.
 $\sum_{n=1}^{\infty} -b_n = (-1) \sum_{n=1}^{\infty} b_n$ which converges since $\sum_{n=1}^{\infty} b_n$ converges.
 Thus, by the Comparison Test, $\sum_{n=1}^{\infty} -a_n$ converges, hence $\sum_{n=1}^{\infty} a_n = (-1) \sum_{n=1}^{\infty} (-a_n)$ also converges.

26. True Since $a_n \geq 0$ for all n , $\sum_{n=1}^{\infty} |(-1)^n a_n| = \sum_{n=1}^{\infty} a_n$ so the series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely.

27. True: $\left| \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} - \sum_{n=1}^{99} (-1)^{n+1} \frac{1}{n} \right|$
 $= \left| -\frac{1}{100} + \frac{1}{101} - \frac{1}{102} + \dots \right| < \frac{1}{100} = 0.01$

28. True: Suppose $\sum |a_n|$ converges. Thus, $\sum 2|a_n|$ converges, so $\sum (|a_n| + a_n)$ converges since $0 \leq |a_n| + a_n \leq 2|a_n|$. But by the linearity of convergent series $\sum a_n = \sum (|a_n| + a_n) - \sum |a_n|$ converges, which is a contradiction.

29. True: $|3 - (-1.1)| = 4.1$, so the radius of convergence of the series is at least 4.1.
 $|3 - 7| = 4 < 4.1$ so $x = 7$ is within the interval of convergence.

30. False: If the radius of convergence is 2, then the convergence at $x = 2$ is independent of the convergence at $x = -2$.

31. True: The radius of convergence is at least 1.5, so 1 is within the interval of convergence.

Thus $\int_0^1 f(x) dx = \left[\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \right]_0^1$
 $= \sum_{n=0}^{\infty} \frac{a_n}{n+1}$

32. False: The convergence set of a power series may consist of a single point.

33. False: Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The Maclaurin series for this function represents the function only at $x = 0$. (See Problem 40 of Section 10.8.)

34. True: On $(-1, 1)$, $f(x) = \frac{1}{1-x}$.

$$f'(x) = \frac{1}{(1-x)^2} = [f(x)]^2$$

35. True: $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = e^{-x}$, $\frac{d}{dx} e^{-x} + e^{-x} = 0$

Sample Test Problems

1. $\lim_{n \rightarrow \infty} \frac{9n}{\sqrt{9n^2 + 1}} = \lim_{n \rightarrow \infty} \frac{9}{\sqrt{9 + \frac{1}{n^2}}} = 3$

The sequence converges to 3.

2. Using l'Hôpital's Rule,

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{2\sqrt{n}}{n} \\ &= \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n}} = 0.\end{aligned}$$

$$3. \lim_{n \rightarrow \infty} \left(1 + \frac{4}{n}\right)^n = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{4}{n}\right)^{n/4}\right]^4 = e^4$$

The sequence converges to e^4 .

$$4. a_{n+1} = \frac{n+1}{3}a_n \text{ thus for } n > 3, \text{ since } \frac{n+1}{3} > 1, \\ a_{n+1} > a_n \text{ and the sequence diverges.}$$

$$5. \text{ Let } y = \sqrt[n]{n} = n^{1/n} \text{ then } \ln y = \frac{1}{n} \ln n.$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \text{ by}$$

using l'Hôpital's Rule. Thus,

$\lim_{n \rightarrow \infty} \sqrt[n]{n} = \lim_{n \rightarrow \infty} e^{\ln y} = 1$. The sequence converges to 1.

$$6. \lim_{n \rightarrow \infty} \frac{1}{\sqrt[3]{n}} = 0 \text{ while } \frac{1}{\sqrt[3]{3}} = \left(\frac{1}{3}\right)^{1/3}. \text{ As } n \rightarrow \infty, \\ \frac{1}{n} \rightarrow 0 \text{ so}$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{3}\right)^{1/n} = \lim_{\frac{1}{n} \rightarrow 0} \left(\frac{1}{3}\right)^{1/n} = \left(\frac{1}{3}\right)^0 = 1.$$

The sequence converges to 1.

$$7. a_n \geq 0; \lim_{n \rightarrow \infty} \frac{\sin^2 n}{\sqrt{n}} \leq \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

The sequence converges to 0.

8. The sequence does not converge, since whenever n is an even multiple of 6, $a_n = 1$, while whenever n is an odd multiple of 6, $a_n = -1$.

$$9. S_n = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \dots + \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}, \text{ so}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{\sqrt{n+1}}\right) = 1. \text{ The series converges to 1.}$$

$$10. S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}, \text{ so}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}. \text{ The series converges to } \frac{3}{2}.$$

$$11. \ln \frac{1}{2} + \ln \frac{2}{3} + \ln \frac{3}{4} + \dots = \sum_{n=1}^{\infty} \ln \frac{n}{n+1} = \sum_{n=1}^{\infty} [\ln n - \ln(n+1)]$$

$$S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \dots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) = \ln 1 - \ln(n+1) = \ln \frac{1}{n+1}$$

$$\text{As } n \rightarrow \infty, \frac{1}{n+1} \rightarrow 0 \text{ so } \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \ln \frac{1}{n+1} = -\infty.$$

The series diverges.

$$12. \cos k\pi = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases} \text{ so} \\ \sum_{k=0}^{\infty} \cos k\pi = \sum_{k=0}^{\infty} (-1)^k \text{ which diverges since}$$

$$S_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \text{ so } \{S_n\} \text{ does not} \\ \text{converge.}$$

$$13. \sum_{k=0}^{\infty} e^{-2k} = \sum_{k=0}^{\infty} \left(\frac{1}{e^2}\right)^k = \frac{1}{1 - \frac{1}{e^2}} = \frac{e^2}{e^2 - 1} \approx 1.1565$$

$$\text{since } \frac{1}{e^2} < 1.$$

$$14. \sum_{k=0}^{\infty} \frac{3}{2^k} = 3 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{3}{1-\frac{1}{2}} = 6$$

$$\sum_{k=0}^{\infty} \frac{4}{3^k} = 4 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{4}{1-\frac{1}{3}} = 6$$

Since both series converge, their sum converges to $6 + 6 = 12$.

$$15. \sum_{k=1}^{\infty} 91 \left(\frac{1}{100}\right)^k = \frac{91}{1-\frac{1}{100}} - 91 = \frac{9100}{99} - 91 = \frac{91}{99}$$

The series converges since $\left|\frac{1}{100}\right| < 1$.

$$16. \sum_{k=1}^{\infty} \left(\frac{1}{\ln 2}\right)^k \text{ diverges since } \left|\frac{1}{\ln 2}\right| > 1.$$

$$17. \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \text{ so}$$

$$1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \dots \text{ converges to}$$

$$\cos 2 \approx -0.41615.$$

$$18. e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \text{ so}$$

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots = e^{-1} \approx 0.3679.$$

$$19. \text{ Let } a_n = \frac{n}{1+n^2} \text{ and } b_n = \frac{1}{n}.$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{1+n^2} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n^2} + 1} = 1;$$

$$0 < 1 < \infty.$$

By the Limit Comparison Test, since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{1+n^2} \text{ also}$$

diverges.

$$20. \text{ Let } a_n = \frac{n+5}{1+n^3} \text{ and } b_n = \frac{1}{n^2}.$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^3 + 5n^2}{1+n^3} = \lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n}}{\frac{1}{n^3} + 1} = 1;$$

$$0 < 1 < \infty.$$

By the Limit Comparison Test, since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges, } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+5}{1+n^3}$$

also converges.

$$21. \text{ Since the series alternates, } \frac{1}{\sqrt[3]{n}} > \frac{1}{\sqrt[3]{n+1}} > 0, \text{ and}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt[3]{n}} = 0, \text{ the series converges by the}$$

Alternating Series Test.

$$22. \text{ The series diverges since}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt[4]{3}} = \lim_{n \rightarrow \infty} 3^{-1/n} = 1.$$

$$23. \sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n \right)$$

$$= \left(\frac{1}{1-\frac{1}{2}} - 1 \right) + \left(\frac{1}{1-\frac{3}{4}} - 1 \right) = 1 + 3 = 4$$

The series converges to 4. The 1's must be subtracted since the index starts with $n = 1$.

$$24. \rho = \lim_{n \rightarrow \infty} \left(\frac{n+1}{e^{(n+1)^2}} \div \frac{n}{e^{n^2}} \right) = \lim_{n \rightarrow \infty} \left(\frac{n+1}{ne^{2n+1}} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1 + \frac{1}{n}}{e^{2n+1}} \right) = 0 < 1. \text{ so the series converges.}$$

$$25. \lim_{n \rightarrow \infty} \frac{n+1}{10n+12} = \frac{1}{10} \neq 0, \text{ so the series diverges.}$$

$$26. \text{ Let } a_n = \frac{\sqrt{n}}{n^2+7} \text{ and } b_n = \frac{1}{n^{3/2}}.$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+7} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{7}{n^2}} = 1;$$

$$0 < 1 < \infty.$$

By the Limit Comparison Test, since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges } \left(\frac{3}{2} > 1 \right),$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+7} \text{ also converges.}$$

$$27. \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(n+1)!} \div \frac{n^2}{n!} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{n^2} \right| = 0 < 1, \text{ so}$$

the series converges.

$$28. \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3 3^{n+1}}{(n+2)!} \div \frac{n^3 3^n}{(n+1)!} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{3(n+1)^3}{n^3(n+2)} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{3}{n} \left(1 + \frac{1}{n}\right)^3}{1 + \frac{2}{n}} \right| = 0 < 1$$

The series converges.

$$29. \rho = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}(n+1)!}{(n+3)!} \div \frac{2^n n!}{(n+2)!} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{2(n+1)}{n+3} \right| = 2 > 1$$

The series diverges.

$$30. \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right)^n = \frac{1}{e} \neq 0, \text{ so the series does not converge.}$$

$$31. \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 \left(\frac{2}{3} \right)^{n+1}}{n^2 \left(\frac{2}{3} \right)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2}{3} \right| \left| \frac{(n+1)^2}{n^2} \right|$$

$$= \frac{2}{3} < 1, \text{ so the series converges.}$$

$$32. \text{ Since the series alternates, } \frac{1}{1 + \ln n} > \frac{1}{1 + \ln(n+1)},$$

and $\lim_{n \rightarrow \infty} \frac{1}{1 + \ln n} = 0$, the series converges by the Alternating Series Test.

$$33. a_n = \frac{1}{3n-1}; \frac{1}{3n-1} > \frac{1}{3n+2} \text{ so } a_n > a_{n+1};$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{3n-1} = 0, \text{ so the series}$$

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{3n-1}$ converges by the Alternating Series Test.

$$\text{Let } b_n = \frac{1}{n}, \text{ then}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{3n-1} = \lim_{n \rightarrow \infty} \frac{1}{3 - \frac{1}{n}} = \frac{1}{3};$$

$0 < \frac{1}{3} < \infty$. By the Limit Comparison Test, since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{3n-1} \text{ also diverges.}$$

The series is conditionally convergent.

$$34. \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{2^{n+1}} \div \frac{n^3}{2^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{2n^3} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{1}{n} \right)^3}{2} \right| = \frac{1}{2} < 1$$

The series is absolutely convergent.

$$35. \frac{3^n}{2^{n+8}} = \frac{1}{2^8} \left(\frac{3}{2} \right)^n;$$

$$\lim_{n \rightarrow \infty} \frac{1}{2^8} \left(\frac{3}{2} \right)^n = \frac{1}{2^8} \lim_{n \rightarrow \infty} \left(\frac{3}{2} \right)^n = \infty \text{ since } \frac{3}{2} > 1.$$

The series is divergent.

$$36. \text{ Let } f(x) = \frac{\sqrt[3]{x}}{\ln x}, \text{ then}$$

$$f'(x) = \frac{1}{(\ln x)^2} \left[\frac{x^{1/3}}{x^2} (1 - \ln x) \ln x - \frac{x^{1/3}}{x} \right]$$

$$= \frac{x^{1/3}}{(x \ln x)^2} [\ln x - (\ln x)^2 - x], \text{ for } x \geq 3, \ln x > 1$$

so $(\ln x)^2 > \ln x$ hence $f(x)$ is decreasing on

$[3, \infty)$. Thus, if $a_n = \frac{\sqrt[3]{n}}{\ln n}$, $a_n > a_{n+1}$.

$$\text{Let } y = \sqrt[3]{n} = n^{1/3}, \text{ so } \ln y = \frac{1}{3} \ln n.$$

Using l'Hôpital's Rule,

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \text{ thus}$$

$$\lim_{n \rightarrow \infty} \sqrt[3]{n} = \lim_{n \rightarrow \infty} e^{\ln y} = e^0 = 1. \text{ Hence, } \lim_{n \rightarrow \infty} \frac{\sqrt[3]{n}}{\ln n}$$

is of the form $\frac{0}{0}$ so $\lim_{n \rightarrow \infty} \frac{\sqrt[3]{n}}{\ln n} = 0$.

Thus, by the Alternating Series Test,

$$\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt[3]{n}}{\ln n} \text{ converges.}$$

$$\ln n < n, \text{ so } \frac{1}{\ln n} > \frac{1}{n} \text{ hence } \sum_{n=2}^{\infty} \frac{\sqrt[3]{n}}{n} < \sum_{n=2}^{\infty} \frac{\sqrt[3]{n}}{\ln n}.$$

$$\text{Thus if } \sum_{n=2}^{\infty} \frac{\sqrt[3]{n}}{n} \text{ diverges, } \sum_{n=2}^{\infty} \frac{\sqrt[3]{n}}{\ln n} \text{ also diverges.}$$

$$\text{Let } a_n = \frac{\sqrt[3]{n}}{n} \text{ and } b_n = \frac{1}{n}. \text{ Then}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \sqrt[3]{n} = 1 \text{ as shown above:}$$

$$0 < 1 < \infty. \text{ Since } \sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges,}$$

$$\sum_{n=2}^{\infty} \frac{\sqrt[3]{n}}{n} \text{ also diverges by the Limit Comparison}$$

$$\text{Test, hence } \sum_{n=2}^{\infty} \frac{\sqrt[3]{n}}{\ln n} \text{ also diverges.}$$

The series is conditionally convergent.

$$37. \rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)^3 + 1} \div \frac{x^n}{n^3 + 1} \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| \frac{n^3 + 1}{(n+1)^3 + 1} \right| = |x|$$

When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{1}{n^3 + 1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \leq 1 + \sum_{n=1}^{\infty} \frac{1}{n^3}, \text{ which}$$

converges.

When $x = -1$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3 + 1}$ which

converges absolutely since $\sum_{n=0}^{\infty} \frac{1}{n^3 + 1}$ converges.

The series converges on $-1 \leq x \leq 1$.

$$38. \rho = \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+2} x^{n+1}}{2n+5} \div \frac{(-2)^{n+1} x^n}{2n+3} \right|$$

$$= \lim_{n \rightarrow \infty} |2x| \left| \frac{2n+3}{2n+5} \right| = |2x|; |2x| < 1 \text{ when}$$

$$-\frac{1}{2} < x < \frac{1}{2}.$$

When $x = \frac{1}{2}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1} \left(\frac{1}{2}\right)^n}{2n+3} = \sum_{n=0}^{\infty} \frac{(-2) \left(\frac{-2}{2}\right)^n}{2n+3}$$

$$= \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{2n+3} \cdot a_n = \frac{2}{2n+3}; \frac{2}{2n+3} > \frac{2}{2n+5}, \text{ so}$$

$$a_n > a_{n+1}; \lim_{n \rightarrow \infty} \frac{2}{2n+3} = 0 \text{ so } \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{2n+3}$$

converges by the Alternating Series Test.

When $x = -\frac{1}{2}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1} \left(-\frac{1}{2}\right)^n}{2n+3} = \sum_{n=0}^{\infty} \frac{(-2) \left(\frac{-2}{2}\right)^n}{2n+3} = -\sum_{n=0}^{\infty} \frac{2}{2n+3}$$

$$a_n = \frac{2}{2n+3}, \text{ let } b_n = \frac{1}{n} \text{ then}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2n}{2n+3} = \lim_{n \rightarrow \infty} \frac{2}{2 + \frac{3}{n}} = 1;$$

$0 < 1 < \infty$ hence since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\sum_{n=0}^{\infty} \frac{2}{2n+3} \text{ and also } -\sum_{n=0}^{\infty} \frac{2}{2n+3} \text{ diverges.}$$

The series converges on $-\frac{1}{2} < x \leq \frac{1}{2}$.

$$39. \rho = \lim_{n \rightarrow \infty} \left| \frac{(x-4)^{n+1}}{n+2} \div \frac{(x-4)^n}{n+1} \right|$$

$$= \lim_{n \rightarrow \infty} |x-4| \left| \frac{n+1}{n+2} \right| = |x-4|; |x-4| < 1 \text{ when}$$

$$3 < x < 5.$$

When $x = 5$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n (1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

$$a_n = \frac{1}{n+1}; \frac{1}{n+1} > \frac{1}{n+2}, \text{ so } a_n > a_{n+1};$$

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 \text{ so } \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \text{ converges by the}$$

Alternating Series Test.

When $x = 3$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}. a_n = \frac{1}{n+1}, \text{ let}$$

$$b_n = \frac{1}{n} \text{ then } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1; 0 < 1 < \infty$$

hence since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=0}^{\infty} \frac{1}{n+1}$

also diverges.

The series converges on $3 < x \leq 5$.

$$40. \rho = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1} x^{3n+3}}{(3n+3)!} \div \frac{3^n x^{3n}}{(3n)!} \right|$$

$$= \lim_{n \rightarrow \infty} |3x^3| \left| \frac{1}{(3n+3)(3n+2)(3n+1)} \right| = 0$$

The series converges for all x .

$$41. \rho = \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{2^{n+1} + 1} \div \frac{(x-3)^n}{2^n + 1} \right|$$

$$= \lim_{n \rightarrow \infty} |x-3| \left| \frac{1 + \frac{1}{2^n}}{2 + \frac{1}{2^n}} \right| = \frac{|x-3|}{2}; \frac{|x-3|}{2} < 1$$

when $1 < x < 5$.

When $x = 5$, the series is

$$\sum_{n=0}^{\infty} \frac{2^n}{2^n + 1} = \sum_{n=0}^{\infty} \frac{1}{1 + \left(\frac{1}{2}\right)^n}; \lim_{n \rightarrow \infty} \frac{1}{1 + \left(\frac{1}{2}\right)^n} = 1 \neq 0$$

so the series diverges.

When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n + 1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + \left(\frac{1}{2}\right)^n}; \lim_{n \rightarrow \infty} \frac{1}{1 + \left(\frac{1}{2}\right)^n} = 1 \neq 0$$

so the series diverges.

The series converges on $1 < x < 5$.

$$42. \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x+1)^{n+1}}{3^{n+1}} \div \frac{n!(x+1)^n}{3^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x+1}{3} \right| |n+1| = \infty \text{ unless } x = -1.$$

When $x = -1$, the series is $\sum_{n=0}^{\infty} \frac{n!0^n}{3^n} = 1$ since

$0^0 = 1$. The series converges only for $x = -1$.

$$43. \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \text{ for } -1 < x < 1.$$

If $f(x) = \frac{1}{1+x}$, then $f'(x) = -\frac{1}{(1+x)^2}$. Thus,

differentiating the series for $\frac{1}{1+x}$ and

multiplying by -1 yields

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots. \text{ The series}$$

converges on $-1 < x < 1$.

$$44. \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \text{ for } -1 < x < 1. \text{ If}$$

$$f(x) = \frac{1}{1+x}, \text{ then } f''(x) = \frac{2}{(1+x)^3}.$$

Differentiating the series for $\frac{1}{1+x}$ twice and dividing by 2 gives

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{1}{2}(4 \cdot 3)x^2 - \frac{1}{2}(5 \cdot 4)x^3 + \dots$$

$$= 1 - 3x + 6x^2 - 10x^3 + \dots.$$

The series converges on $-1 < x < 1$.

$$45. \sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right)^2$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \dots$$

Since the series for $\sin x$ converges for all x , so does the series for $\sin^2 x$.

46. If $f(x) = e^x$, then $f^{(n)}(x) = e^x$. Thus,

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^3}{3!}(x-2)^3 + \frac{e^4}{4!}(x-2)^4 + \dots.$$

$$47. \sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \dots$$

Since the series for $\sin x$ and $\cos x$ converge for all x , so does the series for $\sin x + \cos x$.

$$48. \cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \dots$$

$$\text{Thus } \int_0^1 \cos x^2 dx = \left[x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \frac{x^{17}}{17 \cdot 8!} - \dots \right]_0^1 \approx 0.9045.$$

Four terms are required to compute this value correct to four decimal places.

$$49. \frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots$$

$$\text{Thus, } \int_0^{0.2} \frac{e^x - 1}{x} dx = \left[x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \dots \right]_0^{0.2} = 0.2 + \frac{0.2^2}{2 \cdot 2!} + \frac{0.2^3}{3 \cdot 3!} + \frac{0.2^4}{4 \cdot 4!} + \dots \approx 0.21046.$$

50. One million terms are needed to approximate the sum to within 0.001 since $\frac{1}{\sqrt{n+1}} < 0.001$ is equivalent to $999,999 < n$.

$$51. 1 - \frac{x^2}{2} \text{ is the Maclaurin polynomial of order 3 for } \cos x. \text{ so } |R_3(x)| = \left| \frac{\cos c}{4!} x^4 \right| \leq \frac{0.1^4}{4!} \approx 0.000004167.$$

52. a. From the Maclaurin series for $\frac{1}{1-x}$, we have $\frac{1}{1-x^3} = 1 + x^3 + x^6 + \dots$.

b. In Example 6 of Section 10.8 it is shown that $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots$ so

$$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \dots$$

c. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots$, so $e^{-x} - 1 + x = \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$.

d. Using division with the Maclaurin series for $\cos x$, we get $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \dots$.

$$\text{Thus, } x \sec x = x + \frac{x^3}{2} + \frac{5x^5}{4!} + \dots$$

$$\text{e. } e^{-x} \sin x = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) = x - x^2 + \frac{x^3}{3} - \dots$$

$$\text{f. } 1 + \sin x = 1 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\text{Using division, we get } \frac{1}{1 + \sin x} = 1 - x + x^2 - \dots$$

11.1 Concepts Review

1. $f(1)$; $f'(1)$; $f''(1)$
2. $\frac{f^{(6)}(0)}{6!}$
3. error of the method; error of calculation
4. increase; decrease

Problem Set 11.1

1. $f(x) = e^{2x}$ $f(0) = 1$
 $f'(x) = 2e^{2x}$ $f'(0) = 2$
 $f''(x) = 4e^{2x}$ $f''(0) = 4$
 $f^{(3)}(x) = 8e^{2x}$ $f^{(3)}(0) = 8$
 $f^{(4)}(x) = 16e^{2x}$ $f^{(4)}(0) = 16$
 $f(x) \approx 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$
 $f(0.12) \approx 1 + 2(0.12) + 2(0.12)^2 + \frac{4}{3}(0.12)^3 + \frac{2}{3}(0.12)^4 \approx 1.2712$
2. $f(x) = e^{-3x}$ $f(0) = 1$
 $f'(x) = -3e^{-3x}$ $f'(0) = -3$
 $f''(x) = 9e^{-3x}$ $f''(0) = 9$
 $f^{(3)}(x) = -27e^{-3x}$ $f^{(3)}(0) = -27$
 $f^{(4)}(x) = 81e^{-3x}$ $f^{(4)}(0) = 81$
 $f(x) \approx 1 - 3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \frac{81}{4!}x^4 = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{27}{8}x^4$
 $f(0.12) \approx 1 - 3(0.12) + \frac{9}{2}(0.12)^2 - \frac{9}{2}(0.12)^3 + \frac{27}{8}(0.12)^4 \approx 0.6977$
3. $f(x) = \sin 2x$ $f(0) = 0$
 $f'(x) = 2 \cos 2x$ $f'(0) = 2$
 $f''(x) = -4 \sin 2x$ $f''(0) = 0$
 $f^{(3)}(x) = -8 \cos 2x$ $f^{(3)}(0) = -8$
 $f^{(4)}(x) = 16 \sin 2x$ $f^{(4)}(0) = 0$

$$f(x) \approx 2x - \frac{8}{3!}x^3 = 2x - \frac{4}{3}x^3$$

$$f(0.12) \approx 2(0.12) - \frac{4}{3}(0.12)^3 \approx 0.2377$$

$$\begin{aligned}
 4. \quad f(x) &= \tan x & f(0) &= 0 \\
 f'(x) &= \sec^2 x & f'(0) &= 1 \\
 f''(x) &= 2 \sec^2 x \tan x & f''(0) &= 0 \\
 f^{(3)}(x) &= 2 \sec^4 x + 4 \sec^2 x \tan^2 x & f^{(3)}(0) &= 2 \\
 f^{(4)}(x) &= 16 \sec^4 x \tan x + 8 \sec^2 x \tan^3 x & f^{(4)}(0) &= 0 \\
 f(x) &\approx x + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3 \\
 f(0.12) &\approx 0.12 + \frac{1}{3}(0.12)^3 \approx 0.1206
 \end{aligned}$$

$$\begin{aligned}
 5. \quad f(x) &= \ln(1+x) & f(0) &= 0 \\
 f'(x) &= \frac{1}{1+x} & f'(0) &= 1 \\
 f''(x) &= -\frac{1}{(1+x)^2} & f''(0) &= -1 \\
 f^{(3)}(x) &= \frac{2}{(1+x)^3} & f^{(3)}(0) &= 2 \\
 f^{(4)}(x) &= -\frac{6}{(1+x)^4} & f^{(4)}(0) &= -6 \\
 f(x) &\approx x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 \\
 &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \\
 f(0.12) &\approx 0.12 - \frac{1}{2}(0.12)^2 + \frac{1}{3}(0.12)^3 - \frac{1}{4}(0.12)^4 \\
 &\approx 0.1133
 \end{aligned}$$

$$\begin{aligned}
 6. \quad f(x) &= \sqrt{1+x} & f(0) &= 1 \\
 f'(x) &= \frac{1}{2}(1+x)^{-1/2} & f'(0) &= \frac{1}{2} \\
 f''(x) &= -\frac{1}{4}(1+x)^{-3/2} & f''(0) &= -\frac{1}{4} \\
 f^{(3)}(x) &= \frac{3}{8}(1+x)^{-5/2} & f^{(3)}(0) &= \frac{3}{8} \\
 f^{(4)}(x) &= -\frac{15}{16}(1+x)^{-7/2} & f^{(4)}(0) &= -\frac{15}{16} \\
 f(x) &\approx 1 + \frac{1}{2}x - \frac{1}{2!}x^2 + \frac{3}{3!}x^3 - \frac{15}{4!}x^4 \\
 &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 \\
 f(0.12) &\approx 1 + \frac{1}{2}(0.12) - \frac{1}{8}(0.12)^2 + \frac{1}{16}(0.12)^3 \\
 &\quad - \frac{5}{128}(0.12)^4 \approx 1.0583
 \end{aligned}$$

$$\begin{aligned}
 7. \quad f(x) &= \tan^{-1} x & f(0) &= 0 \\
 f'(x) &= \frac{1}{1+x^2} & f'(0) &= 1 \\
 f''(x) &= -\frac{2x}{(1+x^2)^2} & f''(0) &= 0 \\
 f'''(x) &= \frac{6x^2-2}{(1+x^2)^3} & f'''(0) &= -2 \\
 f^{(4)}(x) &= \frac{-24x^3+24x}{(1+x^2)^4} & f^{(4)}(0) &= 0 \\
 f(x) &\approx x - \frac{2}{3!}x^3 = x - \frac{1}{3}x^3 \\
 f(0.12) &\approx 0.12 - \frac{1}{3}(0.12)^3 \approx 0.1194
 \end{aligned}$$

$$\begin{aligned}
 8. \quad f(x) &= \sinh x & f(0) &= 0 \\
 f'(x) &= \cosh x & f'(0) &= 1 \\
 f''(x) &= \sinh x & f''(0) &= 0 \\
 f'''(x) &= \cosh x & f'''(0) &= 1 \\
 f^{(4)}(x) &= \sinh x & f^{(4)}(0) &= 0 \\
 f(x) &\approx x + \frac{1}{3!}x^3 = x + \frac{1}{6}x^3 \\
 f(0.12) &\approx 0.12 + \frac{1}{6}(0.12)^3 \approx 0.1203
 \end{aligned}$$

$$\begin{aligned}
 9. \quad f(x) &= e^x & f(1) &= e \\
 f'(x) &= e^x & f'(1) &= e \\
 f''(x) &= e^x & f''(1) &= e \\
 f'''(x) &= e^x & f'''(1) &= e \\
 P_3(x) &= e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3
 \end{aligned}$$

$$\begin{aligned}
 10. \quad f(x) &= \sin x & f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\
 f'(x) &= \cos x & f'\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\
 f''(x) &= -\sin x & f''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\
 f'''(x) &= -\cos x & f'''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\
 P_3(x) &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 \\
 &\quad - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3
 \end{aligned}$$

$$11. f(x) = \tan x; \quad f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

$$f'(x) = \sec^2 x; \quad f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$$

$$f''(x) = 2 \sec^2 x \tan x; \quad f''\left(\frac{\pi}{6}\right) = \frac{8\sqrt{3}}{9}$$

$$f'''(x) = 2 \sec^4 x + 4 \sec^2 x \tan^2 x; \quad f'''\left(\frac{\pi}{6}\right) = \frac{16}{3}$$

$$P_3(x) = \frac{\sqrt{3}}{3} + \frac{4}{3}\left(x - \frac{\pi}{6}\right) + \frac{4\sqrt{3}}{9}\left(x - \frac{\pi}{6}\right)^2 + \frac{8}{9}\left(x - \frac{\pi}{6}\right)^3$$

$$12. f(x) = \sec x; \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$f'(x) = \sec x \tan x; \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$f''(x) = \sec^3 x + \sec x \tan^2 x; \quad f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$$

$$f'''(x) = 5 \sec^3 x \tan x + \sec x \tan^3 x;$$

$$f'''\left(\frac{\pi}{4}\right) = 11\sqrt{2}$$

$$P_3(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{11\sqrt{2}}{6}\left(x - \frac{\pi}{4}\right)^3$$

$$13. f(x) = \cot^{-1} x; \quad f(1) = \frac{\pi}{4}$$

$$f'(x) = -\frac{1}{1+x^2}; \quad f'(1) = -\frac{1}{2}$$

$$f''(x) = \frac{2x}{(1+x^2)^2}; \quad f''(1) = \frac{1}{2}$$

$$f'''(x) = \frac{-6x^2 + 2}{(1+x^2)^3}; \quad f'''(1) = -\frac{1}{2}$$

$$P_3(x) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{12}(x-1)^3$$

$$14. f(x) = \sqrt{x}; \quad f(2) = \sqrt{2}$$

$$f'(x) = \frac{1}{2}x^{-1/2}; \quad f'(2) = \frac{\sqrt{2}}{4}$$

$$f''(x) = -\frac{1}{4}x^{-3/2}; \quad f''(2) = -\frac{\sqrt{2}}{16}$$

$$f'''(x) = \frac{3}{8}x^{-5/2}; \quad f'''(2) = \frac{3\sqrt{2}}{64}$$

$$P_3(x) = \sqrt{2} + \frac{\sqrt{2}}{4}(x-2) - \frac{\sqrt{2}}{32}(x-2)^2 + \frac{\sqrt{2}}{128}(x-2)^3$$

$$15. f(x) = x^3 - 2x^2 + 3x + 5; \quad f(1) = 7$$

$$f'(x) = 3x^2 - 4x + 3; \quad f'(1) = 2$$

$$f''(x) = 6x - 4; \quad f''(1) = 2$$

$$f^{(3)}(x) = 6; \quad f^{(3)}(1) = 6$$

$$P_3(x) = 7 + 2(x-1) + (x-1)^2 + (x-1)^3 = 5 + 3x - 2x^2 + x^3 = f(x)$$

$$16. f(x) = x^4; \quad f(2) = 16$$

$$f'(x) = 4x^3; \quad f'(2) = 32$$

$$f''(x) = 12x^2; \quad f''(2) = 48$$

$$f^{(3)}(x) = 24x; \quad f^{(3)}(2) = 48$$

$$f^{(4)}(x) = 24; \quad f^{(4)}(2) = 24$$

$$P_4(x) = 16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4 = x^4 = f(x)$$

$$17. f(x) = \frac{1}{1-x}; \quad f(0) = 1$$

$$f'(x) = \frac{1}{(1-x)^2}; \quad f'(0) = 1$$

$$f''(x) = \frac{2}{(1-x)^3}; \quad f''(0) = 2$$

$$f^{(3)}(x) = \frac{6}{(1-x)^4}; \quad f^{(3)}(0) = 6$$

$$f^{(4)}(x) = \frac{24}{(1-x)^5}; \quad f^{(4)}(0) = 24$$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}; \quad f^{(n)}(0) = n!$$

$$f(x) \approx 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \dots + \frac{n!}{n!}x^n = 1 + x + x^2 + x^3 + \dots + x^n$$

$$\text{Using } n = 4, f(x) \approx 1 + x + x^2 + x^3 + x^4$$

$$\text{a. } f(0.1) \approx 1.1111$$

$$\text{b. } f(0.5) \approx 1.9375$$

$$\text{c. } f(0.9) \approx 4.0951$$

$$\text{d. } f(2) \approx 31$$

18. $f(x) = \sin x$; $f(0) = 0$
 $f'(x) = \cos x$; $f'(0) = 1$
 $f''(x) = -\sin x$; $f''(0) = 0$
 $f^{(3)}(x) = -\cos x$; $f^{(3)}(0) = -1$
 $f^{(4)}(x) = \sin x$; $f^{(4)}(0) = 0$

When n is odd,

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{(n-1)/2} x^n}{n!}$$

Using $n = 5$, $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

a. $\sin(0.1) \approx 0.0998$

b. $\sin(0.5) \approx 0.4794$

c. $\sin(1) \approx 0.8417$

d. $\sin(10) \approx 676.67$

19. The area of the sector with angle t is $\frac{1}{2}tr^2$. The

area of the triangle is

$$\frac{1}{2} \left(r \sin \frac{t}{2} \right) \left(2r \cos \frac{t}{2} \right) = r^2 \sin \frac{t}{2} \cos \frac{t}{2} = \frac{1}{2} r^2 \sin t$$

$$A = \frac{1}{2} tr^2 - \frac{1}{2} r^2 \sin t$$

Using $n = 3$, $\sin t \approx t - \frac{1}{6}t^3$.

$$A \approx \frac{1}{2} tr^2 - \frac{1}{2} r^2 \left(t - \frac{1}{6}t^3 \right) = \frac{1}{12} r^2 t^3$$

20. $m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ $m(0) = m_0$

$$m'(v) = \frac{m_0 v}{c^2 \left(1 - \frac{v^2}{c^2} \right)^{3/2}}$$
 $m'(0) = 0$

$$m''(v) = \frac{2m_0 v^2 + m_0 c^2}{c^4 \left(1 - \frac{v^2}{c^2} \right)^{5/2}}$$
 $m''(0) = \frac{m_0}{c^2}$

The Maclaurin polynomial of order 2 is:

$$m(v) \approx m_0 + \frac{1}{2} \frac{m_0}{c^2} v^2 = m_0 + \frac{m_0}{2} \left(\frac{v}{c} \right)^2.$$

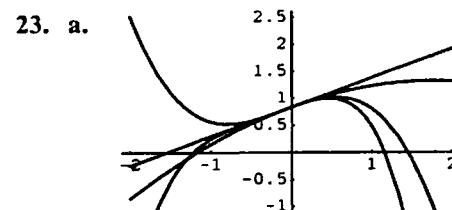
21. a. $\ln \left(1 + \frac{r}{12} \right)^{12n} = \ln 2$
 $12n \ln \left(1 + \frac{r}{12} \right) = \ln 2$
 $n = \frac{\ln 2}{12 \ln \left(1 + \frac{r}{12} \right)}$

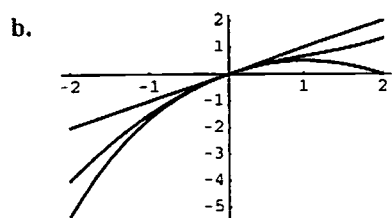
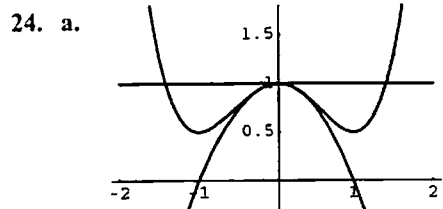
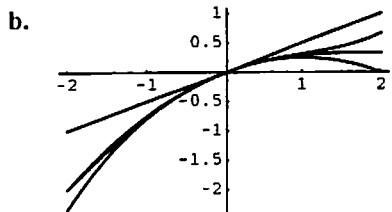
b. $f(x) = \ln(1+x)$; $f(0) = 0$
 $f'(x) = \frac{1}{1+x}$; $f'(0) = 1$
 $f''(x) = -\frac{1}{(1+x)^2}$; $f''(0) = -1$
 $\ln(1+x) \approx x - \frac{x^2}{2}$
 $n \approx \frac{\ln 2}{r - \frac{r^2}{24}} = \left[\frac{24}{r(24-r)} \right] \ln 2$
 $= \frac{\ln 2}{r} + \frac{\ln 2}{24-r}$
 $\approx \frac{\ln 2}{r} + \frac{\ln 2}{24} \approx \frac{0.693}{r} + 0.029$

We let $24 - r \approx 24$ since the interest rate r is going to be close to 0.

c.	r	n (exact)	n (approx.)	n (rule 72)
	0.05	13.8918	13.889	14.4
	0.10	6.9603	6.959	7.2
	0.15	4.6498	4.649	4.8
	0.20	3.4945	3.494	3.6

22. $f(x) = 1 - e^{-(1+k)x}$; $f(0) = 0$
 $f'(x) = (1+k)e^{-(1+k)x}$; $f'(0) = (1+k)$
 $f''(x) = -(1+k)^2 e^{-(1+k)x}$; $f''(0) = -(1+k)^2$
 $1 - e^{-(1+k)x} \approx (1+k)x - \frac{(1+k)^2}{2} x^2$
For $x = 2k$, the polynomial is
 $2k - 4k^3 - 2k^4 \approx 2k$ when k is very small.
 $1 - e^{-(1+0.01)(0.02)} \approx 0.019997 \approx 0.02$





25. $|e^{2c} + e^{-2c}| \leq |e^{2c}| + \left| \frac{1}{e^{2c}} \right| \leq e^6 + 1$

26. $|\tan c + \sec c| \leq |\tan c| + |\sec c| \leq 1 + 1 = 2$

27. $\left| \frac{4c}{\sin c} \right| = \frac{|4c|}{|\sin c|} \leq \frac{2\pi}{\frac{1}{\sqrt{2}}} = 2\sqrt{2}\pi$

28. $\left| \frac{4c}{c+4} \right| = \frac{|4c|}{|c+4|} \leq \frac{4}{4} = 1$

29. $\left| \frac{e^c}{c+5} \right| = \frac{|e^c|}{|c+5|} \leq \frac{e^4}{3}$

30. $\left| \frac{\cos c}{c+2} \right| = \frac{|\cos c|}{|c+2|} \leq \frac{1}{2}$

31. $\left| \frac{c^2 + \sin c}{10 \ln c} \right| = \frac{|c^2 + \sin c|}{|10 \ln c|} \leq \frac{|c^2| + |\sin c|}{|10 \ln c|}$
 $\leq \frac{16+1}{10 \ln 2} = \frac{17}{10 \ln 2}$

32. $\left| \frac{c^2 - c}{\cos c} \right| = \frac{|c^2 - c|}{|\cos c|} \leq \sqrt{2} |c^2 - c| \leq \sqrt{2} \left| \left(\frac{1}{2} \right)^2 - \frac{1}{2} \right|$
 $= \frac{\sqrt{2}}{4}$ (Note that $|x^2 - x|$ is maximum at $\frac{1}{2}$ in $[0, 1]$.)

33. $f(x) = \ln(2+x); f'(x) = \frac{1}{2+x};$
 $f''(x) = -\frac{1}{(2+x)^2}; f^{(3)}(x) = \frac{2}{(2+x)^3};$
 $f^{(4)}(x) = -\frac{6}{(2+x)^4}; f^{(5)}(x) = \frac{24}{(2+x)^5};$
 $f^{(6)}(x) = -\frac{120}{(2+x)^6}; f^{(7)}(x) = \frac{720}{(2+x)^7}$
 $R_6(x) = \frac{1}{7!} \cdot \frac{720}{(2+c)^7} x^7 = \frac{x^7}{7(2+c)^7}$
 $|R_6(0.5)| \leq \left| \frac{0.5^7}{7 \cdot 2^7} \right| \approx 8.719 \times 10^{-6}$

34. $f(x) = e^{-x}; f'(x) = -e^{-x};$
 $f^{(n)}(x) = \begin{cases} e^{-x} & \text{if } n \text{ is even} \\ -e^{-x} & \text{if } n \text{ is odd} \end{cases}$
 $R_6(x) = \frac{-e^{-c}}{7!} (x-1)^7 = -\frac{(x-1)^7}{5040e^c}$
 $|R_6(0.5)| \leq \left| \frac{(-0.5)^7}{5040e^{0.5}} \right| \approx 9.402 \times 10^{-7}$

35. $f(x) = \sin x; f^{(7)}(x) = -\cos x$
 $R_6(x) = \frac{-\cos c}{7!} \left(x - \frac{\pi}{4} \right)^7 = \frac{-\cos c \left(x - \frac{\pi}{4} \right)^7}{5040}$
 $|R_6(0.5)| \leq \left| \frac{\cos 0.5 \left(0.5 - \frac{\pi}{4} \right)^7}{5040} \right| \approx 2.685 \times 10^{-8}$

$$\begin{aligned}
 36. \quad f(x) &= \frac{1}{x-3}; \quad f'(x) = -\frac{1}{(x-3)^2}; \quad f''(x) = \frac{2}{(x-3)^3}; \quad f^{(3)}(x) = -\frac{6}{(x-3)^4}; \quad f^{(4)}(x) = \frac{24}{(x-3)^5}; \\
 f^{(5)}(x) &= -\frac{120}{(x-3)^6}; \quad f^{(6)}(x) = \frac{720}{(x-3)^7}; \quad f^{(7)}(x) = -\frac{5040}{(x-3)^8} \\
 R_6(x) &= \frac{1}{7!} \cdot -\frac{5040}{(c-3)^8} (x-1)^7 = -\frac{(x-1)^7}{(c-3)^8} \\
 |R_6(0.5)| &\leq \left| \frac{(0.5-1)^7}{(1-3)^8} \right| = \left| \frac{0.5^7}{2^8} \right| \approx 3.052 \times 10^{-5}
 \end{aligned}$$

$$\begin{aligned}
 37. \quad f(x) &= e^x & f(0) &= 1 \\
 f'(x) &= e^x & f'(0) &= 1 \\
 f''(x) &= e^x & f''(0) &= 1 \\
 f^{(3)}(x) &= e^x & f^{(3)}(0) &= 1 \\
 f^{(4)}(x) &= e^x & f^{(4)}(c) &= e^c \\
 e^x &\approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \\
 R_3(x) &= \frac{e^c}{4!} x^4 \\
 R_3(-0.1) &= \frac{e^c}{240,000} \leq \frac{1}{240,000} \approx 4.17 \times 10^{-6} \\
 R_3(-0.1) &\geq \frac{e^{-0.1}}{240,000} \approx 3.77 \times 10^{-6} \\
 e^{-0.1} - \left[1 + (-0.1) + \frac{1}{2}(-0.1)^2 + \frac{1}{6}(-0.1)^3 \right] &\approx 4.08 \times 10^{-6}
 \end{aligned}$$

$$\begin{aligned}
 38. \quad f(x) &= \sin x; \quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\
 f'(x) &= \cos x; \quad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\
 f''(x) &= -\sin x; \quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\
 f^{(3)}(x) &= -\cos x; \quad f^{(3)}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\
 f^{(4)}(x) &= \sin x; \quad f^{(4)}(c) = \sin c \\
 \sin x &\approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 \\
 R_3(x) &= \frac{\sin c}{4!} \left(x - \frac{\pi}{4}\right)^4 \\
 R_3\left(\frac{\pi}{8}\right) &= \frac{\sin c}{24} \left(-\frac{\pi}{8}\right)^4 \leq \frac{\sin\left(\frac{\pi}{4}\right)}{24} \left(-\frac{\pi}{8}\right)^4 \approx 7.01 \times 10^{-4}
 \end{aligned}$$

$$R_3\left(\frac{\pi}{8}\right) \geq \frac{\sin\left(\frac{\pi}{8}\right)}{24} \left(-\frac{\pi}{8}\right)^4 \approx 3.79 \times 10^{-4}$$

$$\sin\left(\frac{\pi}{8}\right) - \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(-\frac{\pi}{8}\right) - \frac{\sqrt{2}}{4} \left(-\frac{\pi}{8}\right)^2 - \frac{\sqrt{2}}{12} \left(-\frac{\pi}{8}\right)^3 \right] \approx 6.42 \times 10^{-4}$$

$$39. R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$$

Note that $e^1 < 3$.

$$|R_n(1)| < \frac{3}{(n+1)!}$$

$$\frac{3}{(n+1)!} < 0.000005 \text{ or } 600000 < (n+1)! \text{ when } n \geq 9.$$

$$\begin{aligned} 40. f(x) &= (1+x)^{3/2} & f(0) &= 1 \\ f'(x) &= \frac{3}{2}(1+x)^{1/2} & f'(0) &= \frac{3}{2} \\ f''(x) &= \frac{3}{4}(1+x)^{-1/2} & f''(0) &= \frac{3}{4} \\ f^{(3)}(x) &= -\frac{3}{8}(1+x)^{-3/2} & f^{(3)}(0) &= -\frac{3}{8} \\ f^{(4)}(x) &= \frac{9}{16}(1+x)^{-5/2} & f^{(4)}(c) &= \frac{9}{16}(1+c)^{-5/2} \\ (1+x)^{3/2} &\approx 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 \\ R_3(x) &= \frac{3}{128}(1+c)^{-5/2}x^4 \\ |R_3(x)| &\leq \left| \frac{3}{128}(0.9)^{-5/2}(-0.1)^4 \right| \approx 3.05 \times 10^{-6} \end{aligned}$$

$$\begin{aligned} 41. f(x) &= (1+x)^{-1/2} & f(0) &= 1 \\ f'(x) &= -\frac{1}{2}(1+x)^{-3/2} & f'(0) &= -\frac{1}{2} \\ f''(x) &= \frac{3}{4}(1+x)^{-5/2} & f''(0) &= \frac{3}{4} \\ f^{(3)}(x) &= -\frac{15}{8}(1+x)^{-7/2} & f^{(3)}(0) &= -\frac{15}{8} \\ f^{(4)}(x) &= \frac{105}{16}(1+x)^{-9/2} & f^{(4)}(c) &= \frac{105}{16}(1+c)^{-9/2} \\ (1+x)^{-1/2} &\approx 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \\ R_3(x) &= \frac{35}{128}(1+c)^{-9/2}x^4 \\ |R_3(x)| &\leq \left| \frac{35}{128}(0.95)^{-9/2}(0.05)^4 \right| \approx 2.15 \times 10^{-6} \end{aligned}$$

$$42. f(x) = \ln\left(\frac{1+x}{1-x}\right) \quad f(0) = 0$$

$$f'(x) = \frac{2}{1-x^2} \quad f'(0) = 2$$

$$f''(x) = \frac{4x}{(1-x^2)^2} \quad f''(0) = 0$$

$$f^{(3)}(x) = \frac{4(1+3x^2)}{(1-x^2)^3} \quad f^{(3)}(0) = 4$$

$$f^{(4)}(x) = \frac{48x(1+x^2)}{(1-x^2)^4} \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \frac{48(1+10x^2+5x^4)}{(1-x^2)^5}$$

$$f^{(5)}(c) = \frac{48(1+10c^2+5c^4)}{(1-c^2)^5}$$

$$\ln\left(\frac{1+x}{1-x}\right) \approx 2x + \frac{2}{3}x^3$$

$$R_4(x) = \frac{2}{5} \left[\frac{1+10c^2+5c^4}{(1-c^2)^5} \right] x^5$$

$$|R_4(x)| < \frac{2}{5} \left[\frac{1+10(0.5)^2+5(0.5)^4}{(1-(0.5)^2)^5} \right] (0.5)^5 \approx 0.201$$

$$43. R_4(x) = \frac{\cos c}{5!} x^5$$

$$|R_4(x)| \leq \frac{(0.5)^5}{5!} \approx 0.00026042 \leq 0.0002605$$

$$\int_0^{0.5} \sin x \, dx \approx \int_0^{0.5} \left(x - \frac{1}{6}x^3 \right) dx$$

$$= \left[\frac{1}{2}x^2 - \frac{1}{24}x^4 \right]_0^{0.5} \approx 0.1224$$

$$\text{Error} \leq 0.0002605(0.5 - 0) = 0.00013025$$

$$44. R_5(x) = -\frac{\cos c}{6!} x^6$$

$$|R_5(x)| \leq \frac{1}{6!} \approx 0.001389$$

$$\int_0^1 \cos x \, dx \approx \int_0^1 \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) dx$$

$$= \left[x - \frac{x^3}{6} + \frac{x^5}{120} \right]_0^1 \approx 0.8417$$

$$\text{Error} \leq 0.001389(1 - 0) = 0.001389$$

$$45. f(x) = x^4 - 3x^3 + 2x^2 + x - 2; f(1) = -1$$

$$f'(x) = 4x^3 - 9x^2 + 4x + 1; f'(1) = 0$$

$$f''(x) = 12x^2 - 18x + 4; f''(1) = -2$$

$$f^{(3)}(x) = 24x - 18; f^{(3)}(1) = 6$$

$$f^{(4)}(x) = 24; f^{(4)}(1) = 24$$

$$f^{(5)}(x) = 0$$

$$\text{Since } f^{(5)}(x) = 0, R_5(x) = 0.$$

$$x^4 - 3x^3 + 2x^2 + x - 2$$

$$= -1 - (x-1)^2 + (x-1)^3 + (x-1)^4$$

$$46. P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$P_n'(x) = f'(a) + \frac{f''(a)}{2!} 2(x-a) + \frac{f'''(a)}{3!} 3(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} n(x-a)^{n-1}$$

$$= f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$$

$$P_n'(a) = f'(a) + 0 + 0 + \cdots + 0 = f'(a)$$

$$P_n'' = 0 + f''(a) + \frac{f'''(a)}{2!} 2(x-a) + \cdots + \frac{f^{(n)}(a)}{(n-1)!} (n-1)(x-a)^{n-2}$$

$$= f''(a) + f'''(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n-2}$$

$$P_n'' = f''(a) + 0 + 0 + \cdots + 0 = f''(a)$$

$$\vdots$$

$$P_n^{(n)}(x) = \frac{f^{(n)}(a)}{0!}(x-a)^0 = f^{(n)}(a)$$

$$P_n^{(n)}(a) = f^{(n)}(a)$$

$$47. f(x) = \sin x; f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = \cos x; f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f''(x) = -\sin x; f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f^{(3)}(x) = -\cos x; f^{(3)}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \sin x; f^{(4)}(c) = \sin c$$

$$43^\circ = \frac{\pi}{4} - \frac{\pi}{90} \text{ radians}$$

$$\sin x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + R_3(x)$$

$$\sin\left(\frac{\pi}{4} - \frac{\pi}{90}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(-\frac{\pi}{90}\right) - \frac{\sqrt{2}}{4}\left(-\frac{\pi}{90}\right)^2 - \frac{\sqrt{2}}{12}\left(-\frac{\pi}{90}\right)^3 + R_3\left(\frac{\pi}{4} - \frac{\pi}{90}\right)$$

$$\approx 0.681998 + R_3$$

$$|R_3| = \left| \frac{\sin c}{4!} \left(-\frac{\pi}{90}\right)^4 \right| < \frac{1}{24} \left(\frac{\pi}{90}\right)^4 \approx 6.19 \times 10^{-8}$$

48. $63^\circ = \frac{\pi}{3} + \frac{60}{\pi}$ radians

Since $f^{(n)}(x)$ is $\pm \sin x$ or $\pm \cos x$,

$$|R_n(x)| \leq \frac{1}{(n+1)!} \left(x - \frac{3}{\pi} \right)^{n+1}$$

$$\left| R_n \left(\frac{\pi}{3} + \frac{60}{\pi} \right) \right| \leq \frac{1}{(n+1)!} \left(\frac{60}{\pi} \right)^{n+1}$$

$$\frac{1}{(n+1)!} \left(\frac{60}{\pi} \right)^{n+1} \leq 0.0005 \text{ when } n \geq 2.$$

$$f(x) = \cos x : f \left(\frac{\pi}{3} \right) = \frac{1}{2}$$

$$f'(x) = -\sin x : f' \left(\frac{\pi}{3} \right) = -\frac{\sqrt{3}}{2}$$

$$f''(x) = -\cos x : f'' \left(\frac{\pi}{3} \right) = -\frac{1}{2}$$

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{3} \left(x - \frac{\pi}{3} \right) + \frac{1}{4} \left(x - \frac{\pi}{3} \right)^2 + R_3(x)$$

$$\cos 63^\circ \approx \frac{1}{2} - \frac{\sqrt{3}}{3} \left(\frac{2}{\pi} \right) + \frac{1}{4} \left(\frac{2}{\pi} \right)^2 \approx 0.45397$$

49. $|R_9(x)| \leq \frac{1}{10!} x^{10} \leq \frac{1}{10!} \left(\frac{2}{\pi} \right)^{10} \approx 2.5202 \times 10^{-5}$

50. a. $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{720}x^6$

$$\lim_{x \rightarrow 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x \rightarrow 0} \left(\frac{1}{120} - \frac{\sin x}{720} x \right) = \frac{1}{120}$$

b. $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{5040}x^7$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}}{x^6} = \lim_{x \rightarrow 0} \left(-\frac{\sin x}{720} + \frac{5040}{x^7} \right) = -\frac{1}{720}$$

51. The k th derivative of $h(x)f(x)$ is

$$\sum_{i=0}^k \binom{k}{i} h^{(i)}(x) f^{(k-i)}(x). \text{ If } h(x) = x^{n+1},$$

$$h^{(i)}(x) = \frac{(n+1)!}{(n+1-i)!} x^{n+1-i}.$$

Thus for $i \leq n+1$, $h^{(i)}(0) = 0$. Let

$q(x) = x^{n+1} f(x)$. Then

$$q^{(k)}(0) = \sum_{i=0}^k \binom{k}{i} h^{(i)}(0) f^{(k-i)}(0) = 0$$

for $k \leq n+1$.

$$g^{(k)}(x) = p^{(k)}(x) + q^{(k)}(x), \text{ so}$$

$$g^{(k)}(0) = p^{(k)}(0) + q^{(k)}(0) = p^{(k)}(0)$$

for $k \leq n+1$.

The Maclaurin polynomial of order n for g is

$$p(0) + p'(0)x + \frac{p''(0)}{2!}x^2 + \dots + \frac{p^{(n)}(0)}{n!}x^n \text{ which}$$

is the Maclaurin polynomial of order n for $p(x)$.

Since $p(x)$ is a polynomial of degree at most n ,

the remainder $R_n(x)$ of Maclaurin's Formula for

$p(x)$ is 0, so the Maclaurin polynomial of order n

52. Using Taylor's formula,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

$$+ \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

Since $f'(c) = f''(c) = f'''(c) = \dots = f^{(n)}(c) = 0$,

$$f(x) = f(c) + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!} (a)(x-c)^{n+1} \text{ where } a \text{ is}$$

between x and c .

(i) Since $f^{(n+1)}(x)$ is continuous near c , then

$$f^{(n+1)}(a) > 0 \text{ when } a \text{ is near } c.$$

Thus $R_n(x) < 0$ when x is near c , so

$f(x) < f(c)$ when x is near c . $f(c)$ is a local

maximum.

(ii) Since $f^{(n+1)}(x)$ is continuous near c , then

$$f^{(n+1)}(a) > 0 \text{ when } a \text{ is near } c.$$

Thus $R_n(x) > 0$ when x is near c , so

$f(x) > f(c)$ when x is near c .

$f(c)$ is a local minimum.

Suppose $f(x) = x^4$. $f(x) > 0$ when $x > 0$ and

$f(x) < 0$ when $x < 0$. Thus $x = 0$ is a local

minimum.

$$f'(0) = f''(0) = f'''(0) = 0, f^{(4)}(0) = 24 > 0$$

53. a. $L_{51}(x)$ is of degree 4 since it is the product of four linear factors.

$$L_{51}(x_1) = \frac{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)} = 1$$

$$L_{51}(x_j) = 0 \text{ for } j = 2, 3, 4, 5 \text{ since } x - x_j \text{ is a linear factor of } L_{51}(x).$$

$$\text{b. } L_{52}(x) = \frac{(x - x_1)(x - x_3)(x - x_4)(x - x_5)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)}$$

$$L_{53}(x) = \frac{(x - x_1)(x - x_2)(x - x_4)(x - x_5)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5)}$$

$$L_{54}(x) = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_5)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)(x_4 - x_5)}$$

$$L_{55}(x) = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)}$$

- c. Since $L_{51}, L_{52}, L_{53}, L_{54},$ and L_{55} are of degree 4, L_5 is of degree less than or equal to 4.
Since $L_{5j}(x_i) = 0$ for $i \neq j$ and $L_{5i}(x_i) = 1$, $L_5(x_i) = y_i$.

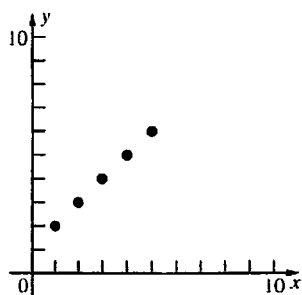
$$\text{d. } L_{31}(x) = \frac{(x - 2)(x - 0)}{(1 - 2)(1 - 0)} = -(x - 2)(x) = -x^2 + 2x$$

$$L_{32}(x) = \frac{(x - 1)(x - 0)}{(2 - 1)(2 - 0)} = \frac{1}{2}(x - 1)(x) = \frac{1}{2}x^2 - \frac{1}{2}x$$

$$L_{33}(x) = \frac{(x - 1)(x - 2)}{(0 - 1)(0 - 2)} = \frac{1}{2}(x - 1)(x - 2) = \frac{1}{2}x^2 - \frac{3}{2}x + 1$$

$$L_3(x) = (-x^2 + 2x)(2) + \left(\frac{1}{2}x^2 - \frac{1}{2}x\right)(2.5) + \left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right)(0) = -0.75x^2 + 2.75x$$

54.



$$L_{51}(x) = \frac{(x - 2)(x - 3)(x - 4)(x - 5)}{(1 - 2)(1 - 3)(1 - 4)(1 - 5)} = \frac{1}{24}(x - 2)(x - 3)(x - 4)(x - 5)$$

$$L_{52}(x) = \frac{(x - 1)(x - 3)(x - 4)(x - 5)}{(2 - 1)(2 - 3)(2 - 4)(2 - 5)} = -\frac{1}{6}(x - 1)(x - 3)(x - 4)(x - 5)$$

$$L_{53}(x) = \frac{(x - 1)(x - 2)(x - 4)(x - 5)}{(3 - 1)(3 - 2)(3 - 4)(3 - 5)} = \frac{1}{4}(x - 1)(x - 2)(x - 4)(x - 5)$$

$$L_{54}(x) = \frac{(x - 1)(x - 2)(x - 3)(x - 5)}{(4 - 1)(4 - 2)(4 - 3)(4 - 5)} = -\frac{1}{6}(x - 1)(x - 2)(x - 3)(x - 5)$$

$$L_{55}(x) = \frac{(x - 1)(x - 2)(x - 3)(x - 4)}{(5 - 1)(5 - 2)(5 - 3)(5 - 4)} = \frac{1}{24}(x - 1)(x - 2)(x - 3)(x - 4)$$

$$L_5(x) = L_{51}(x) \cdot 2 + L_{52}(x) \cdot 3 + L_{53}(x) \cdot 4 + L_{54}(x) \cdot 5 + L_{55}(x) \cdot 6$$

$$= \frac{1}{12}(x-2)(x-3)(x-4)(x-5) - \frac{1}{2}(x-1)(x-3)(x-4)(x-5) + (x-1)(x-2)(x-4)(x-5) \\ - \frac{5}{6}(x-1)(x-2)(x-3)(x-5) + \frac{1}{4}(x-1)(x-2)(x-3)(x-4) = x+1$$

55. $x_1 = 1; y_1 = 0$

$x_2 = 3; y_2 = 1.099$

$x_3 = 5; y_3 = 1.609$

$$L_{31}(x) = \frac{(x-3)(x-5)}{(1-3)(1-5)} = \frac{1}{8}(x-3)(x-5)$$

$$L_{32}(x) = \frac{(x-1)(x-5)}{(3-1)(3-5)} = -\frac{1}{4}(x-1)(x-5)$$

$$L_{33}(x) = \frac{(x-1)(x-3)}{(5-1)(5-3)} = \frac{1}{8}(x-1)(x-3)$$

$$L_3(x) = L_{31}(x) \cdot 0 + L_{32}(x) \cdot 1.099 + L_{33}(x) \cdot 1.609$$

$$= -\frac{1.099}{4}(x-1)(x-5) + \frac{1.609}{8}(x-1)(x-3)$$

$$= \frac{1}{4}(x-1)[-1.099(x-5) + 0.8045(x-3)]$$

$$= \frac{1}{4}(x-1)(-0.2945x + 3.0815)$$

Thus,

$$\ln 2 \approx \frac{1}{4}(2-1)(-0.2945x + 3.0815) \approx 0.623$$

The estimate for the maximum error is

$$R_2(x) = \frac{(x-1)(x-3)(x-5)}{3!} f^{(3)}(\alpha)$$

$$f(x) = \ln x; \quad f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}; \quad f'''(x) = \frac{2}{x^3}$$

$$|R_2(2)| = \left| \frac{(2-1)(2-3)(2-5)}{3!} \right| |f^{(3)}(\alpha)|$$

$$= \left| \frac{1 \cdot (-1) \cdot (-3)}{6} \cdot \frac{2}{\alpha^3} \right| = \frac{1}{\alpha^3} \leq 1 \text{ for } \alpha \in [1, 5].$$

A calculator gives $\ln 2 \approx 0.693$, so the actual error is approximately $0.693 - 0.623 \approx 0.07$.

56. $x_1 = 0; y_1 = 1$

$x_2 = 0.2; y_2 = 1.221$

$x_3 = 0.3; y_3 = 1.350$

$$L_{31}(x) = \frac{(x-0.2)(x-0.3)}{(0-0.2)(0-0.3)} \\ = \frac{50}{3}(x-0.2)(x-0.3)$$

$$L_{32}(x) = \frac{(x-0)(x-0.3)}{(0.2-0)(0.2-0.3)} \\ = -50x(x-0.3)$$

$$L_{33}(x) = \frac{(x-0)(x-0.2)}{(0.3-0)(0.3-0.2)} \\ = \frac{100}{3}x(x-0.2)$$

$$L(x) = L_{31}(x) \cdot 1 + L_{32}(x) \cdot 1.221 + L_{33}(x) \cdot 1.350 \\ = \frac{50}{3}(x-0.2)(x-0.3) - 61.05x(x-0.3) \\ + 45x(x-0.2)$$

$$L(0.25) \approx \frac{50}{3}(0.25-0.2)(0.25-0.3) \\ - 61.05 \cdot 0.25(0.25-0.3) \\ + 45 \cdot 0.25(0.25-0.2) \\ \approx 1.284$$

The error is

$$R_2(x) = \frac{x(x-0.2)(x-0.3)}{3!} f^{(3)}(\alpha)$$

The derivatives of $f(x)$ are

$$f(x) = e^x \quad f'(x) = e^x$$

$$f''(x) = e^x \quad f'''(x) = e^x$$

Thus,

$$R_2(0.25) = \frac{0.25(0.25-0.2)(0.25-0.3)}{3!} \cdot f^{(3)}(\alpha)$$

$$\approx -0.0001042e^\alpha$$

$$\text{Thus } |R_2(0.25)| \leq 0.0001042e^{0.3} \approx 0.0001407$$

$$\text{and } |R_2(0.25)| \geq 0.0001042e^0 \approx 0.0001042.$$

A calculator gives $e^{0.25} \approx 1.2840$ so the actual error is $1.284 - 1.284 = 0$, to three decimal places.

57. The second order Maclaurin polynomial is

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

From Problem 56, we know that the interpolating polynomial is $L(x) = \frac{50}{3}(x-0.2)(x-0.3)$

$$- 61.05x(x-0.3) \\ + 45x(x-0.2)$$

For the Maclaurin polynomial

$$R_2(x) = \frac{f^{(3)}(c)}{3!} x^3 = \frac{1}{6} e^c x^3$$

Thus,

$$|R_2(x)| = \frac{1}{6} e^c |x^3| \leq \frac{1}{6} e^{0.3} 0.3^3 \approx 0.0060744$$

For the interpolating polynomial,

$$|R_2(x)| = \left| \frac{x(x-0.2)(x-0.3)}{6} \right| e^c$$

The expression in absolute values reaches a maximum of 0.0003521 when $x = 0.078475$. On the interval $[0, 0.3]$, the expression e^c reaches a maximum of $e^{0.3} \approx 1.350$. Thus, the largest

possible error for the interpolating polynomial on $[0, 0.3]$ is

$$R_2(x) \leq 0.0003521 \cdot 1.350 \approx 0.0004753$$

A calculator gives $e^{0.1} \approx 1.105$. When $x = 0.1$, the error for the Maclaurin polynomial is

$$|1.105 - P_2(0.1)| = |1.105 - 1.105| = 0$$

and the error for the interpolating polynomial is

$$|1.105 - L_2(0.1)| = |1.105 - 1.104| = 0.001$$

11.2 Concepts Review

1. $1, 2, 2, 2, \dots, 2, 1$

2. $1, 4, 2, 4, 2, \dots, 4, 1$

3. n^4

4. large

Problem Set 11.2

1. $f(x) = \frac{1}{x^2}; h = \frac{3-1}{8} = 0.25$

$x_0 = 1.00$

$f(x_0) = 1$

$x_5 = 2.25$

$f(x_5) \approx 0.1975$

$x_1 = 1.25$

$f(x_1) = 0.64$

$x_6 = 2.50$

$f(x_6) = 0.16$

$x_2 = 1.50$

$f(x_2) \approx 0.4444$

$x_7 = 2.75$

$f(x_7) \approx 0.1322$

$x_3 = 1.75$

$f(x_3) \approx 0.3265$

$x_8 = 3.00$

$f(x_8) \approx 0.1111$

$x_4 = 2.00$

$f(x_4) = 0.25$

Left Riemann Sum: $\int_1^3 \frac{1}{x^2} dx \approx 0.25[f(x_0) + f(x_1) + \dots + f(x_7)] \approx 0.7846$

Trapezoidal Rule: $\int_1^3 \frac{1}{x^2} dx \approx \frac{0.25}{2}[f(x_0) + 2f(x_1) + \dots + 2f(x_7) + f(x_8)] \approx 0.6766$

Parabolic Rule: $\int_1^3 \frac{1}{x^2} dx \approx \frac{0.25}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_7) + f(x_8)] \approx 0.6671$

Fundamental Theorem of Calculus: $\int_1^3 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^3 = -\frac{1}{3} + 1 = \frac{2}{3} \approx 0.6667$

2. $f(x) = \frac{1}{x}; h = \frac{3-1}{8} = 0.25$

$x_0 = 1.00$

$f(x_0) = 1$

$x_5 = 2.25$

$f(x_5) \approx 0.4444$

$x_1 = 1.25$

$f(x_1) = 0.8$

$x_6 = 2.50$

$f(x_6) = 0.4$

$x_2 = 1.50$

$f(x_2) \approx 0.6667$

$x_7 = 2.75$

$f(x_7) \approx 0.3636$

$x_3 = 1.75$

$f(x_3) \approx 0.5714$

$x_8 = 3.00$

$f(x_8) \approx 0.3333$

$x_4 = 2.00$

$f(x_4) = 0.5$

Left Riemann Sum: $\int_1^3 \frac{1}{x} dx \approx 0.25[f(x_0) + f(x_1) + \dots + f(x_7)] \approx 1.1865$

Trapezoidal Rule: $\int_1^3 \frac{1}{x} dx \approx \frac{0.25}{2}[f(x_0) + 2f(x_1) + \dots + 2f(x_7) + f(x_8)] \approx 1.1032$

$$\text{Parabolic Rule: } \int_1^3 \frac{1}{x} dx \approx \frac{0.25}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_7) + f(x_8)] \approx 1.0987$$

$$\text{Fundamental Theorem of Calculus: } \int_1^3 \frac{1}{x} dx = [\ln|x|]_1^3 = \ln 3 \approx 1.0986$$

$$3. \quad f(x) = \sqrt{x}; h = \frac{2-0}{8} = 0.25$$

$x_0 = 0.00$	$f(x_0) = 0$	$x_5 = 1.25$	$f(x_5) \approx 1.1180$
$x_1 = 0.25$	$f(x_1) = 0.5$	$x_6 = 1.50$	$f(x_6) \approx 1.2247$
$x_2 = 0.50$	$f(x_2) \approx 0.7071$	$x_7 = 1.75$	$f(x_7) \approx 1.3229$
$x_3 = 0.75$	$f(x_3) \approx 0.8660$	$x_8 = 2.00$	$f(x_8) \approx 1.4142$
$x_4 = 1.00$	$f(x_4) = 1$		

$$\text{Left Riemann Sum: } \int_0^2 \sqrt{x} dx \approx 0.25[f(x_0) + f(x_1) + \dots + f(x_7)] \approx 1.6847$$

$$\text{Trapezoidal Rule: } \int_0^2 \sqrt{x} dx \approx \frac{0.25}{2} [f(x_0) + 2f(x_1) + \dots + 2f(x_7) + f(x_8)] \approx 1.8615$$

$$\text{Parabolic Rule: } \int_0^2 \sqrt{x} dx \approx \frac{0.25}{3} [f(x_1) + 4f(x_2) + 2f(x_3) + \dots + 4f(x_7) + f(x_8)] \approx 1.8755$$

$$\text{Fundamental Theorem of Calculus: } \int_0^2 \sqrt{x} dx = \left[\frac{2}{3} x^{3/2} \right]_0^2 = \frac{4\sqrt{2}}{3} \approx 1.8856$$

$$4. \quad f(x) = x\sqrt{x^2 + 1}; h = \frac{3-1}{8} = 0.25$$

$x_0 = 1.00$	$f(x_0) \approx 1.4142$	$x_5 = 2.25$	$f(x_5) \approx 5.5400$
$x_1 = 1.25$	$f(x_1) \approx 2.0010$	$x_6 = 2.50$	$f(x_6) \approx 6.7315$
$x_2 = 1.50$	$f(x_2) \approx 2.7042$	$x_7 = 2.75$	$f(x_7) \approx 8.0470$
$x_3 = 1.75$	$f(x_3) \approx 3.5272$	$x_8 = 3.00$	$f(x_8) \approx 9.4868$
$x_4 = 2.00$	$f(x_4) \approx 4.4721$		

$$\text{Left Riemann Sum: } \int_1^3 x\sqrt{x^2 + 1} dx \approx 0.25[f(x_0) + f(x_1) + \dots + f(x_7)] \approx 8.6903$$

$$\text{Trapezoidal Rule: } \int_1^3 x\sqrt{x^2 + 1} dx \approx \frac{0.25}{2} [f(x_0) + 2f(x_1) + \dots + 2f(x_7) + f(x_8)] \approx 9.6184$$

$$\text{Parabolic Rule: } \int_1^3 x\sqrt{x^2 + 1} dx \approx \frac{0.25}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_7) + f(x_8)] \approx 9.5981$$

$$\text{Fundamental Theorem of Calculus: } \int_1^3 x\sqrt{x^2 + 1} dx = \left[\frac{1}{3} (x^2 + 1)^{3/2} \right]_1^3 = \frac{1}{3} (10\sqrt{10} - 2\sqrt{2}) \approx 9.5981$$

$$5. \quad f(x) = \sin x$$

$$n = 2: h = \frac{\pi}{2}$$

$$x_0 = 0 \quad f(x_0) = 0$$

$$x_1 = \frac{\pi}{2} \quad f(x_1) = 1$$

$$x_2 = \pi \quad f(x_2) = 0$$

$$\int_0^\pi \sin x dx \approx \frac{\pi}{4} [f(x_0) + 2f(x_1) + f(x_2)] = \frac{\pi}{2} \approx 1.5708$$

$$n = 6: h = \frac{\pi}{6}$$

$x_0 = 0$	$f(x_0) = 0$	$x_4 = \frac{2\pi}{3}$	$f(x_4) = \frac{\sqrt{3}}{2}$
$x_1 = \frac{\pi}{6}$	$f(x_1) = \frac{1}{2}$	$x_5 = \frac{5\pi}{6}$	$f(x_5) = \frac{1}{2}$
$x_2 = \frac{\pi}{3}$	$f(x_2) = \frac{\sqrt{3}}{2}$	$x_6 = \pi$	$f(x_6) = 0$
$x_3 = \frac{\pi}{2}$	$f(x_3) = 1$		

$$\int_0^{\pi} \sin x \, dx \approx \frac{\pi}{12} [f(x_0) + 2f(x_1) + \dots + 2f(x_5) + f(x_6)] = \frac{\pi}{12} (4 + 2\sqrt{3}) \approx 1.9541$$

$$n = 12: h = \frac{\pi}{12}$$

$x_0 = 0$	$f(x_0) = 0$	$x_7 = \frac{7\pi}{12}$	$f(x_7) = \frac{\sqrt{2}}{4}(\sqrt{3} + 1)$
$x_1 = \frac{\pi}{12}$	$f(x_1) = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$	$x_8 = \frac{2\pi}{3}$	$f(x_8) = \frac{\sqrt{3}}{2}$
$x_2 = \frac{\pi}{6}$	$f(x_2) = \frac{1}{2}$	$x_9 = \frac{3\pi}{4}$	$f(x_9) = \frac{\sqrt{2}}{2}$
$x_3 = \frac{\pi}{4}$	$f(x_3) = \frac{\sqrt{2}}{2}$	$x_{10} = \frac{5\pi}{6}$	$f(x_{10}) = \frac{1}{2}$
$x_4 = \frac{\pi}{3}$	$f(x_4) = \frac{\sqrt{3}}{2}$	$x_{11} = \frac{11\pi}{12}$	$f(x_{11}) = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$
$x_5 = \frac{5\pi}{12}$	$f(x_5) = \frac{\sqrt{2}}{4}(\sqrt{3} + 1)$	$x_{12} = \pi$	$f(x_{12}) = 0$
$x_6 = \frac{\pi}{2}$	$f(x_6) = 1$		

$$\int_0^{\pi} \sin x \, dx \approx \frac{\pi}{24} [f(x_0) + 2f(x_1) + \dots + 2f(x_{11}) + f(x_{12})] \approx \frac{\pi}{24} (4 + 2\sqrt{3} + 2\sqrt{2} + 2\sqrt{6}) \approx 1.9886$$

6. Use the calculations in Problem 5.

$$n = 2:$$

$$\int_0^{\pi} \sin x \, dx \approx \frac{\pi}{6} [f(x_0) + 4f(x_1) + f(x_2)] = \frac{2\pi}{3} \approx 2.0944$$

$$n = 6:$$

$$\int_0^{\pi} \sin x \, dx \approx \frac{\pi}{18} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_5) + f(x_6)] \approx 2.0009$$

$$n = 12:$$

$$\int_0^{\pi} \sin x \, dx \approx \frac{\pi}{36} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_{11}) + f(x_{12})] \approx 2.0001$$

7. $f(x) = \frac{4}{1+x^2}; h = \frac{1}{10}$

$x_0 = 0.0$	$f(x_0) = 4$	$x_6 = 0.6$	$f(x_6) \approx 2.9412$
$x_1 = 0.1$	$f(x_1) \approx 3.9604$	$x_7 = 0.7$	$f(x_7) \approx 2.6846$
$x_2 = 0.2$	$f(x_2) \approx 3.8462$	$x_8 = 0.8$	$f(x_8) \approx 2.4390$
$x_3 = 0.3$	$f(x_3) \approx 3.6697$	$x_9 = 0.9$	$f(x_9) \approx 2.2099$
$x_4 = 0.4$	$f(x_4) \approx 3.4483$	$x_{10} = 1.0$	$f(x_{10}) = 2$
$x_5 = 0.5$	$f(x_5) = 3.2$		

$$\int_0^1 \frac{4}{1+x^2} dx \approx \frac{1}{30} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_9) + f(x_{10})] \approx 3.1416$$

8. $f(x) = \cos(\sin x)$; $h = \frac{1}{10}$

$x_0 = 0.0$	$f(x_0) = 1$	$x_6 = 0.6$	$f(x_6) \approx 0.8448$
$x_1 = 0.1$	$f(x_1) \approx 0.9950$	$x_7 = 0.7$	$f(x_7) \approx 0.7996$
$x_2 = 0.2$	$f(x_2) \approx 0.9803$	$x_8 = 0.8$	$f(x_8) \approx 0.7535$
$x_3 = 0.3$	$f(x_3) \approx 0.9567$	$x_9 = 0.9$	$f(x_9) \approx 0.7086$
$x_4 = 0.4$	$f(x_4) \approx 0.9251$	$x_{10} = 1.0$	$f(x_{10}) \approx 0.6664$
$x_5 = 0.5$	$f(x_5) \approx 0.8873$		

$$\int_0^1 \cos(\sin x) dx \approx \frac{1}{20} [f(x_0) + 2f(x_1) + \dots + 2f(x_9) + f(x_{10})] \approx 0.8684$$

9. $f(x) = e^{-x^2}$

$$f'(x) = -2xe^{-x^2}$$

$$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = e^{-x^2}(4x^2 - 2)$$

$$|E_n| = \frac{1}{12n^2} \left| e^{-c^2}(4c^2 - 2) \right| \leq \frac{1}{12n^2} \cdot 2 = \frac{1}{6n^2}$$

$$\frac{1}{6n^2} \leq 0.01 \text{ when } n \geq 5.$$

$$h = \frac{1}{5} = 0.2$$

$x_0 = 0.0$	$f(x_0) = 1$	$x_3 = 0.6$	$f(x_3) \approx 0.6977$
$x_1 = 0.2$	$f(x_1) \approx 0.9608$	$x_4 = 0.8$	$f(x_4) \approx 0.5273$
$x_2 = 0.4$	$f(x_2) \approx 0.8521$	$x_5 = 1.0$	$f(x_5) \approx 0.3679$

$$\int_0^1 e^{-x^2} dx \approx \frac{0.2}{2} [f(x_0) + 2f(x_1) + \dots + 2f(x_4) + f(x_5)] \approx 0.74$$

10. $f(x) = e^{x^2}$

$$f'(x) = 2xe^{x^2}$$

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = e^{x^2}(4x^2 + 2)$$

$$|E_n| = \frac{(0.6)^3}{12n^2} \left| e^{c^2}(4c^2 + 2) \right|$$

$$\leq \frac{(0.6)^3}{12n^2} e^{0.36} [4(0.6)^2 + 2] \leq \frac{1.06502}{12n^2}$$

$$\frac{1.06502}{12n^2} \leq 0.01 \text{ when } n \geq 3.$$

$$h = \frac{0.6}{3} = 0.2$$

$x_0 = 0.0$	$f(x_0) = 1$
$x_1 = 0.2$	$f(x_1) \approx 1.0408$
$x_2 = 0.4$	$f(x_2) \approx 1.1735$
$x_3 = 0.6$	$f(x_3) \approx 1.4333$

$$\int_0^{0.6} e^{x^2} dx \approx \frac{0.2}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)] \approx 0.69$$

11. $f(x) = \sqrt{\cos x}$

$$f'(x) = -\frac{\sin x}{2\sqrt{\cos x}}$$

$$f''(x) = -\frac{2\cos^2 x + \sin^2 x}{4\cos^{3/2} x}$$

$$|E_n| = \frac{(0.5)^3}{12n^2} \left| \frac{2\cos^2 x + \sin^2 x}{4\cos^{3/2} x} \right|$$

$$\leq \frac{(0.5)^3}{12n^2} \left| \frac{2\cos^2 1 + \sin^2 1.5}{4\cos^{3/2} 1.5} \right| \leq \frac{2.6225}{12n^2}$$

$$\frac{2.6225}{12n^2} \leq 0.01 \text{ when } n \geq 5.$$

$$h = \frac{0.5}{5} = 0.1$$

$$\begin{aligned}
 x_0 &= 1.0 & f(x_0) &\approx 0.7351 \\
 x_1 &= 1.1 & f(x_1) &\approx 0.6735 \\
 x_2 &= 1.2 & f(x_2) &\approx 0.6020 \\
 x_3 &= 1.3 & f(x_3) &\approx 0.5172 \\
 x_4 &= 1.4 & f(x_4) &\approx 0.4123 \\
 x_5 &= 1.5 & f(x_5) &\approx 0.2660
 \end{aligned}$$

$$\int_1^{1.5} \sqrt{\cos x} \, dx \approx \frac{0.1}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)] \approx 0.27$$

$$12. \quad f(x) = \cos \sqrt{x}$$

$$f'(x) = -\frac{\sin \sqrt{x}}{2\sqrt{x}}$$

$$f''(x) = -\frac{\cos \sqrt{x}}{4x} + \frac{\sin \sqrt{x}}{4x^{3/2}}$$

$$|E_n| = \frac{1}{12n^2} \left| -\frac{\cos \sqrt{c}}{4c} + \frac{\sin \sqrt{c}}{4c^{3/2}} \right|$$

$$\leq \frac{1}{12n^2} \left(\left| \frac{\cos \sqrt{c}}{4c} \right| + \left| \frac{\sin \sqrt{c}}{4c^{3/2}} \right| \right) \leq \frac{1}{12n^2} \left(\frac{0.6}{4} + \frac{1}{4} \right)$$

$$= \frac{0.4}{12n^2}$$

$$\frac{0.4}{12n^2} \leq 0.01 \quad \text{when } n \geq 2.$$

$$h = \frac{1}{2} = 0.5$$

$$14. \quad f(x) = \ln x$$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f^{(3)}(x) = \frac{2}{x^3}$$

$$f^{(4)}(x) = -\frac{6}{x^4}$$

$$|E_n| = \frac{2^5}{180n^4} \left| \frac{6}{c^4} \right| \leq \frac{2^5}{180n^4} \left(\frac{6}{1} \right) = \frac{16}{15n^4}$$

$$\frac{16}{15n^4} \leq 0.005 \quad \text{when } n \geq 4.$$

$$h = \frac{2}{4} = 0.5$$

$$\int_1^3 \ln x \, dx \approx \frac{0.5}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \approx 1.295$$

$$\begin{aligned}
 x_0 &= 1.0 & f(x_0) &\approx 0.5403 \\
 x_1 &= 1.5 & f(x_1) &\approx 0.3392 \\
 x_2 &= 2.0 & f(x_2) &\approx 0.1559
 \end{aligned}$$

$$\int_1^2 \cos \sqrt{x} \, dx \approx \frac{0.5}{2} [f(x_0) + 2f(x_1) + f(x_2)] \approx 0.34$$

$$13. \quad f(x) = \frac{1+x}{1-x}$$

$$f'(x) = \frac{2}{(1-x)^2}$$

$$f''(x) = \frac{4}{(1-x)^3}$$

$$f^{(3)}(x) = \frac{12}{(1-x)^4}$$

$$f^{(4)}(x) = \frac{48}{(1-x)^5}$$

$$|E_n| = \frac{4^5}{180n^4} \left| \frac{48}{(1-c)^5} \right| \leq \frac{4^5}{180n^4} \left(\frac{48}{1} \right) = \frac{4096}{15n^4}$$

$$\frac{4096}{15n^4} \leq 0.005 \quad \text{when } n \geq 16.$$

$$h = \frac{4}{16} = 0.25$$

$$\begin{aligned}
 \int_2^6 \frac{1+x}{1-x} \, dx &\approx \frac{0.25}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots \\
 &\quad + 4f(x_{15}) + f(x_{16})] \approx -7.219
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \int_{m-h}^{m+h} (ax^2 + bx + c) dx &= \left[\frac{a}{3} x^3 + \frac{b}{2} x^2 + cx \right]_{m-h}^{m+h} \\
 &= \frac{a}{3} (m+h)^3 + \frac{b}{2} (m+h)^2 + c(m+h) - \frac{a}{3} (m-h)^3 - \frac{b}{2} (m-h)^2 - c(m-h) \\
 &= \frac{a}{3} (6m^2h + 2h^3) + \frac{b}{2} (4mh) + c(2h) = \frac{h}{3} [a(6m^2 + 2h^2) + b(6m) + 6] \\
 &= \frac{h}{3} [f(m-h) + 4f(m) + f(m+h)] \\
 &= \frac{h}{3} [a(m-h)^2 + b(m-h) + c + 4am^2 + 4bm + 4c + a(m+h)^2 + b(m+h) + c] \\
 &= \frac{h}{3} [a(6m^2 + 2h^2) + b(6m) + 6c]
 \end{aligned}$$

16. a. To show that the Parabolic Rule is exact, examine it on the interval $[m-h, m+h]$.

Let $f(x) = ax^3 + bx^2 + cx + d$, then

$$\begin{aligned}
 \int_{m-h}^{m+h} f(x) dx &= \frac{a}{4} [(m+h)^4 - (m-h)^4] + \frac{b}{3} [(m+h)^3 - (m-h)^3] + \frac{c}{2} [(m+h)^2 - (m-h)^2] + d[(m+h) - (m-h)] \\
 &= \frac{a}{4} (8m^3h + 8h^3m) + \frac{b}{3} (6m^2h + 2h^3) + \frac{c}{2} (4mh) + d(2h).
 \end{aligned}$$

The Parabolic Rule with $n = 2$ gives

$$\begin{aligned}
 \int_{m-h}^{m+h} f(x) dx &= \frac{h}{3} [f(m-h) + 4f(m) + f(m+h)] = 2am^3h + 2amh^3 + 2bm^2h + \frac{2}{3}bh^3 + 2chm + 2dh \\
 &= \frac{a}{4} (8m^3h + 8mh^3) + \frac{b}{3} (6m^2h + 2h^3) + \frac{c}{2} (4mh) + d(2h)
 \end{aligned}$$

which agrees with the direct computation. Thus, the Parabolic Rule is exact for any cubic polynomial.

- b. The error in using the Parabolic Rule is given by $E_n = -\frac{(l-k)^5}{180n^4} f^{(4)}(m)$ for some m between l and k .

However, $f'(x) = 3ax^2 + 2bx + c$, $f''(x) = 6ax + 2b$, $f^{(3)}(x) = 6a$, and $f^{(4)}(x) = 0$, so $E_n = 0$.

$$17. \quad f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$|E_n| = \frac{1}{12n^2} \left(\frac{2}{c^3} \right) \leq \frac{1}{6n^2}$$

$$\frac{1}{6n^2} \leq 10^{-10} \text{ when } n \geq 40825.$$

$$18. \quad f^{(3)}(x) = -\frac{6}{x^4}$$

$$f^{(4)}(x) = \frac{24}{x^5}$$

$$|E_n| = \frac{1}{180n^4} \left(\frac{24}{c^5} \right) \leq \frac{2}{15n^4}$$

$$\frac{2}{15n^4} \leq 10^{-10} \text{ when } n \geq 192.$$

19. Let $n = 2$.

$$f(x) = x^k; \quad h = a$$

$$x_0 = -a \quad f(x_0) = -a^k$$

$$x_1 = 0 \quad f(x_1) = 0$$

$$x_2 = a \quad f(x_2) = a^k$$

$$\int_{-a}^a x^k dx \approx \frac{a}{2} [-a^k + 2 \cdot 0 + a^k] = 0$$

$$\int_{-a}^a x^k dx = \left[\frac{1}{k+1} x^{k+1} \right]_{-a}^a = \frac{1}{k+1} [a^{k+1} - (-a)^{k+1}] = \frac{1}{k+1} [a^{k+1} - a^{k+1}] = 0$$

20. a. From Example 1, $T \approx 48.9414$.

$$f'(x) = 4x^3$$

$$T - \frac{[4(3)^3 - 4(1)^3](0.25)^2}{12} \approx 48.9414 - 0.5417 = 48.3997$$

- b. From Problem 5, $T \approx 1.9886$.

$$f'(x) = \cos x$$

$$T - \frac{[\cos \pi - \cos 0] \left(\frac{\pi}{12} \right)^2}{12} \approx 1.9886 + 0.0114 = 2.0000$$

$$21. \quad A \approx \frac{10}{2} [75 + 2 \cdot 71 + 2 \cdot 60 + 2 \cdot 45 + 2 \cdot 45 + 2 \cdot 52 + 2 \cdot 57 + 2 \cdot 60 + 59] = 4570 \text{ ft}^2$$

$$22. \quad A \approx \frac{3}{2} [23 + 4 \cdot 24 + 2 \cdot 23 + 4 \cdot 21 + 2 \cdot 18 + 4 \cdot 15 + 2 \cdot 12 + 4 \cdot 11 + 2 \cdot 10 + 4 \cdot 8 + 0] = 465 \text{ ft}^2$$

$$V = A \cdot 6 \approx 2790 \text{ ft}^3$$

$$23. \quad A \approx \frac{20}{3} [0 + 4 \cdot 7 + 2 \cdot 12 + 4 \cdot 18 + 2 \cdot 20 + 4 \cdot 20 + 2 \cdot 17 + 4 \cdot 10 + 0] = 2120 \text{ ft}^2$$

$$4 \text{ mi/h} = 21,120 \text{ ft/h}$$

$$(2120)(21,120)(24) = 1,074,585,600 \text{ ft}^3$$

$$24. \quad h = \frac{210 - 0}{21} = 10 \text{ minutes} = \frac{1}{6} \text{ hour}$$

Suppose t is time measured in hours and $v(t)$ is the velocity at time t . Then the distance traveled is

$$\int_0^{3.5} v(t) dt \approx \frac{1/6}{2} [0 + 2 \cdot 55 + 2 \cdot 57 + 2 \cdot 60 + 2 \cdot 70 + 2 \cdot 70 + 2 \cdot 70 + 2 \cdot 70 + 2 \cdot 19 + 2 \cdot 0$$

$$+ 2 \cdot 59 + 2 \cdot 63 + 2 \cdot 65 + 2 \cdot 62 + 2 \cdot 0 + 2 \cdot 0 + 2 \cdot 0 + 2 \cdot 22 + 2 \cdot 38 + 2 \cdot 35 + 2 \cdot 25 + 0]$$

$$= 140$$

$$25. \quad h = \frac{24 - 0}{8} = 3 \text{ minutes} = \frac{1}{20} \text{ hour}$$

Suppose t is time measured in hours and $v(t)$ is the velocity at time t .

- a. Using the Trapezoidal Rule, the distance traveled is

$$\int_0^{0.4} v(t) dt \approx \frac{1/20}{2} [0 + 2 \cdot 31 + 2 \cdot 54 + 2 \cdot 53 + 2 \cdot 52 + 2 \cdot 35 + 2 \cdot 31 + 2 \cdot 28 + 0]$$

$$= \frac{464}{40} = 11.6 \text{ miles}$$

- b. Using the Parabolic Rule, the distance traveled is

$$\int_0^{0.4} v(t) dt \approx \frac{1/20}{3} [1 \cdot 0 + 4 \cdot 31 + 2 \cdot 54 + 4 \cdot 53 + 2 \cdot 52 + 4 \cdot 35 + 2 \cdot 31 + 4 \cdot 28 + 1 \cdot 0]$$

$$= \frac{568}{40} = 14.2 \text{ miles.}$$

26. Suppose t is time measured in hours and $r(t)$ is the rate of water usage at time t .

$$h = \frac{120}{10} \text{ minutes} = \frac{1}{5} \text{ hour}$$

- a. Using the Trapezoidal Rule, the total water used is

$$\int_0^2 r(t) dt \approx \frac{1/5}{2} [65 + 2 \cdot 71 + 2 \cdot 68 + 2 \cdot 78 + 2 \cdot 105 + 2 \cdot 111 + 2 \cdot 108 + 2 \cdot 144 + 2 \cdot 160 + 2 \cdot 152 + 148]$$

$$= \frac{1}{10} \cdot 2207 = 220.7 \text{ gallons}$$

- b. Using the Parabolic Rule, the total water used is

$$\int_0^2 r(t) dt \approx \frac{1/5}{2} [65 + 4 \cdot 71 + 2 \cdot 68 + 4 \cdot 78 + 2 \cdot 105 + 4 \cdot 111 + 2 \cdot 108 + 4 \cdot 144 + 2 \cdot 160 + 4 \cdot 152 + 148]$$

$$= \frac{3319}{15} \approx 221.27 \text{ gallons.}$$

27. a. Lay the part with the long flat side along the x -axis, with the upper left corner (as shown in Figure 22) at the origin. Let $h(x)$ denote the height of the lamina at the value of x . Then, using the Trapezoidal Rule,

$$m = \int_0^{40} h(x) dx \approx \frac{5}{2} [5 + 2(6.5 + 8 + 9 + 10 + 10.5 + 10.5 + 10) + 8] = 355$$

$$M_y = \int_0^{40} x h(x) dx \approx \frac{5}{2} [0 \cdot 5 + 2(5 \cdot 6.5 + 10 \cdot 8 + 15 \cdot 9 + 20 \cdot 10 + 25 \cdot 10.5 + 30 \cdot 10.5 + 35 \cdot 10) + 40 \cdot 8] = 7675$$

$$\bar{x} = \frac{M_y}{m} \approx \frac{7675}{355} \approx 21.62$$

$$M_x = \frac{1}{2} \int_0^{40} h^2(x) dx \approx \frac{1}{2} \frac{5}{2} [5^2 + 2(6.5^2 + 8^2 + 9^2 + 10^2 + 10.5^2 + 10.5^2 + 10^2) + 8^2] = 1630.625$$

$$\bar{y} = \frac{M_x}{m} \approx \frac{1630.625}{355} = 4.59$$

- b. Using the Parabolic Rule,

$$m = \int_0^{40} h(x) dx \approx \frac{5}{3} [5 + 4 \cdot 6.5 + 2 \cdot 8 + 4 \cdot 9 + 2 \cdot 10 + 4 \cdot 10.5 + 2 \cdot 10.5 + 4 \cdot 10 + 8] = \frac{5}{3} 214 \approx 356.67$$

$$M_y = \int_0^{40} x h(x) dx$$

$$\approx \frac{5}{3} [0 \cdot 5 + 4(5 \cdot 6.5) + 2(10 \cdot 8) + 4(15 \cdot 9) + 2(20 \cdot 10) + 4(25 \cdot 10.5) + 2(30 \cdot 10.5) + 4(35 \cdot 10) + 40 \cdot 8]$$

$$= \frac{5}{3} 4630 \approx 7716.67$$

$$\bar{x} = \frac{M_y}{m} \approx \frac{7716.67}{356.67} \approx 21.64$$

$$M_x = \frac{1}{2} \int_0^{40} h^2(x) dx$$

$$\approx \frac{1}{2} \frac{5}{3} [5^2 + 4(6.5^2) + 2(8^2) + 4(9^2) + 2(10^2) + 4(10.5^2) + 2(10.5^2) + 4(10^2) + 8^2]$$

$$= \frac{5}{6} 1971.5 \approx 1642.9$$

$$\bar{y} = \frac{M_x}{m} \approx \frac{1642.9}{356.67} = 4.61$$

28. a. Place the lamina so that the origin is the center of the hole, and the long straight side is parallel to the x -axis. Let $h(x)$ denote the height of the lamina at the point x . Let R_1 be the lamina with the hole drilled in it, let R_2 be the lamina consisting of just the hole, and let R_3 be the lamina from Problem 27 without the hole. Using the Trapezoidal Rule,

$$m(R_1) = m(R_3) - m(R_2) \approx 355 - \pi(2.5^2) \approx 335.37$$

$$M_y(R_3) = \int_{-30}^{10} x h(x) dx$$

$$\approx \frac{5}{2}[-30 \cdot 5 + 2(-25 \cdot 6.5 - 20 \cdot 8 - 15 \cdot 9 - 10 \cdot 10 - 5 \cdot 10.5 + 0 \cdot 10.5 + 5 \cdot 10) + 10 \cdot 8] = -2975$$

$$M_x(R_3) = \frac{1}{2} \int_{-30}^{10} [f^2(x) - g^2(x)] dx = \frac{1}{2} \int_{-30}^{10} [(h(x) - 4)^2 - (-4)^2] dx = \frac{1}{2} \int_{-30}^{10} [h^2(x) - 8h(x)] dx$$

$$\approx \frac{5}{4} \left[5^2 + 2(6.5^2 + 8^2 + 9^2 + 10^2 + 10.5^2 + 10.5^2 + 10^2) + 8^2 \right] - 8 \cdot \frac{5}{4} [5 + 2(6.5 + 8 + 10 + 10.5 + 10.5 + 10) + 8]$$

$$\approx 1630.625 - 4 \cdot 355 = 210.625$$

By symmetry, the centroid of R_2 is $(0,0)$. Thus,

$$M_y(R_2) = 0, \text{ and } M_x(R_2) = 0$$

$$M_y(R_1) = M_y(R_3) - M_y(R_2) \approx -2975 - 0 = -2975$$

$$M_x(R_1) = M_x(R_3) - M_x(R_2) \approx 210.625 - 0 = 210.625$$

$$\bar{x}(R_3) = \frac{M_y}{m} \approx \frac{-2975}{335.37} \approx -8.87$$

$$\bar{y}(R_3) = \frac{M_x(R_3)}{m(R_3)} \approx \frac{210.625}{335.37} \approx 0.63$$

- b. Using the Parabolic Rule,

$$m(R_1) = m(R_3) - m(R_2) \approx 356.67 - \pi(2.5^2) \approx 337.04$$

$$M_y(R_3) = \int_{-30}^{10} x h(x) dx$$

$$\approx \frac{5}{6}[-30 \cdot 5 + 4(-25 \cdot 6.5) + 2(-20 \cdot 8) + 4(-15 \cdot 9) + 2(-10 \cdot 10) + 4(-5 \cdot 10.5) + 2(0 \cdot 10.5) + 4(5 \cdot 10) + 10 \cdot 8]$$

$$= \frac{5}{6}(-1790) \approx -1491.67$$

$$M_x(R_3) = \frac{1}{2} \int_{-30}^{10} [f^2(x) - g^2(x)] dx = \frac{1}{2} \int_{-30}^{10} [(h(x) - 4)^2 - (-4)^2] dx = \frac{1}{2} \int_{-30}^{10} [h^2(x) - 8h(x)] dx$$

$$\approx \frac{5}{6} \left[5^2 + 4(6.5^2) + 2(8^2) + 4(9^2) + 2(10^2) + 4(10.5^2) + 2(10.5^2) + 4(10^2) + 8^2 \right]$$

$$- 8 \cdot \frac{5}{6} [5 + 4 \cdot 6.5 + 2 \cdot 8 + 4 \cdot 10 + 2 \cdot 10.5 + 4 \cdot 10.5 + 2 \cdot 10 + 8]$$

$$\approx 1642.92 - 1426.67 = 216.25$$

By symmetry, the centroid of R_2 is $(0,0)$. Thus,

$$M_y(R_2) = 0, \text{ and } M_x(R_2) = 0$$

$$M_y(R_1) = M_y(R_3) - M_y(R_2) \approx -1491.67$$

$$M_x(R_1) = M_x(R_3) - M_x(R_2) \approx 216.25 - 0 = 216.25$$

$$\bar{x}(R_3) = \frac{M_y}{m} \approx \frac{-1491.67}{337.04} \approx -4.43$$

$$\bar{y}(R_3) = \frac{M_x(R_3)}{m(R_3)} \approx \frac{216.25}{337.04} \approx 0.64$$

29. Rotate the map 90° clockwise and put the x -axis along the bottom, with the origin below the southernmost tip of Illinois. Let $f(x)$ denote the "upper" function, and $g(x)$ the "lower" function. Then the east/west dimensions given

on the map are $f(x)-g(x)$, and $g(x)$ is equal to 85, 50, 30, 25, 15, and 10 at the southern end of the state, and 10 and 13 at the northern end.

$$m = \int_0^{380} [f(x) - g(x)] dx \approx \frac{20}{2} [0 + 2(58 + 79 + \dots + 139 + 132) + 140] = 57,000$$

$$M_y = \int_0^{380} x[f(x) - g(x)] dx \approx \frac{20}{2} [0 \cdot 0 + 2(20 \cdot 58 + 40 \cdot 79 + \dots + 340 \cdot 139 + 360 \cdot 132) + 380 \cdot 140] \approx 12,002,800$$

$$M_x = \frac{1}{2} \int_0^{380} [f^2(x) - g^2(x)] dx \approx \frac{1}{2} \frac{20}{2} \left\{ (85^2 - 85^2) + 2[(108^2 - 50^2) + (109^2 - 30^2) + \dots + (142^2 - 10^2)] \right\}$$

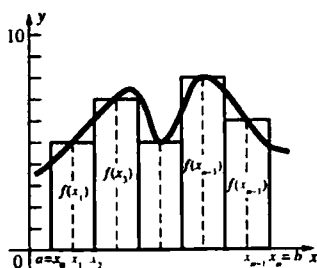
$$\approx 4,994,480$$

$$\bar{x} = \frac{M_y}{m} \approx \frac{12,002,800}{57,000} \approx 210.6$$

$$\bar{y} = \frac{M_x}{m} \approx \frac{4,994,480}{57,000} \approx 87.6$$

This is a point just southeast of Lincoln, IL and it is about 30 miles northeast of Springfield.

30. a.



b. $n = 16$, $h = \frac{3-1}{16} = 0.125$

$$x_1 = 1.125 \quad f(x_1) \approx 1.6018$$

$$x_3 = 1.375 \quad f(x_3) \approx 3.5745$$

$$x_5 = 1.625 \quad f(x_5) \approx 6.9729$$

$$x_7 = 1.875 \quad f(x_7) \approx 12.3596$$

$$x_9 = 2.125 \quad f(x_9) \approx 20.3909$$

$$x_{11} = 2.375 \quad f(x_{11}) \approx 31.8167$$

$$x_{13} = 2.625 \quad f(x_{13}) \approx 47.4807$$

$$x_{15} = 2.875 \quad f(x_{15}) \approx 68.3206$$

$$\int_1^3 x^4 dx \approx 0.25[f(x_1) + f(x_3) + \dots + f(x_{15})] \approx 48.1294$$

31. a. Since $x > X > 0$, $x^2 > Xx$. Thus $e^{-x^2} < e^{-Xx}$.

$$\int_X^\infty e^{-x^2} dx < \int_X^\infty e^{-Xx} dx = \left[-\frac{1}{X} e^{-Xx} \right]_X^\infty = \frac{1}{X} e^{-X^2}$$

b. $\int_5^\infty e^{-x^2} dx < \int_5^\infty e^{-5x} dx = \frac{1}{5} e^{-25} \approx 2.78 \times 10^{-12} < 10^{-11}$

c. $n = 10$, $h = \frac{5-0}{10} = 0.5$

$$x_0 = 0.0 \quad f(x_0) = 1$$

$$x_1 = 0.5 \quad f(x_1) \approx 0.7788$$

$$x_2 = 1.0 \quad f(x_2) \approx 0.3679$$

$$x_3 = 1.5 \quad f(x_3) \approx 0.1054$$

$$x_4 = 2.0 \quad f(x_4) \approx 0.0183$$

$$x_5 = 2.5 \quad f(x_5) \approx 0.0019$$

$$x_6 = 3.0 \quad f(x_6) \approx 1.234 \times 10^{-4}$$

$$x_7 = 3.5 \quad f(x_7) \approx 4.785 \times 10^{-6}$$

$$x_8 = 4.0 \quad f(x_8) \approx 1.125 \times 10^{-7}$$

$$x_9 = 4.5 \quad f(x_9) \approx 1.605 \times 10^{-9}$$

$$x_{10} = 5.0 \quad f(x_{10}) \approx 1.389 \times 10^{-11}$$

$$\int_0^5 e^{-x^2} dx \approx \frac{0.5}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_9) + f(x_{10})] \approx 0.8862$$

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx \approx \frac{2}{\sqrt{\pi}} (0.8862) \approx 0.99997$$

$$f(x) = e^{-x^2}$$

$$f'(x) = -2xe^{-x^2}$$

$$f''(x) = e^{-x^2} (4x^2 - 2)$$

$$f^{(3)}(x) = xe^{-x^2} (12 - 8x^2)$$

$$f^{(4)}(x) = e^{-x^2} (16x^4 - 48x^2 + 12)$$

$$|E_{10}| = \frac{5^5}{180 \times 10^4} \left| \frac{16c^4 - 48c^2 + 12}{e^{c^2}} \right| \leq \frac{5^5}{1,800,000} \cdot 12$$

A graphing utility or computer shows that

$$\left| \frac{16c^4 - 48c^2 + 12}{e^{c^2}} \right| \leq 12 \text{ on } [0, 5].$$

$$|E_{10}| \leq 0.0209$$

Thus, the error in computing

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx \text{ is less than } \frac{2}{\sqrt{\pi}} (0.0209 + 10^{-11}) \approx 0.0236.$$

32. a. First consider $t = \frac{x}{1+x}$. At $x = 0$, $t = 0$. $\frac{dt}{dx} = \frac{1}{(1+x)^2} > 0$ on $[0, \infty)$. $\lim_{x \rightarrow \infty} \frac{x}{1+x} = 1$. Since $t = \frac{x}{1+x}$ is increasing to 1 on $[0, \infty)$, the substitution transforms $[0, \infty)$ into $[0, 1)$.

Now consider $t = e^{-x}$. At $x = 0$, $t = 1$. $\frac{dt}{dx} = -e^{-x} < 0$ on $[0, \infty)$. $\lim_{x \rightarrow \infty} e^{-x} = 0$. Since $t = e^{-x}$ is decreasing to 0 on $[0, \infty)$, the substitution transforms $[0, \infty)$ into $(0, 1]$.

- b. $\frac{dt}{dx} = \frac{2e^x}{(e^x + 1)^2} > 0$ on $(-\infty, \infty)$. $\lim_{x \rightarrow -\infty} \frac{e^x - 1}{e^x + 1} = \frac{-1}{1} = -1$. $\lim_{x \rightarrow \infty} \frac{e^x - 1}{e^x + 1} = \lim_{x \rightarrow \infty} \frac{e^x}{e^x} = 1$. Since $t = \frac{e^x - 1}{e^x + 1}$ is increasing from -1 to 1 on $(-\infty, \infty)$, the substitution transforms $(-\infty, \infty)$ into $(-1, 1)$.

- c. $t = e^{-x}$, $dt = -e^{-x} dx$

$$\int_0^\infty \frac{e^{-x}}{\sqrt{1+e^{-2x}}} dx = \int_1^0 -\frac{1}{\sqrt{1+t^2}} dt = \int_0^1 \frac{1}{\sqrt{1+t^2}} dt$$

$$f(t) = \frac{1}{\sqrt{1+t^2}}, h = 0.1$$

i	t_i	$f(t_i)$
0	0.0	1.
1	0.1	0.99504
2	0.2	0.98058
3	0.3	0.95783
4	0.4	0.92848
5	0.5	0.89443
6	0.6	0.85749
7	0.7	0.81923
8	0.8	0.78087
9	0.9	0.74329
10	1.0	0.70711

$$\int_0^1 \frac{1}{\sqrt{1+t^2}} dt \approx \frac{0.1}{3} [f(t_0) + 4f(t_1) + 2f(t_2) + \dots + 4f(t_9) + f(t_{10})]$$

$$\approx 0.881374$$

$$f^{(4)}(t) = \frac{3(3 - 24t^2 + 8t^4)}{(1+t^2)^{9/2}}$$

$$|E_{10}| \leq \frac{1}{180(10^4)} \frac{3(3+24+8)}{1} < 5.84 \times 10^{-5}$$

d. $\frac{1}{t} = x, -\frac{1}{t^2} dt = dx$

$$\int_1^\infty \frac{x}{1+x^3} dx = \int_1^0 \frac{\frac{1}{t}}{1+(\frac{1}{t})^3} \left(-\frac{1}{t^2}\right) dt = \int_0^1 \frac{1}{t^3+1} dt \quad f(t) = \frac{1}{t^3+1}, \quad h = 0.1$$

i	t_i	$f(t_i)$
0	0.0	1
1	0.1	0.99900
2	0.2	0.99206
3	0.3	0.97371
4	0.4	0.93985
5	0.5	0.88889
6	0.6	0.82237
7	0.7	0.74460
8	0.8	0.66138
9	0.9	0.57837
10	1.0	0.5

$$\int_0^1 \frac{1}{t^3+1} dt \approx \frac{0.1}{3} [f(t_0) + 4f(t_1) + 2f(t_2) + \dots + 4f(t_9) + f(t_{10})]$$

$$\approx 0.835653$$

33. a. $u = \frac{2}{\sqrt{4+x}} \quad dv = \frac{1}{2\sqrt{x}} dx$

$du = -\frac{1}{(4+x)^{3/2}} dx \quad v = \sqrt{x}$

$\int_0^2 \frac{dx}{\sqrt{4x+x^2}} = \left[\frac{2\sqrt{x}}{\sqrt{4+x}} \right]_0^2 + \int_0^2 \frac{\sqrt{x}}{(4+x)^{3/2}} dx = \frac{2}{\sqrt{3}} + \int_0^2 \frac{\sqrt{x}}{(4+x)^{3/2}} dx$

b. $u = \frac{1}{x} \quad dv = \sin x dx$

$du = -\frac{1}{x^2} dx \quad v = -\cos x$

$\int_1^\infty \frac{\sin x}{x} dx = \left[-\frac{\cos x}{x} \right]_1^\infty - \int_1^\infty \frac{\cos x}{x^2} dx = \cos 1 - \int_1^\infty \frac{\cos x}{x^2} dx$

$|\cos x| \leq 1$ for all x , so

$\int_1^\infty \frac{\cos x}{x^2} dx \leq \int_1^\infty \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^\infty = 1.$

Therefore, $\int_1^\infty \frac{\sin x}{x} dx$ exists.

c. $u = \frac{1}{4+x} \quad dv = \frac{1}{\sqrt{x}} dx$

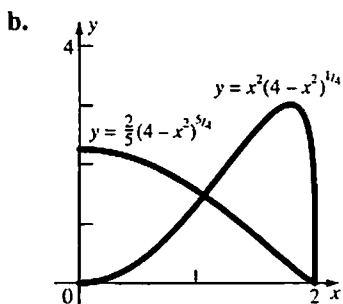
$du = -\frac{1}{(4+x)^2} dx \quad v = 2\sqrt{x}$

$\int_0^1 \frac{1}{\sqrt{x}} \frac{1}{4+x} dx = \left[\frac{2\sqrt{x}}{4+x} \right]_0^1 + \int_0^1 \frac{2\sqrt{x}}{(4+x)^2} dx = \frac{2}{5} + \int_0^1 \frac{2\sqrt{x}}{(4+x)^2} dx$

34. a. $u = x \quad dv = x(4-x^2)^{1/4} dx$

$du = dx \quad v = -\frac{2}{5}(4-x^2)^{5/4}$

$\int_0^2 x^2(4-x^2)^{1/4} dx = \left[-\frac{2}{5}x(4-x^2)^{5/4} \right]_0^2 + \int_0^2 \frac{2}{5}(4-x^2)^{5/4} dx = \int_0^2 \frac{2}{5}(4-x^2)^{5/4} dx$



c. First use the Trapezoidal Rule on the first part.

$f(x) = \frac{2}{5}(4-x^2)^{5/4}$

$x_0 = 0.00 \quad f(x_0) \approx 2.26274$

$x_1 = 0.25 \quad f(x_1) \approx 2.21863$

$$x_2 = 0.50 \quad f(x_2) \approx 2.08737$$

$$x_3 = 0.75 \quad f(x_3) \approx 1.87225$$

$$x_4 = 1.00 \quad f(x_4) \approx 1.57929$$

$$\int_0^1 \frac{2}{5} (4-x^2)^{5/4} dx \approx \frac{0.25}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)] \approx 2.02482$$

$$f''(x) = \frac{3x^2 - 8}{2(4-x^2)^{3/4}}$$

$$|E_4| = \frac{1}{12 \cdot 4^2} |f''(c)| \leq \frac{1}{192} \frac{8}{2(3)^{3/4}} < 0.00914$$

Next use integration by parts on the second part.

$$u = \frac{1}{x} \quad dv = x(4-x^2)^{5/4}$$

$$du = -\frac{1}{x^2} dx \quad v = -\frac{2}{9} (4-x^2)^{9/4}$$

$$= \frac{2}{5} \left(2\sqrt[4]{3} - \int_1^2 \frac{2}{9x^2} (4-x^2)^{9/4} dx \right) = \frac{4\sqrt[4]{3}}{5} - \int_1^2 \frac{4}{45x^2} (4-x^2)^{9/4} dx$$

Use the Trapezoidal Rule on the integral.

$$f(x) = \frac{4}{45x^2} (4-x^2)^{9/4}$$

$$x_0 = 1.00 \quad f(x_0) \approx 1.05286$$

$$x_1 = 1.25 \quad f(x_1) \approx 0.42233$$

$$x_2 = 1.50 \quad f(x_2) \approx 0.13916$$

$$x_3 = 1.75 \quad f(x_3) \approx 0.02510$$

$$x_4 = 2.00 \quad f(x_4) = 0$$

$$\int_1^2 \frac{4}{45x^2} (4-x^2)^{9/4} dx \approx \frac{0.25}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)] \approx 0.27825$$

$$f''(x) = \frac{(5x^4 + 8x^2 + 128)(4-x^2)^{1/4}}{15x^4}$$

$$|E_4| = \frac{1}{12 \cdot 4^2} |f''(c)| \leq \frac{1}{192} \frac{240(3)^{1/4}}{15} < 0.10968$$

Now estimate the value of the original integral.

$$\int_0^2 \frac{2}{5} (4-x^2)^{5/4} dx = \int_0^1 \frac{2}{5} (4-x^2)^{5/4} dx + \frac{4\sqrt[4]{3}}{5} - \int_1^2 \frac{4}{45x^2} (4-x^2)^{9/4} dx \approx 2.02482 + \frac{4\sqrt[4]{3}}{5} - 0.27825 \approx 2.7994$$

$$|\text{Error}| \leq 0.00914 + 0.10968 = 0.11882$$

The error estimate for the original integral would fail since $f''(x)$ is not defined at 2, so the error estimate would be unbounded.

$$35. \int_0^{\pi/2} \ln(\sin x) dx = \int_0^{\pi/2} \ln x dx + \int_0^{\pi/2} \ln\left(\frac{\sin x}{x}\right) dx$$

$$\int_0^{\pi/2} \ln x dx = \lim_{a \rightarrow 0} \int_a^{\pi/2} \ln x dx = \lim_{a \rightarrow 0} [x \ln x - x]_a^{\pi/2} = \left(\frac{\pi}{2} \ln \frac{\pi}{2} - \frac{\pi}{2} \right) - \lim_{a \rightarrow 0} a \ln a = \frac{\pi}{2} \ln \frac{\pi}{2} - \frac{\pi}{2} \approx -0.8615$$

$$f(x) = \ln\left(\frac{\sin x}{x}\right), h = \frac{\pi}{8} \approx 0.3927$$

$$\text{Note that } \lim_{x \rightarrow 0} \ln\left(\frac{\sin x}{x}\right) = \ln 1 = 0.$$

i	x_i	$f(x_i)$
0	0.0000	0
1	0.3927	-0.02584
2	0.7854	-0.10501
3	1.1781	-0.24307
4	1.5708	-0.45158

$$\int_0^{\pi/2} \ln\left(\frac{\sin x}{x}\right) dx \approx \frac{\pi}{24} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \approx -0.2274$$

$$\int_0^{\pi/2} \ln(\sin x) dx \approx -1.0889$$

36. Suppose $X = \frac{(2n+1)\pi}{2}$. Since the cosine function alternates in sign and $\left| \frac{\cos x}{1+x^4} \right|$ is decreasing,

$$\left| \int_X^{\infty} \frac{\cos x}{1+x^4} dx \right| < \left| \int_X^{X+\pi} \frac{\cos x}{1+x^4} dx \right| < (X + \pi - X) \frac{1}{1 + \left[\left(n + \frac{1}{2} \right) \pi \right]^4} = \frac{\pi}{1 + \pi^4 \left(n + \frac{1}{2} \right)^4}$$

$$\frac{\pi}{1 + \pi^4 \left(n + \frac{1}{2} \right)^4} < 10^{-5} \text{ when } n \geq 8. \text{ Let } X = \frac{17\pi}{2}$$

$$\int_0^{\infty} \frac{\cos x}{1+x^4} dx = \int_0^{17\pi/2} \frac{\cos x}{1+x^4} dx + \int_{17\pi/2}^{\infty} \frac{\cos x}{1+x^4} dx$$

Use the Parabolic Rule on the first part with $h = \frac{\pi}{8}$, so $n = 68$.

Calculating with a computer,

$$\int_0^{17\pi/2} \frac{\cos x}{1+x^4} dx \approx \frac{\pi}{24} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_{67}) + f(x_{68})] \approx 0.77335$$

Using a computer to plot $|f^{(4)}(x)|$, we see that $|f^{(4)}(x)| < 65$ on $\left[0, \frac{17\pi}{2}\right]$.

$$|E_{68}| < \frac{\left(\frac{17\pi}{2}\right)^5}{180 \cdot 68^4} 65 < 0.22933$$

$$\text{Hence, } |\text{Error}| < 0.22933 + 10^{-5} = 0.22934$$

11.3 Concepts Review

1. slowness of convergence
2. root; Intermediate Value
3. algorithms
4. $x_1, j; f'(r) = 0$

Problem Set 11.3

1. Let $f(x) = x^3 + 2x - 6$.
 $f(1) = -3, f(2) = 6$

n	h_n	m_n	$f(m_n)$
1	0.5	1.5	0.375
2	0.25	1.25	-1.546875
3	0.125	1.375	-0.650391
4	0.0625	1.4375	-0.154541
5	0.03125	1.46875	0.105927
6	0.015625	1.45312	-0.0253716
7	0.0078125	1.46094	0.04001
8	0.00390625	1.45703	0.00725670
9	0.00195312	1.45508	-0.00907617

$r \approx 1.46$

2. Let $f(x) = x^4 + 5x^3 + 1$.
 $f(-1) = -3, f(0) = 1$

n	h_n	m_n	$f(m_n)$
1	0.5	-0.5	0.4375
2	0.25	-0.75	-0.792969
3	0.125	-0.625	-0.0681152
4	0.0625	-0.5625	0.21022
5	0.03125	-0.59375	0.0776834
6	0.015625	-0.609375	0.00647169
7	0.0078125	-0.617187	-0.0303962
8	0.00390625	-0.613281	-0.011854
9	0.00195312	-0.611328	-0.00266589

$r \approx -0.61$

3. Let $f(x) = 2 \cos x - e^{-x}$.
 $f(1) \approx 0.712725, f(2) \approx -0.967629$

n	h_n	m_n	$f(m_n)$
1	0.5	1.5	-0.0816558
2	0.25	1.25	0.34414
3	0.125	1.375	0.136256
4	0.0625	1.4375	0.0282831
5	0.03125	1.46875	-0.0264745
6	0.015625	1.45313	0.000961516
7	0.0078125	1.46094	-0.0127427

8	0.00390625	1.45703	-0.00588708
9	0.00195312	1.45508	-0.0024619
10	0.000976562	1.4541	-0.000749968
11	0.000488281	1.45361	0.00010583

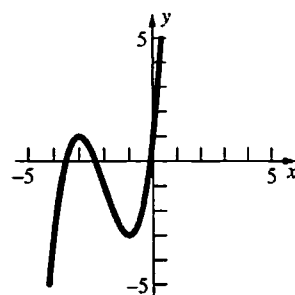
$r \approx 1.45$

4. Let $f(x) = x - 2 + 2 \ln x$.
 $f(1) = -1, f(2) \approx 1.38629$

n	h_n	m_n	$f(m_n)$
1	0.5	1.5	0.31093
2	0.25	1.25	-0.303713
3	0.125	1.375	0.0119075
4	0.0625	1.3125	-0.143633
5	0.03125	1.34375	-0.0653216
6	0.015625	1.35938	-0.0265749
7	0.0078125	1.36719	-0.00730108
8	0.00390625	1.37109	0.00231131
9	0.00195312	1.36914	-0.00249285

$r \approx 1.37$

5. Let $f(x) = x^3 + 6x^2 + 9x + 1 = 0$.

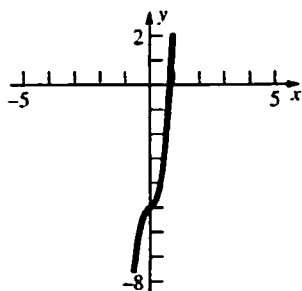


$$f'(x) = 3x^2 + 12x + 9$$

n	x_n
1	0
2	-0.111111
3	-0.1205484
4	-0.1206148
5	-0.1206148

$r \approx -0.12061$

6. Let $f(x) = 7x^3 + x - 5$

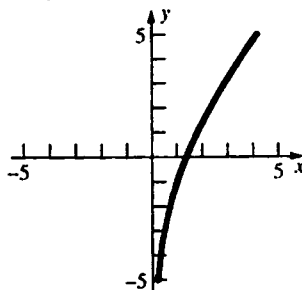


$$f'(x) = 21x^2 + 1$$

n	x_n
1	1
2	0.8636364
3	0.8412670
4	0.8406998
5	0.8406994
6	0.8406994

$$r \approx 0.84070$$

7. Let $f(x) = x - 2 + 2 \ln x$.

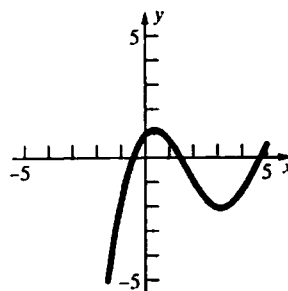


$$f'(x) = 1 + \frac{2}{x}$$

n	x_n
1	1.5
2	1.366744
3	1.370151
4	1.370154
5	1.370154

$$r \approx 1.37015$$

8. Let $f(x) = 2 \cos x - e^{-x}$.

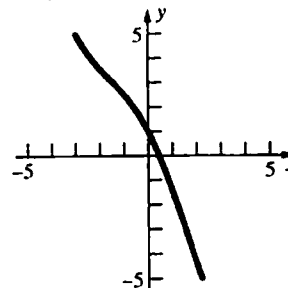


$$f'(x) = -2 \sin x + e^{-x}$$

n	x_n
1	1.5
2	1.453915
3	1.453674
4	1.453674

$$r \approx 1.45367$$

9. Let $f(x) = \cos x - 2x$.

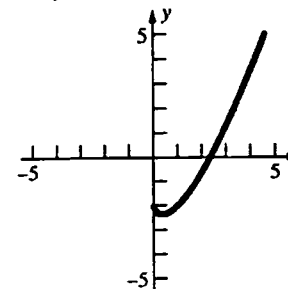


$$f'(x) = -\sin x - 2$$

n	x_n
1	0.5
2	0.4506267
3	0.4501836
4	0.4501836

$$r \approx 0.45018$$

10. Let $f(x) = x \ln x - 2$.

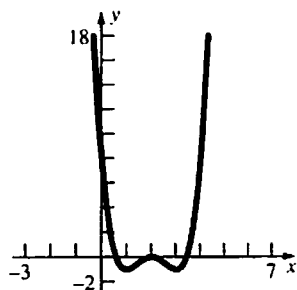


$$f'(x) = \ln x + 1$$

n	x_n
1	2.5
2	2.348287
3	2.345751
4	2.345751

$r \approx 2.34575$

11. Let $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 8$.



$$f'(x) = 4x^3 - 24x^2 + 44x - 24$$

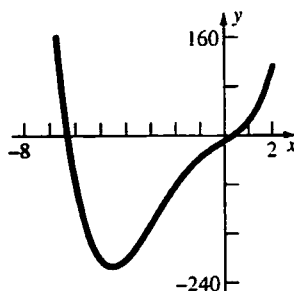
Note that $f(2) = 0$.

n	x_n
1	0.5
2	0.575
3	0.585586
4	0.585786

n	x_n
1	3.5
2	3.425
3	3.414414
4	3.414214
5	3.414214

$r = 2, r \approx 0.58579, r \approx 3.41421$

12. Let $f(x) = x^4 + 6x^3 + 2x^2 + 24x - 8$.



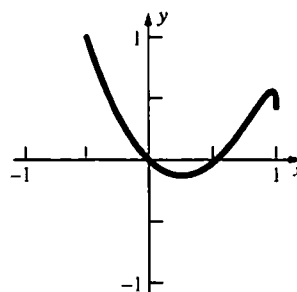
$$f'(x) = 4x^3 + 18x^2 + 4x + 24$$

n	x_n
1	-6.5
2	-6.3299632
3	-6.3167022
4	-6.3166248
5	-6.3166248

n	x_n
1	0.5
2	0.3286290
3	0.3166694
4	0.3166248
5	0.3166248

$r \approx -6.31662, r \approx 0.31662$

13. Let $f(x) = 2x^2 - \sin^{-1} x$.

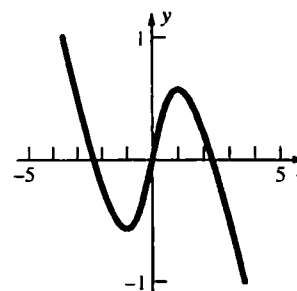


$$f'(x) = 4x - \frac{1}{\sqrt{1-x^2}}$$

n	x_n
1	0.5
2	0.527918
3	0.526583
4	0.526580
5	0.526580

$r \approx 0.52658$

14. Let $f(x) = 2 \tan^{-1} x - x$.



$$f'(x) = \frac{2}{1+x^2} - 1$$

n	x_n
1	2.5
2	2.335087
3	2.331125
4	2.331122
5	2.331122

$$r \approx 2.33112$$

15. Let $f(x) = x^3 - 6$.

$$f'(x) = 3x^2$$

n	x_n
1	1.5
2	1.888889
3	1.819813
4	1.817125
5	1.817121
6	1.817121

$$\sqrt[3]{6} \approx 1.81712$$

16. Let $f(x) = x^4 - 47$.

$$f'(x) = 4x^3$$

n	x_n
1	2.5
2	2.627
3	2.618373
4	2.618330
5	2.618330

$$\sqrt[4]{47} \approx 2.61833$$

17. Let $g(x) = \frac{\sin x}{x}$.

$$f(x) = g'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

$$f'(x) = -\frac{\sin x}{x} - \frac{2 \cos x}{x^2} + \frac{2 \sin x}{x^3}$$

n	x_n
1	4.5
2	4.4934
3	4.49341
4	4.49341

Minimum at $x \approx 4.49341$

Minimum value is ≈ -0.21723 .

18. Suppose $n = 1$, then $\frac{2m}{M} \left(\frac{M}{2m} |x_1 - r| \right)^{2^{n-1}} = |x_1 - r|$, so

$$|x_1 - r| \leq \frac{2m}{M} \left(\frac{M}{2m} |x_1 - r| \right)^{2^{1-1}}$$

Suppose the statement is true for $k \leq n$.

$$|x_{n+1} - r| \leq \frac{M}{2m} (x_n - r)^2$$

$$\leq \frac{M}{2m} \left[\frac{2m}{M} \left(\frac{M}{2m} |x_1 - r| \right)^{2^{n-1}} \right]^2$$

$$= \frac{2m}{M} \left(\frac{M}{2m} |x_1 - r| \right)^{2^n} = \frac{2m}{M} \left(\frac{M}{2m} |x_1 - r| \right)^{2^{n+1-1}}$$

Thus, the statement is true for $n + 1$.

19. Let $f(x) = x^2 - 2$.

$$f'(x) = 2x, f''(x) = 2$$

$$|f'(x)| \geq 2 \text{ on } [1, 2], \text{ so } m = 2.$$

$$|f''(x)| \leq 2 \text{ on } [1, 2], \text{ so } M = 2.$$

$$|x_6 - \sqrt{2}| \leq \frac{2(2)}{2} \left(\frac{2}{2(2)} |1.5 - r| \right)^{2^{6-1}}$$

$$\leq 2(0.25)^{32} \approx 1.08 \times 10^{-19}$$

20. From Problem 19, $m = 2$ and $M = 2$.

$$|x_n - \sqrt{2}| \leq \frac{2(2)}{2} \left(\frac{2}{2(2)} |1.5 - r| \right)^{2^{n-1}} \leq 2(0.25)^{2^{n-1}}$$

$$2(0.25)^{2^{n-1}} \leq 5 \times 10^{-41}$$

$$(0.25)^{2^{n-1}} \leq 2.5 \times 10^{-41}$$

$$2^{n-1} \geq \frac{\log(2.5 \times 10^{-41})}{\log(0.25)}$$

$$n \geq \frac{\log \left[\frac{\log(2.5 \times 10^{-41})}{\log(0.25)} \right]}{\log 2} + 1 \approx 7.08$$

$$n \geq 8$$

21. Let $f(x) = \frac{1 + \ln x}{x}$.

$$f'(x) = -\frac{\ln x}{x^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + \frac{x_n}{\ln x_n} + x_n$$

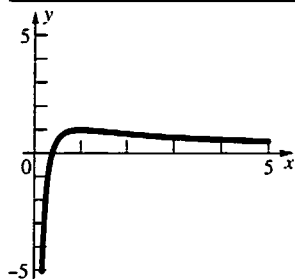
$$= 2x_n + \frac{x_n}{\ln x_n}$$

Suppose $x_1 = 1.2$.

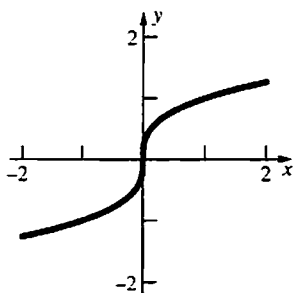
n	x_n
1	1.2
2	8.981778
3	22.05511
4	51.23963
5	115.4958
6	255.3103
7	556.6849
8	1201.425

Suppose $x_1 = 0.5$.

n	x_n
1	0.5
2	0.2786525
3	0.3392312
4	0.3646713
5	0.3678377
6	0.3678794
7	0.3678794



22.



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}}$$

$$= x_n - 3x_n = -2x_n$$

Thus, every iteration of Newton's Method gets further from zero. Note that

$$x_{n+1} = (-2)^{n+1} x_0.$$

23. a. For Tom's car, $P = 2000$, $R = 100$, and $k = 24$, thus

$$2000 = \frac{100}{i} \left[1 - \frac{1}{(1+i)^{24}} \right] \text{ or } 20i = 1 - \frac{1}{(1+i)^{24}},$$

which is equivalent to

$$20i(1+i)^{24} - (1+i)^{24} + 1 = 0.$$

b. Let

$$f(i) = 20i(1+i)^{24} - (1+i)^{24} + 1$$

$$= (1+i)^{24}(20i-1) + 1.$$

Then

$$f'(i) = 20(1+i)^{24} + 480i(1+i)^{23} - 24(1+i)^{23}$$

$$= (1+i)^{23}(500i-4), \text{ so}$$

$$i_{n+1} = i_n - \frac{f(i_n)}{f'(i_n)} = i_n - \frac{(1+i_n)^{24}(20i_n-1)+1}{(1+i_n)^{23}(500i_n-4)}$$

$$= i_n - \left[\frac{20i_n^2 + 19i_n - 1 + (1+i_n)^{-23}}{500i_n - 4} \right].$$

c.

n	i_n
1	0.012
2	0.0165297
3	0.0152651
4	0.0151323
5	0.0151308
6	0.0151308

$$i = 0.0151308$$

$$r = 18.157\%$$

24. From Newton's algorithm, $x_{n+1} - x_n = \frac{f(x_n)}{f'(x_n)}$.

$$\lim_{x_n \rightarrow \bar{x}} (x_{n+1} - x_n) = \lim_{x_n \rightarrow \bar{x}} x_{n+1} - \lim_{x_n \rightarrow \bar{x}} x_n$$

$$= \bar{x} - \bar{x} = 0$$

$$\lim_{x_n \rightarrow \bar{x}} \frac{f(x_n)}{f'(x_n)} \text{ exists if } f \text{ and } f' \text{ are continuous at } \bar{x}$$

and $f'(\bar{x}) \neq 0$.

$$\text{Thus, } \lim_{x_n \rightarrow \bar{x}} \frac{f(x_n)}{f'(x_n)} = \frac{f(\bar{x})}{f'(\bar{x})} = 0, \text{ so } f(\bar{x}) = 0.$$

$\bar{x}$ is a solution of $f(x) = 0$.

25. a. The algorithm computes the root of $\frac{1}{x} - a = 0$ for

$$x_1 \text{ close to } \frac{1}{a}.$$

- b. Let $f(x) = \frac{1}{x} - a$.

$$f'(x) = -\frac{1}{x^2}$$

$$\frac{f(x)}{f'(x)} = -x + ax^2$$

The recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 2x_n - ax_n^2.$$

26. $r \approx -1.25992, r \approx 1.25992$

27. $r \approx -1.87939, r \approx 0.34730, r \approx 1.53209$

28. $r \approx -0.28603, r \approx 1.03208, r \approx 1.08934,$
 $r \approx 2.32816$

29. $r \approx -2.08204, r \approx 0.09251, r \approx 0.91314,$
 $r \approx 1.62015, r \approx 1.85411$

30. $r = 1$

11.4 Concepts Review

1. fixed point
2. x_{n+1}
3. $|g'(x)| \leq M < 1$
4. $|2x| > 1$ near $x = 2$

Problem Set 11.4

1. $x_{n+1} = \frac{1}{9}e^{-2x_n}$

n	x_n
1	1
2	0.015037
3	0.107819
4	0.089559
5	0.092890
6	0.092273
7	0.092387
8	0.092366
9	0.092370
10	0.092369
11	0.092369

$x \approx 0.09237$

2. $x_{n+1} = 2 \tan^{-1} x_n$

n	x_n
1	2
2	2.214297
3	2.293208

4	2.319173
5	2.327392
6	2.329961
7	2.330761
8	2.331010
9	2.331087
10	2.331112
11	2.331119
12	2.331121
13	2.331122
14	2.331122

$x \approx 2.33112$

3. $x_{n+1} = \sqrt{2.7 + x_n}$

n	x_n
1	1
2	1.923538
3	2.150241
4	2.202326
5	2.214120
6	2.216781
7	2.217382
8	2.217517
9	2.217548
10	2.217554
11	2.217556
12	2.217556

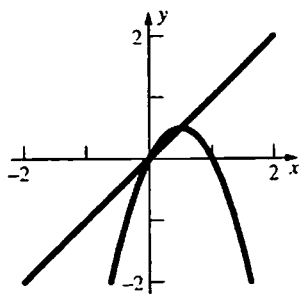
$x \approx 2.21756$

4. $x_{n+1} = \sqrt{3.2 + x_n}$

n	x_n
1	47
2	7.085196
3	3.207054
4	2.531216
5	2.393996
6	2.365163
7	2.359060
8	2.357766
9	2.357491
10	2.357433
11	2.357421
12	2.357418
13	2.357418

$x \approx 2.35742$

5. a.



$x \approx 0.5$

b. $x_{n+1} = 2(x_n - x_n^2)$

n	x_n
1	0.7
2	0.42
3	0.4872
4	0.4996723
5	0.4999998
6	0.5
7	0.5

c. $x = 2(x - x^2)$

$2x^2 - x = 0$

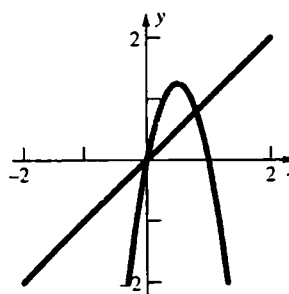
$x(2x - 1) = 0$

$x = 0, x = \frac{1}{2}$

d. $g'(x) = 2 - 4x$

$g'\left(\frac{1}{2}\right) = 0$

6. a.



$x \approx 0.8$

b. $x_{n+1} = 5(x_n - x_n^2)$

n	x_n
1	0.7
2	1.05
3	-0.2625
4	-1.657031
5	-22.01392
6	-2533.133

c. $x = 5(x - x^2)$

$5x^2 - 4x = 0$

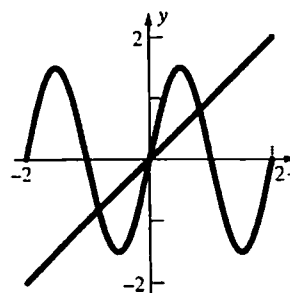
$x(5x - 4) = 0$

$x = 0, x = \frac{4}{5}$

d. $g'(x) = 5 - 10x$

$g'\left(\frac{4}{5}\right) = -3$

7. a.

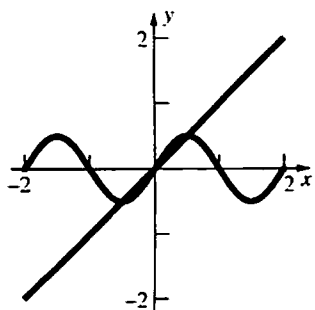


b. The algorithm does not yield a convergent sequence.

c. $g'(x) = \frac{3\pi}{2} \cos \pi x$

$|g'(x)| > 1$ in a neighborhood of the fixed points.

8. a.



b. The algorithm yields convergent sequences to 0.5 or -0.5.

c. $g'(x) = \frac{\pi}{2} \cos \pi x$

$|g'(x)| < 1$ in a neighborhood of the fixed points.

9. a. $x = \frac{3}{2} \sin \pi x$

$$6x = 5x + \frac{3}{2} \sin \pi x$$

$$x = \frac{5}{6}x + \frac{1}{4} \sin \pi x$$

b. Let $x_1 = 0.8$.

n	x_n
1	0.8
2	0.813613
3	0.816176
4	0.816629
5	0.816709
6	0.816723
7	0.816725
8	0.816726
9	0.816726

Let $x_1 = -0.8$.

n	x_n
1	-0.8
2	-0.813613
3	-0.816176
4	-0.816629
5	-0.816709
6	-0.816723

7	-0.816725
8	-0.816726
9	-0.816726

c. $g'(x) = \frac{5}{6} + \frac{\pi}{4} \cos \pi x$

$$g'(0.81673) \approx 0.17456$$

$$g'(-0.81673) \approx 0.17456$$

10. a. $x = 5(x - x^2)$

$$6x = 10x - 5x^2$$

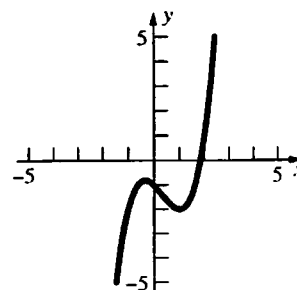
$$x = \frac{5}{3}x - \frac{5}{6}x^2$$

Let $g(x) = \frac{5}{3}x - \frac{5}{6}x^2$.

b. Let $x_1 = 0.75$.

n	x_n
1	0.75
2	0.78125
3	0.793457
4	0.797783
5	0.799257
6	0.799752
7	0.799917
8	0.799972
9	0.799991
10	0.799997
11	0.799999
12	0.8
13	0.8

11. Graph $y = x^3 - x^2 - x - 1$.



The positive root is near 2.

Rewrite the equation as $x = 1 + \frac{1}{x} + \frac{1}{x^2} = g(x)$.

Let $x_1 = 1.8$

n	x_n
1	1.8
2	1.8642
3	1.8242
4	1.8487
5	1.8335
6	1.8429
7	1.8371
8	1.8406
9	1.8384
10	1.8398
11	1.8390
12	1.8395
13	1.8392
14	1.8394
15	1.8392
16	1.8393
17	1.8393

$r \approx 1.839$

12. a. $x_1 = 0$
 $x_2 = \sqrt{5} \approx 2.236068$
 $x_3 = \sqrt{5 + \sqrt{5}} \approx 2.689994$
 $x_4 = \sqrt{5 + \sqrt{5 + \sqrt{5}}} \approx 2.7730839$
 $x_5 = \sqrt{5 + \sqrt{5 + \sqrt{5 + \sqrt{5}}}} \approx 2.7880251$
- b. $x = \sqrt{5 + x}$, and x must satisfy $x \geq 0$
 $x^2 = 5 + x$
 $x^2 - x - 5 = 0$

$$x = \frac{1 \pm \sqrt{1 + 4 \cdot 1 \cdot 5}}{2} = \frac{1 \pm \sqrt{21}}{2}$$
Taking the minus sign gives a negative solution for x , violating the requirement that $x \geq 0$. Hence,
 $x = (1 + \sqrt{21})/2 \approx 2.7912878$
- c. Let $x = \sqrt{5 + \sqrt{5 + \sqrt{5 + \dots}}}$. Then x satisfies the equation $x = \sqrt{5 + x}$. From part b we know that x must equal $(1 + \sqrt{21})/2 \approx 2.7912878$

13. a. $x_1 = 0$

$$x_2 = \sqrt{1} = 1$$

$$x_3 = \sqrt{1 + \sqrt{1}} = \sqrt{2} \approx 1.4142136$$

$$x_4 = \sqrt{1 + \sqrt{1 + \sqrt{1}}} \approx 1.553774$$

$$x_5 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}} \approx 1.5980532$$

- b. $x = \sqrt{1 + x}$
 $x^2 = 1 + x$
 $x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{1 + 4 \cdot 1 \cdot 1}}{2} = \frac{1 \pm \sqrt{5}}{2}$$
Taking the minus sign gives a negative solution for x , violating the requirement that $x \geq 0$.
Hence, $x = \frac{1 + \sqrt{5}}{2} \approx 1.618034$.
- c. Let $x = \sqrt{5 + \sqrt{5 + \sqrt{5 + \dots}}}$. Then x satisfies the equation $x = \sqrt{5 + x}$. From part b we know that x must equal $(1 + \sqrt{5})/2 \approx 1.618034$.

14. a. $x_1 = 1$
 $x_2 = 1 + \frac{1}{1} = 2$
 $x_3 = 1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2} = 1.5$
 $x_4 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = \frac{5}{3} \approx 1.6666667$
 $x_5 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}} = \frac{8}{5} = 1.6$
- b. $x = 1 + \frac{1}{x}$
 $x^2 = x + 1$
 $x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{1 + 4 \cdot 1 \cdot 1}}{2} = \frac{1 \pm \sqrt{5}}{2}$$
Taking the minus sign gives a negative solution for x , violating the requirement that $x \geq 0$.
Hence, $x = \frac{1 + \sqrt{5}}{2} \approx 1.618034$.
- c. Let

$$x = 1 + \frac{1}{1 + \frac{1}{1 + \dots}}$$

Then x satisfies the equation

$$x = 1 + \frac{1}{x}. \text{ From part b we know that}$$

x must equal

$$(1 + \sqrt{5})/2 \approx 1.618034.$$

15. Let $a = \pi$ and $x_1 = 2$.

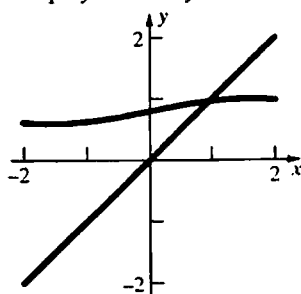
n	x_n
1	2
2	1.785398
3	1.772501
4	1.772454
5	1.772454

$$\sqrt{\pi} \approx 1.77245$$

$$g'(x) = \frac{1}{2} \left(1 - \frac{a}{x^2} \right)$$

$$g'(a) = \frac{1}{2} \left(1 - \frac{1}{a} \right) < 1 \text{ for } a > 0.$$

16. Graph $y = x$ and $y = 0.8 + 0.2 \sin x$.



$$x_{n+1} = 0.8 + 0.2 \sin x_n$$

Let $x_1 = 1$.

n	x_n
1	1
2	0.96829
3	0.96478
4	0.96439
5	0.96434
6	0.96433
7	0.96433

$$x \approx 0.9643$$

17. a. $10,000 = \frac{R}{0.015} [1 - (1.015)^{-48}]$

$$R = \frac{150}{1 - (1.015)^{-48}} \approx 293.75$$

b. $10,000 = \frac{300}{i} [1 - (1+i)^{-48}]$

$$i = \frac{3}{100} [1 - (1+i)^{-48}]$$

Let $i_1 = 0.015$.

n	i_n
1	0.015
2	0.015319
3	0.015539
4	0.015689
5	0.015789
6	0.015857
7	0.015902
8	0.015932
9	0.015952
10	0.15965
11	0.015974
12	0.015980
13	0.015983
14	0.015986
15	0.015988
16	0.015989
17	0.015989

$$i \approx 0.01599$$

18. $500 = \frac{30}{i} [1 - (1+i)^{-24}]$

$$i = \frac{3}{50} [1 - (1+i)^{-24}]$$

Let $i_1 = 0.03$.

n	i_n
1	0.03
2	0.030484
3	0.030815
4	0.031039
5	0.031190
6	0.031290
7	0.031358
8	0.031403
9	0.031432
10	0.031452
11	0.031465
12	0.031474
13	0.031480
14	0.031484
15	0.031486

16	0.031488
17	0.031489
18	0.031490
19	0.031490

$$i \approx 0.03149$$

19. a. Suppose r is a root. Then

$$r = r - \frac{f(r)}{f'(r)}.$$

$$\frac{f(r)}{f'(r)} = 0, \text{ so } f(r) = 0.$$

Suppose $f(r) = 0$. Then

$$r - \frac{f(r)}{f'(r)} = r - 0 = r, \text{ so } r \text{ is a root of}$$

$$x = x - \frac{f(x)}{f'(x)}.$$

- b. If we want to solve $f(x) = 0$ and $f'(x) \neq 0$ in $[a, b]$, then $\frac{f(x)}{f'(x)} = 0$ or

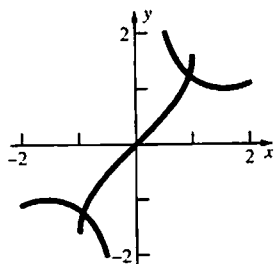
$$x = x - \frac{f(x)}{f'(x)} = g(x).$$

$$g'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)}{[f'(x)]^2} f''(x)$$

$$= \frac{f(x)f''(x)}{[f'(x)]^2}$$

$$\text{and } g'(r) = \frac{f(r)f''(r)}{[f'(r)]^2} = 0.$$

20. a.



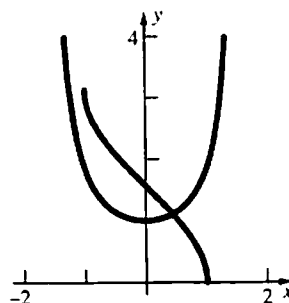
$$f(x) = \sin^{-1} x - \frac{1}{\sin x}$$

Use Newton's Method.

n	x_n
1	0.9
2	0.947423
3	0.944075
4	0.944039
5	0.944039

$$x \approx 0.94404$$

- b.



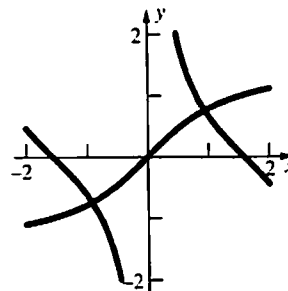
$$f(x) = \cos^{-1} x - \frac{1}{\cos x}$$

Use Newton's Method.

n	x_n
1	0.4
2	0.447464
3	0.446049
4	0.446048
5	0.446048

$$x \approx 0.44605$$

- c.



$$f(x) = \tan^{-1} x - \frac{1}{\tan x}$$

Use Newton's Method.

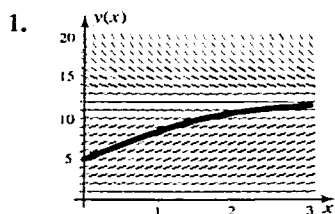
n	x_n
1	0.9
2	0.9278324
3	0.9283946
4	0.9283949
5	0.9283949

$$x \approx 0.92839$$

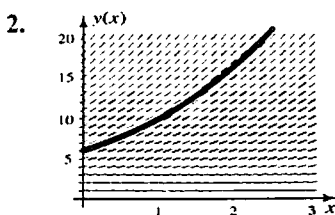
11.5 Concepts Review

1. slope field
2. tangent line, or Taylor polynomial of order 1 based at x_0 .
3. $y_{n-1} + hf(x_{n-1}, y_{n-1})$
4. average

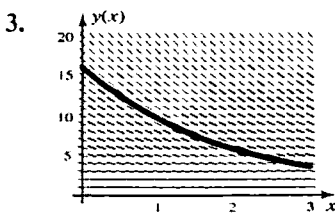
Problem Set 11.5



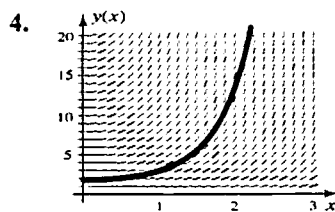
$$\lim_{t \rightarrow \infty} y(t) = 12 \text{ and } y(2) \approx 10.5$$



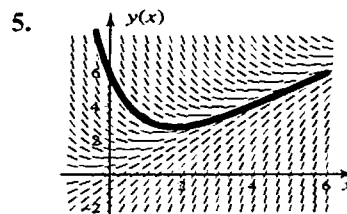
$$\lim_{t \rightarrow \infty} y(t) = \infty \text{ and } y(2) \approx 16$$



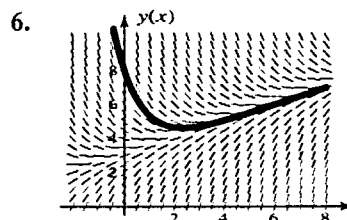
$$\lim_{t \rightarrow \infty} y(t) = 0 \text{ and } y(2) \approx 6$$



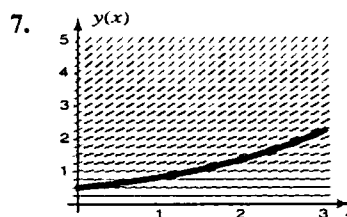
$$\lim_{t \rightarrow \infty} y(t) = \infty \text{ and } y(2) \approx 13$$



The oblique asymptote is $y = x$.



The oblique asymptote is $y = 3 + x/2$.



$$\frac{dy}{dx} = \frac{1}{2}y; \quad y(0) = \frac{1}{2}$$

$$\frac{dy}{y} = \frac{1}{2}dx$$

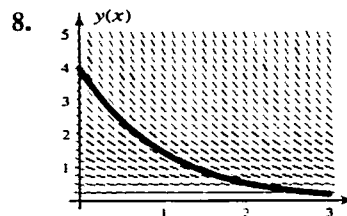
$$\ln y = \frac{x}{2} + C$$

$$y = C_1 e^{x/2}$$

To find C_1 , apply the initial condition:

$$\frac{1}{2} = y(0) = C_1 e^0 = C_1$$

$$y = \frac{1}{2} e^{x/2}$$



$$\frac{dy}{dx} = -y; \quad y(0) = 4$$

$$\frac{dy}{y} = -dx$$

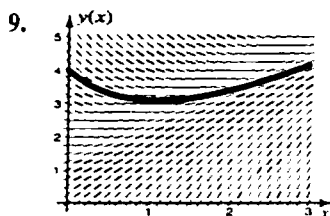
$$\ln y = -x + C$$

$$y = C_1 e^{-x}$$

To find C_1 , apply the initial condition:

$$4 = y(0) = C_1 e^{-0} = C_1$$

$$y = 4e^{-x}$$



$$y' + y = x + 2$$

The integrating factor is $e^{\int 1 dx} = e^x$.

$$e^x y' + ye^x = e^x(x + 2)$$

$$\frac{d}{dx}(e^x y) = (x + 2)e^x$$

$$e^x y = \int (x + 2)e^x dx$$

Integrate by parts: let $u = x + 2$, $dv = e^x dx$.

Then $du = dx$ and $v = e^x$. Thus

$$e^x y = (x + 2)e^x - \int e^x dx$$

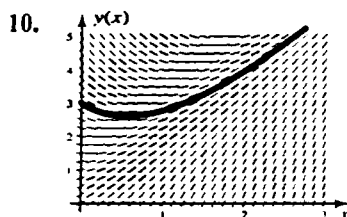
$$e^x y = (x + 2)e^x - e^x + C$$

$$y = x + 2 - 1 + Ce^{-x}$$

To find C , apply the initial condition:

$$4 = y(0) = 0 + 1 + Ce^{-0} = 1 + C$$

Thus, $y = x + 1 + e^{-x}$.



$$y' + y = 2x + \frac{3}{2}$$

$$e^x y' + ye^x = \left(2x + \frac{3}{2}\right)e^x$$

$$\frac{d}{dx}(e^x y) = \left(2x + \frac{3}{2}\right)e^x$$

$$e^x y = \int \left(2x + \frac{3}{2}\right)e^x dx$$

Integrate by parts: let $u = 2x + \frac{3}{2}$,

$dv = e^x dx$. Then $du = 2dx$ and $v = e^x$.

Thus,

$$e^x y = \left(2x + \frac{3}{2}\right)e^x - \int 2e^x dx$$

$$e^x y = \left(2x + \frac{3}{2}\right)e^x - 2e^x + C$$

$$y = 2x - \frac{1}{2} + Ce^{-x}$$

To find C , apply the initial condition:

$$3 = y(0) = 0 - \frac{1}{2} + Ce^{-0} = C - \frac{1}{2}$$

Thus $C = \frac{7}{2}$, so the solution is

$$y = 2x - \frac{1}{2} + \frac{7}{2}e^{-x}$$

Note: Solutions to Problems 17-26 are given along with the corresponding solutions to 11-16.

11., 17.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	3.0	3.0
0.2	4.2	4.44
0.4	5.88	6.5712
0.6	8.232	9.72538
0.8	11.5248	14.39356
1.0	16.1347	21.30246

12., 18.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	2.0	2.0
0.2	1.6	1.64
0.4	1.28	1.3448
0.6	1.024	1.10274
0.8	0.8195	0.90424
1.0	0.65536	0.74148

13., 19.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	0.0	0.0
0.2	0.0	0.2
0.4	0.04	0.8
0.6	0.12	0.18
0.8	0.24	0.032
1.0	0.40	0.05

14., 20.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	0.0	0.0
0.2	0.0	0.004
0.4	0.008	0.024
0.6	0.040	0.076
0.8	0.112	0.176
1.0	0.240	0.340

15., 21.

x_n	Euler's Method y_n	Improved Euler Method y_n
1.0	1.0	1.0
1.2	1.2	1.244
1.4	1.488	1.60924
1.6	1.90464	2.16410
1.8	2.51412	3.02455
2.0	3.41921	4.391765

16., 22.

x_n	Euler's Method y_n	Improved Euler Method y_n
1.0	2.0	2.0

24.

$h = 0.2$		
x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	1.0	1.0
0.2	1.0	1.0
0.4	0.95946	0.96028
0.6	0.87833	0.88251
0.8	0.75815	0.77002
1.0	0.60202	0.62778
1.2	0.41450	0.46269
1.4	0.20127	0.28589
$h = 0.1$		
0.0	1.0	1.0
0.1	1.0	1.0
0.2	0.98997	0.99002
0.3	0.96990	0.97015
0.4	0.93990	0.94061
0.5	0.90016	0.90168
0.6	0.85098	0.85376
0.7	0.79276	0.79735

1.2	1.2	1.312
1.4	0.624	0.80609
1.6	0.27456	0.46689
1.8	0.09884	0.25698
2.0	0.02768	0.13568

23.

h	Error from Euler's Method	Error from Improved Euler Method
0.2	0.229962	0.015574
0.1	0.124539	0.004201
0.05	0.064984	0.001091
0.01	0.013468	0.000045
0.005	0.006765	0.000011

For Euler's method, the error is halved as the step size h is halved. Thus, the error is proportional to h . For the improved Euler method, when h is halved, the error decreases to approximately one-fourth of what it was. Hence, for the improved Euler method, the error is proportional to h^2 .

0.8	0.72599	0.73302
0.9	0.65124	0.66143
1.0	0.56917	0.58333
1.1	0.48053	0.49956
1.2	0.38612	0.41105
1.3	0.28680	0.31892
1.4	0.18349	0.22473
1.5	0.07711	0.13221

$h = 0.05$		
0.0	1.0	1.0
0.05	1.0	1.0
0.10	0.99750	0.99750
0.15	0.99249	0.99251
0.20	0.98499	0.98504
0.25	0.97501	0.97510
0.30	0.96256	0.96273
0.35	0.94767	0.94796
0.40	0.93038	0.93082
0.45	0.91071	0.91135
0.50	0.88871	0.88960

0.55	0.86444	0.86563
0.60	0.83794	0.83950
0.65	0.80928	0.81128
0.70	0.77851	0.78103
0.75	0.74573	0.74883
0.80	0.71099	0.71476
0.85	0.67439	0.67891
0.90	0.63600	0.64137
0.95	0.59593	0.60223
1.00	0.55426	0.56159
1.05	0.51110	0.51957
1.10	0.46655	0.47625
1.15	0.42072	0.43176
1.20	0.37371	0.38622
1.25	0.32565	0.33974
1.30	0.276650	0.29247
1.35	0.22682	0.24453
1.40	0.17630	0.19613
1.45	0.12519	0.14751
1.50	0.07362	0.09927
1.55	0.02171	0.05395

For this example, Euler's method seems to be more accurate than the improved Euler method.

25. a. $y_0 = 1$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= y_0 + hy_0 = (1+h)y_0$$

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + hy_1$$

$$= (1+h)y_1 = (1+h)^2 y_0$$

$$y_3 = y_2 + hf(x_2, y_2) = y_2 + hy_2$$

$$= (1+h)y_2 = (1+h)^3 y_0$$

$\vdots$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) = y_{n-1} + hy_{n-1}$$

$$= (1+h)y_{n-1} = (1+h)^n y_0$$

- b. Let $N = 1/h$. Then y_N is an approximation to the solution at $x = Nh = (1/h)h = 1$. The exact solution is $y(1) = e$. Thus, $(1 + 1/N)^N \approx e$ for large N . From Chapter 7, we know that
- $$\lim_{N \rightarrow \infty} (1 + 1/N)^N = e.$$

26. $y_0 = y(x_0) = 0$

$$y_1 = y_0 + hf(x_0) = 0 + hf(x_0) = hf(x_0)$$

$$y_2 = y_1 + hf(x_1) = hf(x_0) + hf(x_1)$$

$$= h(f(x_0) + f(x_1))$$

$$y_3 = y_2 + hf(x_2)$$

$$= h[f(x_0) + f(x_1)] + hf(x_2)$$

$$= h[f(x_0) + f(x_1) + f(x_2)] = h \sum_{i=0}^{3-1} f(x_i)$$

At the n th step of Euler's method,

$$y_n = y_{n-1} + hf(x_{n-1}) = h \sum_{i=0}^{n-1} f(x_i)$$

27. a. $\int_{x_0}^{x_1} y'(x) dx = \int_{x_0}^{x_1} \sin x^2 dx$

$$y(x_1) - y(x_0) \approx (x_1 - x_0) \sin x_0^2$$

$$y(x_1) - y(0) = h \sin x_0^2$$

$$y(x_1) - 0 \approx 0.1 \sin 0^2$$

$$y(x_1) \approx 0$$

b. $\int_{x_0}^{x_2} y'(x) dx = \int_{x_0}^{x_2} \sin x^2 dx$

$$y(x_2) - y(x_0) \approx (x_1 - x_0) \sin x_0^2$$

$$+ (x_2 - x_1) \sin x_1^2$$

$$y(x_2) - y(0) = h \sin x_0^2 + h \sin x_1^2$$

$$y(x_2) - 0 \approx 0.1 \sin 0^2 + 0.1 \sin 0.1^2$$

$$y(x_2) \approx 0.00099998$$

c. $\int_{x_0}^{x_3} y'(x) dx = \int_{x_0}^{x_3} \sin x^2 dx$

$$y(x_3) - y(x_0) \approx (x_1 - x_0) \sin x_0^2$$

$$+ (x_2 - x_1) \sin x_1^2 + (x_3 - x_2) \sin x_2^2$$

$$y(x_3) - y(0) = h \sin x_0^2 + h \sin x_1^2 + h \sin x_2^2$$

$$y(x_3) - 0 \approx 0.1 \sin 0^2 + 0.1 \sin 0.1^2$$

$$+ 0.1 \sin 0.2^2$$

$$y(x_3) \approx 0.004999$$

Continuing in this fashion, we have

$$\int_{x_0}^{x_n} y'(x) dx = \int_{x_0}^{x_n} \sin x^2 dx$$

$$y(x_n) - y(x_0) \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sin x_i^2$$

$$y(x_n) \approx h \sum_{i=0}^{n-1} f(x_{i-1})$$

When $n = 10$, this becomes

$$y(x_{10}) = y(1) \approx 0.269097$$

- d. The result $y(x_n) \approx h \sum_{i=0}^{n-1} f(x_{i-1})$ is the same

as that given in Problem 26. Thus, when $f(x, y)$ depends only on x , then the two methods (1) Euler's method for

approximating the solution to $y' = f(x)$ at x_n , and (2) the left-endpoint Riemann sum for approximating $\int_0^{x_n} f(x) dx$, are equivalent.

28. a. $\int_0^{x_1} y'(x) dx = \int_0^{x_1} \sqrt{x+1} dx$

$$y(x_1) - y(x_0) \approx (x_1 - x_0) \sqrt{x_0 + 1}$$

$$y(x_1) - y(0) = h \sqrt{x_0 + 1}$$

$$y(x_1) - 0 \approx 0.1 \sqrt{0+1}$$

$$y(x_1) \approx 0.1$$

b. $\int_0^{x_2} y'(x) dx = \int_0^{x_2} \sqrt{x+1} dx$

$$y(x_2) - y(x_0) \approx (x_1 - x_0) \sqrt{x_0 + 1} + (x_2 - x_1) \sqrt{x_1 + 1}$$

$$y(x_2) - y(0) = h \sqrt{x_0 + 1} + h \sqrt{x_1 + 1}$$

$$y(x_2) - 0 \approx 0.1 \sqrt{0+1} + 0.1 \sqrt{0.1+1}$$

$$y(x_2) \approx 0.204881$$

c. $\int_0^{x_3} y'(x) dx = \int_0^{x_3} \sqrt{x+1} dx$

$$y(x_3) - y(x_0) \approx (x_1 - x_0) \sqrt{x_0 + 1}$$

$$+ (x_2 - x_1) \sqrt{x_1 + 1} + (x_3 - x_2) \sqrt{x_2 + 1}$$

$$y(x_3) - y(0) = 0.1 \sqrt{0+1} + 0.1 \sqrt{0.1+1} + 0.1 \sqrt{0.2+1}$$

$$y(x_3) \approx 0.314425$$

Continuing in this fashion, we have

$$\int_0^{x_n} y'(x) dx = \int_0^{x_n} \sqrt{x+1} dx$$

$$y(x_n) - y(x_0) \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sqrt{x_i + 1}$$

$$y(x_n) \approx h \sum_{i=0}^{n-1} \sqrt{x_i + 1}$$

When $n = 10$, this becomes

$$y(x_{10}) = y(1) \approx 1.198119$$

11.6 Chapter Review

Concepts Test

1. True: $P(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

2. True: If $p(x)$ and $q(x)$ are polynomials of degree less than or equal to n , satisfying $p(a) = q(a) = f(a)$ and $p^{(k)}(a) = q^{(k)}(a) = f^{(k)}(a)$ for $k \leq n$, then $p(x) = q(x)$.

3. True: $f(0) = f'(0) = f''(0) = 0$, its second order Maclaurin polynomial is 0.

4. True: After simplifying, $P_3(x) = f(x)$.

5. True: Any Maclaurin polynomial for $\cos x$ involves only even powers of x .

6. True: The Maclaurin polynomial of an even function involves only even powers of x , so $f'(0) = 0$ if $f(x)$ is an even function.

7. True: Taylor's Formula with Remainder for $n = 0$ is $f(x) = f(a) + f'(c)(x - a)$ which is equivalent to the Mean Value Theorem.

8. False: A calculator can be limited to a certain number of significant digits.

9. False: For example $\int \frac{\sin x}{x} dx$ cannot be expressed in terms of elementary functions.

10. False: $E_{10} = -\frac{5^3}{12 \cdot 10^2} 6c < 0$, so the Trapezoidal Rule will give a value greater than the true value.

11. True: $E_{10} = 0$ since the fourth derivative of x^3 is 0.

12. False: A computer can be limited.

13. True:
$$\left| e^{-x^2} + x^2 + \sin(x+1) \right| \leq \left| e^{-x^2} \right| + \left| x^2 \right| + \left| \sin(x+1) \right| \leq 1 + 4 + 1 = 6$$

14. True: This is the Parabolic Rule for $n = 2$. Since $f^{(4)}(x) = 0$, $E_2 = 0$, so the rule is exact.

15. True: Intermediate Value Theorem

16. False: See Example 1 of Section 10.4.

17. False: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n$. (See Problem 22 of Problem Set 10.4.)

18. True: If $f'(x) > 1$, the Fixed Point Theorem fails.

19. False: $g(x) = 5(x - x^2) + 0.01$,
 $g'(x) = 5 - 10x$; $|g'(x)| > 1$ in a neighborhood of the fixed point.

20. True: Note that the fixed point is $\sqrt{a}$.
 $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$;
 $|g'(x)| = \left|\frac{1}{2}\left(1 - \frac{a}{x^2}\right)\right| < 1$ when
 $\sqrt{\frac{a}{3}} < x$.

21. True: At (2, 1) the slope is $y' = 2 \cdot 1 = 2$

22. False: $y' = 2y$, $y'' = 2 > 0$. Thus, the solution is concave up. The estimate from Euler's method will underestimate the solution.

Sample Test Problems

1. $f(0) = 0$
 $f'(x) = \cos^2 x - 2x^2 \sin^2 x$
 $f'(0) = 1$
 $p(x) = x$; $p(0.2) = 0.2$; $f(0.2) = 0.1998$

2. a. $f(x) = xe^x$ $f(0) = 0$
 $f'(x) = e^x + xe^x$ $f'(0) = 1$
 $f''(x) = 2e^x + xe^x$ $f''(0) = 2$
 $f^{(3)}(x) = 3e^x + xe^x$ $f^{(3)}(0) = 3$
 $f^{(4)}(x) = 4e^x + xe^x$ $f^{(4)}(0) = 4$
 $f(x) \approx x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$
 $f(0.1) \approx 0.11052$

b. $f(x) = \cosh x$ $f(0) = 1$
 $f'(x) = \sinh x$ $f'(0) = 0$
 $f''(x) = \cosh x$ $f''(0) = 1$
 $f^{(3)}(x) = \sinh x$ $f^{(3)}(0) = 0$
 $f^{(4)}(x) = \cosh x$ $f^{(4)}(0) = 1$

$$f(x) \approx 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$f(0.1) \approx 1.0050042$$

3. $g(x) = x^3 - 2x^2 + 5x - 7$ $g(2) = 3$
 $g'(x) = 3x^2 - 4x + 5$ $g'(2) = 9$
 $g''(x) = 6x - 4$ $g''(2) = 8$
 $g^{(3)}(x) = 6$ $g^{(3)}(2) = 6$

Since $g^{(4)}(x) = 0$, $R_3(x) = 0$, so the Taylor polynomial of order 3 based at 2 is an exact representation.

$$g(x) = P_4(x) = 3 + 9(x - 2) + 4(x - 2)^2 + (x - 2)^3$$

$$4. g(2.1) = 3 + 9(0.1) + 4(0.1)^2 + (0.1)^3 = 3.941$$

5. $f(x) = \frac{1}{x+1}$ $f(1) = \frac{1}{2}$
 $f'(x) = -\frac{1}{(x+1)^2}$ $f'(1) = -\frac{1}{4}$
 $f''(x) = \frac{2}{(x+1)^3}$ $f''(1) = \frac{1}{4}$
 $f^{(3)}(x) = -\frac{6}{(x+1)^4}$ $f^{(3)}(1) = -\frac{3}{8}$
 $f^{(4)}(x) = \frac{24}{(x+1)^5}$ $f^{(4)}(1) = \frac{3}{4}$

$$f(x) \approx \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4$$

6. $f^{(5)}(x) = -\frac{120}{(x+1)^6}$, so $R_4(x) = -\frac{(x-1)^5}{(c+1)^6}$.
 $|R_4(1.2)| = \frac{(0.2)^5}{(c+1)^6} \leq \frac{(0.2)^5}{(2)^6} = 0.000005$

7. $f(x) = \frac{1}{2}(1 - \cos 2x)$ $f(0) = 0$
 $f'(x) = \sin 2x$ $f'(0) = 0$
 $f''(x) = 2 \cos 2x$ $f''(0) = 2$
 $f^{(3)}(x) = -4 \sin 2x$ $f^{(3)}(0) = 0$
 $f^{(4)}(x) = -8 \cos 2x$ $f^{(4)}(0) = -8$
 $f^{(5)}(x) = 16 \sin 2x$ $f^{(5)}(0) = 0$
 $f^{(6)}(x) = 32 \cos 2x$ $f^{(6)}(c) = 32 \cos 2c$
 $\sin^2 x \approx \frac{2}{2!}x^2 - \frac{8}{4!}x^4 = x^2 - \frac{1}{3}x^4$

$$|R_4(x)| = |R_5(x)| = \left| \frac{32}{6!} (\cos 2c)x^6 \right| \leq \frac{2}{45} (0.2)^6 < 2.85 \times 10^{-6}$$

$$8. f^{(n+1)}(x) = \frac{(-1)^n n!}{x^{n+1}}$$

$$|R_n(x)| = \left| \frac{(-1)^n}{(n+1)c^{n+1}} (x-1)^{n+1} \right| \leq \frac{0.2^{n+1}}{(n+1)0.8^{n+1}} = \frac{(0.25)^{n+1}}{(n+1)} < 0.00005 \text{ when } n \geq 5.$$

9. From Problem 8,

$$\begin{aligned} \ln x &\approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 \\ |R_5(x)| &= \left| \frac{1}{6c^6} (x-1)^6 \right| \leq \frac{0.2^6}{6 \cdot 0.8^6} < 4.07 \times 10^{-5} \\ \int_{0.8}^{1.2} \ln x \, dx &\approx \int_{0.8}^{1.2} \left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 \right] dx \\ &= \left[\frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{30}(x-1)^6 \right]_{0.8}^{1.2} \\ &\approx -0.00269867 \\ \text{Error} &\leq (1.2 - 0.8)4.07 \times 10^{-5} < 1.63 \times 10^{-5} \end{aligned}$$

10. $f(x) = \ln x$, $h = 0.05$

n	x_n	$f(x_n)$
0	0.80	-0.22314
1	0.85	-0.16252
2	0.90	-0.10536
3	0.95	-0.051293
4	1.00	0
5	1.05	0.04879
6	1.10	0.09531
7	1.15	0.13976
8	1.20	0.18232

$$\int_{0.8}^{1.2} \ln x \, dx \approx \frac{0.05}{2} [f(x_0) + 2f(x_1) + \dots + 2f(x_7) + f(x_8)]$$

$$\approx -0.00278607$$

$$|E_8| = \left| \frac{0.4^3}{12 \cdot 8^2} \frac{1}{c^2} \right| \leq \frac{0.4^3}{12 \cdot 8^2 \cdot 0.8^2} < 1.31 \times 10^{-4}$$

11. $f(x) = \ln x$, $h = 0.05$

n	x_n	$f(x_n)$
0	0.80	-0.22314
1	0.85	-0.16252
2	0.90	-0.10536
3	0.95	-0.051293
4	1.00	0
5	1.05	0.04879
6	1.10	0.09531
7	1.15	0.13976
8	1.20	0.18232

$$\int_{0.8}^{1.2} \ln x \, dx \approx \frac{0.05}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_7) + f(x_8)]$$

$$\approx -0.00269939$$

$$|E_8| = \left| \frac{0.4^5}{180 \cdot 8^4} \frac{6}{c^4} \right| \leq \frac{0.4^5 \cdot 6}{180 \cdot 8^4 \cdot 0.8^4} < 2.04 \times 10^{-7}$$

$$12. \int_{0.8}^{1.2} \ln x \, dx = [x \ln x - x]_{0.8}^{1.2} \approx -0.00269929$$

$$13. f(x) = 3x - \cos 2x, \quad f'(x) = 3 + 2 \sin 2x$$

Let $x_1 = 0.5$.

n	x_n
1	0.5
2	0.2950652
3	0.2818563
4	0.2817846
5	0.2817846

$$x \approx 0.281785$$

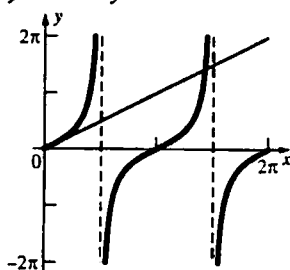
$$14. x_{n+1} = \frac{\cos 2x_n}{3}$$

n	x_n
1	0.5
2	0.18010
3	0.311942
4	0.270539
5	0.285718

6	0.280375
7	0.282285
8	0.281606
9	0.281848
10	0.281762
11	0.281793
12	0.281782
13	0.281786
14	0.281784
15	0.281785
16	0.281785

$$x \approx 0.2818$$

15. $y = x$ and $y = \tan x$



$$\text{Let } x_1 = \frac{11\pi}{8}.$$

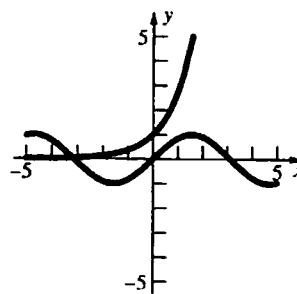
$$f(x) = x - \tan x, \quad f'(x) = 1 - \sec^2 x.$$

n	x_n
1	$\frac{11\pi}{8}$
2	4.64661795
3	4.60091050
4	4.54662258
5	4.50658016
6	4.49422443
7	4.49341259
8	4.49340946

$$x \approx 4.4934$$

16. The Fixed-Point Method does not work because if $g(x) = \tan x$, $g'(x) = \sec^2 x$ and $|g'(x)| > 1$ in a neighborhood of the root.

17. Graph $y = e^x$ and $y = \sin x$



$$\text{Let } x_1 = -3.$$

$$f(x) = e^x - \sin x, \quad f'(x) = e^x - \cos x$$

$$x_{n+1} = x_n - \frac{e^{x_n} - \sin x_n}{e^{x_n} - \cos x_n}$$

n	x_n
1	-3
2	-3.183603
3	-3.183063
4	-3.183063

$$x \approx -3.18306$$

18. Euler's Method:

x_n	y_n
1	2
1.2	2.4
1.4	2.976
1.6	3.80928
1.8	5.02825
2.0	6.83842

19. Improved Euler Method

x_n	y_n
0	2
0.2	3.56
0.4	6.3368
0.6	11.2795
0.8	20.07752
1.0	35.73798
1.2	63.61361
1.4	113.2322
1.6	201.5533
1.8	358.7650
2.0	638.6016

11.7 Additional Problem Set

1. Using a computer, the table is as follows:

i	(x_i, y_i)
0	$(\pi, 1)$
1	(3.4558, 0.9511)
2	(3.7699, 0.809)
3	(4.0841, 0.5878)
4	(4.3982, 0.309)
5	(4.7124, 0)
6	(5.0265, -0.309)
7	(5.3407, -0.5878)
8	(5.6549, -0.809)
9	(5.9690, -0.9511)
10	(6.2832, -1)

2. a. $y_{n+1} = y_n + 0.1(x_n + y_n) = 0.1x_n + 1.1y_n$;
 $x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.1	1.1
2	0.2	1.22
3	0.3	1.362
4	0.4	1.5282

- b. $y_{n+1} = y_n + 0.25(y_n) = 1.25y_n$;
 $x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.25	1.25
2	0.5	1.5625
3	0.75	1.953125
4	1.00	2.44140625

- c. $y_{n+1} = y_n + 0.1(y_n) = 1.1y_n$; $x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.1	1.1
2	0.2	1.21
3	0.3	1.331
4	0.4	1.4641
5	0.5	1.61051
6	0.6	1.771561
7	0.7	1.9487171
8	0.8	2.14358881
9	0.9	2.357947691
10	1.0	2.5937424601

3. a. $y_{n+1}^{\text{predicted}} = y_n + 0.25(y_n) = 1.25y_n$
 $y_{n+1} = y_n + \frac{0.25}{2}(y_n + 1.25y_n) = 1.28125y_n$;
 $x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.25	1.28125
2	0.50	1.641602
3	0.75	2.103302
4	1.00	2.694856

- $y_{n+1}^{\text{predicted}} = y_n + 0.1(y_n) = 1.1y_n$
 $y_{n+1} = y_n + \frac{0.1}{2}(y_n + 1.1y_n) = 1.105y_n$;
 $x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.1	1.105
2	0.2	1.221025
3	0.3	1.349233
4	0.4	1.490902
5	0.5	1.647447
6	0.6	1.820429
7	0.7	2.011574
8	0.8	2.222789
9	0.9	2.456182
10	1.0	2.714081

b. $y_{n+1} = y_n + 0.05(x_n + y_n)$

$$= 0.05x_n + 1.05y_n;$$

$$x_0 = 0, y_0 = 1$$

n	x_n	y_n
1	0.05	1.05
2	0.10	1.105
3	0.15	1.16525
4	0.20	1.23101
5	0.25	1.30256
6	0.30	1.38019
7	0.35	1.46420
8	0.40	1.55491
9	0.45	1.65266
10	0.50	1.75779
11	0.55	1.87068
12	0.60	1.99171
13	0.65	2.12130
14	0.70	2.25986
15	0.75	2.40786
16	0.80	2.56575
17	0.85	2.73404
18	0.90	2.91324
19	0.95	3.10390
20	1.00	3.30660

$$y_{n+1}^{\text{predicted}} = y_n + 0.1(x_n + y_n)$$

$$= 0.1x_n + 1.1y_n$$

$$y_{n+1} = y_n + \frac{0.1}{2}(x_n + y_n + x_{n+1} + 0.1x_n + 1.1y_n)$$

$$= y_n + 0.05(1.1x_n + 2.1y_n + x_n + 0.1)$$

$$= 0.105x_n + 1.105y_n + 0.005;$$

$$x_0 = 0, y_0 = 1$$

n	x_n	y_n
1	0.1	1.11
2	0.2	1.24205
3	0.3	1.39847
4	0.4	1.58180
5	0.5	1.79489
6	0.6	2.04086
7	0.7	2.32315
8	0.8	2.64558
9	0.9	3.01236
10	1.0	3.42816

$$y(1) = 2e - 1 - 1 = 3.4365$$

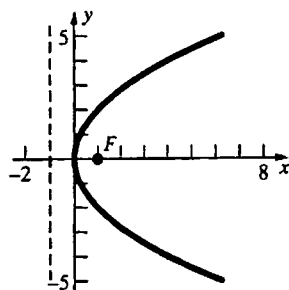
12.1 Concepts Review

1. $e < 1$; $e = 1$; $e > 1$
2. $y^2 = 4px$
3. $(0, 1)$; $y = -1$
4. parallel to the axis

Problem Set 12.1

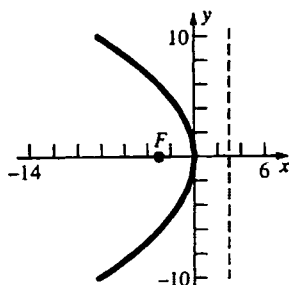
1. $y^2 = 4(1)x$

Focus at $(1, 0)$
Directrix: $x = -1$



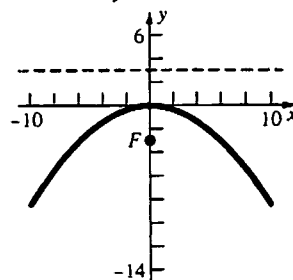
2. $y^2 = -4(3)x$

Focus at $(-3, 0)$
Directrix: $x = 3$



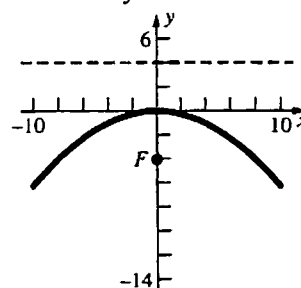
3. $x^2 = -4(3)y$

Focus at $(0, -3)$
Directrix: $y = 3$



4. $x^2 = -4(4)y$

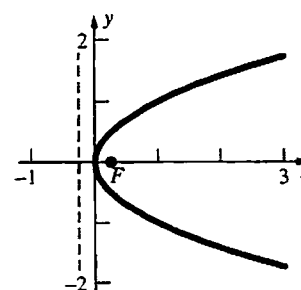
Focus at $(0, -4)$
Directrix: $y = 4$



5. $y^2 = 4\left(\frac{1}{4}\right)x$

Focus at $\left(\frac{1}{4}, 0\right)$

Directrix: $x = -\frac{1}{4}$

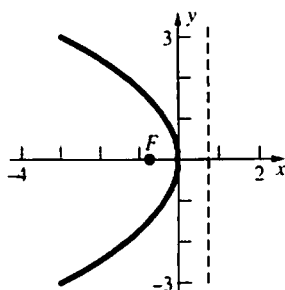


6. $y^2 = -3x$

$$y^2 = -4\left(\frac{3}{4}\right)x$$

Focus at $\left(-\frac{3}{4}, 0\right)$

Directrix: $x = \frac{3}{4}$

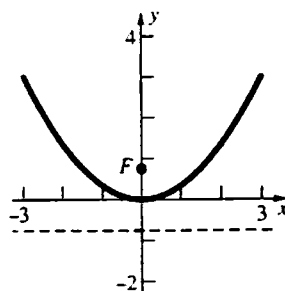


7. $2x^2 = 6y$

$$x^2 = 4\left(\frac{3}{4}\right)y$$

Focus at $\left(0, \frac{3}{4}\right)$

Directrix: $y = -\frac{3}{4}$

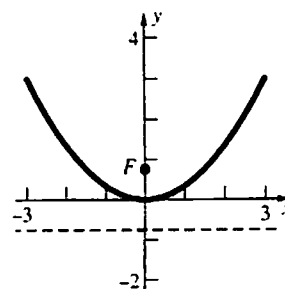


8. $3x^2 = 9y$

$$x^2 = 4\left(\frac{3}{4}\right)y$$

Focus at $\left(0, \frac{3}{4}\right)$

Directrix: $y = -\frac{3}{4}$



9. The parabola opens to the right, and $p = 2$.

$$y^2 = 8x$$

10. The parabola opens to the left, and $p = 3$.

$$y^2 = -12x$$

11. The parabola opens downward, and $p = 2$.

$$x^2 = -8y$$

12. The parabola opens downward, and $p = \frac{1}{9}$.

$$x^2 = -\frac{4}{9}y$$

13. The parabola opens to the left, and $p = 4$.

$$y^2 = -16x$$

14. The parabola opens downward, and $p = \frac{7}{2}$.

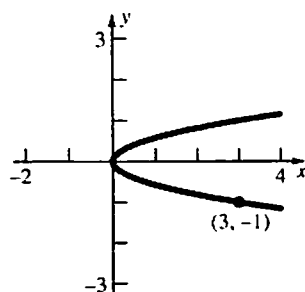
$$x^2 = -14y$$

15. The equation has the form $y^2 = cx$, so

$$(-1)^2 = 3c.$$

$$c = \frac{1}{3}$$

$$y^2 = \frac{1}{3}x$$

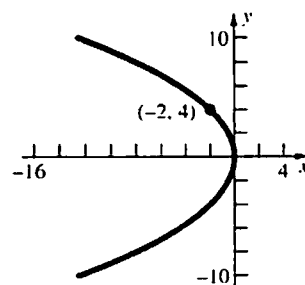


16. The equation has the form $y^2 = cx$, so

$$(4)^2 = -2c.$$

$$c = -8$$

$$y^2 = -8x$$

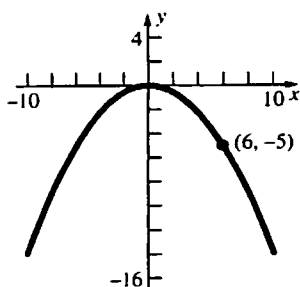


17. The equation has the form $x^2 = cy$, so

$$(6)^2 = -5c.$$

$$c = -\frac{36}{5}$$

$$x^2 = -\frac{36}{5}y$$

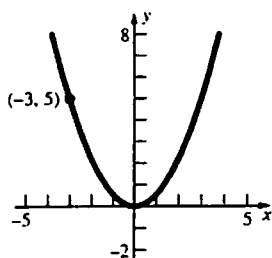


18. The equation has the form $x^2 = cy$, so

$$(-3)^2 = 5c.$$

$$c = \frac{9}{5}$$

$$x^2 = \frac{9}{5}y$$



19. $y^2 = 16x$

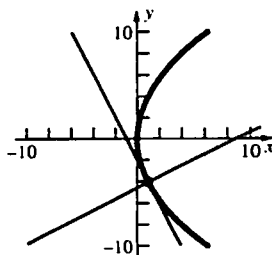
$$2yy' = 16$$

$$y' = \frac{16}{2y}$$

$$\text{At } (1, -4), y' = -2.$$

$$\text{Tangent: } y + 4 = -2(x - 1) \text{ or } y = -2x - 2$$

$$\text{Normal: } y + 4 = \frac{1}{2}(x - 1) \text{ or } y = \frac{1}{2}x - \frac{9}{2}$$



20. $x^2 = -10y$

$$2x = -10y'$$

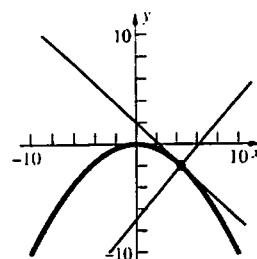
$$y' = -\frac{x}{5}$$

$$\text{At } (2\sqrt{5}, -2), y' = -\frac{2\sqrt{5}}{5}.$$

$$\text{Tangent: } y + 2 = -\frac{2\sqrt{5}}{5}(x - 2\sqrt{5}) \text{ or}$$

$$y = -\frac{2\sqrt{5}}{5}x + 2$$

$$\text{Normal: } y + 2 = \frac{\sqrt{5}}{2}(x - 2\sqrt{5}) \text{ or } y = \frac{\sqrt{5}}{2}x - 7$$



21. $x^2 = 2y$

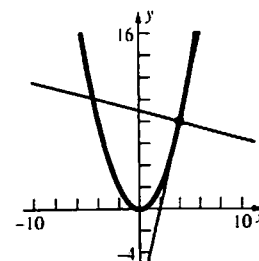
$$2x = 2y'$$

$$y' = x$$

$$\text{At } (4, 8), y' = 4.$$

$$\text{Tangent: } y - 8 = 4(x - 4) \text{ or } y = 4x - 8$$

$$\text{Normal: } y - 8 = -\frac{1}{4}(x - 4) \text{ or } y = -\frac{1}{4}x + 9$$



22. $y^2 = -9x$

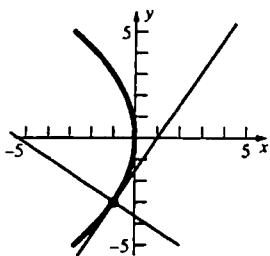
$$2yy' = -9$$

$$y' = -\frac{9}{2y}$$

$$\text{At } (-1, -3), y' = \frac{3}{2}$$

$$\text{Tangent: } y + 3 = \frac{3}{2}(x + 1) \text{ or } y = \frac{3}{2}x - \frac{3}{2}$$

Normal: $y+3 = -\frac{2}{3}(x+1)$ or $y = -\frac{2}{3}x - \frac{11}{3}$



23. $y^2 = -15x$

$2yy' = -15$

$y' = -\frac{15}{2y}$

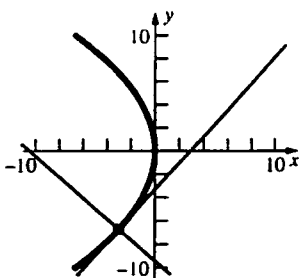
At $(-3, -3\sqrt{5})$, $y' = \frac{\sqrt{5}}{2}$.

Tangent: $y+3\sqrt{5} = \frac{\sqrt{5}}{2}(x+3)$ or

$y = \frac{\sqrt{5}}{2}x - \frac{3\sqrt{5}}{2}$

Normal: $y+3\sqrt{5} = -\frac{2\sqrt{5}}{5}(x+3)$ or

$y = -\frac{2\sqrt{5}}{5}x - \frac{21\sqrt{5}}{5}$



24. $x^2 = 4y$

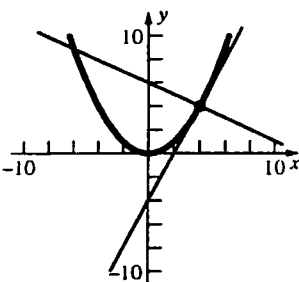
$2x = 4y'$

$y' = \frac{x}{2}$

At $(4, 4)$, $y' = 2$.

Tangent: $y-4 = 2(x-4)$ or $y = 2x-4$

Normal: $y-4 = -\frac{1}{2}(x-4)$ or $y = -\frac{1}{2}x+6$



25. $x^2 = -6y$

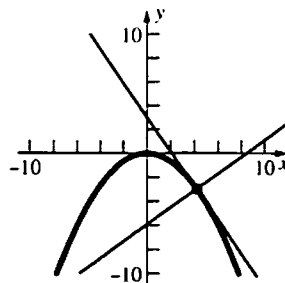
$2x = -6y'$

$y' = -\frac{x}{3}$

At $(3\sqrt{2}, -3)$, $y' = -\sqrt{2}$.

Tangent: $y+3 = -\sqrt{2}(x-3\sqrt{2})$ or $y = -\sqrt{2}x+3$

Normal: $y+3 = \frac{\sqrt{2}}{2}(x-3\sqrt{2})$ or $y = \frac{\sqrt{2}}{2}x-6$



26. $y^2 = 20x$

$2yy' = 20$

$y' = \frac{10}{y}$

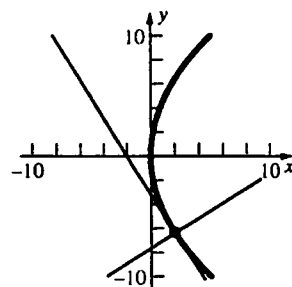
At $(2, -2\sqrt{10})$, $y' = -\frac{\sqrt{10}}{2}$.

Tangent: $y+2\sqrt{10} = -\frac{\sqrt{10}}{2}(x-2)$ or

$y = -\frac{\sqrt{10}}{2}x - \sqrt{10}$

Normal: $y+2\sqrt{10} = \frac{\sqrt{10}}{5}(x-2)$ or

$y = \frac{\sqrt{10}}{5}x - \frac{12\sqrt{10}}{5}$



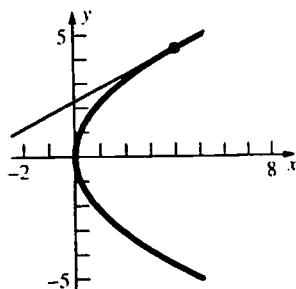
27. $y^2 = 5x$

$$2yy' = 5$$

$$y' = \frac{5}{2y}$$

$$y' = \frac{\sqrt{5}}{4} \text{ when } y = 2\sqrt{5}, \text{ so } x = 4.$$

The point is $(4, 2\sqrt{5})$.



28. $x^2 = -14y$

$$2x = -14y'$$

$$y' = -\frac{x}{7}$$

$$y' = -\frac{2\sqrt{7}}{7} \text{ when } x = 2\sqrt{7}, \text{ so } y = -2.$$

The point is $(2\sqrt{7}, -2)$.

29. The slope of the line is $\frac{3}{2}$.

$$y^2 = -18x; 2yy' = -18$$

$$2y\left(\frac{3}{2}\right) = -18; y = -6$$

$$(-6)^2 = -18x; x = -2$$

The equation of the tangent line is

$$y + 6 = \frac{3}{2}(x + 2) \text{ or } y = \frac{3}{2}x - 3.$$

30. Place the x -axis along the axis of the parabola such that the equation $y^2 = 4px$ describes the

parabola. Let $\left(\frac{y_0^2}{4p}, y_0\right)$ be one of the

extremities and $\left(\frac{y_1^2}{4p}, y_1\right)$ be the other.

First solve for y_1 in terms of y_0 and p . Since the focal chord passes through the focus $(p, 0)$, we have the following relation.

$$\frac{y_1}{\frac{y_1^2}{4p} - p} = \frac{y_0}{\frac{y_0^2}{4p} - p}$$

$$y_1(y_0^2 - 4p^2) = y_0(y_1^2 - 4p^2)$$

$$y_0y_1^2 - (y_0^2 - 4p^2)y_1 - 4p^2y_0 = 0$$

$$(y_1 - y_0)(y_0y_1 + 4p^2) = 0$$

$$y_1 = y_0 \text{ or } y_1 = -\frac{4p^2}{y_0}$$

Thus, the other extremity is $\left(\frac{4p^3}{y_0^2}, -\frac{4p^2}{y_0}\right)$.

Implicitly differentiate $y^2 = 4px$ to get

$$2yy' = 4p, \text{ so } y' = \frac{2p}{y}.$$

At $\left(\frac{y_0^2}{4p}, y_0\right)$, $y' = \frac{2p}{y_0}$. The equation of the

tangent line is $y - y_0 = \frac{2p}{y_0}\left(x - \frac{y_0^2}{4p}\right)$. When

$$x = -p, y = -\frac{2p^2}{y_0} + \frac{y_0}{2}.$$

At $\left(\frac{4p^3}{y_0^2}, -\frac{4p^2}{y_0}\right)$, $y' = -\frac{y_0}{2p}$. The equation of

the tangent line is $y + \frac{4p^2}{y_0} = -\frac{y_0}{2p}\left(x - \frac{4p^3}{y_0^2}\right)$.

$$\text{When } x = -p, y = \frac{y_0}{2} - \frac{2p^2}{y_0}.$$

Thus, the two tangent lines intersect on the

directrix at $\left(-p, \frac{y_0}{2} - \frac{2p^2}{y_0}\right)$.

31. From Problem 30, if the parabola is described by the equation $y^2 = 4px$, the slopes of the tangent

lines are $\frac{2p}{y_0}$ and $-\frac{y_0}{2p}$. Since they are negative

reciprocals, the tangent lines are perpendicular.

32. Place the x -axis along the axis of the parabola such that the equation $y^2 = 4px$ describes the

parabola. The endpoints of the chord are $\left(1, \frac{1}{2}\right)$

and $\left(1, -\frac{1}{2}\right)$, so $\left(\frac{1}{2}\right)^2 = 4(1)p$ or $p = \frac{1}{16}$. The

distance from the vertex to the focus is $\frac{1}{16}$.

33. Assume that the x - and y -axes are positioned such that the axis of the parabola is the y -axis with the vertex at the origin and opening upward. Then the equation of the parabola is $x^2 = 4py$ and $(0, p)$ is the focus. Let D be the distance from a point on the parabola to the focus.

$$D = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + \left(\frac{x^2}{4p} - p\right)^2}$$

$$= \sqrt{\frac{x^4}{16p^2} + \frac{x^2}{2} + p^2} = \frac{x^2}{4p} + p$$

$$D' = \frac{x}{2p}; \frac{x}{2p} = 0, x = 0$$

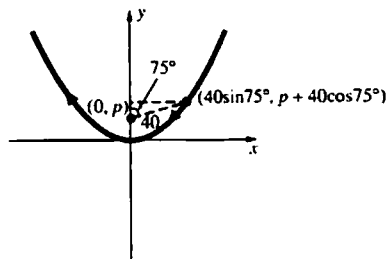
$D'' > 0$ so at $x = 0$, D is minimum. $y = 0$
Therefore, the vertex $(0, 0)$ is the point closest to the focus.

34. Let the y -axis be the axis of the parabola, so Earth's coordinates are $(0, p)$ and the equation of the path is $x^2 = 4py$, where the coordinates are in millions of miles. When the line from Earth to the asteroid makes an angle of 90° with the axis of the parabola, the asteroid is at $(40, p)$.

$$(40)^2 = 4p(p), p = 20$$

The closest point to Earth is $(0, 0)$, so the asteroid will come within 20 million miles of Earth.

35. Let the y -axis be the axis of the parabola, so the Earth's coordinates are $(0, p)$ and the equation of the path is $x^2 = 4py$, where the coordinates are in millions of miles. When the line from Earth to the asteroid makes an angle of 75° with the axis of the parabola, the asteroid is at $(40 \sin 75^\circ, p + 40 \cos 75^\circ)$. (See figure.)



$$(40 \sin 75^\circ)^2 = 4p(p + 40 \cos 75^\circ)$$

$$p^2 + 40p \cos 75^\circ - 400 \sin^2 75^\circ = 0$$

$$p = \frac{-40 \cos 75^\circ \pm \sqrt{1600 \cos^2 75^\circ + 1600 \sin^2 75^\circ}}{2}$$

$$= -20 \cos 75^\circ \pm 20$$

$$p = 20 - 20 \cos 75^\circ \approx 14.8 (p > 0)$$

The closest point to Earth is $(0, 0)$, so the asteroid will come within 14.8 million miles of Earth.

36. Let the equation $x^2 = 4py$ describe the cables.

The cables are attached to the towers at $(\pm 400, 400)$.

$$(400)^2 = 4p(400), p = 100$$

The vertical struts are at $x = \pm 300$.

$$(300)^2 = 4(100)y, y = 225$$

The struts must be 225 m long.

37. Let $|RL|$ be the distance from R to the directrix.

Observe that the distance from the latus rectum to the directrix is $2p$ so $|RG| = 2p - |RL|$. From the

definition of a parabola, $|RL| = |FR|$. Thus,

$$|FR| + |RG| = |RL| + 2p - |RL| = 2p.$$

38. Let C denote the center of the circle and r the radius. Observe that the distance from a point P to the circle is $|PC| - r$. Let l be the line and $|PL|$ the distance from the point to the line. Thus, $|PC| - r = |PL|$. Let l' be the line parallel to l , r units away and on the side opposite from the circle. Then $|PL'|$, the distance from P to l' , is $|PL| + r$; so $|PL| = |PL'| - r$. Therefore, $|PC| - r = |PL'| - r$ or $|PC| = |PL'|$. The set of points is a parabola by definition.

39. Let P_1 and P_2 denote (x_1, y_1) and (x_2, y_2) , respectively. $|P_1 P_2| = |P_1 F| + |P_2 F|$ since the focal chord passes through the focus. By definition of a parabola, $|P_1 F| = p + x_1$ and $|P_2 F| = p + x_2$. Thus, the length of the chord is $|P_1 P_2| = x_1 + x_2 + 2p$.
 $L = p + p + 2p = 4p$

40. Let the coordinates of P be (x_0, y_0) .

$$2yy' = 4p, \text{ so } y' = \frac{2p}{y}. \text{ Thus the slope of the}$$

$$\text{normal line at } P \text{ is } -\frac{y_0}{2p}.$$

The equation of the normal line is

$$y - y_0 = -\frac{y_0}{2p}(x - x_0). \text{ When } y = 0,$$

$$x = 2p + x_0, \text{ so } B \text{ is at } (2p + x_0, 0). A \text{ is at } (x_0, 0). \text{ Thus, } |AB| = 2p + x_0 - x_0 = 2p.$$

$$41. \frac{dy}{dx} = \frac{\delta x}{H}$$

$$y = \frac{\delta x^2}{2H} + C$$

$$y(0) = 0 \text{ implies that } C = 0. y = \frac{\delta x^2}{2H}$$

This is an equation for a parabola.

42. a. $A(T_1)$ is the area of the trapezoid formed by

$(a, 0)$, P , Q , $(b, 0)$ minus the area the two trapezoids formed by $(a, 0)$, P , (c, c^2) , $(c, 0)$ and by $(c, 0)$, (c, c^2) ,

Q , $(b, 0)$. Observe that since $c = \frac{a+b}{2}$, $\frac{b-c}{2} = \frac{c-a}{2} = \frac{b-a}{4}$.

$$A(T_1) = \frac{b-a}{2}[a^2 + b^2] - \frac{c-a}{2}[a^2 + c^2] - \frac{b-c}{2}[c^2 + b^2] = \frac{b-a}{2}[a^2 + b^2] - \frac{b-a}{4}[a^2 + 2c^2 + b^2]$$

$$= \frac{b-a}{2}[a^2 + b^2] - \frac{b-a}{4}\left[a^2 + 2\left(\frac{a+b}{2}\right)^2 + b^2\right] = \frac{b-a}{4}\left[a^2 + b^2 - \frac{a^2}{2} - ab - \frac{b^2}{2}\right]$$

$$= \frac{b-a}{4}\left(\frac{a^2}{2} - ab + \frac{b^2}{2}\right) = \frac{(b-a)^3}{8}$$

$$b. A(T_2) = \frac{(c-a)^3}{8} + \frac{(b-c)^3}{8} = \frac{\left(\frac{b-a}{2}\right)^3}{8} + \frac{\left(\frac{b-a}{2}\right)^3}{8} = \frac{(b-a)^3}{32} = \frac{A(T_1)}{4}$$

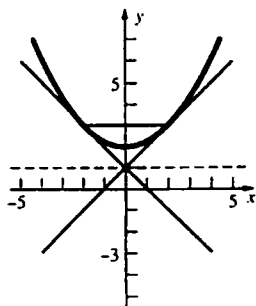
c. Using reasoning similar to part b, $A(T_n) = \frac{A(T_{n-1})}{4}$, so $A(T_n) = \frac{A(T_1)}{4^{n-1}}$.

$$A(S) = A(T_1) + A(T_2) + A(T_3) + \dots = \sum_{n=1}^{\infty} \frac{A(T_1)}{4^{n-1}} = A(T_1) \left(\frac{1}{1 - \frac{1}{4}} \right) = \frac{4}{3} A(T_1)$$

$$d. \text{ Area} = \frac{b-a}{2}[a^2 + b^2] - A(S) = \frac{(b-a)(a^2 + b^2)}{2} - \frac{(b-a)^3}{6}$$

$$= \frac{b-a}{6}[3a^2 + 3b^2 - (b^2 - 2ab + a^2)] = \frac{(b-a)}{6}[2a^2 + 2ab + 2b^2] = \frac{1}{3}(b^3 - a^3) = \frac{b^3}{3} - \frac{a^3}{3}$$

43.



12.2 Concepts Review

$$1. \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$3. \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$2. \frac{x^2}{9} + \frac{y^2}{16} = 1$$

$$4. y = \pm \frac{2}{3}x$$

Problem Set 12.2

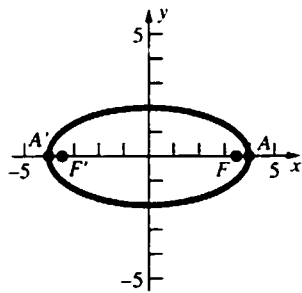
- Horizontal ellipse
- Horizontal hyperbola
- Vertical hyperbola
- Horizontal hyperbola
- Vertical parabola
- Vertical parabola
- Vertical ellipse
- Horizontal hyperbola

9. $\frac{x^2}{16} + \frac{y^2}{4} = 1$; horizontal ellipse

$a = 4, b = 2, c = 2\sqrt{3}$

Vertices: $(\pm 4, 0)$

Foci: $(\pm 2\sqrt{3}, 0)$



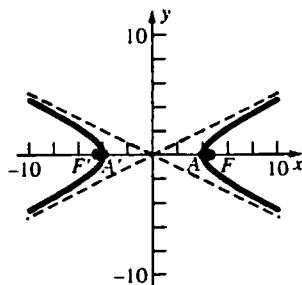
10. $\frac{x^2}{16} - \frac{y^2}{4} = 1$; horizontal hyperbola

$a = 4, b = 2, c = 2\sqrt{5}$

Vertices: $(\pm 4, 0)$

Foci: $(\pm 2\sqrt{5}, 0)$

Asymptotes: $y = \pm \frac{1}{2}x$



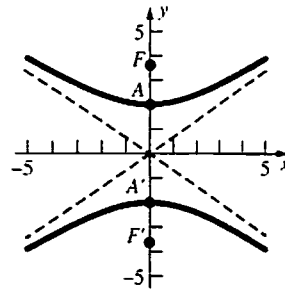
11. $\frac{y^2}{4} - \frac{x^2}{9} = 1$; vertical hyperbola

$a = 2, b = 3, c = \sqrt{13}$

Vertices: $(0, \pm 2)$

Foci: $(0, \pm \sqrt{13})$

Asymptotes: $y = \pm \frac{2}{3}x$

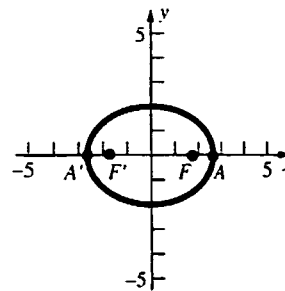


12. $\frac{x^2}{7} + \frac{y^2}{4} = 1$; horizontal ellipse

$a = \sqrt{7}, b = 2, c = \sqrt{3}$

Vertices: $(\pm \sqrt{7}, 0)$

Foci: $(\pm \sqrt{3}, 0)$

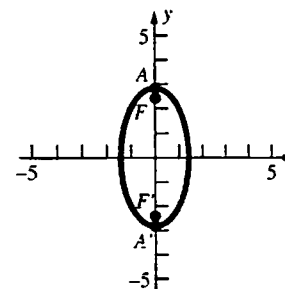


13. $\frac{x^2}{2} + \frac{y^2}{8} = 1$; vertical ellipse

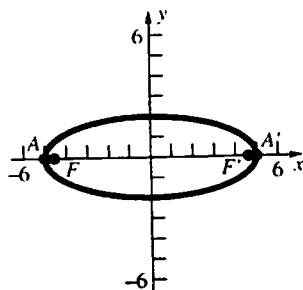
$a = 2\sqrt{2}, b = \sqrt{2}, c = \sqrt{6}$

Vertices: $(0, \pm 2\sqrt{2})$

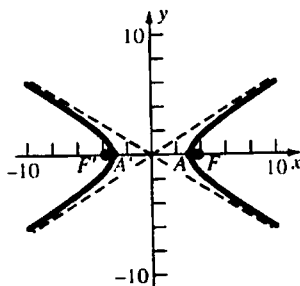
Foci: $(0, \pm \sqrt{6})$



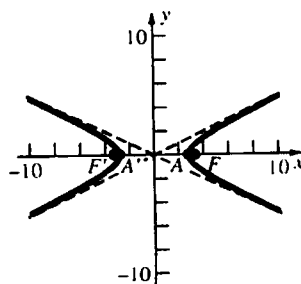
14. $\frac{x^2}{25} + \frac{y^2}{4} = 1$; horizontal ellipse
 $a = 5, b = 2, c = \sqrt{21}$
 Vertices: $(\pm 5, 0)$
 Foci: $(\pm \sqrt{21}, 0)$



15. $\frac{x^2}{10} - \frac{y^2}{4} = 1$; horizontal hyperbola
 $a = \sqrt{10}, b = 2, c = \sqrt{14}$
 Vertices: $(\pm \sqrt{10}, 0)$
 Foci: $(\pm \sqrt{14}, 0)$
 Asymptotes: $y = \pm \frac{2}{\sqrt{10}}x$



16. $\frac{x^2}{8} - \frac{y^2}{2} = 1$; horizontal hyperbola
 $a = 2\sqrt{2}, b = \sqrt{2}, c = \sqrt{10}$
 Vertices: $(\pm 2\sqrt{2}, 0)$
 Foci: $(\pm \sqrt{10}, 0)$
 Asymptotes: $y = \pm \frac{1}{2}x$



17. This is a horizontal ellipse with $a = 6$ and $c = 3$.
 $b = \sqrt{36 - 9} = \sqrt{27}$
 $\frac{x^2}{36} + \frac{y^2}{27} = 1$

18. This is a horizontal ellipse with $c = 6$.
 $a = \frac{c}{e} = \frac{6}{\frac{2}{3}} = 9, b = \sqrt{81 - 36} = \sqrt{45}$
 $\frac{x^2}{81} + \frac{y^2}{45} = 1$

19. This is a vertical ellipse with $c = 5$.
 $a = \frac{c}{e} = \frac{5}{\frac{1}{3}} = 15, b = \sqrt{225 - 25} = \sqrt{200}$
 $\frac{x^2}{200} + \frac{y^2}{225} = 1$

20. This is a vertical ellipse with $b = 4$ and $c = 3$.
 $a = \sqrt{16 + 9} = 5$
 $\frac{x^2}{16} + \frac{y^2}{25} = 1$

21. This is a horizontal ellipse with $a = 5$.
 $\frac{x^2}{25} + \frac{y^2}{b^2} = 1$
 $\frac{4}{25} + \frac{9}{b^2} = 1$
 $b^2 = \frac{225}{21}$
 $\frac{x^2}{25} + \frac{y^2}{\frac{225}{21}} = 1$

22. This is a horizontal hyperbola with $a = 4$ and $c = 5$.
 $b = \sqrt{25 - 16} = 3$
 $\frac{x^2}{16} - \frac{y^2}{9} = 1$

23. This is a vertical hyperbola with $a = 4$ and $c = 5$.
 $b = \sqrt{25 - 16} = 3$
 $\frac{y^2}{16} - \frac{x^2}{9} = 1$

24. This is a vertical hyperbola with $a = 3$.
 $c = ae = 3\left(\frac{3}{2}\right) = \frac{9}{2}, b = \sqrt{\frac{81}{4} - 9} = \frac{\sqrt{45}}{2}$
 $\frac{y^2}{9} - \frac{x^2}{\frac{45}{4}} = 1$

25. This is a horizontal hyperbola with $a = 8$.

The asymptotes are $y = \pm \frac{1}{2}x$, so $\frac{b}{8} = \frac{1}{2}$ or $b = 4$.

$$\frac{x^2}{64} - \frac{y^2}{16} = 1$$

26. $c = ae = \frac{\sqrt{6}}{2}a$, $b^2 = c^2 - a^2 = \frac{3}{2}a^2 - a^2 = \frac{1}{2}a^2$

$$\frac{y^2}{a^2} - \frac{x^2}{\frac{1}{2}a^2} = 1$$

$$\frac{16}{a^2} - \frac{4}{\frac{1}{2}a^2} = 1$$

$$a^2 = 8$$

$$\frac{y^2}{8} - \frac{x^2}{4} = 1$$

27. This is a horizontal ellipse with $c = 2$.

$$8 = \frac{a}{e}, 8 = \frac{a}{\frac{c}{a}}, \text{ so } a^2 = 8c = 16.$$

$$b = \sqrt{16 - 4} = \sqrt{12}$$

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

28. This is a horizontal hyperbola with $c = 4$.

$$1 = \frac{a}{e}, 1 = \frac{a}{\frac{c}{a}}, \text{ so } a^2 = c = 4.$$

$$b = \sqrt{16 - 4} = \sqrt{12}$$

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

29. The asymptotes are $y = \pm \frac{1}{2}x$. If the hyperbola is

horizontal $\frac{b}{a} = \frac{1}{2}$ or $a = 2b$. If the hyperbola is

vertical, $\frac{a}{b} = \frac{1}{2}$ or $b = 2a$.

Suppose the hyperbola is horizontal.

$$\frac{x^2}{4b^2} - \frac{y^2}{b^2} = 1$$

$$\frac{16}{4b^2} - \frac{9}{b^2} = 1$$

$$b^2 = -5$$

This is not possible.

Suppose the hyperbola is vertical.

$$\frac{y^2}{a^2} - \frac{x^2}{4a^2} = 1$$

$$\frac{9}{a^2} - \frac{16}{4a^2} = 1$$

$$a^2 = 5$$

$$\frac{y^2}{5} - \frac{x^2}{20} = 1$$

$$30. \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\left. \begin{aligned} \frac{25}{a^2} + \frac{1}{b^2} &= 1 \\ \frac{16}{a^2} + \frac{4}{b^2} &= 1 \end{aligned} \right\} - \frac{84}{a^2} = -3$$

$$a^2 = 28$$

$$b^2 = \frac{28}{3}$$

$$\frac{x^2}{28} + \frac{y^2}{\frac{28}{3}} = 1$$

31. Let the y -axis run through the center of the arch and the x -axis lie on the floor. Thus $a = 5$ and

$b = 4$ and the equation of the arch is $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

$$\text{When } y = 2, \frac{x^2}{25} + \frac{(2)^2}{16} = 1, \text{ so } x = \pm \frac{5\sqrt{3}}{2}.$$

The width of the box can at most be

$$5\sqrt{3} \approx 8.66 \text{ ft.}$$

32. Let the y -axis run through the center of the arch and the x -axis lie on the floor.

The equation of the arch is $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

$$\text{When } x = 2, \frac{(2)^2}{25} + \frac{y^2}{16} = 1, \text{ so } y = \pm \frac{4\sqrt{21}}{5}.$$

The height at a distance of 2 feet to the right of the

$$\text{center is } \frac{4\sqrt{21}}{5} \approx 3.67 \text{ ft.}$$

33. The foci are at $(\pm c, 0)$.

$$c = \sqrt{a^2 - b^2}$$

$$\frac{a^2 - b^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = \frac{b^4}{a^2}, y = \pm \frac{b^2}{a}$$

Thus, the length of the latus rectum is $\frac{2b^2}{a}$.

34. The foci are at $(\pm c, 0)$

$$c = \sqrt{a^2 + b^2}$$

$$\frac{a^2 + b^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$y^2 = \frac{b^4}{a^2}, y = \pm \frac{b^2}{a}$$

Thus, the length of the latus rectum is $\frac{2b^2}{a}$.

35. $a = 18.09, b = 4.56,$

$$c = \sqrt{(18.09)^2 - (4.56)^2} \approx 17.51$$

The comet's minimum distance from the sun is $18.09 - 17.51 \approx 0.58$ AU.

36. $a - c = 0.13, c = ae, a(1 - e) = 0.13,$

$$a = \frac{0.13}{1 - 0.999925} \approx 1733$$

$$a + c = a(1 + e) \approx 1733(1 + 0.999925) \approx 3466 \text{ AU}$$

37. $a - c = 4132; a + c = 4583$

$$2a = 8715; a = 4357.5$$

$$c = 4357.5 - 4132 = 225.5$$

$$e = \frac{c}{a} = \frac{225.5}{4357.5} \approx 0.05175$$

38. $\lim_{x \rightarrow \infty} (\sqrt{x^2 - a^2} - x)$

$$= \lim_{x \rightarrow \infty} \left[\frac{(\sqrt{x^2 - a^2} - x)(\sqrt{x^2 - a^2} + x)}{(\sqrt{x^2 - a^2} + x)} \right]$$

$$= \lim_{x \rightarrow \infty} \frac{-a^2}{\sqrt{x^2 - a^2} + x} = 0$$

39. $2a = p + q, 2c = |p - q|$

$$b^2 = a^2 - c^2 = \frac{(p + q)^2}{4} - \frac{(p - q)^2}{4} = pq$$

$$b = \sqrt{pq}$$

40. $x = a \cos t, y = a \sin t - b \sin t = (a - b) \sin t$

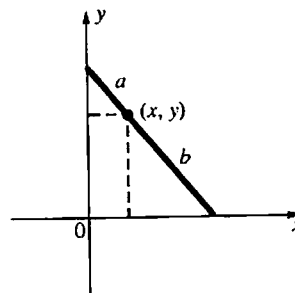
$$\cos t = \frac{x}{a}, \sin t = \frac{y}{a - b}$$

$$\frac{x^2}{a^2} + \frac{y^2}{(a - b)^2} = 1$$

Thus the coordinates of R at time t lie on an ellipse.

41. Let (x, y) be the coordinates of P as the ladder slides. Using a property of similar triangles,

$$\frac{x}{a} = \frac{\sqrt{b^2 - y^2}}{b}$$



Square both sides to get

$$\frac{x^2}{a^2} = \frac{b^2 - y^2}{b^2} \text{ or } b^2 x^2 + a^2 y^2 = a^2 b^2 \text{ or}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

42. Place the x -axis on the axis of the hyperbola such

that the equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. One focus is at

$(c, 0)$ and the asymptotes are $y = \pm \frac{b}{a}x$. The

equations of the lines through the focus, perpendicular to the asymptotes, are

$y = \pm \frac{a}{b}(x - c)$. Then solve for x in

$$\frac{b}{a}x = -\frac{a}{b}(x - c).$$

$$\frac{a^2 + b^2}{ab}x = \frac{ac}{b}$$

$$x = \frac{a^2 c}{a^2 + b^2}$$

Since $c^2 = a^2 + b^2, x = \frac{a^2}{c}$. The equation of the

directrix nearest the focus is $x = \frac{a^2}{c}$, so the line

through a focus and perpendicular to an asymptote intersects that asymptote on the directrix nearest the focus.

43. The equations of the hyperbolas are $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

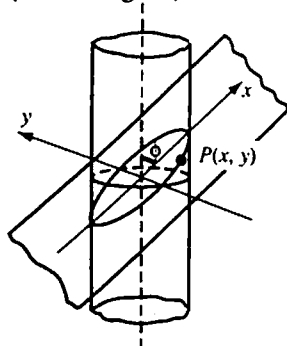
$$\text{and } \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$$

$$E = \frac{c}{b} = \frac{\sqrt{a^2 + b^2}}{b}$$

$$e^{-2} + E^{-2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

44. Position the x -axis on the plane so that it makes the angle ϕ with the axis of the cylinder and the y -axis is perpendicular to the axis of the cylinder. (See the figure.)



If $P(x, y)$ is a point on C , $(x \sin \phi)^2 + y^2 = r^2$

12.3 Concepts Review

- foci
- $\|PF\| - \|PF'\| = 2a$
- to the other focus
- directly away from the other focus

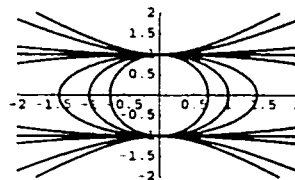
Problem Set 12.3

- This is a vertical ellipse with $a = 13$ and $c = 9$.
 $b = \sqrt{169 - 81} = 2\sqrt{22}$
 $\frac{x^2}{88} + \frac{y^2}{169} = 1$
- This is a horizontal ellipse with $a = 7$ and $c = 4$.
 $b = \sqrt{49 - 16} = \sqrt{33}$
 $\frac{x^2}{49} + \frac{y^2}{33} = 1$
- This is a horizontal hyperbola with $a = 6$ and $c = 7$.
 $b = \sqrt{49 - 36} = \sqrt{13}$
 $\frac{x^2}{36} - \frac{y^2}{13} = 1$

where r is the radius of the cylinder. Then

$$\frac{x^2}{\frac{r^2}{\sin^2 \phi}} + \frac{y^2}{r^2} = 1.$$

45.



When $a < 0$, the conic is an ellipse. When $a > 0$, the conic is a hyperbola. When $a = 0$, the graph is two parallel lines.

4. This is a vertical hyperbola with $a = 5$ and $c = 6$.

$$b = \sqrt{36 - 25} = \sqrt{11}$$

$$\frac{y^2}{25} - \frac{x^2}{11} = 1$$

5. $a^2 = 27, b^2 = 9$

$$\frac{3x}{27} + \frac{\sqrt{6}y}{9} = 1$$

$$x + \sqrt{6}y = 9$$

6. $a^2 = 24, b^2 = 16$

$$\frac{3\sqrt{2}x}{24} + \frac{-2y}{16} = 1$$

$$\sqrt{2}x - y = 8$$

7. $a^2 = 27, b^2 = 9$

$$\frac{3x}{27} + \frac{-\sqrt{6}y}{9} = 1$$

$$x - \sqrt{6}y = 9$$

8. $a^2 = 2, b^2 = 4$

$$\frac{\sqrt{3}x}{2} - \frac{\sqrt{2}y}{4} = 1$$

$$2\sqrt{3}x - \sqrt{2}y = 4$$

9. $a^2 = 169, b^2 = 169$

$$\frac{5x}{169} + \frac{12y}{169} = 1$$

$$5x + 12y = 169$$

$$10. \quad y^2 - x^2 = 1$$

$$a^2 = 1, b^2 = 1$$

$$\frac{\sqrt{3}y}{1} - \frac{\sqrt{2}x}{1} = 1$$

$$\sqrt{3}y - \sqrt{2}x = 1$$

$$11. \quad a^2 = 169, b^2 = 88$$

$$\frac{13y}{169} = 1$$

$$y = 13$$

$$12. \quad a^2 = 49, b^2 = 33$$

$$\frac{7x}{49} = 1$$

$$x = 7$$

$$13. \quad \frac{x^2}{4} + \frac{y^2}{9} = 1$$

Equation of tangent line at (x_0, y_0) :

$$\frac{xx_0}{4} + \frac{yy_0}{9} = 1$$

At $(0, 6)$, $y_0 = \frac{3}{2}$.

When $y = \frac{3}{2}$, $\frac{x^2}{4} + \frac{1}{4} = 1$, $x = \pm\sqrt{3}$.

The points of tangency are $\left(\sqrt{3}, \frac{3}{2}\right)$ and $\left(-\sqrt{3}, \frac{3}{2}\right)$

$$14. \quad \frac{x^2}{4} - \frac{y^2}{36} = 1$$

Equation of tangent line at (x_0, y_0) :

$$\frac{xx_0}{4} - \frac{yy_0}{36} = 1$$

At $(0, 6)$, $y_0 = -6$.

When $y = -6$, $\frac{x^2}{4} - \frac{36}{36} = 1$, $x = \pm 2\sqrt{2}$.

The points of tangency are $(2\sqrt{2}, -6)$ and $(-2\sqrt{2}, -6)$.

$$15. \quad 2x^2 - 7y^2 - 35 = 0; 4x - 14yy' = 0$$

$$y' = \frac{2x}{7y}; -\frac{2}{3} = \frac{2x}{7y}; x = -\frac{7y}{3}$$

Substitute $x = -\frac{7y}{3}$ into the equation of the hyperbola.

$$\frac{98}{9}y^2 - 7y^2 - 35 = 0, y = \pm 3$$

The coordinates of the points of tangency are $(-7, 3)$ and $(7, -3)$.

$$16. \quad \text{The slope of the line is } \frac{1}{\sqrt{2}}.$$

$$x^2 + 2y^2 - 2 = 0; 2x + 4yy' = 0$$

$$y' = -\frac{x}{2y}; \frac{1}{\sqrt{2}} = -\frac{x}{2y}; x = -\sqrt{2}y$$

Substitute $x = -\sqrt{2}y$ into the equation of the ellipse.

$$2y^2 + 2y^2 - 2 = 0; y = \pm \frac{1}{\sqrt{2}}$$

The tangent lines are tangent at $\left(-1, \frac{1}{\sqrt{2}}\right)$ and

$\left(1, -\frac{1}{\sqrt{2}}\right)$. The equations of the tangent lines are $y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x+1)$ and $y + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x-1)$ or $x - \sqrt{2}y + 2 = 0$ and $x - \sqrt{2}y - 2 = 0$.

$$17. \quad y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

$$A = 4b \int_0^a \sqrt{1 - \frac{x^2}{a^2}} dx$$

Let $x = a \sin t$ then $dx = a \cos t dt$. Then the limits are 0 and $\frac{\pi}{2}$.

$$A = 4b \int_0^a \sqrt{1 - \frac{x^2}{a^2}} dx = 4ab \int_0^{\pi/2} \cos^2 t dt$$

$$= 2ab \int_0^{\pi/2} (1 + \cos 2t) dt = 2ab \left[t + \frac{\sin 2t}{2} \right]_0^{\pi/2}$$

$$= \pi ab$$

$$18. \quad x = \pm a\sqrt{1 - \frac{y^2}{b^2}}$$

$$V = 2 \cdot \pi \int_0^b a^2 \left(1 - \frac{y^2}{b^2}\right) dy = 2\pi a^2 \left[y - \frac{y^3}{3b^2} \right]_0^b$$

$$= \frac{4\pi a^2 b}{3}$$

$$19. \quad y = \pm b\sqrt{\frac{x^2}{a^2} - 1}$$

The vertical line at one focus is $x = \sqrt{a^2 + b^2}$.

$$\begin{aligned}
 V &= \pi \int_a^{\sqrt{a^2+b^2}} \left(b \sqrt{\frac{x^2}{a^2} - 1} \right)^2 dx \\
 &= b^2 \pi \int_a^{\sqrt{a^2+b^2}} \left(\frac{x^2}{a^2} - 1 \right) dx \\
 &= b^2 \pi \left[\frac{x^3}{3a^2} - x \right]_a^{\sqrt{a^2+b^2}} \\
 &= b^2 \pi \left[\frac{(a^2+b^2)^{3/2}}{3a^2} - \sqrt{a^2+b^2} + \frac{2}{3}a \right] \\
 &= \frac{\pi b^2}{3a^2} \left[(a^2+b^2)^{3/2} - 3a^2 \sqrt{a^2+b^2} + 2a^3 \right]
 \end{aligned}$$

20. $y = \pm b \sqrt{1 - \frac{x^2}{a^2}}$

$$\begin{aligned}
 V &= 2 \cdot \pi \int_0^a \left(b \sqrt{1 - \frac{x^2}{a^2}} \right)^2 dx \\
 &= 2\pi b^2 \int_0^a \left(1 - \frac{x^2}{a^2} \right) dx = 2\pi b^2 \left[x - \frac{x^3}{3a^2} \right]_0^a \\
 &= \frac{4}{3} \pi a b^2
 \end{aligned}$$

21. If one corner of the rectangle is at (x, y) the sides have length $2x$ and $2y$.

$$x = \pm a \sqrt{1 - \frac{y^2}{b^2}}$$

$$A = 4xy = 4ya \sqrt{1 - \frac{y^2}{b^2}} = 4a \sqrt{y^2 - \frac{y^4}{b^2}}$$

$$\frac{dA}{dy} = \frac{2a \left(2y - \frac{4y^3}{b^2} \right)}{\sqrt{y^2 - \frac{y^4}{b^2}}}; \frac{dA}{dy} = 0 \text{ when}$$

$$y - \frac{2y^3}{b^2} = 0$$

$$y \left(1 - \frac{2y^2}{b^2} \right) = 0$$

$$y = 0 \text{ or } y = \pm \frac{b}{\sqrt{2}}$$

The Second Derivative Test shows that $y = \frac{b}{\sqrt{2}}$ is a maximum.

$$x = a \sqrt{1 - \frac{\left(\frac{b}{\sqrt{2}}\right)^2}{b^2}} = \frac{a}{\sqrt{2}}$$

Therefore, the rectangle is $a\sqrt{2}$ by $b\sqrt{2}$.

22. Position the x -axis on the axis of the hyperbola such that the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ describes the hyperbola. The equation of the tangent line at (x_0, y_0) is $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$. The equations of the asymptotes are $y = \pm \frac{b}{a} x$.

Substitute $y = \frac{b}{a} x$ and $y = -\frac{b}{a} x$ into the equation of the tangent line.

$$\begin{aligned}
 \frac{x_0 x}{a^2} - \frac{y_0 x}{ab} &= 1 \quad \frac{x_0 x}{a^2} + \frac{y_0 x}{ab} = 1 \\
 x \left(\frac{bx_0 - ay_0}{a^2 b} \right) &= 1 \quad x \left(\frac{bx_0 + ay_0}{a^2 b} \right) = 1 \\
 x &= \frac{a^2 b}{bx_0 - ay_0} \quad x = \frac{a^2 b}{bx_0 + ay_0}
 \end{aligned}$$

Thus the tangent line intersects the asymptotes at

$$\left(\frac{a^2 b}{bx_0 - ay_0}, \frac{ab^2}{bx_0 - ay_0} \right) \text{ and } \left(\frac{a^2 b}{bx_0 + ay_0}, -\frac{ab^2}{bx_0 + ay_0} \right)$$

Observe that $b^2 x_0 - a^2 y_0 = a^2 b^2$.

$$\begin{aligned}
 \frac{1}{2} \left(\frac{a^2 b}{bx_0 - ay_0} + \frac{a^2 b}{bx_0 + ay_0} \right) &= \frac{a^2 b^2 x_0}{b^2 x_0^2 - a^2 y_0^2} = x_0 \\
 \frac{1}{2} \left(\frac{ab^2}{bx_0 - ay_0} - \frac{ab^2}{bx_0 + ay_0} \right) &= \frac{a^2 b^2 y_0}{b^2 x_0^2 - a^2 y_0^2} = y_0
 \end{aligned}$$

Thus, the point of contact is midway between the two points of intersection.

23. Add the two equations to get $9y^2 = 675$.

$$y = \pm 5\sqrt{3}$$

Substitute $y = 5\sqrt{3}$ into either of the two equations and solve for x .

$$x = \pm 6$$

The point in the first quadrant is $(6, 5\sqrt{3})$.

24. Substitute $x = 6 - 2y$ into $x^2 + 4y^2 = 20$.

$$(6 - 2y)^2 + 4y^2 = 20$$

$$8y^2 - 24y + 16 = 0$$

$$y^2 - 3y + 2 = 0$$

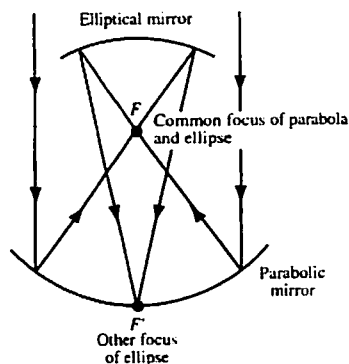
$$(y - 1)(y - 2) = 0$$

$$y = 1 \text{ or } y = 2$$

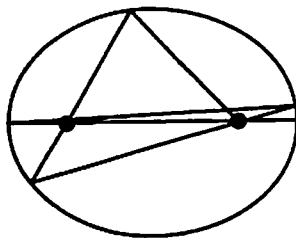
$$x = 4 \text{ or } x = 2$$

Thus, the points of intersection are (1, 4) and (2, 2).

25.

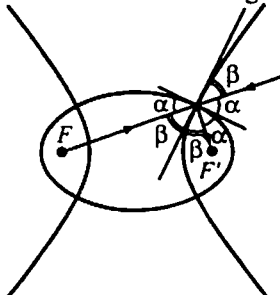


26. If the original path is not along the major axis, the ultimate path will approach the major axis.



27. Written response. Possible answer: the ball will follow a path that does not go between the foci.

28. Consider the following figure.



Observe that $2(\alpha + \beta) = 180^\circ$, so $\alpha + \beta = 90^\circ$. The ellipse and hyperbola meet at right angles.

29. Possible answer: Attach one end of a string to F and attach one end of another string to F' . Place a spool at a vertex. Tightly wrap both strings in the same direction around the spool. Insert a pencil through the spool. Then trace out a branch of the hyperbola by unspooling the strings while keeping both strings taut.

$$30. \frac{|AP|}{u} = \frac{|AB|}{v} + \frac{|BP|}{u}$$

$$|AP| - |BP| = \frac{2uc}{v}$$

Thus the curve is the right branch of the horizontal

hyperbola with $a = \frac{uc}{v}$, so $b = \sqrt{\left(1 - \frac{u^2}{v^2}\right)c}$.

The equation of the curve is

$$\frac{x^2}{\frac{u^2 c^2}{v^2}} - \frac{y^2}{\left(1 - \frac{u^2}{v^2}\right)c^2} = 1 \left(x \geq \frac{uc}{v} \right).$$

31. Let $P(x, y)$ be the location of the explosion.

$$3|AP| = 3|BP| + 12$$

$$|AP| - |BP| = 4$$

Thus, P lies on the right branch of the horizontal hyperbola with $a = 2$ and $c = 8$, so $b = 2\sqrt{15}$.

$$\frac{x^2}{4} - \frac{y^2}{60} = 1$$

Since $|BP| = |CP|$, the y -coordinate of P is 5.

$$\frac{x^2}{4} - \frac{25}{60} = 1, x = \pm \sqrt{\frac{17}{3}}$$

$$P \text{ is at } \left(\sqrt{\frac{17}{3}}, 5 \right).$$

12.4 Concepts Review

1. $\frac{a^2}{4}$
2. 14; ellipse
3. a line, parallel lines, intersecting lines, a point, the empty set
4. intersecting lines

Problem Set 12.4

1. $x^2 + y^2 - 2x + 2y + 1 = 0$
 $(x^2 - 2x + 1) + (y^2 + 2y + 1) = -1 + 1 + 1$
 $(x - 1)^2 + (y + 1)^2 = 1$
 This is a circle.
2. $x^2 + y^2 + 6x - 2y + 6 = 0$
 $(x^2 + 6x + 9) + (y^2 - 2y + 1) = -6 + 9 + 1$
 $(x + 3)^2 + (y - 1)^2 = 4$
 This is a circle.
3. $9x^2 + 4y^2 + 72x - 16y + 124 = 0$
 $9(x^2 + 8x + 16) + 4(y^2 - 4y + 4) = -124 + 144 + 16$
 $9(x + 4)^2 + 4(y - 2)^2 = 36$
 This is an ellipse.
4. $16x^2 - 9y^2 + 192x + 90y - 495 = 0$
 $16(x^2 + 12x + 36) - 9(y^2 - 10y + 25) = 495 + 576 - 225$
 $16(x + 6)^2 - 9(y - 5)^2 = 846$
 This is a hyperbola.
5. $9x^2 + 4y^2 + 72x - 16y + 160 = 0$
 $9(x^2 + 8x + 16) + 4(y^2 - 4y + 4) = -160 + 144 + 16$
 $9(x + 4)^2 + 4(y - 2)^2 = 0$
 This is a point.
6. $16x^2 + 9y^2 + 192x + 90y + 1000 = 0$
 $16(x^2 + 12x + 36) + 9(y^2 + 10y + 25) = -1000 + 576 + 225$
 $16(x + 6)^2 + 9(y + 5)^2 = -199$
 This is the empty set.
7. $y^2 - 5x - 4y - 6 = 0$
 $(y^2 - 4y + 4) = 5x + 6 + 4$
 $(y - 2)^2 = 5(x + 2)$
 This is a parabola.
8. $4x^2 + 4y^2 + 8x - 28y - 11 = 0$
 $4(x^2 + 2x + 1) + 4\left(y^2 - 7y + \frac{49}{4}\right) = 11 + 4 + 49$
 $4(x + 1)^2 + 4\left(y - \frac{7}{2}\right)^2 = 64$
 This is a circle.
9. $3x^2 + 3y^2 - 6x + 12y + 60 = 0$
 $3(x^2 - 2x + 1) + 3(y^2 + 4y + 4) = -60 + 3 + 12$
 $3(x - 1)^2 + 3(y + 2)^2 = -45$
 This is the empty set.
10. $4x^2 - 4y^2 - 2x + 2y + 1 = 0$
 $4\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) - 4\left(y^2 - \frac{1}{2}y + \frac{1}{16}\right) = -1 + \frac{1}{4} - \frac{1}{4}$
 $4\left(x - \frac{1}{4}\right)^2 - 4\left(y - \frac{1}{4}\right)^2 = -1$
 $4\left(y - \frac{1}{4}\right)^2 - 4\left(x - \frac{1}{4}\right)^2 = 1$
 This is a hyperbola.
11. $4x^2 - 4y^2 + 8x + 12y - 5 = 0$
 $4(x^2 + 2x + 1) - 4\left(y^2 - 3y + \frac{9}{4}\right) = 5 + 4 - 9$
 $4(x + 1)^2 - 4\left(y - \frac{3}{2}\right)^2 = 0$
 This is two intersecting lines.
12. $4x^2 - 4y^2 + 8x + 12y - 6 = 0$
 $4(x^2 + 2x + 1) - 4\left(y^2 - 3y + \frac{9}{4}\right) = 6 + 4 - 9$
 $4(x + 1)^2 - 4\left(y - \frac{3}{2}\right)^2 = 1$
 This is a hyperbola.
13. $4x^2 - 24x + 36 = 0$
 $4(x^2 - 6x + 9) = -36 + 36$
 $4(x - 3)^2 = 0$
 This is a line.

14. $4x^2 - 24x + 35 = 0$

$$4(x^2 - 6x + 9) = -35 + 36$$

$$4(x-3)^2 = 1$$

This is two parallel lines.

15. $25x^2 + 4y^2 + 150x - 8y + 129 = 0$

$$25(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -129 + 225 + 4$$

$$25(x+3)^2 + 4(y-1)^2 = 100$$

This is an ellipse.

16. $4x^2 - 25y^2 - 8x + 150y + 129 = 0$

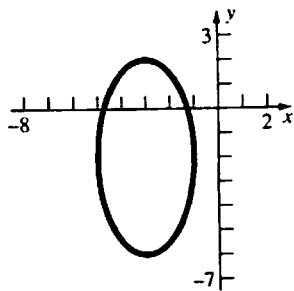
$$4(x^2 - 2x + 1) - 25(y^2 - 6y + 9) = -129 + 4 - 225$$

$$4(x-1)^2 - 25(y-3)^2 = -350$$

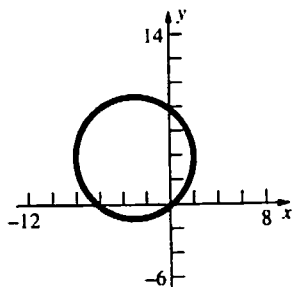
$$25(y-3)^2 - 4(x-1)^2 = 350$$

This is a hyperbola.

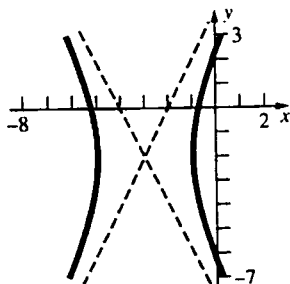
17. $\frac{(x+3)^2}{4} + \frac{(y+2)^2}{16} = 1$



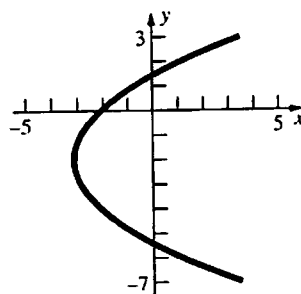
18. $(x+3)^2 + (y-4)^2 = 25$



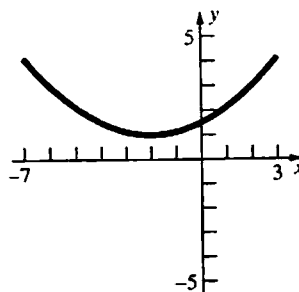
19. $\frac{(x+3)^2}{4} - \frac{(y+2)^2}{16} = 1$



20. $4(x+3) = (y+2)^2$



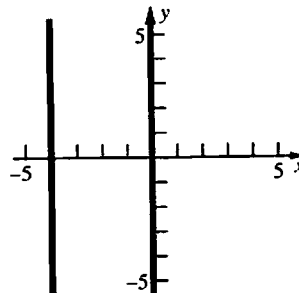
21. $(x+2)^2 = 8(y-1)$



22. $(x+2)^2 = 4$

$$x+2 = \pm 2$$

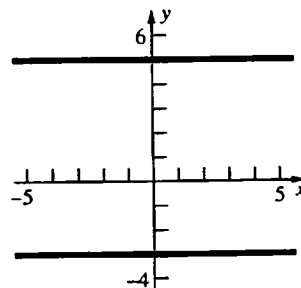
$$x = -4, x = 0$$



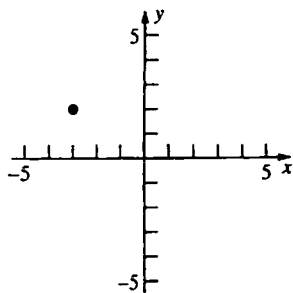
23. $(y-1)^2 = 16$

$$y-1 = \pm 4$$

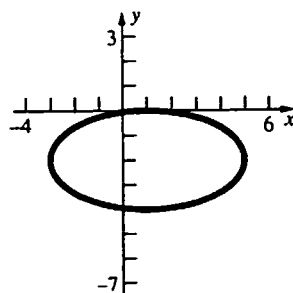
$$y = 5, y = -3$$



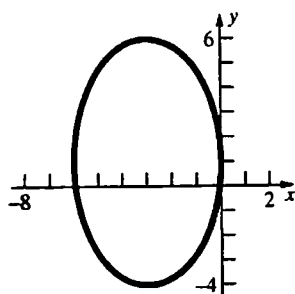
24. $\frac{(x+3)^2}{4} + \frac{(y-2)^2}{8} = 0$
 $(-3, 2)$



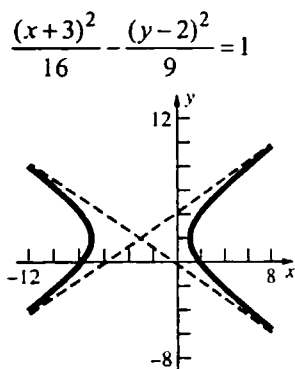
$$\begin{aligned}
 25. \quad & x^2 + 4y^2 - 2x + 16y + 1 = 0 \\
 & (x^2 - 2x + 1) + 4(y^2 + 4y + 4) = -1 + 1 + 16 \\
 & (x-1)^2 + 4(y+2)^2 = 16 \\
 & \frac{(x-1)^2}{16} + \frac{(y+2)^2}{4} = 1
 \end{aligned}$$



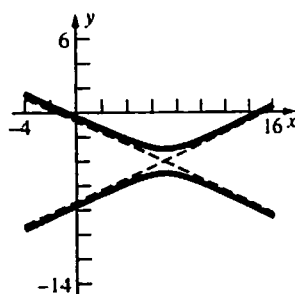
$$\begin{aligned}
 26. \quad & 25x^2 + 9y^2 + 150x - 18y + 9 = 0 \\
 & 25(x^2 + 6x + 9) + 9(y^2 - 2y + 1) = -9 + 225 + 9 \\
 & 25(x+3)^2 + 9(y-1)^2 = 225 \\
 & \frac{(x+3)^2}{9} + \frac{(y-1)^2}{25} = 1
 \end{aligned}$$



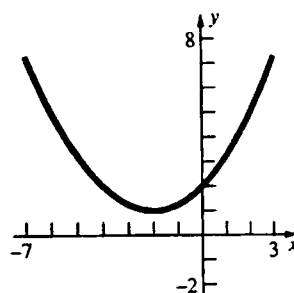
$$\begin{aligned}
 27. \quad & 9x^2 - 16y^2 + 54x + 64y - 127 = 0 \\
 & 9(x^2 + 6x + 9) - 16(y^2 - 4y + 4) = 127 + 81 - 64 \\
 & 9(x+3)^2 - 16(y-2)^2 = 144
 \end{aligned}$$



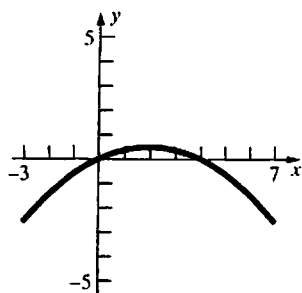
$$\begin{aligned}
 28. \quad & x^2 - 4y^2 - 14x - 32y - 11 = 0 \\
 & (x^2 - 14x + 49) - 4(y^2 + 8y + 16) = 11 + 49 - 64 \\
 & 4(y+4)^2 - (x-7)^2 = 4 \\
 & (y+4)^2 - \frac{(x-7)^2}{4} = 1
 \end{aligned}$$



$$\begin{aligned}
 29. \quad & 4x^2 + 16x - 16y + 32 = 0 \\
 & 4(x^2 + 4x + 4) = 16y - 32 + 16 \\
 & 4(x+2)^2 = 16(y-1) \\
 & (x+2)^2 = 4(y-1)
 \end{aligned}$$



$$\begin{aligned}
 30. \quad & x^2 - 4x + 8y = 0 \\
 & x^2 - 4x + 4 = -8y + 4 \\
 & (x-2)^2 = -8\left(y - \frac{1}{2}\right)
 \end{aligned}$$



31. $2y^2 - 4y - 10x = 0$

$$2(y^2 - 2y + 1) = 10x + 2$$

$$(y-1)^2 = 5\left(x + \frac{1}{5}\right)$$

$$(y-1)^2 = 4\left(\frac{5}{4}\right)\left(x + \frac{1}{5}\right)$$

Horizontal parabola, $p = \frac{5}{4}$

Vertex $\left(-\frac{1}{5}, 1\right)$

Focus is at $\left(\frac{21}{20}, 1\right)$ and directrix is at $x = -\frac{29}{20}$.

32. $-9x^2 + 18x + 4y^2 + 24y = 9$

$$-9(x^2 - 2x + 1) + 4(y^2 + 6y + 9) = 9 - 9 + 36$$

$$4(y+3)^2 - 9(x-1)^2 = 36$$

$$\frac{(y+3)^2}{9} - \frac{(x-1)^2}{4} = 1$$

$$a^2 = 9, a = 3$$

The distance between the vertices is $2a = 6$.

33. $16(x-1)^2 + 25(y+2)^2 = 400$

$$\frac{(x-1)^2}{25} + \frac{(y+2)^2}{16} = 1$$

Horizontal ellipse, center $(1, -2)$, $a = 5$, $b = 4$,

$$c = \sqrt{25 - 16} = 3$$

Foci are at $(-2, -2)$ and $(4, -2)$.

34. $x^2 - 6x + 4y + 3 = 0$

$$x^2 - 6x + 9 = -4y - 3 + 9$$

$$(x-3)^2 = -4\left(y - \frac{3}{2}\right)$$

Vertical parabola, opens downward. vertex

$$\left(3, \frac{3}{2}\right), p = 1$$

Focus is at $\left(3, \frac{1}{2}\right)$ and directrix is $y = \frac{5}{2}$.

35. $a = 5, b = 4$

$$\frac{(x-5)^2}{25} + \frac{(y-1)^2}{16} = 1$$

36. Horizontal hyperbola, $a = 2, c = 3$,

$$b = \sqrt{9 - 4} = \sqrt{5}$$

$$\frac{(x-2)^2}{4} - \frac{(y+1)^2}{5} = 1$$

37. Vertical parabola, opens upward, $p = 5 - 3 = 2$

$$(x-2)^2 = 4(2)(y-3)$$

$$(x-2)^2 = 8(y-3)$$

38. An equation for the ellipse can be written in the

$$\text{form } \frac{(x-2)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1.$$

Substitute the points into the equation.

$$\frac{16}{a^2} = 1, \frac{4}{b^2} = 1$$

Therefore, $a = 4$ and $b = 2$.

$$\frac{(x-2)^2}{16} + \frac{(y-3)^2}{4} = 1$$

39. Vertical hyperbola, center $(0, 3)$, $2a = 6, a = 3$,

$$c = 5, b = \sqrt{25 - 9} = 4$$

$$\frac{(y-3)^2}{9} - \frac{x^2}{16} = 1$$

40. Vertical ellipse; center $(2, 6)$, $a = 8, c = 6$,

$$b = \sqrt{64 - 36} = \sqrt{28}$$

$$\frac{(x-2)^2}{28} + \frac{(y-6)^2}{64} = 1$$

41. Horizontal parabola, opens to the left

$$\text{Vertex } (6, 5), p = \frac{10-2}{2} = 4$$

$$(y-5)^2 = -4(4)(x-6)$$

$$(y-5)^2 = -16(x-6)$$

42. Vertical parabola, opens downward, $p = 1$

$$(x-2)^2 = -4(y-6)$$

43. Horizontal ellipse, center $(0, 2)$, $c = 2$

Since it passes through the origin and center is at $(0, 2)$, $b = 2$.

$$a = \sqrt{4 + 4} = \sqrt{8}$$

$$\frac{x^2}{8} + \frac{(y-2)^2}{4} = 1$$

44. Vertical hyperbola, center $(0, 2)$, $c = 2$,

$$b^2 = 4 - a^2$$

An equation for the hyperbola can be written in

the form $\frac{(y-2)^2}{a^2} - \frac{x^2}{4-a^2} = 1$.

Substitute $(12, 9)$ into the equation.

$$\frac{49}{a^2} - \frac{144}{4-a^2} = 1$$

$$49(4-a^2) - 144a^2 = a^2(4-a^2)$$

$$a^4 - 197a^2 + 196 = 0$$

$$(a^2 - 196)(a^2 - 1) = 0$$

$$a^2 = 196, a^2 = 1$$

$$\text{Since } a < c, a = 1, b = \sqrt{4-1} = \sqrt{3}$$

$$(y-2)^2 - \frac{x^2}{3} = 1$$

45. a. If C is a vertical parabola, the equation for C can be written in the form $y = ax^2 + bx + c$.

Substitute the three points into the equation.

$$2 = a - b + c$$

$$0 = c$$

$$6 = 9a + 3b + c$$

Solve the system to get $a = 1, b = -1, c = 0$.

$$y = x^2 - x$$

- b. If C is a horizontal parabola, an equation for C can be written in the form $x = ay^2 + by + c$.

Substitute the three points into the equation.

$$-1 = 4a + 2b + c$$

$$0 = c$$

$$3 = 36a + 6b + c$$

Solve the system to get $a = \frac{1}{4}, b = -1, c = 0$.

$$x = \frac{1}{4}y^2 - y$$

- c. If C is a circle, an equation for C can be written in the form $(x-h)^2 + (y-k)^2 = r^2$.

Substitute the three points into the equation.

$$(-1-h)^2 + (2-k)^2 = r^2$$

$$h^2 + k^2 = r^2$$

$$(3-h)^2 + (6-k)^2 = r^2$$

Solve the system to get $h = \frac{5}{2}, k = \frac{5}{2}$, and

$$r^2 = \frac{25}{2}.$$

$$\left(x - \frac{5}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{2}$$

46. Let (p, q) be the coordinates of P . By properties of similar triangles and since $\alpha = \frac{|KP|}{|AP|}, \alpha = \frac{x-p}{a-p}$

and $\alpha = \frac{y-q}{b-q}$. Solve for p and q to get

$$p = \frac{x-\alpha a}{1-\alpha} \text{ and } q = \frac{y-\alpha b}{1-\alpha}.$$

Since $P(p, q)$ is a point on a circle of radius r centered at $(0, 0)$, $p^2 + q^2 = r^2$

$$\text{Therefore, } \left(\frac{x-\alpha a}{1-\alpha}\right)^2 + \left(\frac{y-\alpha b}{1-\alpha}\right)^2 = r^2 \text{ or}$$

$$(x-\alpha a)^2 + (y-\alpha b)^2 = (1-\alpha)^2 r^2 \text{ is the equation for } C.$$

47. $y^2 = Lx + Kx^2$

$$K\left(x^2 + \frac{L}{K}x + \frac{L^2}{4K^2}\right) - y^2 = \frac{L^2}{4K}$$

$$K\left(x + \frac{L}{2K}\right)^2 - y^2 = \frac{L^2}{4K}$$

$$\frac{\left(x + \frac{L}{2K}\right)^2}{\frac{L^2}{4K^2}} - \frac{y^2}{\frac{L^2}{4K}} = 1$$

If $K < -1$, the conic is a vertical ellipse. If $K = -1$, the conic is a circle. If $-1 < K < 0$, the conic is a horizontal ellipse. If $K = 0$, the original equation is $y^2 = Lx$, so the conic is a horizontal parabola. If $K > 0$, the conic is a horizontal hyperbola.

If $-1 < K < 0$ (a horizontal ellipse) the length of the latus rectum is (see problem 33, Section 12.2)

$$\frac{2b^2}{a} = 2 \frac{L^2}{4|K|} \frac{1}{\frac{|L|}{2|K|}} = |L|$$

From general considerations, the result for a vertical ellipse is the same as the one just obtained. For $K = -1$ (a circle) we have

$$\left(x - \frac{L}{2}\right)^2 + y^2 = \frac{L^2}{4} \Rightarrow 2 \frac{|L|}{2} = |L|$$

If $K = 0$ (a horizontal parabola) we have

$$y^2 = Lx; y^2 = 4 \frac{L}{4} x; p = \frac{L}{4}, \text{ and the latus rectum is}$$

$$2\sqrt{Lp} = 2\sqrt{L \frac{L}{4}} = |L|.$$

If $K > 0$ (a horizontal hyperbola) we can use the

result of Problem 34, Section 12.2. The length of the latus rectum is $\frac{2b^2}{a}$, which is equal to $|L|$.

48. Parabola: horizontal parabola, opens to the right,
 $p = c - a$

$$y^2 = 4(c - a)(x - a)$$

Hyperbola: horizontal hyperbola, $b^2 = c^2 - a^2$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

Now show that y^2 (hyperbola) is greater than y^2 (parabola).

$$\frac{b^2}{a^2}(x^2 - a^2) = \frac{c^2 - a^2}{a^2}(x^2 - a^2)$$

$$= \frac{(c + a)(c - a)}{a^2}(x + a)(x - a)$$

$$\frac{(c + a)(x + a)}{a^2}(c - a)(x - a) > \frac{(2a)(2a)}{a^2}(c - a)(x - a)$$

$$\frac{(2a)(2a)}{a^2}(c - a)(x - a) = 4(c - a)(x - a)$$

$c + a > 2a$ and $x + a > 2a$ since $c > a$ and $x > a$ except at the vertex.

12.5 Concepts Test

1. $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

2. $\frac{A - C}{B}$

3. hyperbola

4. rotation; translation

Problem Set 12.5

1. $x^2 + xy + y^2 = 6$

$$\cot 2\theta = 0$$

$$2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$$

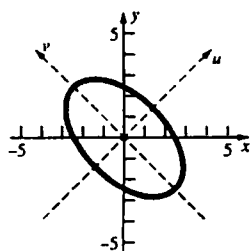
$$x = u \frac{\sqrt{2}}{2} - v \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(u - v)$$

$$y = u \frac{\sqrt{2}}{2} + v \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(u + v)$$

$$\frac{1}{2}(u - v)^2 + \frac{1}{2}(u - v)(u + v) + \frac{1}{2}(u + v)^2 = 6$$

$$\frac{3}{2}u^2 + \frac{1}{2}v^2 = 6$$

$$\frac{u^2}{4} + \frac{v^2}{12} = 1$$



2. $3x^2 + 10xy + 3y^2 + 10 = 0$

$$\cot 2\theta = 0, 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$$

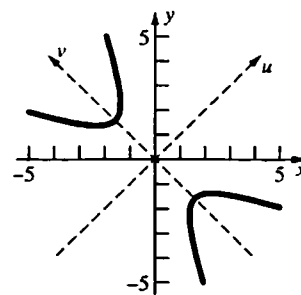
$$x = \frac{\sqrt{2}}{2}(u - v)$$

$$y = \frac{\sqrt{2}}{2}(u + v)$$

$$\frac{3}{2}(u - v)^2 + 5(u - v)(u + v) + \frac{3}{2}(u + v)^2 + 10 = 0$$

$$8u^2 - 2v^2 = -10$$

$$\frac{v^2}{5} - \frac{u^2}{\frac{5}{4}} = 1$$



3. $4x^2 + xy + 4y^2 = 56$

$$\cot 2\theta = 0, 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$$

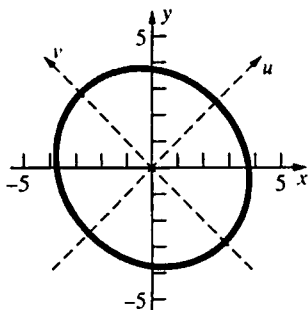
$$x = \frac{\sqrt{2}}{2}(u - v)$$

$$y = \frac{\sqrt{2}}{2}(u + v)$$

$$2(u - v)^2 + \frac{1}{2}(u - v)(u + v) + 2(u + v)^2 = 56$$

$$\frac{9}{2}u^2 + \frac{7}{2}v^2 = 56$$

$$\frac{u^2}{\frac{112}{9}} + \frac{v^2}{16} = 1$$



4. $4xy - 3y^2 = 64$

$$\cot 2\theta = \frac{3}{4}, r = 5$$

$$\cos 2\theta = \frac{3}{5}$$

$$\cos \theta = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}$$

$$\sin \theta = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}$$

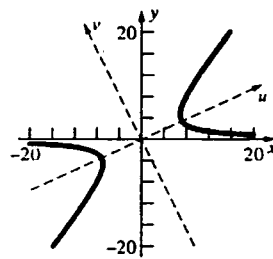
$$x = \frac{1}{\sqrt{5}}(2u - v)$$

$$y = \frac{1}{\sqrt{5}}(u + 2v)$$

$$\frac{4}{5}(2u - v)(u + 2v) - \frac{3}{5}(u + 2v)^2 = 64$$

$$u^2 - 4v^2 = 64$$

$$\frac{u^2}{64} - \frac{v^2}{16} = 1$$



5. $4x^2 - 3xy = 18$

$$\cot 2\theta = -\frac{4}{3}, r = 5$$

$$\cos 2\theta = -\frac{4}{5}$$

$$\cos \theta = \sqrt{\frac{1 - \frac{4}{5}}{2}} = \frac{1}{\sqrt{10}}$$

$$\sin \theta = \sqrt{\frac{1 + \frac{4}{5}}{2}} = \frac{3}{\sqrt{10}}$$

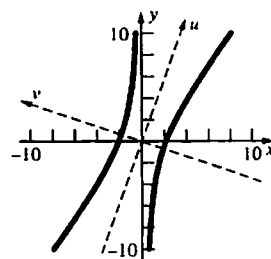
$$x = \frac{1}{\sqrt{10}}(u - 3v)$$

$$y = \frac{1}{\sqrt{10}}(3u + v)$$

$$\frac{2}{5}(u - 3v)^2 - \frac{3}{10}(u - 3v)(3u + v) = 18$$

$$-\frac{u^2}{2} + \frac{9v^2}{2} = 18$$

$$\frac{v^2}{4} - \frac{u^2}{36} = 1$$



6. $11x^2 + 96xy + 39y^2 + 240x + 570y + 875 = 0$

$$\cot 2\theta = -\frac{7}{24}, r = 25$$

$$\cos 2\theta = -\frac{7}{25}$$

$$\cos \theta = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \frac{3}{5}$$

$$\sin \theta = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \frac{4}{5}$$

$$x = \frac{1}{5}(3u - 4v)$$

$$y = \frac{1}{5}(4u + 3v)$$

$$\frac{11}{25}(3u - 4v)^2 + \frac{96}{25}(3u - 4v)(4u + 3v) + \frac{39}{25}(4u + 3v)^2 + 48(3u - 4v) + 114(4u + 3v) + 875 = 0$$

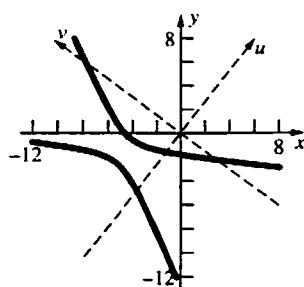
$$75u^2 - 25v^2 + 600u + 150v + 875 = 0$$

$$3u^2 - v^2 + 24u + 6v + 35 = 0$$

$$3(u^2 + 8u + 16) - (v^2 - 6v + 9) = -35 + 48 - 9$$

$$3(u + 4)^2 - (v - 3)^2 = 4$$

$$\frac{(u + 4)^2}{\frac{4}{3}} - \frac{(v - 3)^2}{4} = 1$$



$$7. -\frac{1}{2}x^2 + 7xy - \frac{1}{2}y^2 - 6\sqrt{2}x - 6\sqrt{2}y = 0$$

$$\cot 2\theta = 0, 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$$

$$x = \frac{\sqrt{2}}{2}(u - v)$$

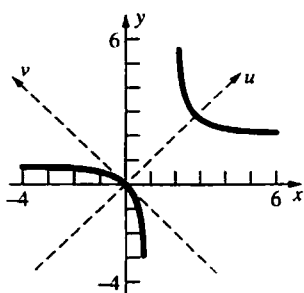
$$y = \frac{\sqrt{2}}{2}(u + v)$$

$$-\frac{1}{4}(u - v)^2 + \frac{7}{2}(u - v)(u + v) - \frac{1}{4}(u + v)^2 - 6(u - v) - 6(u + v) = 0$$

$$3u^2 - 4v^2 - 12u = 0$$

$$3(u^2 - 4u + 4) - 4v^2 = 12$$

$$\frac{(u - 2)^2}{4} - \frac{v^2}{3} = 1$$



$$8. \frac{3}{2}x^2 + xy + \frac{3}{2}y^2 + \sqrt{2}x + \sqrt{2}y = 13$$

$$\cot 2\theta = 0, 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$$

$$x = \frac{\sqrt{2}}{2}(u - v)$$

$$y = \frac{\sqrt{2}}{2}(u + v)$$

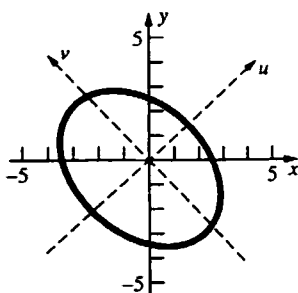
$$\frac{3}{4}(u - v)^2 + \frac{1}{2}(u - v)(u + v) + \frac{3}{4}(u + v)^2 + (u - v) + (u + v) = 13$$

$$2u^2 + v^2 + 2u = 13$$

$$2\left(u^2 + u + \frac{1}{4}\right) + v^2 = 13 + \frac{1}{2}$$

$$2\left(u + \frac{1}{2}\right)^2 + v^2 = \frac{27}{2}$$

$$\frac{\left(u + \frac{1}{2}\right)^2}{\frac{27}{4}} + \frac{v^2}{\frac{27}{2}} = 1$$



$$9. 34x^2 + 24xy + 41y^2 + 250y = -325$$

$$\cot 2\theta = -\frac{7}{24}, r = 25$$

$$\cos 2\theta = -\frac{7}{25}$$

$$\cos \theta = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \frac{3}{5}$$

$$\sin \theta = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \frac{4}{5}$$

$$x = \frac{1}{5}(3u - 4v)$$

$$y = \frac{1}{5}(4u + 3v)$$

$$\frac{34}{25}(3u - 4v)^2 + \frac{24}{25}(3u - 4v)(4u + 3v) + \frac{41}{25}(4u + 3v)^2 + 50(4u + 3v) = -325$$

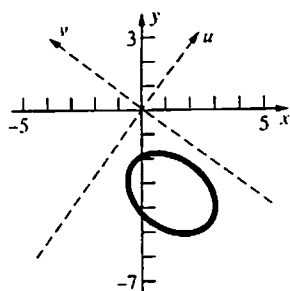
$$50u^2 + 25v^2 + 200u + 150v = -325$$

$$2u^2 + v^2 + 8u + 6v = -13$$

$$2(u^2 + 4u + 4) + (v^2 + 6v + 9) = -13 + 8 + 9$$

$$2(u + 2)^2 + (v + 3)^2 = 4$$

$$\frac{(u+2)^2}{2} + \frac{(v+3)^2}{4} = 1$$



10. $16x^2 + 24xy + 9y^2 - 20x - 15y - 150 = 0$

$$\cot 2\theta = \frac{7}{24}, \quad r = 25$$

$$\cos 2\theta = \frac{7}{25}$$

$$\cos \theta = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \frac{4}{5}$$

$$\sin \theta = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \frac{3}{5}$$

$$x = \frac{1}{5}(4u - 3v)$$

$$y = \frac{1}{5}(3u + 4v)$$

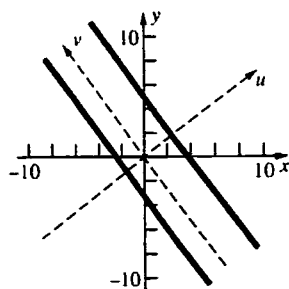
$$\frac{16}{25}(4u - 3v)^2 + \frac{24}{25}(4u - 3v)(3u + 4v) + \frac{9}{25}(3u + 4v)^2 - 4(4u - 3v) - 3(3u + 4v) - 150 = 0$$

$$25u^2 - 25u - 150 = 0$$

$$u^2 - u - 6 = 0$$

$$u^2 - u + \frac{1}{4} = 6 + \frac{1}{4}$$

$$\left(u - \frac{1}{2}\right)^2 = \frac{25}{4}$$



11. $5x^2 - 3xy + y^2 + 65x - 25y + 203 = 0$

$$\cot 2\theta = -\frac{4}{3}, \quad r = 5$$

$$\cos 2\theta = -\frac{4}{5}$$

$$\cos \theta = \sqrt{\frac{1 - \frac{4}{5}}{2}} = \frac{1}{\sqrt{10}}$$

$$\sin \theta = \sqrt{\frac{1 + \frac{4}{5}}{2}} = \frac{3}{\sqrt{10}}$$

$$x = \frac{1}{\sqrt{10}}(u - 3v)$$

$$y = \frac{1}{\sqrt{10}}(3u + v)$$

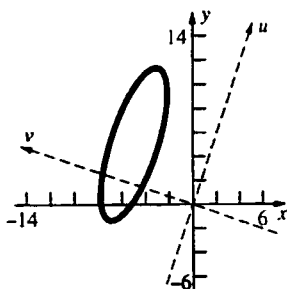
$$\frac{1}{2}(u - 3v)^2 - \frac{3}{10}(u - 3v)(3u + v) + \frac{1}{10}(3u + v)^2 + \frac{65}{\sqrt{10}}(u - 3v) - \frac{25}{\sqrt{10}}(3u + v) + 203 = 0$$

$$\frac{1}{2}u^2 + \frac{11}{2}v^2 - \sqrt{10}u - 22\sqrt{10}v + 203 = 0$$

$$\frac{1}{2}(u^2 - 2\sqrt{10}u + 10) + \frac{11}{2}(v^2 - 4\sqrt{10}v + 40) = -203 + 5 + 220$$

$$\frac{1}{2}(u - \sqrt{10})^2 + \frac{11}{2}(v - 2\sqrt{10})^2 = 22$$

$$\frac{(u - \sqrt{10})^2}{44} + \frac{(v - 2\sqrt{10})^2}{4} = 1$$



12. $6x^2 - 5xy - 6y^2 + 78x + 52y + 26 = 0$

$$\cot 2\theta = -\frac{12}{5}, \quad r = 13$$

$$\cos 2\theta = -\frac{12}{13}$$

$$\cos \theta = \sqrt{\frac{1 - \frac{12}{13}}{2}} = \frac{1}{\sqrt{26}}$$

$$\sin \theta = \sqrt{\frac{1 + \frac{12}{13}}{2}} = \frac{5}{\sqrt{26}}$$

$$x = \frac{1}{\sqrt{26}}(u - 5v)$$

$$y = \frac{1}{\sqrt{26}}(5u + v)$$

$$\frac{6}{26}(u - 5v)^2 - \frac{5}{26}(u - 5v)(5u + v) - \frac{6}{26}(5u + v)^2 + \frac{78}{\sqrt{26}}(u - 5v) + \frac{52}{\sqrt{26}}(5u + v) + 26 = 0$$

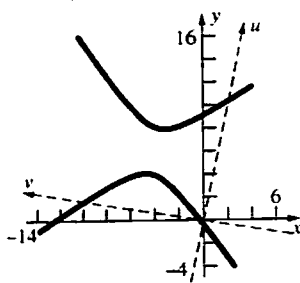
$$-\frac{13}{2}u^2 + \frac{13}{2}v^2 + 13\sqrt{26}u - 13\sqrt{26}v + 26 = 0$$

$$-u^2 + v^2 + 2\sqrt{26}u - 2\sqrt{26}v + 4 = 0$$

$$-(u^2 - 2\sqrt{26}u + 26) + (v^2 - 2\sqrt{26}v + 26) = -4 - 26 + 26$$

$$-(u - \sqrt{26})^2 + (v - \sqrt{26})^2 = -4$$

$$\frac{(v - \sqrt{26})^2}{4} - \frac{(u - \sqrt{26})^2}{4} = 1$$



13. $x = u \cos \alpha - v \sin \alpha$
 $y = u \sin \alpha + v \cos \alpha$
 $(u \cos \alpha - v \sin \alpha) \cos \alpha + (u \sin \alpha + v \cos \alpha) \sin \alpha = d$
 $u(\cos^2 \alpha + \sin^2 \alpha) = d$
 $u = d$
 Thus, the perpendicular distance from the origin is d .

14. $x = \frac{\sqrt{2}}{2}(u - v)$
 $y = \frac{\sqrt{2}}{2}(u + v)$
 $\left[\frac{\sqrt{2}}{2}(u - v) \right]^{1/2} + \left[\frac{\sqrt{2}}{2}(u + v) \right]^{1/2} = a^{1/2}$
 $\frac{\sqrt{2}}{2}(u - v) + 2 \left[\frac{1}{2}(u - v)(u + v) \right]^{1/2} + \frac{\sqrt{2}}{2}(u + v) = a$
 $\sqrt{2}u + \sqrt{2}(u^2 - v^2)^{1/2} = a$
 $\sqrt{2}(u^2 - v^2)^{1/2} = a - \sqrt{2}u$
 $2(u^2 - v^2) = a^2 - 2\sqrt{2}au + 2u^2$
 $v^2 = \sqrt{2}au - \frac{1}{2}a^2$

The corresponding curve is a parabola with $x > 0$ and $y > 0$.

15. $x = u \cos \theta - v \sin \theta$
 $y = u \sin \theta + v \cos \theta$
 $x(\cos \theta) + y(\sin \theta) = (u \cos^2 \theta - v \cos \theta \sin \theta) + (u \sin^2 \theta + v \cos \theta \sin \theta) = u$
 $x(-\sin \theta) + y(\cos \theta) = (-u \cos \theta \sin \theta + v \sin^2 \theta) + (u \cos \theta \sin \theta + v \cos^2 \theta) = v$
 Thus, $u = x \cos \theta + y \sin \theta$ and $v = -x \sin \theta + y \cos \theta$.

16. $u = 5 \cos 60^\circ - 3 \sin 60^\circ = \frac{5}{2} - \frac{3\sqrt{3}}{2}$
 $v = -5 \sin 60^\circ - 3 \cos 60^\circ = -\frac{5\sqrt{3}}{2} - \frac{3}{2}$
 $(u, v) = \left(\frac{5}{2} - \frac{3\sqrt{3}}{2}, -\frac{5\sqrt{3}}{2} - \frac{3}{2} \right)$

17. Rotate to eliminate the xy -term.

$$x^2 + 14xy + 49y^2 = 100$$

$$\cot 2\theta = -\frac{24}{7}$$

$$\cos 2\theta = -\frac{24}{25}$$

$$\cos \theta = \sqrt{\frac{1 - \frac{24}{25}}{2}} = \frac{1}{5\sqrt{2}}$$

$$\sin \theta = \sqrt{\frac{1 + \frac{24}{25}}{2}} = \frac{7}{5\sqrt{2}}$$

$$x = \frac{1}{5\sqrt{2}}(u - 7v)$$

$$y = \frac{1}{5\sqrt{2}}(7u + v)$$

$$\frac{1}{50}(u - 7v)^2 + \frac{14}{50}(u - 7v)(7u + v) + \frac{49}{50}(7u + v)^2 = 100$$

$$50u^2 = 100$$

$$u^2 = 2$$

$$u = \pm\sqrt{2}$$

Thus the points closest to the origin in uv -coordinates are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$.

$$x = \frac{1}{5\sqrt{2}}(\sqrt{2}) = \frac{1}{5} \text{ or } x = \frac{1}{5\sqrt{2}}(-\sqrt{2}) = -\frac{1}{5}$$

$$y = \frac{1}{5\sqrt{2}}(7\sqrt{2}) = \frac{7}{5} \text{ or } y = \frac{1}{5\sqrt{2}}(-7\sqrt{2}) = -\frac{7}{5}$$

The points closest to the origin in xy -coordinates are $(\frac{1}{5}, \frac{7}{5})$ and $(-\frac{1}{5}, -\frac{7}{5})$.

18. $x = u \cos \theta - v \sin \theta$

$$y = u \sin \theta + v \cos \theta$$

$$Ax^2 = A(u \cos \theta - v \sin \theta)^2 = A(u^2 \cos^2 \theta - 2uv \cos \theta \sin \theta + v^2 \sin^2 \theta)$$

$$Bxy = B(u \cos \theta - v \sin \theta)(u \sin \theta + v \cos \theta) = B(u^2 \cos \theta \sin \theta + uv(\cos^2 \theta - \sin^2 \theta) - v^2 \cos \theta \sin \theta)$$

$$Cy^2 = C(u \sin \theta + v \cos \theta)^2 = C(u^2 \sin^2 \theta + 2uv \cos \theta \sin \theta + v^2 \cos^2 \theta)$$

$$Ax^2 + Bxy + Cy^2 = (A \cos^2 \theta + B \cos \theta \sin \theta + C \sin^2 \theta)u^2 + (-2A \cos \theta \sin \theta + B(\cos^2 \theta - \sin^2 \theta) + 2C \cos \theta \sin \theta)uv + (A \sin^2 \theta - B \cos \theta \sin \theta + C \cos^2 \theta)v^2$$

Thus, $a = A \cos^2 \theta + B \cos \theta \sin \theta + C \sin^2 \theta$ and $c = A \sin^2 \theta - B \cos \theta \sin \theta + C \cos^2 \theta$.

$$a + c = A(\cos^2 \theta + \sin^2 \theta) + B(\cos \theta \sin \theta - \cos \theta \sin \theta) + C(\sin^2 \theta + \cos^2 \theta) = A + C$$

19. From Problem 18 $a = A \cos^2 \theta + B \cos \theta \sin \theta + C \sin^2 \theta$,

$$b = -2A \cos \theta \sin \theta + B(\cos^2 \theta - \sin^2 \theta) + 2C \cos \theta \sin \theta, \text{ and}$$

$$c = A \sin^2 \theta - B \cos \theta \sin \theta + C \cos^2 \theta.$$

$$b^2 = B^2 \cos^4 \theta + 4(-AB + BC) \cos^3 \theta \sin \theta + 2(2A^2 - B^2 - 4AC + 2C^2) \cos^2 \theta \sin^2 \theta$$

$$+ 4(AB - BC) \cos \theta \sin^3 \theta + B^2 \sin^4 \theta$$

$$4ac = 4AC \cos^4 \theta + 4(-AB + BC) \cos^3 \theta \sin \theta + 4(A^2 - B^2 + C^2) \cos^2 \theta \sin^2 \theta + 4(AB - BC) \cos \theta \sin^3 \theta + 4AC \sin^4 \theta$$

$$b^2 - 4ac = (B^2 - 4AC) \cos^4 \theta + 2(B^2 - 4AC) \cos^2 \theta \sin^2 \theta + (B^2 - 4AC) \sin^4 \theta$$

$$\begin{aligned}
 &= (B^2 - 4AC)(\cos^2 \theta)(\cos^2 \theta + \sin^2 \theta) + (B^2 - 4AC)(\sin^2 \theta)(\cos^2 \theta + \sin^2 \theta) \\
 &= (B^2 - 4AC)(\cos^2 \theta + \sin^2 \theta) = B^2 - 4AC
 \end{aligned}$$

20. By choosing an appropriate angle of rotation, the second-degree equation can be written in the form $au^2 + cv^2 + du + ev + f = 0$. From Problem 19,

$$-4ac = B^2 - 4AC.$$

- If $B^2 - 4AC = 0$, then $4ac = 0$, so the graph is a parabola or limiting form.
- If $B^2 - 4AC < 0$, then $4ac > 0$, so the graph is an ellipse or limiting form.
- If $B^2 - 4AC > 0$, then $4ac < 0$, so the graph is a hyperbola or limiting form.

21. a. From Problem 19, $-4ac = B^2 - 4AC = -\Delta$ or $\frac{1}{ac} = \frac{4}{\Delta}$.

- b. From Problem 18, $a + c = A + C$.

$$\frac{1}{a} + \frac{1}{c} = \frac{a+c}{ac} = \frac{4(A+C)}{\Delta}$$

- c.
$$\frac{2}{\Delta} \left(A + C \pm \sqrt{(A-C)^2 + B^2} \right)$$

$$= \frac{2}{\Delta} \left[\frac{\Delta}{4} \left(\frac{1}{a} + \frac{1}{c} \right) \pm \sqrt{A^2 + 2AC + C^2 + B^2 - 4AC} \right]$$

$$= \frac{1}{2} \left(\frac{1}{a} + \frac{1}{c} \right) \pm \frac{2}{\Delta} \sqrt{(A+C)^2 - \Delta}$$

$$= \frac{1}{2} \left(\frac{1}{a} + \frac{1}{c} \right) \pm \frac{2}{\Delta} \sqrt{\frac{\Delta^2}{16} \left(\frac{1}{a} + \frac{1}{c} \right)^2 - \Delta}$$

$$= \frac{1}{2} \left(\frac{1}{a} + \frac{1}{c} \right) \pm \frac{1}{2} \sqrt{\left(\frac{1}{a} + \frac{1}{c} \right)^2 - 4 \left(\frac{4}{\Delta} \right)}$$

$$= \frac{1}{2} \left(\frac{1}{a} + \frac{1}{c} \right) \pm \sqrt{\frac{1}{a^2} + \frac{2}{ac} + \frac{1}{c^2} - 4 \left(\frac{1}{ac} \right)}$$

$$= \frac{1}{2} \left(\frac{1}{a} + \frac{1}{c} \right) \pm \frac{1}{2} \sqrt{\left(\frac{1}{a} - \frac{1}{c} \right)^2} = \frac{1}{2} \left(\frac{1}{a} + \frac{1}{c} \pm \left| \frac{1}{a} - \frac{1}{c} \right| \right)$$

The two values of this expression are $\frac{1}{a}$ and $\frac{1}{c}$.

22. $Ax^2 + Bxy + Cy^2 = 1$ can be transformed to $au^2 + cv^2 = 1$. Since $4ac = \Delta > 0$, the graph is an ellipse or a limiting form. $\frac{1}{a} + \frac{1}{c} = \frac{4}{\Delta}(A+C) > 0$, so $a > 0$ and $c > 0$. Thus, the graph is an ellipse (or circle).

The area of $au^2 + cv^2 = 1$ is

$$\pi \frac{1}{\sqrt{ac}} = \pi \sqrt{\frac{4}{\Delta}} = \frac{2\pi}{\sqrt{\Delta}}.$$

23. $\cot 2\theta = 0$, $\theta = \frac{\pi}{4}$

$$x = \frac{\sqrt{2}}{2}(u-v)$$

$$y = \frac{\sqrt{2}}{2}(u+v)$$

$$\frac{1}{2}(u-v)^2 + \frac{B}{2}(u-v)(u+v) + \frac{1}{2}(u+v)^2 = 1$$

$$\frac{2+B}{2}u^2 + \frac{2-B}{2}v^2 = 1$$

- a. The graph is an ellipse if $\frac{2+B}{2} > 0$ and $\frac{2-B}{2} > 0$, so $-2 < B < 2$.

- b. The graph is a circle if $\frac{2+B}{2} = \frac{2-B}{2}$, so $B = 0$.

- c. The graph is a hyperbola if $\frac{2+B}{2} > 0$ and $\frac{2-B}{2} < 0$ or if $\frac{2+B}{2} < 0$ and $\frac{2-B}{2} > 0$, so $B < -2$ or $B > 2$.

- d. The graph is two parallel lines if $\frac{2+B}{2} = 0$ or $\frac{2-B}{2} = 0$, so $B = \pm 2$.

24. $\Delta = 4(25)(1) - 8^2 = 36$

Since $c < a$, $\frac{1}{c} = \frac{2}{\Delta} \left(A + C + \sqrt{(A-C)^2 + B^2} \right)$

$\frac{1}{c} = \frac{1}{18} \left(25 + 1 + \sqrt{24^2 + 8^2} \right) = \frac{1}{9} (13 + 4\sqrt{10})$

$c = \left(\frac{9}{13 + 4\sqrt{10}} \right) \left(\frac{13 - 4\sqrt{10}}{13 - 4\sqrt{10}} \right) = 13 - 4\sqrt{10}$

$\frac{2\pi}{\sqrt{\Delta}} = \frac{2\pi}{6} = \frac{\pi}{3}$

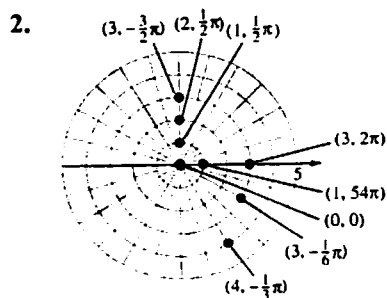
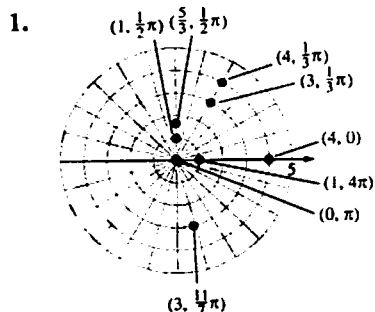
Thus, the distance between the foci is $26 - 8\sqrt{10}$

and the area is $\frac{\pi}{3}$.

12.6 Concepts Review

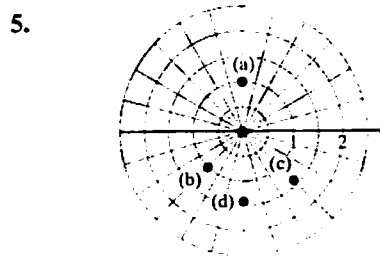
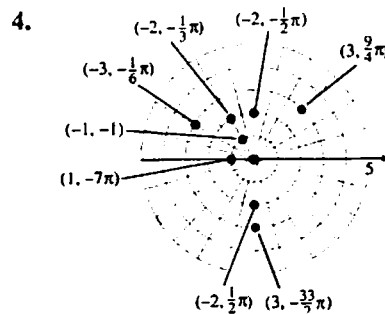
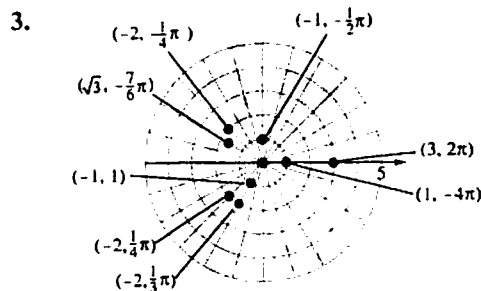
1. infinitely many
2. $r \cos \theta$; $r \sin \theta$; r^2
3. circle; line
4. conic

Problem Set 12.6



25. From Figure 1 it is clear that $v = r \sin \phi$ and $u = r \cos \phi$.

Also noting that $y = r \sin(\theta + \phi)$ leads us to
 $y = r \sin(\theta + \phi) = r \sin \theta \cos \phi + r \cos \theta \sin \phi$
 $= (r \cos \phi)(\sin \theta) + (r \sin \phi)(\cos \theta)$
 $= u \sin \theta + v \cos \theta$



a. $\left(1, -\frac{3}{2} \pi \right), \left(1, \frac{5}{2} \pi \right), \left(-1, -\frac{1}{2} \pi \right), \left(-1, \frac{3}{2} \pi \right)$

b. $\left(1, -\frac{3}{4} \pi \right), \left(1, \frac{5}{4} \pi \right), \left(-1, -\frac{7}{4} \pi \right), \left(-1, \frac{9}{4} \pi \right)$

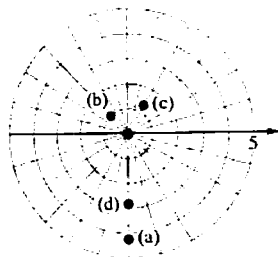
$$\text{c. } \left(\sqrt{2}, -\frac{7}{3}\pi\right), \left(\sqrt{2}, \frac{5}{3}\pi\right), \left(-\sqrt{2}, -\frac{4}{3}\pi\right),$$

$$\left(-\sqrt{2}, \frac{2}{3}\pi\right)$$

$$\text{d. } \left(\sqrt{2}, -\frac{1}{2}\pi\right), \left(\sqrt{2}, \frac{3}{2}\pi\right), \left(-\sqrt{2}, -\frac{3}{2}\pi\right),$$

$$\left(-\sqrt{2}, \frac{1}{2}\pi\right)$$

6.



$$\text{a. } \left(3\sqrt{2}, -\frac{1}{2}\pi\right), \left(3\sqrt{2}, \frac{3}{2}\pi\right), \left(-3\sqrt{2}, -\frac{3}{2}\pi\right),$$

$$\left(-3\sqrt{2}, \frac{1}{2}\pi\right)$$

$$\text{b. } \left(1, -\frac{5}{4}\pi\right), \left(1, \frac{3}{4}\pi\right), \left(-1, -\frac{1}{4}\pi\right), \left(-1, \frac{7}{4}\pi\right)$$

$$\text{c. } \left(\sqrt{2}, -\frac{5}{3}\pi\right), \left(\sqrt{2}, \frac{1}{3}\pi\right), \left(-\sqrt{2}, -\frac{8}{3}\pi\right),$$

$$\left(-\sqrt{2}, \frac{4}{3}\pi\right)$$

$$\text{d. } \left(2\sqrt{2}, -\frac{1}{2}\pi\right), \left(2\sqrt{2}, \frac{3}{2}\pi\right), \left(-2\sqrt{2}, -\frac{3}{2}\pi\right),$$

$$\left(-2\sqrt{2}, \frac{1}{2}\pi\right)$$

$$\text{7. a. } x = 1 \cos \frac{1}{2}\pi = 0$$

$$y = 1 \sin \frac{1}{2}\pi = 1$$

$$(0, 1)$$

$$\text{b. } x = -1 \cos \frac{1}{4}\pi = -\frac{\sqrt{2}}{2}$$

$$y = -1 \sin \frac{1}{4}\pi = -\frac{\sqrt{2}}{2}$$

$$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$\text{c. } x = \sqrt{2} \cos \left(-\frac{1}{3}\pi\right) = \frac{\sqrt{2}}{2}$$

$$y = \sqrt{2} \sin \left(-\frac{1}{3}\pi\right) = -\frac{\sqrt{6}}{2}$$

$$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{2}\right)$$

$$\text{d. } x = -\sqrt{2} \cos \frac{5}{2}\pi = 0$$

$$y = -\sqrt{2} \sin \frac{5}{2}\pi = -\sqrt{2}$$

$$(0, -\sqrt{2})$$

$$\text{8. a. } x = 3\sqrt{2} \cos \frac{7}{2}\pi = 0$$

$$y = 3\sqrt{2} \sin \frac{7}{2}\pi = -3\sqrt{2}$$

$$(0, -3\sqrt{2})$$

$$\text{b. } x = -1 \cos \frac{15}{4}\pi = -\frac{\sqrt{2}}{2}$$

$$y = -1 \sin \frac{15}{4}\pi = \frac{\sqrt{2}}{2}$$

$$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\text{c. } x = -\sqrt{2} \cos \left(-\frac{2}{3}\pi\right) = \frac{\sqrt{2}}{2}$$

$$y = -\sqrt{2} \sin \left(-\frac{2}{3}\pi\right) = \frac{\sqrt{6}}{2}$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)$$

$$\text{d. } x = -2\sqrt{2} \cos \frac{29}{2}\pi = 0$$

$$y = -2\sqrt{2} \sin \frac{29}{2}\pi = -2\sqrt{2}$$

$$(0, -2\sqrt{2})$$

$$\text{9. a. } r^2 = (3\sqrt{3})^2 + 3^2 = 36, r = 6$$

$$\tan \theta = \frac{1}{\sqrt{3}}, \theta = \frac{\pi}{6}$$

$$\left(6, \frac{\pi}{6}\right)$$

$$b. \quad r^2 = (-2\sqrt{3})^2 + 2^2 = 16, \quad r = 4$$

$$\tan \theta = \frac{2}{-2\sqrt{3}}, \quad \theta = \frac{5\pi}{6}$$

$$\left(4, \frac{5\pi}{6}\right)$$

$$c. \quad r^2 = (-\sqrt{2})^2 + (-\sqrt{2})^2 = 4, \quad r = 2$$

$$\tan \theta = \frac{-\sqrt{2}}{-\sqrt{2}}, \quad \theta = \frac{5\pi}{4}$$

$$\left(2, \frac{5\pi}{4}\right)$$

$$d. \quad r^2 = 0^2 + 0^2 = 0, \quad r = 0$$

$$\tan \theta = 0, \quad \theta = 0$$

$$(0, 0)$$

$$10. \quad a. \quad r^2 = \left(-\frac{3}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{10}{3}, \quad r = \frac{\sqrt{10}}{\sqrt{3}}$$

$$\tan \theta = \frac{\frac{1}{\sqrt{3}}}{-\frac{3}{\sqrt{3}}}, \quad \theta = \pi + \tan^{-1}\left(-\frac{1}{3}\right)$$

$$\left(\frac{\sqrt{10}}{\sqrt{3}}, \pi + \tan^{-1}\left(-\frac{1}{3}\right)\right)$$

$$b. \quad r^2 = \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{6}{4}, \quad r = \frac{\sqrt{6}}{2}$$

$$\tan \theta = \frac{\frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2}}, \quad \theta = \frac{3\pi}{4}$$

$$\left(\frac{\sqrt{6}}{2}, \frac{3\pi}{4}\right)$$

$$c. \quad r^2 = 0^2 + (-2)^2 = 4, \quad r = 2$$

$$\tan \theta = -\frac{2}{0}, \quad \theta = \frac{3\pi}{2}$$

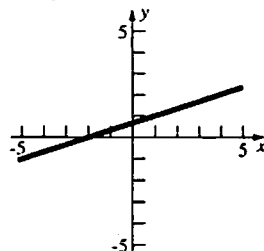
$$\left(2, \frac{3\pi}{2}\right)$$

$$d. \quad r^2 = 3^2 + (-4)^2 = 25, \quad r = 5$$

$$\tan \theta = -\frac{4}{3}, \quad \theta = \tan^{-1}\left(-\frac{4}{3}\right)$$

$$\left(5, \tan^{-1}\left(-\frac{4}{3}\right)\right)$$

$$11. \quad x - 3y + 2 = 0$$

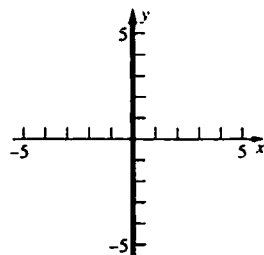


$$r \cos \theta - 3r \sin \theta + 2 = 0$$

$$r = -\frac{2}{\cos \theta - 3 \sin \theta}$$

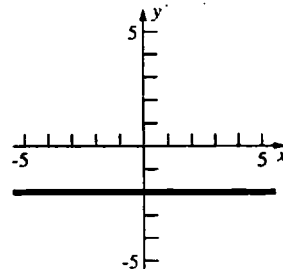
$$r = \frac{2}{3 \sin \theta - \cos \theta}$$

$$12. \quad x = 0$$



$$\theta = \frac{\pi}{2}$$

$$13. \quad y = -2$$

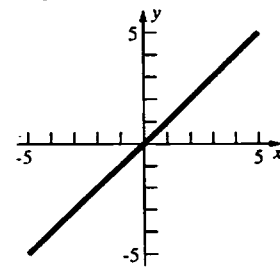


$$r \sin \theta = -2$$

$$r = -\frac{2}{\sin \theta}$$

$$r = -2 \csc \theta$$

$$14. \quad x - y = 0$$

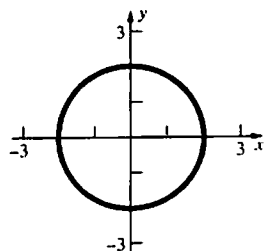


$$r \cos \theta - r \sin \theta = 0$$

$$\tan \theta = 1$$

$$\theta = \frac{\pi}{4}$$

15. $x^2 + y^2 = 4$

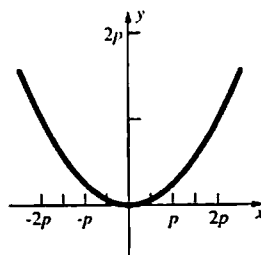


$$(r \cos \theta)^2 + (r \sin \theta)^2 = 4$$

$$r^2 = 4$$

$$r = 2$$

16. $x^2 = 4py$



$$(r \cos \theta)^2 = 4p(r \sin \theta)$$

$$r = \frac{4p \sin \theta}{\cos^2 \theta}$$

$$r = 4p \sec \theta \tan \theta$$

17. $\theta = \frac{\pi}{2}$

$$\cot \theta = 0$$

$$\frac{x}{y} = 0$$

$$x = 0$$

18. $r = 3$

$$r^2 = 9$$

$$x^2 + y^2 = 9$$

19. $r \cos \theta + 3 = 0$

$$x + 3 = 0$$

$$x = -3$$

20. $r - 5 \cos \theta = 0$

$$r^2 - 5r \cos \theta = 0$$

$$x^2 + y^2 - 5x = 0$$

$$\left(x^2 - 5x + \frac{25}{4}\right) + y^2 = \frac{25}{4}$$

$$\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$$

21. $r \sin \theta - 1 = 0$

$$y - 1 = 0$$

$$y = 1$$

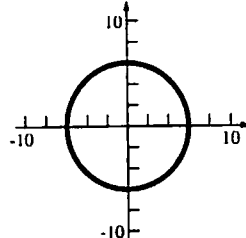
22. $r^2 - 6r \cos \theta - 4r \sin \theta + 9 = 0$

$$x^2 + y^2 - 6x - 4y + 9 = 0$$

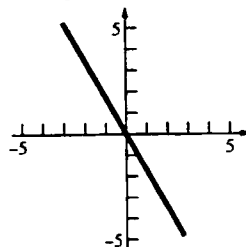
$$(x^2 - 6x + 9) + (y^2 - 4y + 4) = -9 + 9 + 4$$

$$(x - 3)^2 + (y - 2)^2 = 4$$

23. $r = 6$, circle

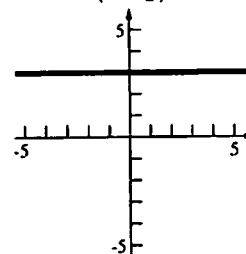


24. $\theta = \frac{2\pi}{3}$, line



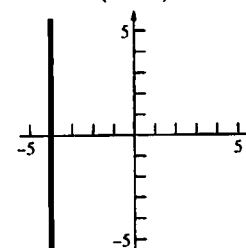
25. $r = \frac{3}{\sin \theta}$

$$r = \frac{3}{\cos\left(\theta - \frac{\pi}{2}\right)}, \text{ line}$$



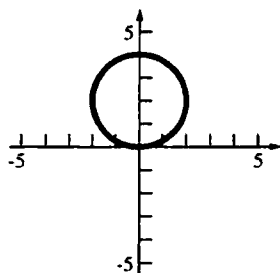
26. $r = -\frac{4}{\cos \theta}$

$$r = \frac{4}{\cos(\theta - \pi)}, \text{ line}$$



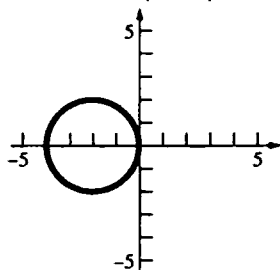
27. $r = 4 \sin \theta$

$r = 2(2) \cos\left(\theta - \frac{\pi}{2}\right)$, circle



28. $r = -4 \cos \theta$

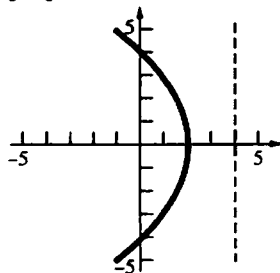
$r = 2(2) \cos(\theta - \pi)$, circle



29. $r = \frac{4}{1 + \cos \theta}$

$r = \frac{(1)(4)}{1 + (1) \cos \theta}$, parabola

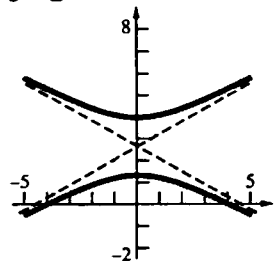
$e = 1$



30. $r = \frac{4}{1 + 2 \sin \theta}$

$r = \frac{(2)(2)}{1 + 2 \cos\left(\theta - \frac{\pi}{2}\right)}$, hyperbola

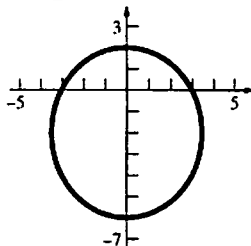
$e = 2$



31. $r = \frac{6}{2 + \sin \theta}$

$r = \frac{\left(\frac{1}{2}\right)6}{1 + \left(\frac{1}{2}\right) \cos\left(\theta - \frac{\pi}{2}\right)}$, ellipse

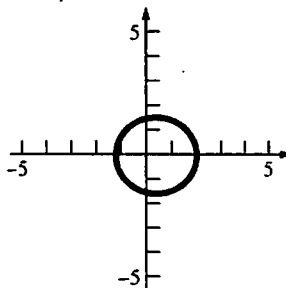
$e = \frac{1}{2}$



32. $r = \frac{6}{4 - \cos \theta}$

$r = \frac{6\left(\frac{1}{4}\right)}{1 + \left(\frac{1}{4}\right) \cos(\theta - \pi)}$, ellipse

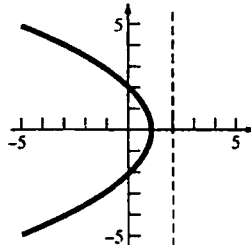
$e = \frac{1}{4}$



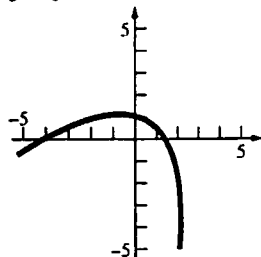
33. $r = \frac{4}{2 + 2 \cos \theta}$

$r = \frac{(1)(2)}{1 + (1) \cos \theta}$, parabola

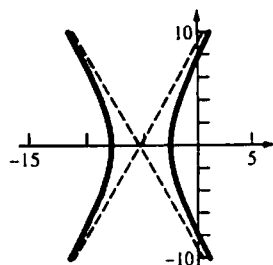
$e = 1$



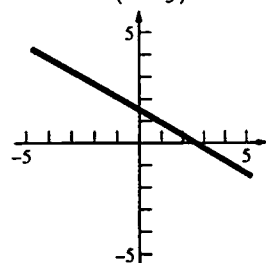
34. $r = \frac{4}{2 + 2\cos(\theta - \frac{\pi}{3})}$
 $r = \frac{2(1)}{1 + (1)\cos(\theta - \frac{\pi}{3})}$, parabola
 $e = 1$



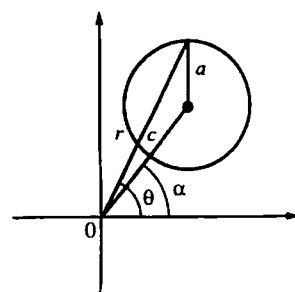
35. $r = \frac{4}{\frac{1}{2} + \cos(\theta - \pi)}$
 $r = \frac{4(2)}{1 + 2\cos(\theta - \pi)}$,
 hyperbola
 $e = 2$



36. $r = \frac{4}{3\cos(\theta - \frac{\pi}{3})}$, line



37. By the Law of Cosines,
 $a^2 = r^2 + c^2 - 2rc\cos(\theta - \alpha)$ (see figure below).



38. $r = a\sin\theta + b\cos\theta$
 $r^2 = ar\sin\theta + br\cos\theta$
 $x^2 + y^2 = ay + bx$
 $x^2 - bx + y^2 - ay = 0$
 $\left(x^2 - bx + \frac{b^2}{4}\right) + \left(y^2 - ay + \frac{a^2}{4}\right) = \frac{a^2 + b^2}{4}$
 $\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4}$
 This is an equation of a circle with radius
 $\frac{\sqrt{a^2 + b^2}}{2}$ and center $\left(\frac{b}{2}, \frac{a}{2}\right)$.

39. Recall that the latus rectum is perpendicular to the axis of the conic through a focus.

$$r\left(\theta_0 + \frac{\pi}{2}\right) = \frac{ed}{1 + e\cos\frac{\pi}{2}} = ed$$

Thus the length of the latus rectum is $2ed$.

40. a. The point closest to the pole is at θ_0 .

$$r_1 = r(\theta_0) = \frac{ed}{1 + e\cos(0)} = \frac{ed}{1 + e}$$

The point furthest from the pole is at $\theta_0 + \pi$.

$$r_2 = r(\theta_0 + \pi) = \frac{ed}{1 + e\cos\pi} = \frac{ed}{1 - e}$$

- b. The length of the major diameter is

$$\begin{aligned} r_1 + r_2 &= \frac{ed}{1 + e} + \frac{ed}{1 - e} = \frac{ed - e^2d}{1 - e^2} + \frac{ed + e^2d}{1 - e^2} \\ &= \frac{2ed}{1 - e^2} \end{aligned}$$

$$a = \frac{ed}{1 - e^2}$$

$$c = ea = \frac{e^2d}{1 - e^2}$$

$$b^2 = a^2 - c^2 = \left(\frac{ed}{1 - e^2}\right)^2 - \left(\frac{e^2d}{1 - e^2}\right)^2$$

$$= \frac{e^2d^2(1 - e^2)}{(1 - e^2)^2} = \frac{e^2d^2}{1 - e^2}$$

$$b = \frac{ed}{\sqrt{1 - e^2}}$$

The length of the minor diameter is $\frac{2ed}{\sqrt{1 - e^2}}$.

41. $a + c = 183, a - c = 17$

$2a = 200, a = 100$

$2c = 166, c = 83$

$e = \frac{c}{a} = 0.83$

42. $a = \frac{185.8}{2} = 92.9,$

$c = ea = (0.0167)92.9 = 1.55143$

Perihelion = $a - c \approx 91.3$ million miles

43. Let sun lie at the pole and the axis of the parabola lie on the pole so that the parabola opens to the left. Then the path is described by the equation

$r = \frac{d}{1 + \cos \theta}$. Substitute $(100, 120^\circ)$ into the equation and solve for d .

$100 = \frac{d}{1 + \cos 120^\circ}$

$d = 50$

The closest distance occurs when $\theta = 0^\circ$.

$r = \frac{50}{1 + \cos 0^\circ} = 25$ million miles

44. a. $4 = \frac{d}{1 + \cos(\frac{\pi}{2} - \theta_0)}$ $3 = \frac{d}{1 + \cos(\frac{\pi}{4} - \theta_0)}$

$4 + 4\left(\cos \frac{\pi}{2} \cos \theta_0 + \sin \frac{\pi}{2} \sin \theta_0\right) = d$ $3 + 3\left(\cos \frac{\pi}{4} \cos \theta_0 + \sin \frac{\pi}{4} \sin \theta_0\right) = d$

$d = 4 + 4 \sin \theta_0$ $d = 3 + \frac{3\sqrt{2}}{2} \cos \theta_0 + \frac{3\sqrt{2}}{2} \sin \theta_0$

$4 + 4 \sin \theta_0 = 3 + \frac{3\sqrt{2}}{2} \cos \theta_0 + \frac{3\sqrt{2}}{2} \sin \theta_0$

$\frac{3\sqrt{2}}{2} \cos \theta_0 + \left(\frac{3\sqrt{2}}{2} - 4\right) \sin \theta_0 - 1 = 0$

$3\sqrt{2} \cos \theta_0 + (3\sqrt{2} - 8) \sin \theta_0 - 2 = 0$

$4.24 \cos \theta_0 - 3.76 \sin \theta_0 - 2 = 0$

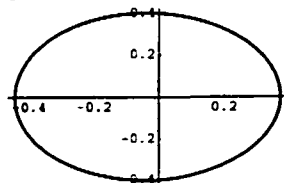
b. A graph shows that a root lies near 0.5. Using Newton's Method, $\theta_0 \approx 0.485$.

c. $d = 4 + 4 \sin \theta_0 \approx 5.86$

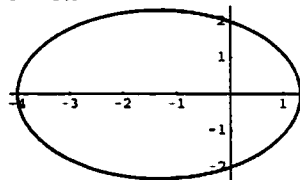
The closest the sun gets is $r = \frac{d}{1 + \cos(\theta_0 - \theta_0)} = \frac{d}{2} \approx 2.93$ AU

45. $x = \frac{4e}{1 + e \cos t} \cos t, y = \frac{4e}{1 + e \cos t} \sin t$

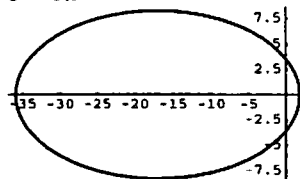
$e = 0.1$

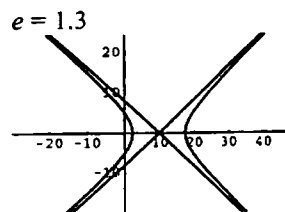
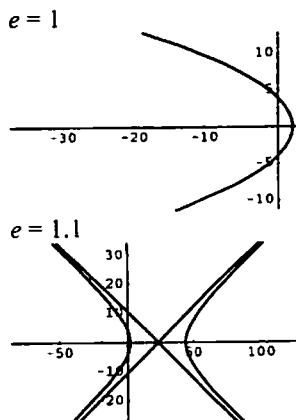


$e = 0.5$



$e = 0.9$





12.7 Concepts Review

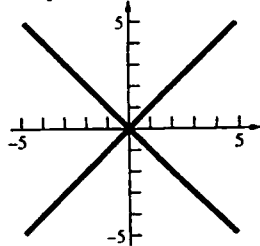
1. limaçon
2. cardioid
3. rose; odd; even
4. spiral

Problem Set 12.7

1. $\theta^2 - \frac{\pi^2}{16} = 0$

$\theta = \pm \frac{\pi}{4}$

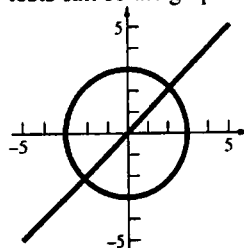
Changing $\theta \rightarrow -\theta$ or $r \rightarrow -r$ yields an equivalent set of equations. Therefore all 3 tests are passed.



2. $(r-3)\left(\theta - \frac{\pi}{4}\right) = 0$

$r = 3$ or $\theta = \frac{\pi}{4}$

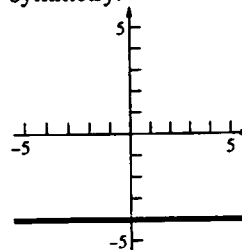
$\theta = \theta_0$ defines a line through the pole. Since a line forms an angle of π radians, changing $\theta \rightarrow \pi + \theta$ results in an equivalent set of equations, thus passing test 3. The other two tests fail so the graph has only origin symmetry.



3. $r \sin \theta + 4 = 0$

$r = -\frac{4}{\sin \theta}$

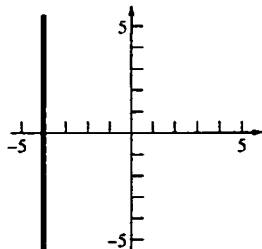
Since $\sin(-\theta) = -\sin \theta$, test 2 is passed. The other two tests fail so the graph has only y-axis symmetry.



4. $r = -4 \sec \theta$

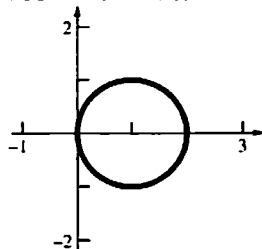
$$r = -\frac{4}{\cos \theta}$$

Since $\cos(-\theta) = \cos \theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



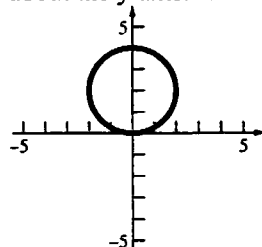
5. $r = 2 \cos \theta$

Since $\cos(-\theta) = \cos \theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



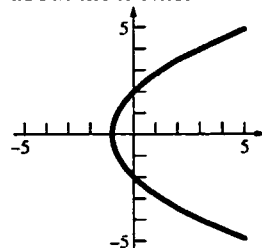
6. $r = 4 \sin \theta$

Since $\sin(-\theta) = -\sin \theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



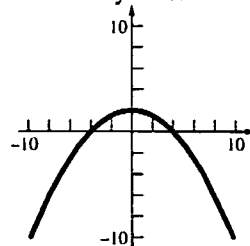
7. $r = \frac{2}{1 - \cos \theta}$

Since $\cos(-\theta) = \cos \theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



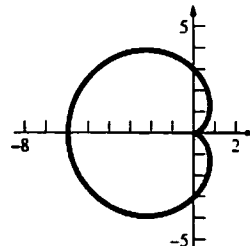
8. $r = \frac{4}{1 + \sin \theta}$

Since $\sin(\pi - \theta) = \sin \theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



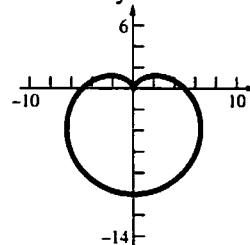
9. $r = 3 - 3 \cos \theta$ (cardioid)

Since $\cos(-\theta) = \cos \theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



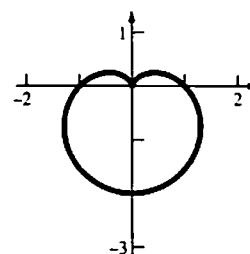
10. $r = 5 - 5 \sin \theta$ (cardioid)

Since $\sin(\pi - \theta) = \sin \theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



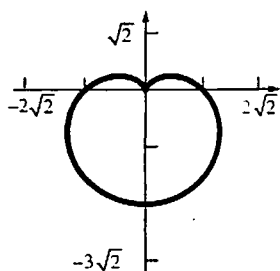
11. $r = 1 - 1 \sin \theta$ (cardioid)

Since $\sin(\pi - \theta) = \sin \theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



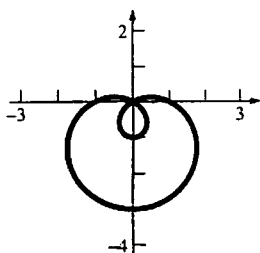
12. $r = \sqrt{2} - \sqrt{2} \sin \theta$ (cardioid)

Since $\sin(\pi - \theta) = \sin \theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



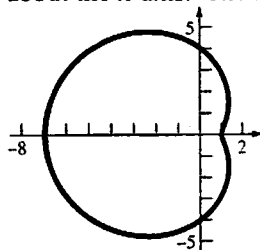
13. $r = 1 - 2 \sin \theta$ (limaçon)

Since $\sin(\pi - \theta) = \sin \theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



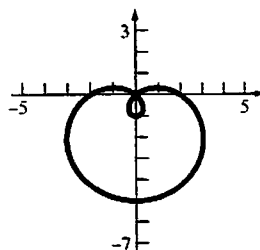
14. $r = 4 - 3 \cos \theta$ (limaçon)

Since $\cos(-\theta) = \cos \theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



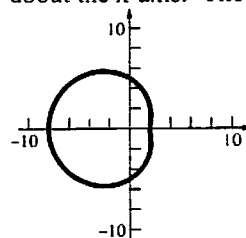
15. $r = 2 - 3 \sin \theta$ (limaçon)

Since $\sin(\pi - \theta) = \sin \theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



16. $r = 5 - 3 \cos \theta$ (limaçon)

Since $\cos(-\theta) = \cos \theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



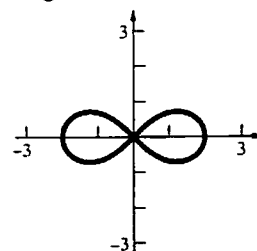
17. $r^2 = 4 \cos 2\theta$ (lemniscate)

$$r = \pm 2\sqrt{\cos 2\theta}$$

Since $\cos(-2\theta) = \cos 2\theta$ and

$$\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$$

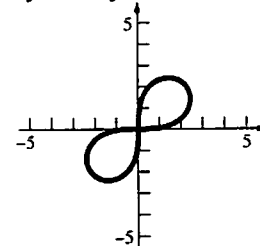
the graph is symmetric about both axes and the origin.



18. $r^2 = 9 \sin 2\theta$ (lemniscate)

$$r = \pm 3\sqrt{\sin(2\theta)}$$

Since $\sin(2(\pi + \theta)) = \sin(2\pi + 2\theta) = \sin 2\theta$, the graph is symmetric about the origin. The other symmetry tests fail.



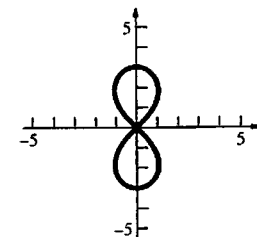
19. $r^2 = -9 \cos 2\theta$ (lemniscate)

$$r = \pm 3\sqrt{-\cos 2\theta}$$

Since $\cos(-2\theta) = \cos 2\theta$ and

$$\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$$

the graph is symmetric about both axes and the origin.



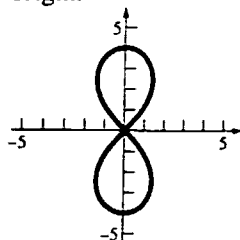
20. $r^2 = -16 \cos 2\theta$ (lemniscate)

$$r = \pm 4\sqrt{-\cos 2\theta}$$

Since $\cos(-2\theta) = \cos 2\theta$ and

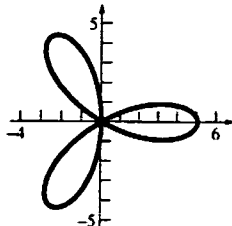
$$\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$$

the graph is symmetric about both axes and the origin.



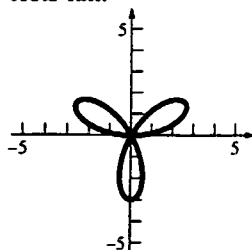
21. $r = 5 \cos 3\theta$ (three-leaved rose)

Since $\cos(-3\theta) = \cos(3\theta)$, the graph is symmetric about the x-axis. The other symmetry tests fail.



22. $r = 3 \sin 3\theta$ (three-leaved rose)

Since $\sin(-3\theta) = -\sin(3\theta)$, the graph is symmetric about the y-axis. The other symmetry tests fail.



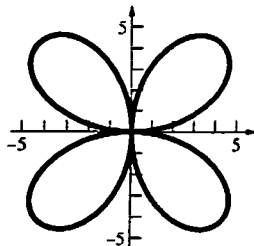
23. $r = 6 \sin 2\theta$ (four-leaved rose)

Since

$$\sin(2(\pi - \theta)) = \sin(2\pi - 2\theta)$$

$$= \sin(-2\theta) = -\sin(2\theta)$$

and $\sin(-2\theta) = -\sin(2\theta)$, the graph is symmetric about both axes and the origin.

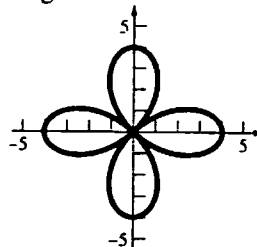


24. $r = 4 \cos 2\theta$ (four-leaved rose)

Since $\cos(-2\theta) = \cos 2\theta$ and

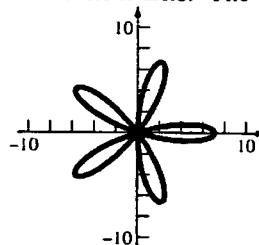
$$\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$$

the graph is symmetric about both axes and the origin.



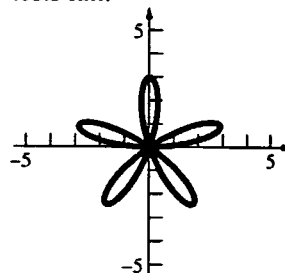
25. $r = 7 \cos 5\theta$ (five-leaved rose)

Since $\cos(-5\theta) = \cos 5\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



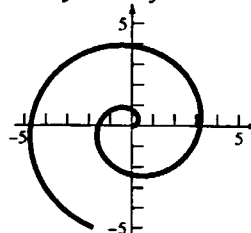
26. $r = 3 \sin 5\theta$ (five-leaved rose)

Since $\sin(-5\theta) = -\sin 5\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



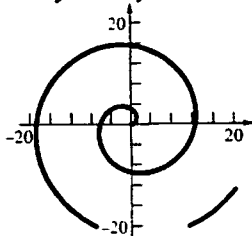
27. $r = \frac{1}{2}\theta, \theta \geq 0$ (spiral of Archimedes)

No symmetry. All three tests fail.



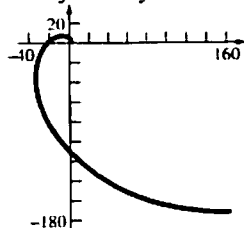
28. $r = 2\theta, \theta \geq 0$ (spiral of Archimedes)

No symmetry. All three tests fail.



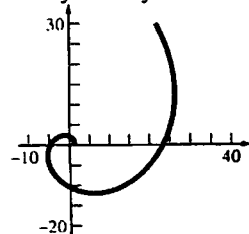
29. $r = e^\theta, \theta \geq 0$ (logarithmic spiral)

No symmetry. All three tests fail.



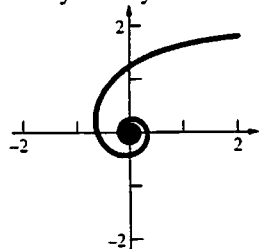
30. $r = e^{\theta/2}, \theta \geq 0$ (logarithmic spiral)

No symmetry. All three tests fail.



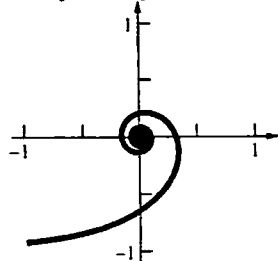
31. $r = \frac{2}{\theta}, \theta > 0$ (reciprocal spiral)

No symmetry. All three tests fail.

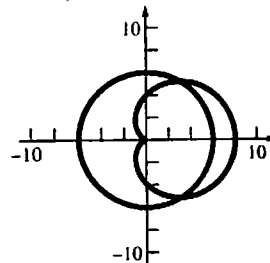


32. $r = -\frac{1}{\theta}, \theta > 0$ (reciprocal spiral)

No symmetry. All three tests fail.



33. $r = 6, r = 4 + 4\cos\theta$



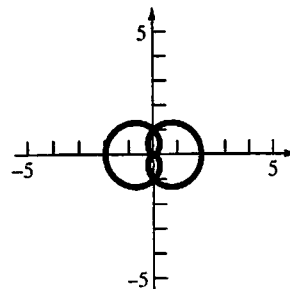
$$6 = 4 + 4\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3}$$

$$\left(6, \frac{\pi}{3}\right), \left(6, \frac{5\pi}{3}\right)$$

34. $r = 1 - \cos\theta, r = 1 + \cos\theta$



$$1 - \cos\theta = 1 + \cos\theta$$

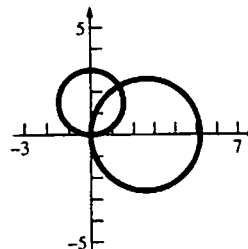
$$\cos\theta = 0$$

$$\theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}$$

$$\left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$$

$(0, 0)$ is also a solution since both graphs include the pole.

35. $r = 3\sqrt{3}\cos\theta, r = 3\sin\theta$



$$3\sqrt{3} \cos \theta = 3 \sin \theta$$

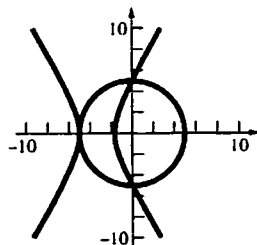
$$\tan \theta = \sqrt{3}$$

$$\theta = \frac{\pi}{3}, \theta = \frac{4\pi}{3}$$

$$\left(\frac{3\sqrt{3}}{2}, \frac{\pi}{3}\right) = \left(-\frac{3\sqrt{3}}{2}, \frac{4\pi}{3}\right)$$

$(0, 0)$ is also a solution since both graphs include the pole.

$$36. \quad r = 5, r = \frac{5}{1-2\cos\theta}$$



$$5 = \frac{5}{1-2\cos\theta}$$

$$\cos \theta = 0$$

$$\theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}; \left(5, \frac{\pi}{2}\right), \left(5, \frac{3\pi}{2}\right)$$

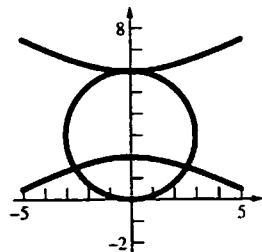
Note that $r = -5$ is equivalent to $r = 5$.

$$-5 = \frac{5}{1-2\cos\theta}$$

$$\cos \theta = 1$$

$$\theta = 0; (-5, 0)$$

$$37. \quad r = 6 \sin \theta, r = \frac{6}{1+2\sin\theta}$$



$$6 \sin \theta = \frac{6}{1+2\sin\theta}$$

$$12 \sin^2 \theta + 6 \sin \theta - 6 = 0$$

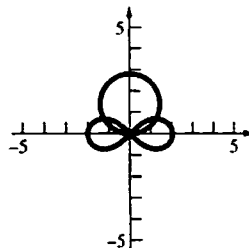
$$6(2 \sin \theta - 1)(\sin \theta + 1) = 0$$

$$\sin \theta = \frac{1}{2}, \sin \theta = -1$$

$$\theta = \frac{\pi}{6}, \theta = \frac{5\pi}{6}, \theta = \frac{3\pi}{2}$$

$$\left(3, \frac{\pi}{6}\right), \left(3, \frac{5\pi}{6}\right), \left(-6, \frac{3\pi}{2}\right) \text{ or } \left(6, \frac{\pi}{2}\right)$$

$$38. \quad r^2 = 4 \cos 2\theta, r = 2\sqrt{2} \sin \theta$$



$$4 \cos 2\theta = (2\sqrt{2} \sin \theta)^2$$

$$4 - 8 \sin^2 \theta = 8 \sin^2 \theta$$

$$\sin^2 \theta = \frac{1}{4} \Rightarrow \sin \theta = \pm \frac{1}{2}$$

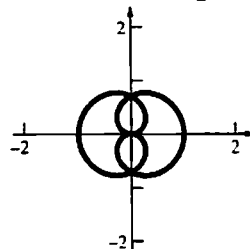
$$\theta = \frac{\pi}{6}, \theta = \frac{5\pi}{6}, \theta = \frac{7\pi}{6}, \theta = \frac{11\pi}{6}$$

$$\left(\sqrt{2}, \frac{\pi}{6}\right) = \left(-\sqrt{2}, \frac{7\pi}{6}\right),$$

$$\left(\sqrt{2}, \frac{5\pi}{6}\right) = \left(-\sqrt{2}, \frac{11\pi}{6}\right)$$

$(0, 0)$ is also a solution since both graphs includes the pole.

$$39. \quad \text{Consider } r = \cos \frac{1}{2} \theta.$$



The graph is clearly symmetric with respect to the y -axis.

Substitute (r, θ) by $(-r, -\theta)$.

$$-r = \cos\left(-\frac{1}{2}\theta\right) = \cos \frac{1}{2}\theta$$

$$r = -\cos \frac{1}{2}\theta$$

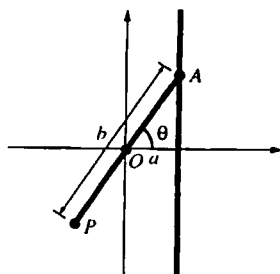
Substitute (r, θ) by $(r, \pi - \theta)$

$$r = \cos \frac{1}{2}(\pi - \theta) = \cos \frac{1}{2}\pi \cos \frac{1}{2}\theta + \sin \frac{1}{2}\pi \sin \frac{1}{2}\theta$$

$$= \sin \frac{1}{2}\theta$$

$$r = \sin \frac{1}{2}\theta$$

40. Consider the following figure.



$$r = \frac{a}{\cos \theta} - b$$

$$r \cos \theta = a - b \cos \theta$$

$$x = a - b \cos \theta$$

$$xr = ar - br \cos \theta$$

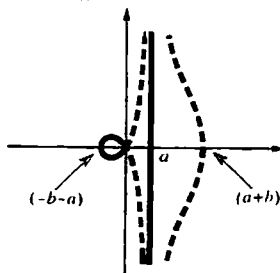
$$(x - a)r = -bx$$

$$(x - a)^2 r^2 = b^2 x^2$$

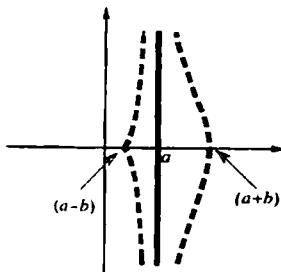
$$(x - a)^2 (x^2 + y^2) = b^2 x^2$$

$$y^2 = \frac{b^2 x^2}{(x - a)^2} - x^2$$

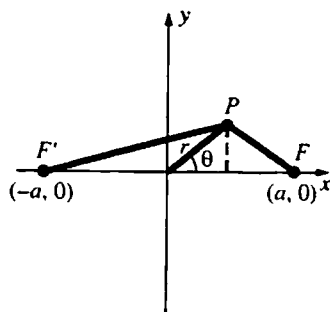
if $b > a$



if $b < a$



41.



$$|PF| = \sqrt{(a - r \cos \theta)^2 + (r \sin \theta)^2}$$

$$= \sqrt{r^2 + a^2 - 2ar \cos \theta}$$

$$|PF'| = \sqrt{(a + r \cos \theta)^2 + (r \sin \theta)^2}$$

$$= \sqrt{r^2 + a^2 + 2ar \cos \theta}$$

$$|PF||PF'| = \sqrt{(r^2 + a^2)^2 - 4a^2 r^2 \cos^2 \theta} = a^2$$

$$(r^4 + 2a^2 r^2 + a^4) - 4a^2 r^2 \cos^2 \theta = a^4$$

$$r^4 - 4a^2 r^2 \cos^2 \theta + 2a^2 r^2 = 0$$

$$r^2 (r^2 - 2a^2 (2 \cos^2 \theta - 1)) = 0$$

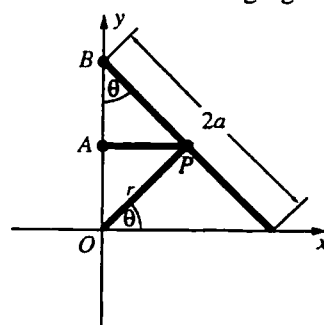
$$r^2 - 2a^2 (2 \cos^2 \theta - 1) = 0$$

$$r^2 = 2a^2 (1 + \cos 2\theta - 1)$$

$$r^2 = 2a^2 \cos 2\theta$$

This is the equation of a lemniscate.

42. Consider the following figure.



Then $\tan \theta = \frac{AP}{BA} = \frac{r \cos \theta}{2a \cos \theta - r \sin \theta}$

$$\frac{\sin \theta}{\cos \theta} = \frac{r \cos \theta}{2a \cos \theta - r \sin \theta}$$

$$2a \sin \theta \cos \theta - r \sin^2 \theta = r \cos^2 \theta$$

$$r \cos^2 \theta + r \sin^2 \theta = 2a \sin \theta \cos \theta$$

$$r = a \sin 2\theta$$

This is a polar equation for a four-leaved rose.

43. a. $y = 45$
 $r \sin \theta = 45$
 $r = \frac{45}{\sin \theta}$

b. $x^2 + y^2 = 36$
 $r^2 = 36$
 $r = 6$

$$\begin{aligned} \text{c. } x^2 - y^2 &= 1 \\ r^2 \cos^2 \theta - r^2 \sin^2 \theta &= 1 \\ r^2 &= \frac{1}{\cos 2\theta} \\ r &= \pm \frac{1}{\sqrt{\cos 2\theta}} \end{aligned}$$

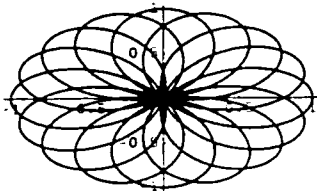
$$\begin{aligned} \text{d. } 4xy &= 1 \\ 4r^2 \cos \theta \sin \theta &= 1 \\ r^2 &= \frac{1}{2 \sin 2\theta} \\ r &= \pm \frac{1}{\sqrt{2 \sin 2\theta}} \end{aligned}$$

$$\begin{aligned} \text{e. } y &= 3x + 2 \\ r \sin \theta &= 3r \cos \theta + 2 \\ r(\sin \theta - 3 \cos \theta) &= 2 \\ r &= \frac{2}{\sin \theta - 3 \cos \theta} \end{aligned}$$

$$\begin{aligned} \text{f. } 3x^2 + 4y &= 2 \\ 3r^2 \cos^2 \theta + 4r \sin \theta &= 2 \\ (3 \cos^2 \theta)r^2 + (4 \sin \theta)r - 2 &= 0 \\ r &= \frac{-4 \sin \theta \pm \sqrt{16 \sin^2 \theta + 24 \cos^2 \theta}}{6 \cos^2 \theta} \\ r &= \frac{-2 \sin \theta \pm \sqrt{4 \sin^2 \theta + 6 \cos^2 \theta}}{3 \cos^2 \theta} \end{aligned}$$

$$\begin{aligned} \text{g. } x^2 + 2x + y^2 - 4y - 25 &= 0 \\ r^2 + 2r \cos \theta - 4r \sin \theta - 25 &= 0 \\ r^2 + (2 \cos \theta - 4 \sin \theta)r - 25 &= 0 \\ r &= \frac{-2 \cos \theta + 4 \sin \theta \pm \sqrt{(2 \cos \theta - 4 \sin \theta)^2 + 100}}{2} \\ r &= -\cos \theta + 2 \sin \theta \pm \sqrt{(\cos \theta - 2 \sin \theta)^2 + 25} \end{aligned}$$

44.



45. a. VII

b. I

c. VIII

d. III

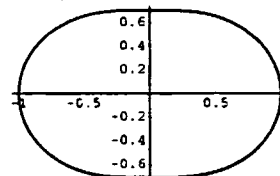
e. V

f. II

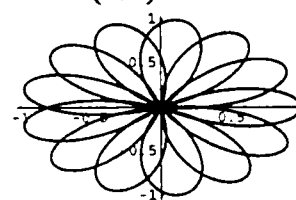
g. VI

h. IV

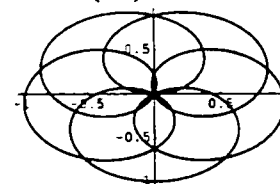
$$46. r = \sqrt{1 - 0.5 \sin^2 \theta}$$



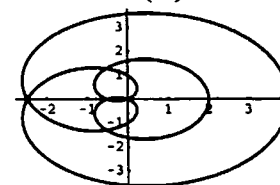
$$47. r = \cos\left(\frac{13\theta}{5}\right)$$



$$48. r = \sin\left(\frac{5\theta}{7}\right)$$



$$49. r = 1 + 3 \cos\left(\frac{\theta}{3}\right)$$



50. a. The graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is the

rotation of the graph of $r = 1 + \sin \theta$ by $\frac{\pi}{3}$

counter-clockwise about the pole. The graph of $r = 1 + \sin\left(\theta + \frac{\pi}{3}\right)$ is the rotation of the

graph of $r = 1 + \sin \theta$ by $\frac{\pi}{3}$ clockwise about the pole.

- b. $r = 1 - \sin \theta = 1 + \sin(\theta - \pi)$
The graph of $r = 1 + \sin \theta$ is the rotation of the graph of $r = 1 - \sin \theta$ by π about the pole.

c. $r = 1 + \cos \theta = 1 + \sin\left(\theta + \frac{\pi}{2}\right)$

The graph of $r = 1 + \sin \theta$ is the rotation of the graph of $r = 1 + \cos \theta$ by $\frac{\pi}{2}$ counter-clockwise about the pole.

- d. The graph of $r = f(\theta)$ is the rotation of the graph of $r = f(\theta - \alpha)$ by α clockwise about the pole.

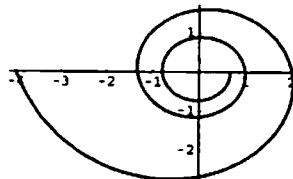
51. a. The graph for $\phi = 0$ is the graph for $\phi \neq 0$ rotated by ϕ counterclockwise about the pole.
b. As n increases, the number of "leaves" increases.
c. If $|a| > |b|$, the graph will not pass through the pole and will not "loop." If $|b| < |a|$, the graph will pass through the pole and will have $2n$ "loops" (n small "loops" and n large "loops"). If $|a| = |b|$, the graph passes

through the pole and will have n "loops." If $ab \neq 0, n > 1$, and $\phi = 0$, the graph will be symmetric about $\theta = \frac{\pi}{n}k$, where $k = 0, n-1$.

52. The number of loops is $2n$.

53. The spiral will unwind clockwise for $c < 0$. The spiral will unwind counter-clockwise for $c > 0$.

54. This is for $c = 4\pi$.



The spiral will *wind* in the counter-clockwise direction.

55. a. III
b. IV
c. I
d. II
e. VI
f. V

12.8 Concepts Review

1. $\frac{1}{2}r^2\theta$

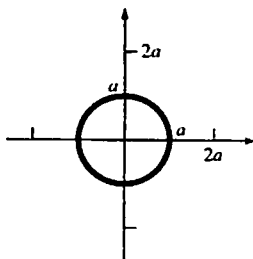
2. $\frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$

3. $\frac{1}{2} \int_0^{2\pi} (2 + 2\cos\theta)^2 d\theta$

4. $f(\theta) = 0$

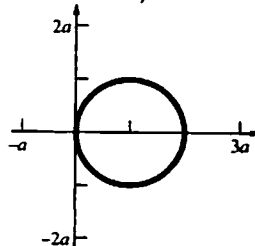
Problem Set 12.8

1. $r = a, a > 0$



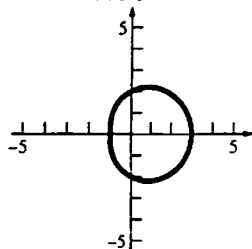
$$A = \frac{1}{2} \int_0^{2\pi} a^2 d\theta = \pi a^2$$

2. $r = 2a \cos \theta, a > 0$



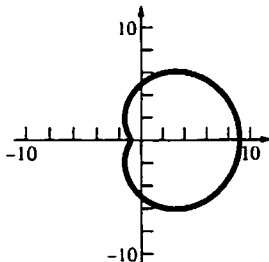
$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi} (2a \cos \theta)^2 d\theta = 2a^2 \int_0^{\pi} \cos^2 \theta d\theta \\ &= a^2 \int_0^{\pi} (1 + \cos 2\theta) d\theta = a^2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi} = \pi a^2 \end{aligned}$$

3. $r = 2 + \cos \theta$



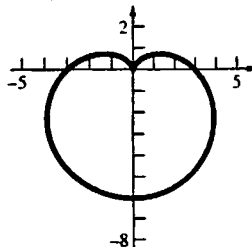
$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (2 + \cos \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (4 + 4\cos \theta + \cos^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left[4 + 4\cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left(\frac{9}{2} + 4\cos \theta + \frac{1}{2}\cos 2\theta \right) d\theta \\ &= \frac{1}{2} \left[\frac{9}{2}\theta + 4\sin \theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = \frac{9}{2}\pi \end{aligned}$$

4. $r = 5 + 4 \cos \theta$



$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (5 + 4\cos \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (25 + 40\cos \theta + 16\cos^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} [25 + 40\cos \theta + 8(1 + \cos 2\theta)] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (33 + 40\cos \theta + 8\cos 2\theta) d\theta \\ &= \frac{1}{2} [33\theta + 40\sin \theta + 4\sin 2\theta]_0^{2\pi} = 33\pi \end{aligned}$$

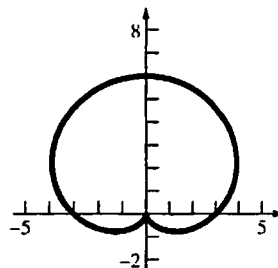
5. $r = 3 - 3 \sin \theta$



$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (3 - 3\sin \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (9 - 18\sin \theta + 9\sin^2 \theta) d\theta \end{aligned}$$

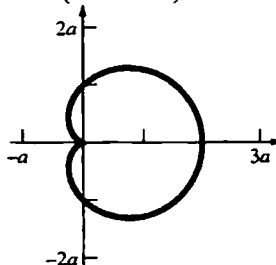
$$\begin{aligned} &= \frac{1}{2} \int_0^{2\pi} \left[9 - 18\sin \theta + \frac{9}{2}(1 - \cos 2\theta) \right] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left(\frac{27}{2} - 18\sin \theta - \frac{9}{2}\cos 2\theta \right) d\theta \\ &= \frac{1}{2} \left[\frac{27}{2}\theta + 18\cos \theta - \frac{9}{4}\sin 2\theta \right]_0^{2\pi} = \frac{27}{2}\pi \end{aligned}$$

6.



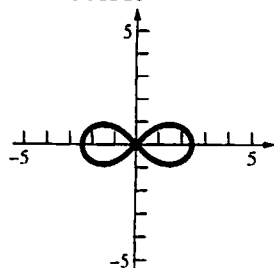
$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (3 + 3\sin \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (9 + 18\sin \theta + 9\sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left[9 + 18\sin \theta + \frac{9}{2}(1 - \cos 2\theta) \right] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left(\frac{27}{2} + 18\sin \theta - \frac{9}{2}\cos 2\theta \right) d\theta \\ &= \frac{1}{2} \left[\frac{27}{2}\theta - 18\cos \theta - \frac{9}{4}\sin 2\theta \right]_0^{2\pi} \\ &= \frac{27}{2}\pi \end{aligned}$$

7. $r = a(1 + \cos \theta)$



$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} [a(1 + \cos \theta)]^2 d\theta \\ &= \frac{a^2}{2} \int_0^{2\pi} (1 + 2\cos \theta + \cos^2 \theta) d\theta \\ &= \frac{a^2}{2} \int_0^{2\pi} \left[1 + 2\cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta \\ &= \frac{a^2}{2} \int_0^{2\pi} \left(\frac{3}{2} + 2\cos \theta + \frac{1}{2}\cos 2\theta \right) d\theta \\ &= \frac{a^2}{2} \left[\frac{3}{2}\theta + 2\sin \theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = \frac{3\pi a^2}{2} \end{aligned}$$

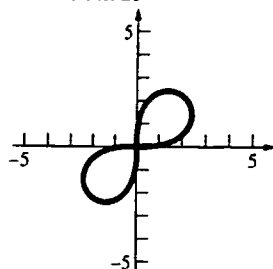
8. $r^2 = 6 \cos 2\theta$



$$A = 2 \cdot \frac{1}{2} \int_{-\pi/4}^{\pi/4} 6 \cos 2\theta \, d\theta = 6 \int_{-\pi/4}^{\pi/4} \cos 2\theta \, d\theta$$

$$= 3[\sin 2\theta]_{-\pi/4}^{\pi/4} = 6$$

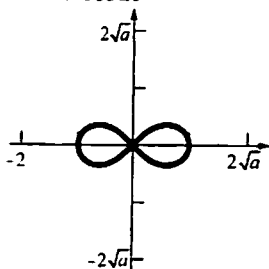
9. $r^2 = 9 \sin 2\theta$



$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/2} 9 \sin 2\theta \, d\theta = 9 \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= \frac{9}{2}[-\cos 2\theta]_0^{\pi/2} = 9$$

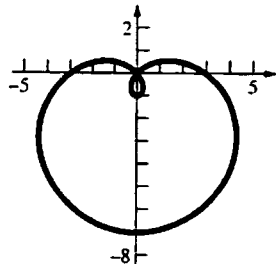
10. $r^2 = a \cos 2\theta$



$$A = 2 \cdot \frac{1}{2} \int_{-\pi/4}^{\pi/4} a \cos 2\theta \, d\theta = a \int_{-\pi/4}^{\pi/4} \cos 2\theta \, d\theta$$

$$= \frac{a}{2}[\sin 2\theta]_{-\pi/4}^{\pi/4} = a$$

11. $r = 3 - 4 \sin \theta$



$$3 - 4 \sin \theta = 0, \theta = \sin^{-1} \frac{3}{4}$$

$$A = 2 \cdot \frac{1}{2} \int_{\sin^{-1} \frac{3}{4}}^{\pi/2} (3 - 4 \sin \theta)^2 \, d\theta$$

$$= \int_{\sin^{-1} \frac{3}{4}}^{\pi/2} (9 - 24 \sin \theta + 16 \sin^2 \theta) \, d\theta$$

$$= \int_{\sin^{-1} \frac{3}{4}}^{\pi/2} [9 - 24 \sin \theta + 8(1 - \cos 2\theta)] \, d\theta$$

$$= \int_{\sin^{-1} \frac{3}{4}}^{\pi/2} (17 - 24 \sin \theta - 8 \cos 2\theta) \, d\theta$$

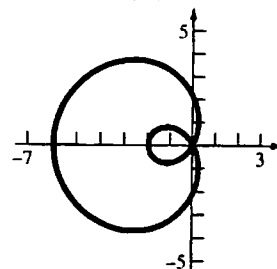
$$= [17\theta + 24 \cos \theta - 4 \sin 2\theta]_{\sin^{-1} \frac{3}{4}}^{\pi/2}$$

$$= [17\theta + 24 \cos \theta - 8 \sin \theta \cos \theta]_{\sin^{-1} \frac{3}{4}}^{\pi/2}$$

$$= \frac{17\pi}{2} - \left[17 \sin^{-1} \frac{3}{4} + 24 \left(\frac{\sqrt{7}}{4} \right) - 8 \left(\frac{3}{4} \right) \left(\frac{\sqrt{7}}{4} \right) \right]$$

$$= \frac{17\pi}{2} - 17 \sin^{-1} \frac{3}{4} - \frac{9\sqrt{7}}{2}$$

12. $r = 2 - 4 \cos \theta$



$$2 - 4 \cos \theta = 0, \theta = \frac{\pi}{3}$$

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/3} (2 - 4 \cos \theta)^2 \, d\theta$$

$$= \int_0^{\pi/3} (4 - 16 \cos \theta + 16 \cos^2 \theta) \, d\theta$$

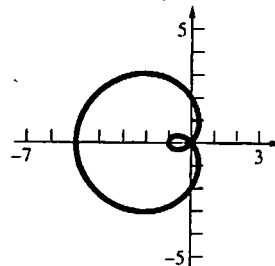
$$= \int_0^{\pi/3} [4 - 16 \cos \theta + 8(1 + \cos 2\theta)] \, d\theta$$

$$= \int_0^{\pi/3} (12 - 16 \cos \theta + 8 \cos 2\theta) \, d\theta$$

$$= [12\theta - 16 \sin \theta + 4 \sin 2\theta]_0^{\pi/3}$$

$$= 4\pi - 6\sqrt{3}$$

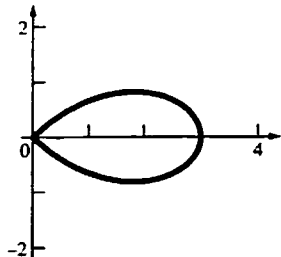
13. $r = 2 - 3 \cos \theta$



$$2 - 3 \cos \theta = 0, \theta = \cos^{-1} \frac{2}{3}$$

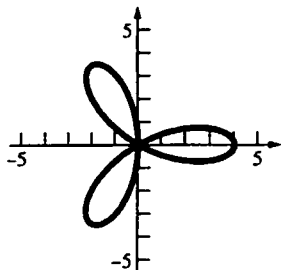
$$\begin{aligned}
 A &= 2 \cdot \frac{1}{2} \int_{\cos^{-1} \frac{2}{3}}^{\pi} (2 - 3 \cos \theta)^2 d\theta \\
 &= \int_{\cos^{-1} \frac{2}{3}}^{\pi} (4 - 12 \cos \theta + 9 \cos^2 \theta) d\theta \\
 &= \int_{\cos^{-1} \frac{2}{3}}^{\pi} \left[4 - 12 \cos \theta + \frac{9}{2} (1 + \cos 2\theta) \right] d\theta \\
 &= \int_{\cos^{-1} \frac{2}{3}}^{\pi} \left(\frac{17}{2} - 12 \cos \theta + \frac{9}{2} \cos 2\theta \right) d\theta \\
 &= \left[\frac{17}{2} \theta - 12 \sin \theta + \frac{9}{4} \sin 2\theta \right]_{\cos^{-1} \frac{2}{3}}^{\pi} \\
 &= \left[\frac{17\theta}{2} - 12 \sin \theta + \frac{9}{2} \sin \theta \cos \theta \right]_{\cos^{-1} \frac{2}{3}}^{\pi} \\
 &= \frac{17\pi}{2} - \left[\frac{17}{2} \cos^{-1} \frac{2}{3} - 12 \left(\frac{\sqrt{5}}{3} \right) + \frac{9}{2} \left(\frac{\sqrt{5}}{3} \right) \left(\frac{2}{3} \right) \right] \\
 &= \frac{17\pi}{2} - \frac{17}{2} \cos^{-1} \frac{2}{3} + 3\sqrt{5}
 \end{aligned}$$

14. $r = 3 \cos 2\theta$



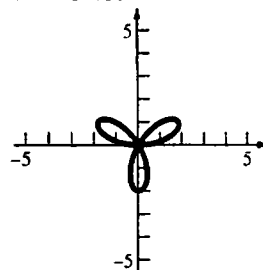
$$\begin{aligned}
 A &= 2 \cdot \frac{1}{2} \int_0^{\pi/4} (3 \cos 2\theta)^2 d\theta = 9 \int_0^{\pi/4} \cos^2 2\theta d\theta \\
 &= 9 \int_0^{\pi/4} \frac{1}{2} (1 + \cos 4\theta) d\theta \\
 &= \frac{9}{2} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} = \frac{9\pi}{8}
 \end{aligned}$$

15. $r = 4 \cos 3\theta$



$$\begin{aligned}
 A &= 6 \cdot \frac{1}{2} \int_0^{\pi/6} (4 \cos 3\theta)^2 d\theta = 48 \int_0^{\pi/6} \cos^2 3\theta d\theta \\
 &= 24 \int_0^{\pi/6} (1 + \cos 6\theta) d\theta = 24 \left[\theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6} = 4\pi
 \end{aligned}$$

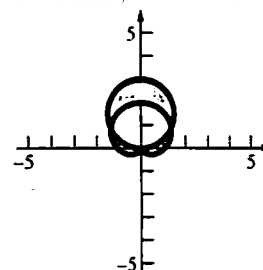
16. $r = 2 \sin 3\theta$



$$\begin{aligned}
 A &= 3 \cdot \frac{1}{2} \int_0^{\pi/3} (2 \sin 3\theta)^2 d\theta = 6 \int_0^{\pi/3} \sin^2 3\theta d\theta \\
 &= 3 \int_0^{\pi/3} (1 - \cos 6\theta) d\theta \\
 &= 3 \left[\theta - \frac{1}{6} \sin 6\theta \right]_0^{\pi/3} = \pi
 \end{aligned}$$

17. $A = \frac{1}{2} \int_0^{2\pi} 100 d\theta - \frac{1}{2} \int_0^{2\pi} 49 d\theta = 51\pi$

18. $r = 3 \sin \theta, r = 1 + \sin \theta$



Solve for the θ -coordinate of the first intersection point.

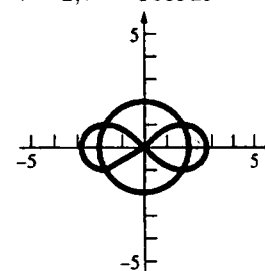
$$3 \sin \theta = 1 + \sin \theta$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

$$\begin{aligned}
 A &= 2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} [(3 \sin \theta)^2 - (1 + \sin \theta)^2] d\theta \\
 &= \int_{\pi/6}^{\pi/2} (8 \sin^2 \theta - 2 \sin \theta - 1) d\theta \\
 &= \int_{\pi/6}^{\pi/2} (3 - 4 \cos 2\theta - 2 \sin \theta) d\theta \\
 &= [3\theta - 2 \sin 2\theta + 2 \cos \theta]_{\pi/6}^{\pi/2} = \pi
 \end{aligned}$$

19. $r = 2, r^2 = 8 \cos 2\theta$



Solve for the θ -coordinate of the first intersection point.

$$4 = 8 \cos 2\theta$$

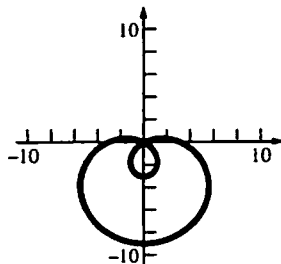
$$\cos 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{3}$$

$$\theta = \frac{\pi}{6}$$

$$\begin{aligned} A &= 4 \cdot \frac{1}{2} \int_0^{\pi/6} (8 \cos 2\theta - 4) d\theta \\ &= 2[4 \sin 2\theta - 4\theta]_0^{\pi/6} \\ &= 4\sqrt{3} - \frac{4\pi}{3} \end{aligned}$$

20. $r = 3 - 6 \sin \theta$



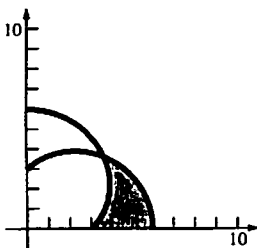
Let A_1 be the area inside the large loop and let A_2 be the area inside the small loop.

$$\begin{aligned} A_1 &= 2 \cdot \frac{1}{2} \int_{-\pi/2}^{\pi/6} (3 - 6 \sin \theta)^2 d\theta \\ &= \int_{-\pi/2}^{\pi/6} (9 - 36 \sin \theta + 36 \sin^2 \theta) d\theta \\ &= \int_{-\pi/2}^{\pi/6} (27 - 36 \sin \theta - 18 \cos 2\theta) d\theta \\ &= [27\theta + 36 \cos \theta - 9 \sin 2\theta]_{-\pi/2}^{\pi/6} = 18\pi + \frac{27\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} A_2 &= 2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} (3 - 6 \sin \theta)^2 d\theta \\ &= [27\theta + 36 \cos \theta - 9 \sin 2\theta]_{\pi/6}^{\pi/2} = 9\pi - \frac{27\sqrt{3}}{2} \end{aligned}$$

$$A = A_1 - A_2 = 9\pi + 27\sqrt{3}$$

21. $r = 3 + 3 \cos \theta$, $r = 3 + 3 \sin \theta$



Solve for the θ -coordinate of the intersection point.

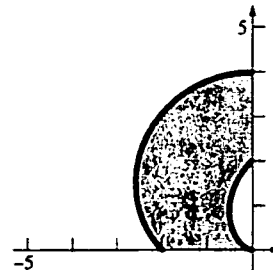
$$3 + 3 \cos \theta = 3 + 3 \sin \theta$$

$$\tan \theta = 1$$

$$\theta = \frac{\pi}{4}$$

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/4} [(3 + 3 \cos \theta)^2 - (3 + 3 \sin \theta)^2] d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} (18 \cos \theta + 9 \cos^2 \theta - 18 \sin \theta - 9 \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} (18 \cos \theta - 18 \sin \theta + 9 \cos 2\theta) d\theta \\ &= \frac{1}{2} \left[18 \sin \theta + 18 \cos \theta + \frac{9}{2} \sin 2\theta \right]_0^{\pi/4} \\ &= 9\sqrt{2} - \frac{27}{4} \end{aligned}$$

22. $r = 2 + 2 \sin \theta$, $r = 2 + 2 \cos \theta$



$$\begin{aligned} A &= \frac{1}{2} \int_{\pi/2}^{\pi} [(2 + 2 \sin \theta)^2 - (2 + 2 \cos \theta)^2] d\theta \\ &= \frac{1}{2} \int_{\pi/2}^{\pi} (8 \sin \theta + 4 \sin^2 \theta - 8 \cos \theta - 4 \cos^2 \theta) d\theta \\ &= \frac{1}{2} \int_{\pi/2}^{\pi} (8 \sin \theta - 8 \cos \theta - 4 \cos 2\theta) d\theta \\ &= \frac{1}{2} [-8 \cos \theta - 8 \sin \theta - 2 \sin 2\theta]_{\pi/2}^{\pi} = 8 \end{aligned}$$

23. a. $f(\theta) = 2 \cos \theta$, $f'(\theta) = -2 \sin \theta$

$$\begin{aligned} m &= \frac{(2 \cos \theta) \cos \theta + (-2 \sin \theta) \sin \theta}{-(2 \cos \theta) \sin \theta + (-2 \sin \theta) \cos \theta} \\ &= \frac{2 \cos^2 \theta - 2 \sin^2 \theta}{-4 \cos \theta \sin \theta} = \frac{\cos 2\theta}{-\sin 2\theta} \end{aligned}$$

$$\text{At } \theta = \frac{\pi}{3}, m = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

b. $f(\theta) = 1 + \sin \theta$, $f'(\theta) = \cos \theta$

$$m = \frac{(1 + \sin \theta) \cos \theta + (\cos \theta) \sin \theta}{-(1 + \sin \theta) \sin \theta + (\cos \theta) \cos \theta}$$

$$= \frac{\cos \theta + 2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta - \sin \theta} = \frac{\cos \theta + \sin 2\theta}{\cos 2\theta - \sin \theta}$$

At $\theta = \frac{\pi}{3}$, $m = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}}{-\frac{1}{2} - \frac{\sqrt{3}}{2}} = -1$.

c. $f(\theta) = \sin 2\theta$, $f'(\theta) = 2 \cos 2\theta$

$$m = \frac{(\sin 2\theta) \cos \theta + (2 \cos 2\theta) \sin \theta}{-(\sin 2\theta) \sin \theta + (2 \cos 2\theta) \cos \theta}$$

At $\theta = \frac{\pi}{3}$, .

$$m = \frac{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right)}{-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + (-1)\left(\frac{1}{2}\right)} = \frac{-\frac{\sqrt{3}}{4}}{-\frac{5}{4}} = \frac{\sqrt{3}}{5}$$

d. $f(\theta) = 4 - 3 \cos \theta$, $f'(\theta) = 3 \sin \theta$

$$m = \frac{(4 - 3 \cos \theta) \cos \theta + (3 \sin \theta) \sin \theta}{-(4 - 3 \cos \theta) \sin \theta + (3 \sin \theta) \cos \theta}$$

$$= \frac{4 \cos \theta - 3 \cos^2 \theta + 3 \sin^2 \theta}{-4 \sin \theta + 6 \sin \theta \cos \theta}$$

$$= \frac{4 \cos \theta - 3 \cos 2\theta}{-4 \sin \theta + 3 \sin 2\theta}$$

At $\theta = \frac{\pi}{3}$,

$$m = \frac{4\left(\frac{1}{2}\right) - 3\left(-\frac{1}{2}\right)}{-4\left(\frac{\sqrt{3}}{2}\right) + 3\left(\frac{\sqrt{3}}{2}\right)} = \frac{\frac{7}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{7}{\sqrt{3}}$$

24. $f(\theta) = a(1 + \cos \theta)$, $f'(\theta) = -a \sin \theta$

$$m = \frac{a(1 + \cos \theta) \cos \theta + (-a \sin \theta) \sin \theta}{-a(1 + \cos \theta) \sin \theta + (-a \sin \theta) \cos \theta}$$

$$= \frac{\cos \theta + \cos^2 \theta - \sin^2 \theta}{-\sin \theta - 2 \sin \theta \cos \theta} = \frac{2 \cos^2 \theta + \cos \theta - 1}{-\sin \theta(1 + 2 \cos \theta)}$$

26. Recall from Chapter 6 that $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ for x and y functions of t and $a \leq t \leq b$.

$$x = r \cos \theta = f(\theta) \cos \theta, y = r \sin \theta = f(\theta) \sin \theta$$

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta, \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta$$

$$L = \int_a^b \sqrt{(f'(\theta) \cos \theta - f(\theta) \sin \theta)^2 + (f'(\theta) \sin \theta + f(\theta) \cos \theta)^2} d\theta$$

$$= \int_a^b \sqrt{[f(\theta)]^2 (\sin^2 \theta + \cos^2 \theta) + [f'(\theta)]^2 (\sin^2 \theta + \cos^2 \theta)} d\theta = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$$

a. $m = 0$ when $2 \cos^2 \theta + \cos \theta - 1 = 0$
 $(2 \cos \theta - 1)(\cos \theta + 1) = 0$.

$$\cos \theta = \frac{1}{2}, \cos \theta = -1$$

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}, \theta = \pi; \text{ when}$$

$$\theta = \pi, f(\theta) = 0, \text{ so } \theta = \pi \text{ is the tangent line.}$$

$$\left(\frac{3a}{2}, \frac{\pi}{3}\right), \left(\frac{3a}{2}, -\frac{\pi}{3}\right), (0, \pi)$$

b. m is undefined when $\sin \theta(1 + 2 \cos \theta) = 0$.

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$(2a, 0), \left(\frac{a}{2}, \frac{2\pi}{3}\right), \left(\frac{a}{2}, \frac{4\pi}{3}\right)$$

25. $f(\theta) = 1 - 2 \sin \theta$, $f'(\theta) = -2 \cos \theta$

$$m = \frac{(1 - 2 \sin \theta) \cos \theta + (-2 \cos \theta) \sin \theta}{-(1 - 2 \sin \theta) \sin \theta + (-2 \cos \theta) \cos \theta}$$

$$= \frac{\cos \theta - 4 \sin \theta \cos \theta}{-\sin \theta + 2 \sin^2 \theta - 2 \cos^2 \theta}$$

$$= \frac{\cos \theta(1 - 4 \sin \theta)}{-\sin \theta + 2 \sin^2 \theta - 2 \cos^2 \theta}$$

$$m = 0 \text{ when } \cos \theta(1 - 4 \sin \theta) = 0$$

$$\cos \theta = 0, 1 - 4 \sin \theta = 0$$

$$\theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}, \theta = \sin^{-1}\left(\frac{1}{4}\right) \approx 0.25,$$

$$\theta = \pi - \sin^{-1}\left(\frac{1}{4}\right) \approx 2.89$$

$$f\left(\frac{\pi}{2}\right) = -1, f\left(\frac{3\pi}{2}\right) = 3, f\left(\sin^{-1}\left(\frac{1}{4}\right)\right) = \frac{1}{2},$$

$$f\left(\pi - \sin^{-1}\left(\frac{1}{4}\right)\right) = \frac{1}{2}$$

$$\left(-1, \frac{\pi}{2}\right), \left(3, \frac{3\pi}{2}\right), \left(\frac{1}{2}, 0.25\right), \left(\frac{1}{2}, 2.89\right)$$

27. $f(\theta) = a(1 + \cos \theta)$, $f'(\theta) = -a \sin \theta$

$$L = \int_0^{2\pi} \sqrt{[a(1 + \cos \theta)]^2 + [-a \sin \theta]^2} d\theta = a \int_0^{2\pi} \sqrt{2 + 2 \cos \theta} d\theta = 2a \int_0^{2\pi} \sqrt{\frac{1 + \cos \theta}{2}} d\theta = 2a \int_0^{2\pi} \left| \cos \frac{\theta}{2} \right| d\theta$$

$$= 2a \left[\int_0^{\pi} \cos \frac{\theta}{2} d\theta - \int_{\pi}^{2\pi} \cos \frac{\theta}{2} d\theta \right] = 2a \left(\left[2 \sin \frac{\theta}{2} \right]_0^{\pi} - \left[2 \sin \frac{\theta}{2} \right]_{\pi}^{2\pi} \right) = 8a$$

28. $f(\theta) = e^{\theta/2}$, $f'(\theta) = \frac{1}{2}e^{\theta/2}$

$$L = \int_0^{2\pi} \sqrt{[e^{\theta/2}]^2 + \left[\frac{1}{2}e^{\theta/2}\right]^2} d\theta = \int_0^{2\pi} \sqrt{\frac{5}{4}}e^{\theta/2} d\theta = \int_0^{2\pi} \frac{\sqrt{5}}{2}e^{\theta/2} d\theta = \left[\sqrt{5}e^{\theta/2} \right]_0^{2\pi} = \sqrt{5}(e^{\pi} - 1) \approx 49.51$$

29. If n is even, there are $2n$ leaves.

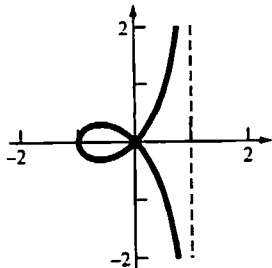
$$A = 2n \cdot \frac{1}{2} \int_{-\pi/2n}^{\pi/2n} (a \cos n\theta)^2 d\theta = na^2 \int_{-\pi/2n}^{\pi/2n} \cos^2 n\theta d\theta = na^2 \int_{-\pi/2n}^{\pi/2n} \frac{1 + \cos 2n\theta}{2} d\theta$$

$$= na^2 \left[\frac{1}{2}\theta + \frac{\sin 2n\theta}{4n} \right]_{-\pi/2n}^{\pi/2n} = \frac{1}{2}a^2\pi$$

If n is odd, there are n leaves.

$$A = n \cdot \frac{1}{2} \int_{-\pi/2n}^{\pi/2n} (a \cos n\theta)^2 d\theta = \frac{na^2}{2} \int_{-\pi/2n}^{\pi/2n} \cos^2 n\theta d\theta = \frac{na^2}{2} \left[\frac{1}{2}\theta + \frac{\sin 2n\theta}{4n} \right]_{-\pi/2n}^{\pi/2n} = \frac{1}{4}a^2\pi$$

30. $r = \sec \theta - 2 \cos \theta$



Solve for the θ -coordinate when $r = 0$.

$$\sec \theta - 2 \cos \theta = 0$$

$$\cos^2 \theta = \frac{1}{2}$$

$$\cos \theta = \pm \frac{1}{\sqrt{2}}$$

$$\theta = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$$

Notice that the loop is produced for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

$$A = \frac{1}{2} \int_{-\pi/4}^{\pi/4} (\sec \theta - 2 \cos \theta)^2 d\theta$$

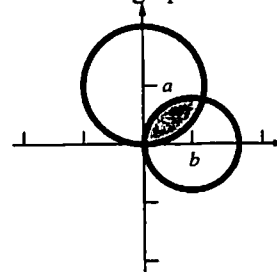
$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} (\sec^2 \theta - 4 + 4 \cos^2 \theta) d\theta$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} (\sec^2 \theta - 2 + 2 \cos 2\theta) d\theta$$

$$= \frac{1}{2} \left[\tan \theta - 2\theta + \sin 2\theta \right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2} \left[\left(1 - \frac{\pi}{2} + 1 \right) - \left(-1 + \frac{\pi}{2} - 1 \right) \right] = 2 - \frac{\pi}{2}$$

31. a. Sketch the graph.



Solve for the θ -coordinate of the intersection.

$$2a \sin \theta = 2b \cos \theta$$

$$\tan \theta = \frac{b}{a}$$

$$\theta = \tan^{-1} \left(\frac{b}{a} \right)$$

$$\text{Let } \theta_0 = \tan^{-1} \left(\frac{b}{a} \right).$$

$$A = \frac{1}{2} \int_0^{\theta_0} (2a \sin \theta)^2 d\theta + \frac{1}{2} \int_{\theta_0}^{\pi/2} (2b \cos \theta)^2 d\theta$$

$$= 2a^2 \int_0^{\theta_0} \sin^2 \theta d\theta + 2b^2 \int_{\theta_0}^{\pi/2} \cos^2 \theta d\theta$$

$$\begin{aligned}
&= a^2 \int_0^{\theta_0} (1 - \cos 2\theta) d\theta + b^2 \int_{\theta_0}^{\pi/2} (1 + \cos 2\theta) d\theta \\
&= a^2 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\theta_0} + b^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_{\theta_0}^{\pi/2} \\
&= a^2 \theta_0 + b^2 \left(\frac{\pi}{2} - \theta_0 \right) - \frac{a^2 + b^2}{2} \sin 2\theta_0 \\
&= a^2 \theta_0 + b^2 \left(\frac{\pi}{2} - \theta_0 \right) - (a^2 + b^2) \sin \theta_0 \cos \theta_0 \\
&= a^2 \tan^{-1} \left(\frac{b}{a} \right) + b^2 \left(\frac{\pi}{2} - \tan^{-1} \left(\frac{b}{a} \right) \right) - ab.
\end{aligned}$$

Note that since $\tan \theta = \frac{b}{a}$, $\cos \theta = \frac{a}{\sqrt{a^2 + b^2}}$

and $\sin \theta = \frac{b}{\sqrt{a^2 + b^2}}$.

b. Let m_1 be the slope of $r = 2a \sin \theta$.

$$\begin{aligned}
m_1 &= \frac{2a \sin \theta \cos \theta + 2a \cos \theta \sin \theta}{-2a \sin \theta \sin \theta + 2a \cos \theta \cos \theta} \\
&= \frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}
\end{aligned}$$

At $\theta = \tan^{-1} \left(\frac{b}{a} \right)$, $m_1 = \frac{2ab}{a^2 - b^2}$.

At $\theta = 0$ (the pole), $m_1 = 0$.

Let m_2 be the slope of $r = 2b \cos \theta$.

$$\begin{aligned}
m_2 &= \frac{2b \cos \theta \cos \theta - 2b \sin \theta \sin \theta}{-2b \cos \theta \sin \theta - 2b \sin \theta \cos \theta} \\
&= \frac{\cos^2 \theta - \sin^2 \theta}{-2 \sin \theta \cos \theta}
\end{aligned}$$

At $\theta = \tan^{-1} \left(\frac{b}{a} \right)$, $m_2 = -\frac{a^2 - b^2}{2ab}$.

At $\theta = \frac{\pi}{2}$ (the pole), m_2 is undefined.

Therefore the two circles intersect at right angles.

32. The area swept from time t_0 to t_1 is

$$A = \int_{\theta(t_0)}^{\theta(t_1)} \frac{1}{2} r^2 d\theta.$$

By the Fundamental Theorem of Calculus,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}.$$

So $\frac{dA}{dt} = \frac{k}{2m}$ where k is the constant angular momentum.

Therefore, $\frac{dA}{dt}$ is a constant so $A = \frac{k}{2m} (t_1 - t_0)$.

Equal areas will be swept out in equal time.

33. The edge of the pond is described by the equation $r = 2a \cos \theta$.

Solve for intersection points of the circles $r = ak$ and $r = 2a \cos \theta$.

$$ak = 2a \cos \theta$$

$$\cos \theta = \frac{k}{2}, \theta = \cos^{-1} \left(\frac{k}{2} \right)$$

Let A be the grazing area.

$$A = \frac{1}{2} \pi (ka)^2 + 2 \cdot \frac{1}{2} \int_{\cos^{-1}(\frac{k}{2})}^{\pi/2} [(ka)^2 - (2a \cos \theta)^2] d\theta = \frac{1}{2} k^2 a^2 \pi + a^2 \int_{\cos^{-1}(\frac{k}{2})}^{\pi/2} (k^2 - 4 \cos^2 \theta) d\theta$$

$$= \frac{1}{2} k^2 a^2 \pi + a^2 \int_{\cos^{-1}(\frac{k}{2})}^{\pi/2} ((k^2 - 2) - 2 \cos 2\theta) d\theta = \frac{1}{2} k^2 a^2 \pi + a^2 \left[(k^2 - 2)\theta - \sin 2\theta \right]_{\cos^{-1}(\frac{k}{2})}^{\pi/2}$$

$$= \frac{1}{2} k^2 a^2 \pi + a^2 \left[k^2 \theta - 2\theta - 2 \sin \theta \cos \theta \right]_{\cos^{-1}(\frac{k}{2})}^{\pi/2}$$

$$= \frac{1}{2} k^2 a^2 \pi + a^2 \left[\frac{k^2 \pi}{2} - \pi - k^2 \cos^{-1} \left(\frac{k}{2} \right) + 2 \cos^{-1} \left(\frac{k}{2} \right) + \frac{k\sqrt{4-k^2}}{2} \right]$$

$$= a^2 \left[(k^2 - 1)\pi + (2 - k^2) \cos^{-1} \left(\frac{k}{2} \right) + \frac{k\sqrt{4-k^2}}{2} \right]$$

34. $|PT| = ka - \phi a$; ϕ goes from 0 to k .

$$A = \frac{1}{2} \int_0^k (ka - \phi a)^2 d\phi = \frac{1}{2} \left[-\frac{1}{3a} (ka - \phi a)^3 \right]_0^k$$

$$= \frac{1}{6} a^2 k^3$$

The grazing area is

$$\frac{1}{2} \pi (ka)^2 + 2A = a^2 \left(\frac{\pi k^2}{2} + \frac{k^3}{3} \right).$$

35. The untethered goat has a grazing area of πa^2 .
From Problem 34, the tethered goat has a grazing

area of $a^2 \left(\frac{\pi k^2}{2} + \frac{k^3}{3} \right)$.

$$\pi a^2 = a^2 \left(\frac{\pi k^2}{2} + \frac{k^3}{3} \right)$$

$$\pi = \frac{\pi k^2}{2} + \frac{k^3}{3}$$

$$2k^3 + 3\pi k^2 - 6\pi = 0$$

Using a numerical method or graphing calculator,
 $k \approx 1.26$. The length of the rope is approximately
 $1.26a$.

36. $f(\theta) = 2 + \cos \theta$, $f'(\theta) = -\sin \theta$

$$L = 2 \int_0^\pi \sqrt{[2 + \cos \theta]^2 + [-\sin \theta]^2} d\theta$$

$$= 2 \int_0^\pi \sqrt{5 + 4 \cos \theta} d\theta \approx 13.36$$

$$f(\theta) = 2 + 4 \cos \theta, f'(\theta) = -4 \sin \theta$$

$$L = 2 \int_0^\pi \sqrt{[2 + 4 \cos \theta]^2 + [-4 \sin \theta]^2} d\theta$$

$$= 4 \int_0^\pi \sqrt{5 + 4 \cos \theta} d\theta \approx 26.73$$

37. $A = 3 \cdot \frac{1}{2} \int_0^{\pi/3} (4 \sin 3\theta)^2 d\theta = 24 \int_0^{\pi/3} \sin^2 3\theta d\theta$

$$= 12 \int_0^{\pi/3} (1 - \cos 6\theta) d\theta$$

$$= 12 \left[\theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/3} = 4\pi$$

$$f(\theta) = 4 \sin 3\theta, f'(\theta) = 12 \cos 3\theta$$

$$L = 3 \int_0^{\pi/3} \sqrt{(4 \sin 3\theta)^2 + (12 \cos 3\theta)^2} d\theta$$

$$= 3 \int_0^{\pi/3} \sqrt{16 \sin^2 3\theta + 144 \cos^2 3\theta} d\theta$$

$$= 12 \int_0^{\pi/3} \sqrt{1 + 8 \cos^2 3\theta} d\theta \approx 26.73$$

38. $A = 4 \cdot \frac{1}{2} \int_0^{\pi/4} 8 \cos 2\theta d\theta = 2[4 \sin 2\theta]_0^{\pi/4} = 8$

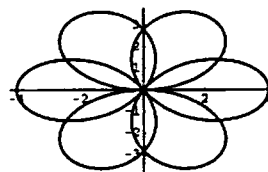
$$f(\theta) = \sqrt{8 \cos 2\theta}, f'(\theta) = \frac{-8 \sin 2\theta}{\sqrt{8 \cos 2\theta}}$$

$$L = 4 \int_0^{\pi/4} \sqrt{8 \cos 2\theta + \frac{8 \sin^2 2\theta}{\cos 2\theta}} d\theta$$

$$= 4 \int_0^{\pi/4} \sqrt{\frac{8}{\cos 2\theta}} d\theta$$

$$= 8\sqrt{2} \int_0^{\pi/4} \frac{1}{\sqrt{\cos 2\theta}} d\theta \approx 14.83$$

39. $r = 4 \sin\left(\frac{3\theta}{2}\right), 0 \leq \theta \leq 4\pi$



$$f(\theta) = 4 \sin\left(\frac{3\theta}{2}\right), f'(\theta) = 6 \cos\left(\frac{3\theta}{2}\right)$$

$$L = \int_0^{4\pi} \sqrt{\left[4 \sin\left(\frac{3\theta}{2}\right)\right]^2 + \left[6 \cos\left(\frac{3\theta}{2}\right)\right]^2} d\theta$$

$$= \int_0^{4\pi} \sqrt{16 \sin^2\left(\frac{3\theta}{2}\right) + 36 \cos^2\left(\frac{3\theta}{2}\right)} d\theta$$

$$= \int_0^{4\pi} \sqrt{16 + 20 \cos^2\left(\frac{3\theta}{2}\right)} d\theta \approx 63.46$$

12.9 Chapter Review

Concepts Test

1. False: If $a = 0$, the graph is a line.

2. True: The defining condition of a parabola is $|PF| = |PL|$. Since the axis of a parabola is perpendicular to the directrix and the

distance from the vertex to the directrix is equal to the distance to the focus, the vertex is midway between the focus and the directrix.

3. False: The defining condition of an ellipse is $|PF| = e|PL|$ where $0 < e < 1$. Hence the distance from the vertex to a directrix is a greater than the distance to a focus.
4. True: See Problem 33 in Section 12.1.
5. True: The asymptotes for both hyperbolas are $y = \pm \frac{b}{a}x$.
6. True: $C = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt$;
 $2\pi b = \int_0^{2\pi} \sqrt{b^2 \sin^2 t + b^2 \cos^2 t} dt$
 $< C < \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} dt = 2\pi a$
7. True: As e approaches 0, the ellipse becomes more circular.
8. False: The equation can be rewritten as $\frac{x^2}{4} + \frac{y^2}{6} = 1$ which is a vertical ellipse with foci on the y -axis.
9. False: The equation $x^2 - y^2 = 0$ represents the two lines $y = \pm x$.
10. True: $(y^2 - 4x + 1)^2 = 0$ implies $y^2 - 4x + 1 = 0$ which is an equation for a parabola.
11. True: If $k > 0$, the equation is a horizontal hyperbola; if $k < 0$, the equation is a vertical hyperbola.
12. False: If $k < 0$, there is no graph.
13. False: If $b > a$, the distance is $2\sqrt{b^2 - a^2}$.
14. True: If $y = 0$, $\frac{x^2}{9} = -2$ which is not possible.
15. True: Since light from one focus reflects to the other focus, light emanating from a point between a focus and the nearest vertex will reflect beyond the other focus.
16. True: This is the set of points equidistant from the point $(0, 0)$ and the line $x = 4$. (See Problem 38 of Section 12.1)
17. True: Since angular momentum $mr^2 \frac{d\theta}{dt}$ is assumed to be constant, $\frac{d\theta}{dt}$ is greatest at the vertex nearest the sun.
18. True: $a = \frac{8}{2} = 4$, $c = \frac{2}{2} = 1$,
 $b = \sqrt{16 - 1} = \sqrt{15}$. The length of the minor diameter is $2b = \sqrt{60}$.
19. True: The equation is equivalent to $\left(x + \frac{C}{2}\right)^2 + \left(y + \frac{D}{2}\right)^2 = -F + \frac{C^2}{4} + \frac{D^2}{4}$.
Thus, the graph is a circle if $-F + \frac{C^2}{4} + \frac{D^2}{4} > 0$, a point if $-F + \frac{C^2}{4} + \frac{D^2}{4} = 0$, or the empty set if $-F + \frac{C^2}{4} + \frac{D^2}{4} < 0$.
20. False: The equation is equivalent to $2\left(x + \frac{C}{4}\right)^2 + \left(y + \frac{D}{2}\right)^2 = -F + \frac{C^2}{8} + \frac{D^2}{4}$.
Thus, the graph can be a point if $-F + \frac{C^2}{8} + \frac{D^2}{4} = 0$.
21. False: The limiting forms of two parallel lines and the empty set cannot be formed in such a manner.
22. True: By definition, these curves are conic sections, which can be expressed by an equation of the form $Ax^2 + Cy^2 + Dx + Ex + F = 0$.
23. False: For example, $xy = 1$ is a hyperbola with coordinates only in the first and third quadrants.
24. False: For example, the graph of $x^2 + 3xy + y^2 = 1$ is a hyperbola that passes through the four points.
25. True: The graph of $r = 4 \cos \theta$ is a circle of radius 2 centered at $(2, 0)$. The graph

$r = 4 \cos\left(\theta - \frac{\pi}{3}\right)$ is the graph of

$r = 4 \cos \theta$ rotated $\frac{\pi}{3}$

counter-clockwise about the pole.

26. True: (r, θ) can be expressed as $(r, \theta + 2\pi n)$ for any integer n .
27. False: For example, if $f(\theta) = \cos \theta$ and $g(\theta) = \sin \theta$, solving the two equations simultaneously does not give the pole (which is $\left(0, \frac{\pi}{2}\right)$ for $f(\theta)$ and $(0, 0)$ for $g(\theta)$).
28. True: Since f is odd $f(-\theta) = -f(\theta)$. Thus, if we replace (r, θ) by $(-r, -\theta)$, the equation $-r = f(-\theta)$ is $-r = -f(\theta)$ or $r = f(\theta)$. Therefore, the graph is symmetric about the y -axis.
29. True: Since f is even $f(-\theta) = f(\theta)$. Thus, if we replace (r, θ) by $(r, -\theta)$, the equation $r = f(-\theta)$ is $r = f(\theta)$. Therefore, the graph is symmetric about the x -axis.
30. True: The graph has 3 leaves and the area is exactly one quarter of the circle $r = 4$. (See Problem 15 of Section 12.8.)

Sample Test Problems

1. a. $x^2 - 4y^2 = 0$; $y = \pm \frac{x}{2}$

(5) Two intersecting lines

b. $x^2 - 4y^2 = 0.01$; $\frac{x^2}{0.01} - \frac{y^2}{0.0025} = 1$

(9) A hyperbola

c. $x^2 - 4 = 0$; $x = \pm 2$

(4) Two parallel lines

d. $x^2 - 4x + 4 = 0$; $x = 2$

(3) A single line

e. $x^2 + 4y^2 = 0$; $(0, 0)$

(2) A single point

f. $x^2 + 4y^2 = x$; $x^2 - x + \frac{1}{4} + 4y^2 = \frac{1}{4}$;

$$\frac{\left(x - \frac{1}{2}\right)^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{16}} = 1$$

(8) An ellipse

g. $x^2 + 4y^2 = -x$; $x^2 + x + \frac{1}{4} + 4y^2 = \frac{1}{4}$;

$$\frac{\left(x + \frac{1}{2}\right)^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{16}} = 1$$

(8) An ellipse

h. $x^2 + 4y^2 = -1$

(1) No graph

i. $(x^2 + 4y - 1)^2 = 0$; $x^2 + 4y - 1 = 0$

(7) A parabola

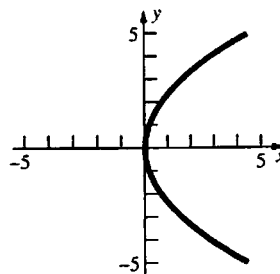
j. $3x^2 + 4y^2 = -x^2 + 1$; $x^2 + y^2 = \frac{1}{4}$

(6) A circle

2. $y^2 - 6x = 0$; $y^2 = 6x$; $y^2 = 4\left(\frac{3}{2}\right)x$

Horizontal parabola; opens to the right; $p = \frac{3}{2}$

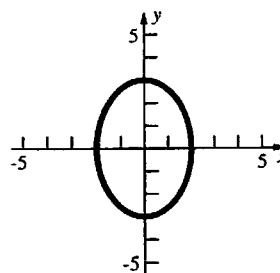
Focus is at $\left(\frac{3}{2}, 0\right)$ and vertex is at $(0, 0)$.



3. $9x^2 + 4y^2 - 36 = 0$; $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Vertical ellipse; $a = 3$, $b = 2$, $c = \sqrt{5}$

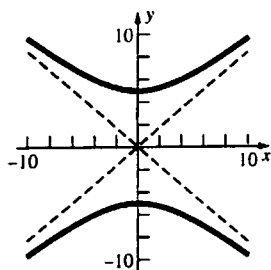
Foci are at $(0, \pm\sqrt{5})$ and vertices are at $(0, \pm 3)$.



4. $25x^2 - 36y^2 + 900 = 0; \frac{y^2}{25} - \frac{x^2}{36} = 1$

Vertical hyperbola; $a = 5$, $b = 6$, $c = \sqrt{61}$

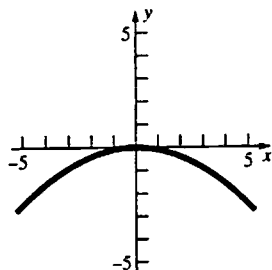
Foci are at $(0, \pm\sqrt{61})$ and vertices are at $(0, \pm 5)$.



5. $x^2 + 9y = 0; x^2 = -9y; x^2 = -4\left(\frac{9}{4}\right)y$

Vertical parabola; opens downward; $p = \frac{9}{4}$

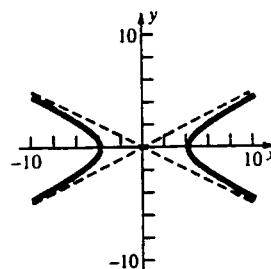
Focus at $\left(0, -\frac{9}{4}\right)$ and vertex at $(0, 0)$.



6. $x^2 - 4y^2 - 16 = 0; \frac{x^2}{16} - \frac{y^2}{4} = 1$

Horizontal hyperbola; $a = 4$, $b = 2$, $c = 2\sqrt{5}$

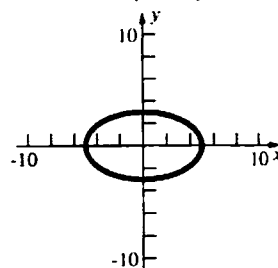
Foci are at $(\pm 2\sqrt{5}, 0)$ and vertices are at $(\pm 4, 0)$.



7. $9x^2 + 25y^2 - 225 = 0; \frac{x^2}{25} + \frac{y^2}{9} = 1$

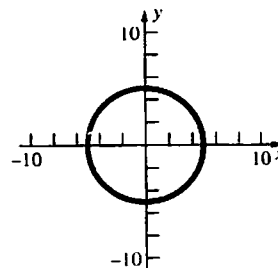
Horizontal ellipse, $a = 5$, $b = 3$, $c = 4$

Foci are at $(\pm 4, 0)$ and vertices are at $(\pm 5, 0)$.



8. $9x^2 + 9y^2 - 225 = 0; x^2 + y^2 = 25$

Circle; $r = 5$

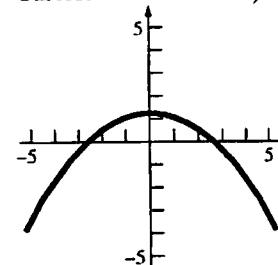


9. $r = \frac{5}{2 + 2\sin\theta} = \frac{\left(\frac{5}{2}\right)(1)}{1 + (1)\cos\left(\theta - \frac{\pi}{2}\right)}$

$e = 1$; parabola

Focus is at $(0, 0)$ and vertex is at $\left(0, \frac{5}{4}\right)$ (in

Cartesian coordinates).



10. $r(2 + \cos\theta) = 3; r = \frac{3\left(\frac{1}{2}\right)}{1 + \frac{1}{2}\cos\theta}$

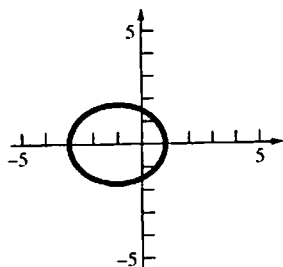
$e = \frac{1}{2}$, ellipse

At $\theta = 0$, $r = 1$. At $\theta = \pi$, $r = 3$.

$a = \frac{1+3}{2} = 2$, $c = ea = 1$

Center is at $(-1, 0)$ (in Cartesian coordinates).

Foci are $(0, 0)$ and $(-2, 0)$ and vertices are at $(1, 0)$ and $(-3, 0)$ (all in Cartesian coordinates).



11. Horizontal ellipse; center at $(0, 0)$, $a = 4$,

$$e = \frac{c}{a} = \frac{1}{2}, c = 2, b = \sqrt{16 - 4} = 2\sqrt{3}$$

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

12. Vertical parabola; opens downward; $p = 3$

$$x^2 = -12y$$

13. Horizontal parabola;

$$y^2 = ax, (3)^2 = a(-1), a = -9$$

$$y^2 = -9x$$

14. Vertical hyperbola; $a = 3$, $c = ae = 5$,

$$b = \sqrt{25 - 9} = 4, \text{ center at } (0, 0)$$

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

15. Horizontal hyperbola, $a = 2$,

$$x = \pm 2y, \frac{a}{b} = 2, b = 1$$

$$\frac{x^2}{4} - \frac{y^2}{1} = 1$$

16. Vertical parabola; opens downward; $p = 1$

$$(x - 3)^2 = -4(y - 3)$$

17. Horizontal ellipse; $2a = 10$, $a = 5$, $c = 4 - 1 = 3$,

$$b = \sqrt{25 - 9} = 4$$

$$\frac{(x - 1)^2}{25} + \frac{(y - 2)^2}{16} = 1$$

18. Vertical hyperbola; $2a = 6$, $a = 3$, $c = ae = 10$,

$$b = \sqrt{100 - 9} = \sqrt{91}, \text{ center at } (2, 3)$$

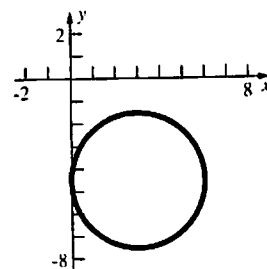
$$\frac{(y - 3)^2}{9} - \frac{(x - 2)^2}{91} = 1$$

19. $4x^2 + 4y^2 - 24x + 36y + 81 = 0$

$$4(x^2 - 6x + 9) + 4\left(y^2 + 9y + \frac{81}{4}\right) = -81 + 36 + 81$$

$$4(x - 3)^2 + 4\left(y + \frac{9}{2}\right)^2 = 36$$

$$(x - 3)^2 + \left(y + \frac{9}{2}\right)^2 = 9; \text{ circle}$$

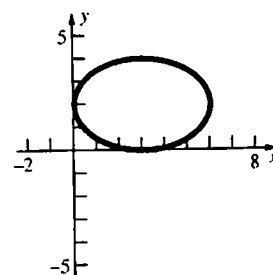


20. $4x^2 + 9y^2 - 24x - 36y + 36 = 0$

$$4(x^2 - 6x + 9) + 9(y^2 - 4y + 4) = -36 + 36 + 36$$

$$4(x - 3)^2 + 9(y - 2)^2 = 36$$

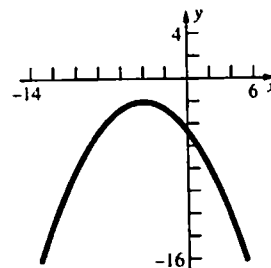
$$\frac{(x - 3)^2}{9} + \frac{(y - 2)^2}{4} = 1; \text{ ellipse}$$



21. $x^2 + 8x + 6y + 28 = 0$

$$(x^2 + 8x + 16) = -6y - 28 + 16$$

$$(x + 4)^2 = -6(y + 2); \text{ parabola}$$

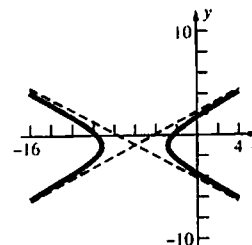


22. $3x^2 - 10y^2 + 36x - 20y + 68 = 0$

$$3(x^2 + 12x + 36) - 10(y^2 + 2y + 1) = -68 + 108 - 10$$

$$3(x + 6)^2 - 10(y + 1)^2 = 30$$

$$\frac{(x + 6)^2}{10} - \frac{(y + 1)^2}{3} = 1; \text{ hyperbola}$$



$$23. \quad x = \frac{\sqrt{2}}{2}(u-v)$$

$$y = \frac{\sqrt{2}}{2}(u+v)$$

$$\frac{1}{2}(u-v)^2 + \frac{3}{2}(u-v)(u+v) + \frac{1}{2}(u+v)^2 = 10$$

$$\frac{5}{2}u^2 - \frac{1}{2}v^2 = 10$$

$$r = \frac{5}{2}, s = -\frac{1}{2}$$

$$\frac{u^2}{4} - \frac{v^2}{20} = 1; \text{ hyperbola}$$

$$a = 2, b = 2\sqrt{5}, c = \sqrt{4+20} = 2\sqrt{6}$$

The distance between foci is $4\sqrt{6}$.

$$24. \quad 7x^2 + 8xy + y^2 = 9$$

$$\cot 2\theta = \frac{3}{4}$$

$$\cos 2\theta = \frac{3}{5}$$

$$\cos \theta = \sqrt{\frac{1+\frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}$$

$$\sin \theta = \sqrt{\frac{1-\frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}$$

$$x = \frac{1}{\sqrt{5}}(2u-v)$$

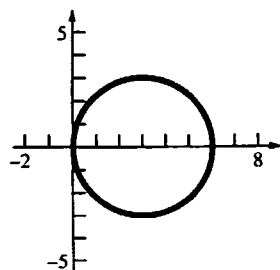
$$y = \frac{1}{\sqrt{5}}(u+2v)$$

$$\frac{7}{5}(2u-v)^2 + \frac{8}{5}(2u-v)(u+2v) + \frac{1}{5}(u+2v)^2 = 9$$

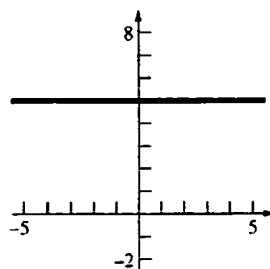
$$9u^2 - v^2 = 9$$

$$u^2 - \frac{v^2}{9} = 1; \text{ hyperbola}$$

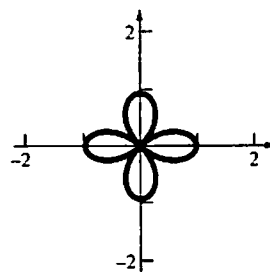
$$25. \quad r = 6 \cos \theta$$



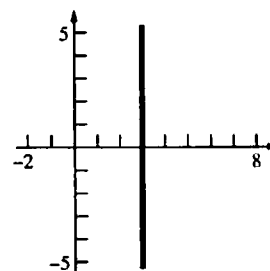
$$26. \quad r = \frac{5}{\sin \theta}$$



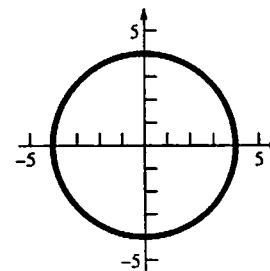
$$27. \quad r = \cos 2\theta$$



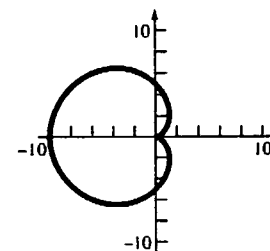
$$28. \quad r = \frac{3}{\cos \theta}$$



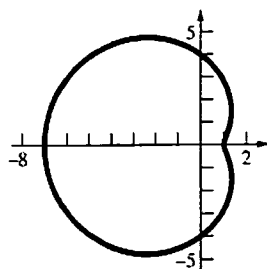
$$29. \quad r = 4$$



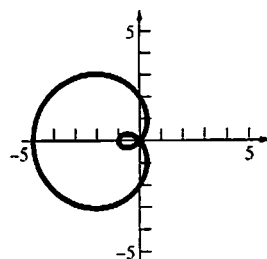
$$30. \quad r = 5 - 5 \cos \theta$$



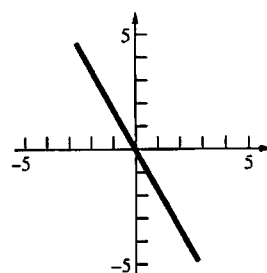
31. $r = 4 - 3\cos\theta$



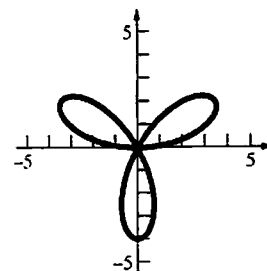
32. $r = 2 - 3\cos\theta$



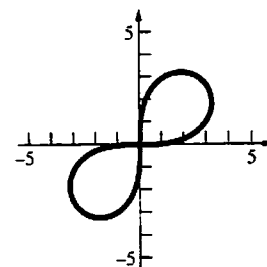
33. $\theta = \frac{2}{3}\pi$



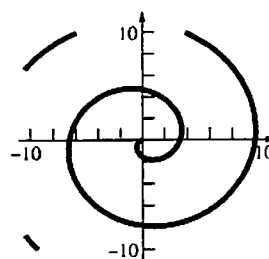
34. $r = 4\sin 3\theta$



35. $r^2 = 16\sin 2\theta$
 $r = \pm 4\sqrt{\sin 2\theta}$



36. $r = -\theta, \theta \geq 0$

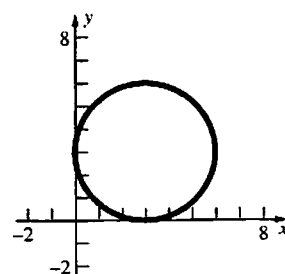


37. $r^2 - 6r(\cos\theta + \sin\theta) + 9 = 0$

$$x^2 + y^2 - 6x - 6y + 9 = 0$$

$$(x^2 - 6x + 9) + (y^2 - 6y + 9) = -9 + 9 + 9$$

$$(x - 3)^2 + (y - 3)^2 = 9$$

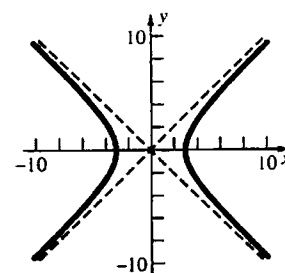


38. $r^2 \cos 2\theta = 9$

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 9$$

$$x^2 - y^2 = 9$$

$$\frac{x^2}{9} - \frac{y^2}{9} = 1$$



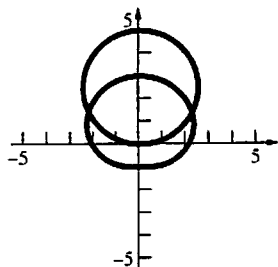
39. $f(\theta) = 3 + 3\cos\theta, f'(\theta) = -3\sin\theta$

$$m = \frac{(3 + 3\cos\theta)\cos\theta + (-3\sin\theta)\sin\theta}{-(3 + 3\cos\theta)\sin\theta + (-3\sin\theta)\cos\theta}$$

$$= \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\sin\theta} = \frac{\cos\theta + \cos 2\theta}{-\sin\theta - \sin 2\theta}$$

$$\text{At } \theta = \frac{\pi}{6}, m = \frac{\cos \frac{\pi}{6} + \cos \frac{\pi}{3}}{-\sin \frac{\pi}{6} - \sin \frac{\pi}{3}} = -1.$$

40. $r = 5 \sin \theta, r = 2 + \sin \theta$



$$5 \sin \theta = 2 + \sin \theta$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\left(\frac{5}{2}, \frac{\pi}{6}\right), \left(\frac{5}{2}, \frac{5\pi}{6}\right)$$

$$\begin{aligned} 41. \quad A &= 2 \cdot \frac{1}{2} \int_0^{\pi} (5 - 5 \cos \theta)^2 d\theta \\ &= 25 \int_0^{\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta \\ &= 25 \int_0^{\pi} \left(\frac{3}{2} - 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta \\ &= 25 \left[\frac{3}{2} \theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{\pi} = \frac{75\pi}{2} \end{aligned}$$

$$\begin{aligned} 42. \quad A &= 2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} [(5 \sin \theta)^2 - (2 + \sin \theta)^2] d\theta \\ &= \int_{\pi/6}^{\pi/2} (24 \sin^2 \theta - 4 \sin \theta - 4) d\theta \\ &= \int_{\pi/6}^{\pi/2} (8 - 12 \cos 2\theta - 4 \sin \theta) d\theta \\ &= [8\theta - 6 \sin 2\theta + 4 \cos \theta]_{\pi/6}^{\pi/2} \\ &= \frac{8}{3} \pi + \sqrt{3} \end{aligned}$$

$$\begin{aligned} 43. \quad \frac{x^2}{400} + \frac{y^2}{100} &= 1; \frac{x}{200} + \frac{yy'}{50} = 0 \\ y' &= -\frac{x}{4y}; y' = -\frac{2}{3} \text{ at } (16, 6) \end{aligned}$$

$$\text{Tangent line: } y - 6 = -\frac{2}{3}(x - 16)$$

$$\text{When } x = 14, y = -\frac{2}{3}(14 - 16) + 6 = \frac{22}{3}.$$

$$k = \frac{22}{3}$$

44. a. III

b. IV

c. I

d. II

45. a. I

b. IV

c. III

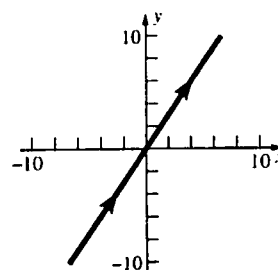
d. II

e. V

13.1 Concepts Review

1. simple; closed; simple
2. parametric; parameter
3. cycloid

4. $\frac{dy}{dt}$
 $\frac{dx}{dt}$



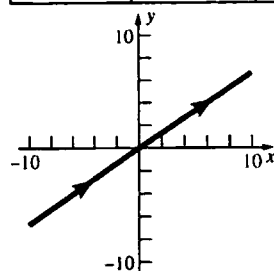
b. Simple; not closed

c. $t = \frac{x}{2}$
 $y = \frac{3}{2}x$

Problem Set 13.1

1. a.

t	x	y
-2	-6	-4
-1	-3	-2
0	0	0
1	3	2
2	6	4



b. Simple; not closed

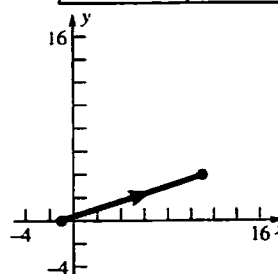
c. $t = \frac{x}{3}$
 $y = \frac{2}{3}x$

2. a.

t	x	y
-2	-4	-6
-1	-2	-3
0	0	0
1	2	3
2	4	6

3. a.

t	x	y
0	-1	0
1	2	1
2	5	2
3	8	3
4	11	4

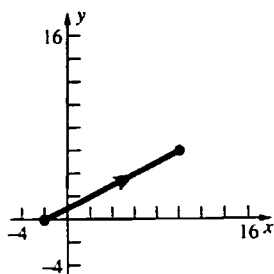


b. Simple; not closed

c. $t = \frac{1}{3}(x+1)$
 $y = \frac{1}{3}(x+1)$

4. a.

t	x	y
0	-2	0
1	2	2
2	6	4
3	10	6

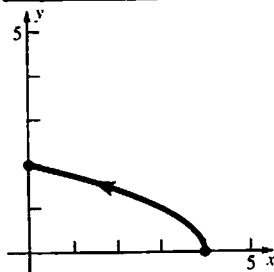


b. Simple; not closed

c. $t = \frac{1}{4}(x+2)$
 $y = \frac{1}{2}(x+2)$

5. a.

t	x	y
0	4	0
1	3	$\frac{1}{\sqrt{2}}$
2	2	$\sqrt{2}$
3	1	$\sqrt{3}$
4	0	2

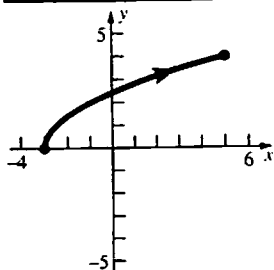


b. Simple; not closed

c. $t = 4 - x$
 $y = \sqrt{4 - x}$

6. a.

t	x	y
0	-3	0
2	-1	2
4	1	$2\sqrt{2}$
6	3	$2\sqrt{3}$
8	5	4

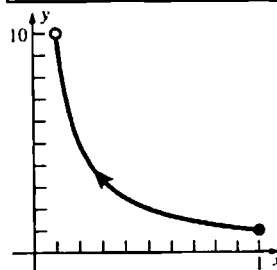


b. Simple; not closed

c. $t = x + 3$
 $y = \sqrt{2x + 6}$

7. a.

s	x	y
1	1	1
3	$\frac{1}{3}$	3
5	$\frac{1}{5}$	5
7	$\frac{1}{7}$	7
9	$\frac{1}{9}$	9

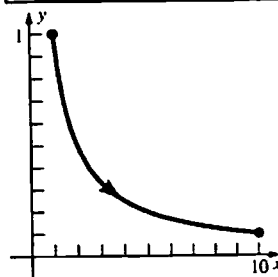


b. Simple; not closed

c. $s = \frac{1}{x}$
 $y = \frac{1}{x}$

8. a.

s	x	y
1	1	1
3	3	$\frac{1}{3}$
5	5	$\frac{1}{5}$
7	7	$\frac{1}{7}$
9	9	$\frac{1}{9}$

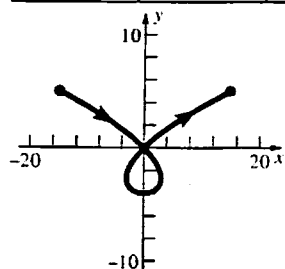


b. Simple; not closed

c. $s = x$
 $y = \frac{1}{x}$

9. a.

t	x	y
-3	-15	5
-2	0	0
-1	3	-3
0	0	-4
1	-3	-3
2	0	0
3	15	5

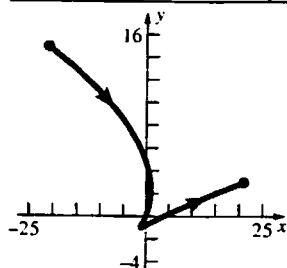


b. Not simple; not closed

c. $x^2 = t^6 - 8t^4 + 16t^2$
 $t^2 = y + 4$
 $x^2 = (y + 4)^3 - 8(y + 4)^2 + 16(y + 4)$
 $x^2 = y^3 + 4y^2$

10. a.

t	x	y
-3	-21	15
-2	-4	8
-1	1	3
0	0	0
1	-1	-1
2	4	0
3	21	3

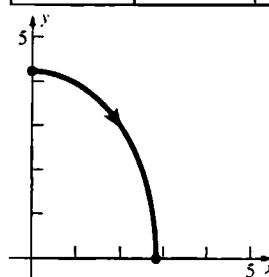


b. Simple; not closed

c. $t^2 - 2t - y = 0$
 $t = 1 \pm \sqrt{1 + y}$
 $x = (1 \pm \sqrt{1 + y})^3 - 2(1 \pm \sqrt{1 + y})$
 $x = 2 + 3y \pm (y + 2)\sqrt{1 + y}$
 $(x - 3y - 2)^2 = (y + 1)(y + 2)^2$

11. a.

t	x	y
2	0	$3\sqrt{2}$
3	2	3
4	$2\sqrt{2}$	0

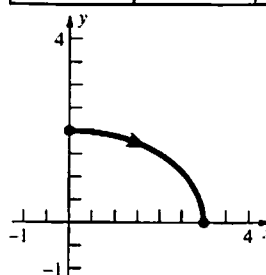


b. Simple; not closed

c. $t = \frac{1}{4}x^2 + 2$
 $t = 4 - \frac{1}{9}y^2$
 $\frac{1}{4}x^2 + 2 = 4 - \frac{1}{9}y^2$
 $\frac{x^2}{8} + \frac{y^2}{18} = 1$

12. a.

t	x	y
3	0	2
$\frac{7}{2}$	$\frac{3}{\sqrt{2}}$	$\sqrt{2}$
4	3	0



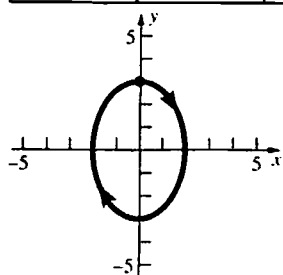
b. Simple; not closed

c. $t = \frac{1}{9}x^2 + 3$
 $t = 4 - \frac{1}{4}y^2$
 $\frac{1}{9}x^2 + 3 = 4 - \frac{1}{4}y^2$
 $\frac{x^2}{9} + \frac{y^2}{4} = 1$

13. a.

t	x	y
0	0	3
$\frac{\pi}{2}$	2	0

π	0	-3
$\frac{3\pi}{2}$	-2	0
2π	0	3



b. Simple; closed

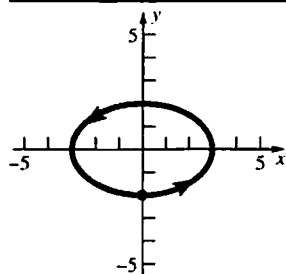
c. $\sin^2 t = \frac{x^2}{4}$

$\cos^2 t = \frac{y^2}{9}$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

14. a.

r	x	y
0	0	-2
$\frac{\pi}{2}$	3	0
π	0	2
$\frac{3\pi}{2}$	-3	0
2π	0	-2



b. Simple; closed

c. $\sin^2 r = \frac{x^2}{9}$

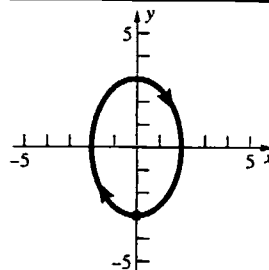
$\cos^2 r = \frac{y^2}{4}$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

15. a.

r	x	y
0	0	-3
$\frac{\pi}{2}$	-2	0
π	0	3

$\frac{3\pi}{2}$	2	0
2π	0	-3
$\frac{5\pi}{2}$	-2	0
3π	0	3
$\frac{7\pi}{2}$	2	0
4π	0	-3



b. Not simple; closed

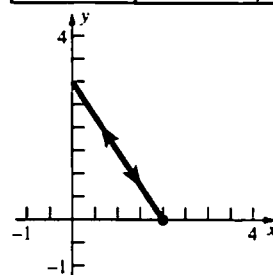
c. $\sin^2 r = \frac{x^2}{4}$

$\cos^2 r = \frac{y^2}{9}$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

16. a.

r	x	y
0	2	0
$\frac{\pi}{2}$	0	3
π	2	0
$\frac{3\pi}{2}$	0	3
2π	2	0



b. Not simple; closed

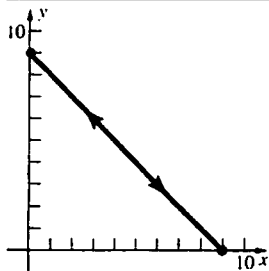
c. $\cos^2 r = \frac{x}{2}$

$\sin^2 r = \frac{y}{3}$

$\frac{x}{2} + \frac{y}{3} = 1$

17. a.

θ	x	y
0	0	9
$\frac{\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
$\frac{\pi}{2}$	0	9
$\frac{3\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
π	0	9



b. Not simple; closed

c. $\sin^2 \theta = \frac{x}{9}$

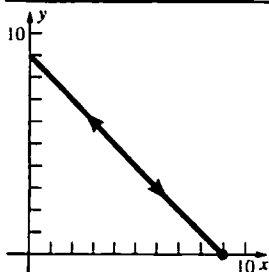
$\cos^2 \theta = \frac{y}{9}$

$\frac{x}{9} + \frac{y}{9} = 1$

$x + y = 9$

18. a.

θ	x	y
0	9	0
$\frac{\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
$\frac{\pi}{2}$	0	9
$\frac{3\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
π	9	0



b. Not simple; closed

c. $\cos^2 \theta = \frac{x}{9}$

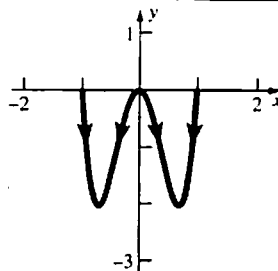
$\sin^2 \theta = \frac{y}{9}$

$\frac{x}{9} + \frac{y}{9} = 1$

$x + y = 9$

19. a.

θ	x	y
$0 + 2\pi n$	1	0
$\frac{\pi}{3} + 2\pi n$	$\frac{1}{2}$	$-\frac{3}{2}$
$\frac{\pi}{2} + 2\pi n$	0	0
$\frac{2\pi}{3} + 2\pi n$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\pi + 2\pi n$	-1	0
$\frac{4\pi}{3} + 2\pi n$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\frac{3\pi}{2} + 2\pi n$	0	0
$\frac{5\pi}{3} + 2\pi n$	$\frac{1}{2}$	$-\frac{3}{2}$



b. Not simple; not closed

c. $\cos \theta = x$

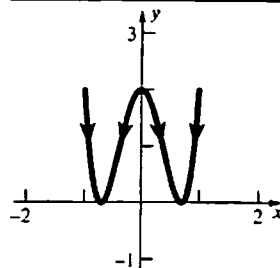
$\sin \theta = \sqrt{1 - x^2}$

$y = -8 \sin^2 \theta \cos^2 \theta$

$y = -8x^2(1 - x^2)$

20. a.

θ	x	y
$0 + 2\pi n$	0	2
$\frac{\pi}{6} + 2\pi n$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{2} + 2\pi n$	1	2
$\frac{5\pi}{6} + 2\pi n$	$\frac{1}{2}$	$\frac{1}{2}$
$\pi + 2\pi n$	0	2
$\frac{7\pi}{6} + 2\pi n$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{3\pi}{2} + 2\pi n$	-1	2
$\frac{11\pi}{6} + 2\pi n$	$-\frac{1}{2}$	$\frac{1}{2}$



b. Not simple; not closed

c. $\sin \theta = x$

$$\cos \theta = \sqrt{1-x^2}$$

$$y = 2(\cos^2 \theta - \sin^2 \theta)^2$$

$$y = 2(2x^2 - 1)^2$$

$$21. \frac{dx}{d\tau} = 6\tau$$

$$\frac{dy}{d\tau} = 12\tau^2$$

$$\frac{dy}{dx} = 2\tau$$

$$\frac{dy'}{d\tau} = 2$$

$$\frac{d^2 y}{dx^2} = \frac{1}{3\tau}$$

$$22. \frac{dx}{ds} = 12s$$

$$\frac{dy}{ds} = -6s^2$$

$$\frac{dy}{dx} = -\frac{1}{2}s$$

$$\frac{dy'}{ds} = -\frac{1}{2}$$

$$\frac{d^2 y}{dx^2} = -\frac{1}{24s}$$

$$23. \frac{dx}{d\theta} = 4\theta$$

$$\frac{dy}{d\theta} = 3\sqrt{5}\theta^2$$

$$\frac{dy}{dx} = \frac{3\sqrt{5}}{4}\theta$$

$$\frac{dy'}{d\theta} = \frac{3\sqrt{5}}{4}$$

$$\frac{d^2 y}{dx^2} = \frac{3\sqrt{5}}{16\theta}$$

$$24. \frac{dx}{d\theta} = 2\sqrt{3}\theta$$

$$\frac{dy}{d\theta} = -3\sqrt{3}\theta^2$$

$$\frac{dy}{dx} = -\frac{3}{2}\theta$$

$$\frac{dy'}{d\theta} = -\frac{3}{2}$$

$$\frac{d^2 y}{dx^2} = -\frac{\sqrt{3}}{4\theta}$$

$$25. \frac{dx}{dt} = \sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\frac{dy}{dx} = \cot t$$

$$\frac{dy'}{dt} = -\csc^2 t$$

$$\frac{d^2 y}{dx^2} = -\csc^3 t$$

$$26. \frac{dx}{dt} = 2 \sin t$$

$$\frac{dy}{dt} = 5 \cos t$$

$$\frac{dy}{dx} = \frac{5}{2} \cot t$$

$$\frac{dy'}{dt} = -\frac{5}{2} \csc^2 t$$

$$\frac{d^2 y}{dx^2} = -\frac{5}{4} \csc^3 t$$

$$27. \frac{dx}{dt} = 3 \sec^2 t$$

$$\frac{dy}{dt} = 5 \sec t \tan t$$

$$\frac{dy}{dx} = \frac{5}{3} \sin t$$

$$\frac{dy'}{dt} = \frac{5}{3} \cos t$$

$$\frac{d^2 y}{dx^2} = \frac{5}{9} \cos^3 t$$

$$28. \frac{dx}{dt} = -\csc^2 t$$

$$\frac{dy}{dt} = 2 \csc t \cot t$$

$$\frac{dy}{dx} = -2 \cos t$$

$$\frac{dy'}{dt} = 2 \sin t$$

$$\frac{d^2 y}{dx^2} = -2 \sin^3 t$$

$$29. \frac{dx}{dt} = -\frac{2t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2t-1}{t^2(1-t)^2}$$

$$\frac{dy}{dx} = \frac{(1-2t)(1+t^2)^2}{2t^3(1-t)^2}$$

$$\frac{dy'}{dt} = -\frac{3t^5 + 7t^4 - 6t^3 + 10t^2 - 9t + 3}{2t^4(1-t)^3}$$

$$\frac{d^2y}{dx^2} = \frac{(3t^5 + 7t^4 - 6t^3 + 10t^2 - 9t + 3)(1+t^2)^2}{4t^5(1-t)^3}$$

$$30. \frac{dx}{dt} = -\frac{4t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = -\frac{2(3t^2+1)}{t^2(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{3t^2+1}{2t^3}$$

$$\frac{dy'}{dt} = -\frac{3(t^2+1)}{2t^4}$$

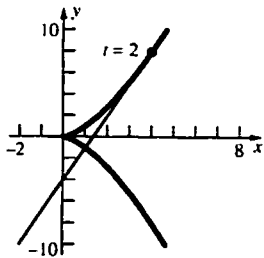
$$\frac{d^2y}{dx^2} = \frac{3(t^2+1)^3}{8t^5}$$

$$31. \frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = \frac{3}{2}t$$

At $t = 2$, $x = 4$, $y = 8$, and $\frac{dy}{dx} = 3$.

Tangent line: $y - 8 = 3(x - 4)$ or $3x - y - 4 = 0$

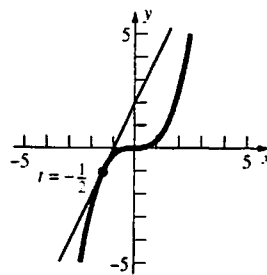


$$32. \frac{dx}{dt} = 3, \frac{dy}{dt} = 24t^2$$

$$\frac{dy}{dx} = 8t^2$$

At $t = -\frac{1}{2}$, $x = -\frac{3}{2}$, $y = -1$, and $\frac{dy}{dx} = 2$.

Tangent line: $y + 1 = 2\left(x + \frac{3}{2}\right)$ or $2x - y + 2 = 0$



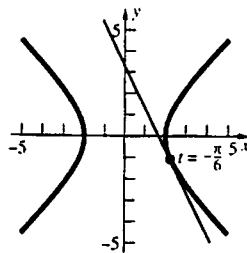
$$33. \frac{dx}{dt} = 2 \sec t \tan t, \frac{dy}{dt} = 2 \sec^2 t$$

$$\frac{dy}{dx} = \csc t$$

At $t = -\frac{\pi}{6}$, $x = \frac{4}{\sqrt{3}}$, $y = -\frac{2}{\sqrt{3}}$, and $\frac{dy}{dx} = -2$.

Tangent line: $y + \frac{2}{\sqrt{3}} = -2\left(x - \frac{4}{\sqrt{3}}\right)$ or

$$2\sqrt{3}x + \sqrt{3}y - 6 = 0$$



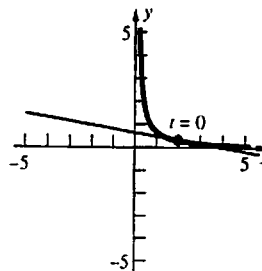
$$34. \frac{dx}{dt} = 2e^t, \frac{dy}{dt} = -\frac{1}{3}e^{-t}$$

$$\frac{dy}{dx} = -\frac{1}{6}e^{-2t}$$

At $t = 0$, $x = 2$, $y = \frac{1}{3}$, and $\frac{dy}{dx} = -\frac{1}{6}$.

Tangent line: $y - \frac{1}{3} = -\frac{1}{6}(x - 2)$ or

$$x + 6y - 4 = 0$$



$$35. \frac{dx}{dt} = 2, \frac{dy}{dt} = 3$$

$$L = \int_0^3 \sqrt{4+9t} dt = \sqrt{13} \int_0^3 dt = \sqrt{13}[t]_0^3 = 3\sqrt{13}$$

$$36. \frac{dx}{dt} = -1, \frac{dy}{dt} = 2$$

$$L = \int_{-3}^3 \sqrt{1+4} dt = \sqrt{5} \int_{-3}^3 dt = \sqrt{5} [t]_{-3}^3 = 6\sqrt{5}$$

$$37. \frac{dx}{dt} = 1, \frac{dy}{dt} = \frac{3}{2} t^{1/2}$$

$$L = \int_0^3 \sqrt{1 + \frac{9}{4} t} dt$$

$$= \frac{1}{2} \int_0^3 \sqrt{4+9t} dt$$

$$= \frac{1}{18} \left[\frac{2}{3} (4+9t)^{3/2} \right]_0^3$$

$$= \frac{1}{27} (31^{3/2} - 8) = \frac{1}{27} (31\sqrt{31} - 8)$$

$$38. \frac{dx}{dt} = 2 \cos t, \frac{dy}{dt} = -2 \sin t$$

$$L = \int_0^\pi \sqrt{4 \cos^2 t + 4 \sin^2 t} dt = 2 \int_0^\pi dt = 2[t]_0^\pi = 2\pi$$

$$39. \frac{dx}{dt} = 6t, \frac{dy}{dt} = 3t^2$$

$$L = \int_0^2 \sqrt{36t^2 + 9t^4} dt = 3 \int_0^2 t \sqrt{4+t^2} dt$$

$$3 \left[\frac{1}{3} (4+t^2)^{3/2} \right]_0^2 = 16\sqrt{2} - 8$$

$$40. \frac{dx}{dt} = 1 - \frac{1}{t^2}, \frac{dy}{dt} = \frac{2}{t}$$

$$L = \int_1^4 \sqrt{\left(1 - \frac{2}{t^2} + \frac{1}{t^4}\right) + \frac{4}{t^2}} dt$$

$$= \int_1^4 \sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}} dt = \int_1^4 \sqrt{\left(1 + \frac{1}{t^2}\right)^2} dt$$

$$= \int_1^4 \left(1 + \frac{1}{t^2}\right) dt = \left[t - \frac{1}{t}\right]_1^4 = \frac{15}{4}$$

$$41. \frac{dx}{dt} = 2e^t, \frac{dy}{dt} = \frac{9}{2} e^{3t/2}$$

$$L = \int_{\ln 3}^{2 \ln 3} \sqrt{4e^{2t} - \frac{81}{4} e^{3t}} dt = \int_{\ln 3}^{2 \ln 3} e^t \sqrt{4 - \frac{81}{4} e^t} dt$$

$$= \left[-\frac{8}{243} \left(4 - \frac{81}{4} e^t\right)^{3/2} \right]_{\ln 3}^{2 \ln 3}$$

$$= \frac{713\sqrt{713} - 227\sqrt{227}}{243}$$

$$42. \frac{dx}{dt} = -\frac{t}{\sqrt{1-t^2}}, \frac{dy}{dt} = -1$$

$$L = \int_0^{1/4} \sqrt{\frac{t^2}{1-t^2} + 1} dt = \int_0^{1/4} \frac{1}{\sqrt{1-t^2}} dt$$

$$= [\sin^{-1} t]_0^{1/4} = \sin^{-1} \frac{1}{4}$$

$$43. \frac{dx}{dt} = \frac{2}{\sqrt{t}}, \frac{dy}{dt} = 2t - \frac{1}{2t^2}$$

$$L = \int_{1/4}^1 \sqrt{\frac{4}{t} + \left(4t^2 - \frac{2}{t} + \frac{1}{4t^4}\right)} dt$$

$$= \int_{1/4}^1 \sqrt{4t^2 + \frac{2}{t} + \frac{1}{4t^4}} dt$$

$$= \int_{1/4}^1 \sqrt{\left(2t + \frac{1}{2t^2}\right)^2} dt$$

$$= \int_{1/4}^1 \left(2t + \frac{1}{2t^2}\right) dt = \left[t^2 - \frac{1}{2t}\right]_{1/4}^1 = \frac{39}{16}$$

$$44. \frac{dx}{dt} = \operatorname{sech}^2 t, \frac{dy}{dt} = 2 \tanh t$$

$$L = \int_{-3}^3 \sqrt{\operatorname{sech}^4 t + 4 \tanh^2 t} dt$$

$$= \int_{-3}^3 \sqrt{4 - 4 \operatorname{sech}^2 t + \operatorname{sech}^4 t} dt$$

$$= \int_{-3}^3 \sqrt{(2 - \operatorname{sech}^2 t)^2} dt = \int_{-3}^3 (2 - \operatorname{sech}^2 t) dt$$

$$= [2t - \tanh t]_{-3}^3 = 12 - 2 \tanh 3$$

$$45. \frac{dx}{dt} = -\sin t,$$

$$\frac{dy}{dt} = \frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} - \cos t = \sec t - \cos t$$

$$L = \int_0^{\pi/4} \sqrt{\sin^2 t + (\sec^2 t - 2 + \cos^2 t)} dt$$

$$= \int_0^{\pi/4} \tan t dt$$

$$= [-\ln |\cos t|]_0^{\pi/4} = -\ln \frac{1}{\sqrt{2}} = \frac{1}{2} \ln 2$$

$$46. \frac{dx}{dt} = t \sin t, \frac{dy}{dt} = t \cos t$$

$$L = \int_{\pi/4}^{\pi/2} \sqrt{t^2 \sin^2 t + t^2 \cos^2 t} dt$$

$$= \int_{\pi/4}^{\pi/2} t dt = \left[\frac{1}{2} t^2\right]_{\pi/4}^{\pi/2} = \frac{3\pi^2}{32}$$

47. a. $\frac{dx}{d\theta} = \cos \theta, \frac{dy}{d\theta} = -\sin \theta$
 $L = \int_0^{2\pi} \sqrt{\cos^2 \theta + \sin^2 \theta} d\theta = \int_0^{2\pi} d\theta$
 $= [\theta]_0^{2\pi} = 2\pi$
- b. $\frac{dx}{d\theta} = 3 \cos 3\theta, \frac{dy}{d\theta} = -3 \sin 3\theta$
 $L = \int_0^{2\pi} \sqrt{9 \cos^2 3\theta + 9 \sin^2 3\theta} d\theta$
 $= 3 \int_0^{2\pi} d\theta = 3[\theta]_0^{2\pi} = 6\pi$
- c. The curve in part a goes around the unit circle once, while the curve in part b goes around the unit circle three times.

48. $\Delta S = 2\pi x \Delta s$
 $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
 $S = \int_a^b 2\pi x ds = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
 See Section 6.4 of the text

49. $\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$
 $S = \int_0^{2\pi} 2\pi(1 + \cos t) \sqrt{\sin^2 t + \cos^2 t} dt$
 $= 2\pi \int_0^{2\pi} (1 + \cos t) dt = 2\pi[t + \sin t]_0^{2\pi} = 4\pi^2$

50. $\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$
 $S = \int_0^{2\pi} 2\pi(3 + \sin t) \sqrt{\sin^2 t + \cos^2 t} dt$
 $= 2\pi \int_0^{2\pi} (3 + \sin t) dt = 2\pi[3t - \cos t]_0^{2\pi} = 12\pi^2$

51. $\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$
 $S = \int_0^{2\pi} 2\pi(1 + \sin t) \sqrt{\sin^2 t + \cos^2 t} dt$
 $= 2\pi \int_0^{2\pi} (1 + \sin t) dt = 2\pi[t - \cos t]_0^{2\pi} = 4\pi^2$

52. $\frac{dx}{dt} = \sqrt{t}, \frac{dy}{dt} = \frac{1}{\sqrt{t}}$
 $S = \int_0^{2\sqrt{3}} 2\pi\left(\frac{2}{3}t^{3/2}\right) \sqrt{t + \frac{1}{t}} dt$

$$= \frac{4\pi}{3} \int_0^{2\sqrt{3}} t \sqrt{t^2 + 1} dt$$

$$= \frac{4\pi}{3} \left[\frac{1}{3} (t^2 + 1)^{3/2} \right]_0^{2\sqrt{3}} = \frac{4\pi}{9} (13\sqrt{13} - 1)$$

53. $\frac{dx}{dt} = 1, \frac{dy}{dt} = t + \sqrt{7}$
 $S = \int_{-\sqrt{7}}^{\sqrt{7}} 2\pi(t + \sqrt{7}) \sqrt{1 + (t + \sqrt{7})^2} dt$
 $= 2\pi \left[\frac{1}{3} (1 + (t + \sqrt{7})^2)^{3/2} \right]_{-\sqrt{7}}^{\sqrt{7}}$
 $= \frac{2\pi}{3} (29\sqrt{29} - 1)$

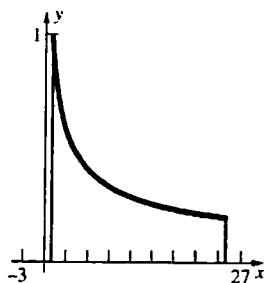
54. $\frac{dx}{dt} = t + a, \frac{dy}{dt} = 1$
 $S = \int_{-\sqrt{a}}^{\sqrt{a}} 2\pi(t + a) \sqrt{(t + a)^2 + 1} dt$
 $= 2\pi \left[\frac{1}{3} ((t + a)^2 + 1)^{3/2} \right]_{-\sqrt{a}}^{\sqrt{a}}$
 $= \frac{2\pi}{3} \left[(a^2 + 2a\sqrt{a} + a + 1)^{3/2} - (a^2 - 2a\sqrt{a} + a + 1)^{3/2} \right]$

55. $dx = dt$; when $x = 0, t = -1$; when $x = 1, t = 0$.
 $\int_0^1 (x^2 - 4y) dx = \int_{-1}^0 [(t + 1)^2 - 4(t^3 + 4)] dt$
 $= \int_{-1}^0 (-4t^3 + t^2 + 2t - 15) dt$
 $= \left[-t^4 + \frac{1}{3}t^3 + t^2 - 15t \right]_{-1}^0 = -\frac{44}{3}$

56. $dy = \sec^2 t dt$; when $y = 1, t = \frac{\pi}{4}$;
 when $y = \sqrt{3}, t = \frac{\pi}{3}$.
 $\int_1^{\sqrt{3}} xy dy = \int_{\pi/4}^{\pi/3} (\sec t)(\tan t) \sec^2 t dt$
 $= \left[\frac{1}{3} \sec^3 t \right]_{\pi/4}^{\pi/3} = \frac{8}{3} - \frac{2\sqrt{2}}{3}$

57. $dx = 2e^{2t} dt$

$$A = \int_1^{25} y dx = \int_0^{\ln 5} 2e^t dt = [2e^t]_0^{\ln 5} = 8$$



58. a. $t = \frac{x}{v_0 \cos \alpha}$

$$y = -16 \left(\frac{x}{v_0 \cos \alpha} \right)^2 + (v_0 \sin \alpha) \left(\frac{x}{v_0 \cos \alpha} \right)$$

$$y = - \left(\frac{16}{v_0^2 \cos^2 \alpha} \right) x^2 + (\tan \alpha) x$$

This is an equation for a parabola.

b. Solve for t when $y = 0$.

$$-16t^2 + (v_0 \sin \alpha)t = 0$$

$$t(-16t + v_0 \sin \alpha) = 0$$

$$t = 0, \frac{v_0 \sin \alpha}{16}$$

The time of flight is $\frac{v_0 \sin \alpha}{16}$ seconds.

c. At $t = \frac{v_0 \sin \alpha}{16}$, $x = (v_0 \cos \alpha) \left(\frac{v_0 \sin \alpha}{16} \right)$

$$= \frac{v_0^2 \sin \alpha \cos \alpha}{16} = \frac{v_0^2 \sin 2\alpha}{32}$$

d. Let R be the range as a function of α .

$$R = \frac{v_0^2 \sin 2\alpha}{32}$$

$$\frac{dR}{d\alpha} = \frac{v_0^2 \cos 2\alpha}{16}$$

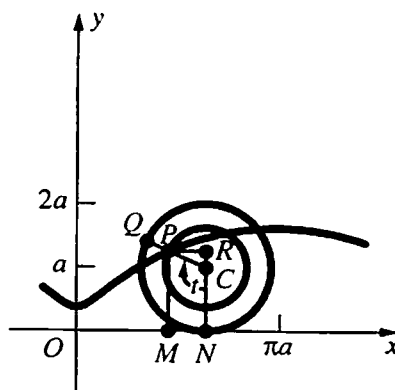
$$\frac{v_0^2 \cos 2\alpha}{16} = 0, \cos 2\alpha = 0, \alpha = \frac{\pi}{4}$$

$$\frac{d^2 R}{d\alpha^2} = -\frac{v_0^2 \sin 2\alpha}{8}; \frac{d^2 R}{d\alpha^2} < 0 \text{ at } \alpha = \frac{\pi}{4}$$

The range is the largest possible when

$$\alpha = \frac{\pi}{4}$$

59.

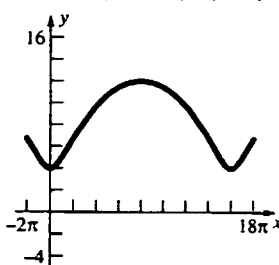


Let the wheel roll along the x -axis with P initially at $(0, a - b)$.

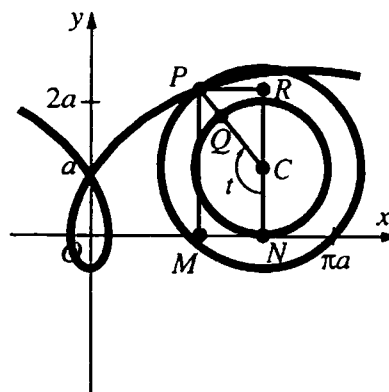
$$|ON| = \text{arc } NQ = at$$

$$x = |OM| = |ON| - |MN| = at - b \sin t$$

$$y = |MP| = |RN| = |NC| + |CR| = a - b \cos t$$



60.

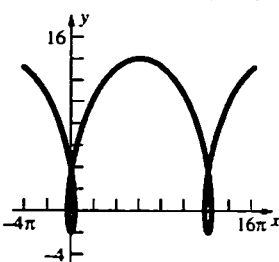


Let the wheel roll along the x -axis with P initially at $(0, a - b)$.

$$|ON| = \text{arc } NQ = at$$

$$x = |OM| = |ON| - |MN| = at - b \sin t$$

$$y = |MP| = |RN| = |NC| + |CR| = a - b \cos t$$



61. The x - and y -coordinates of the center of the circle of radius b are $(a-b)\cos t$ and $(a-b)\sin t$, respectively. The angle measure (in a clockwise

direction) of arc BP is $\frac{a}{b}t$. The horizontal

change from the center of the circle of radius b to

P is $b\cos\left(-\left(\frac{a}{b}t - t\right)\right) = b\cos\left(\frac{a-b}{b}t\right)$ and the

vertical change is

$b\sin\left(-\left(\frac{a}{b}t - t\right)\right) = -b\sin\left(\frac{a-b}{b}t\right)$. Therefore,

$$x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right) \text{ and}$$

$$y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right).$$

62. From Problem 61,

$$x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right) \text{ and}$$

$$y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right).$$

Substitute $b = \frac{a}{4}$.

$$x = \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)\cos(3t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)\cos(2t+t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)(\cos 2t \cos t - \sin 2t \sin t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)(\cos^3 t - \sin^2 t \cos t - 2\sin^2 t \cos t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)\cos^3 t - \left(\frac{3a}{4}\right)\cos t \sin^2 t$$

$$= \left(\frac{3a}{4}\right)(\cos t)(1 - \sin^2 t) + \left(\frac{a}{4}\right)\cos^3 t = a\cos^3 t$$

$$y = \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)\sin(3t)$$

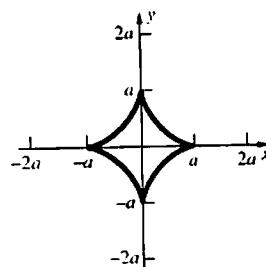
$$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)\sin(2t+t)$$

$$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)(\sin 2t \cos t + \cos 2t \sin t)$$

$$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)(2\sin t \cos^2 t + \cos^2 t \sin t - \sin^3 t)$$

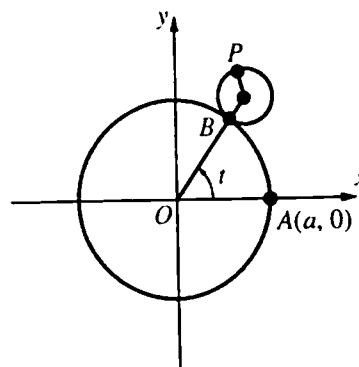
$$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{3a}{4}\right)\sin t \cos^2 t + \left(\frac{a}{4}\right)\sin^3 t$$

$$= \left(\frac{3a}{4}\right)(\sin t)(1 - \cos^2 t) + \left(\frac{a}{4}\right)\sin^3 t = a\sin^3 t$$



$$x^{2/3} + y^{2/3} = a^{2/3} \cos^2 t + a^{2/3} \sin^2 t = a^{2/3}$$

63. Consider the following figure similar to the one in the text for Problem 61.



The x - and y -coordinates of the center of the circle of radius b are $(a+b)\cos t$ and $(a+b)\sin t$, respectively. The angle measure (in a counter-

clockwise direction) of arc PB is $\frac{a}{b}t$. The

horizontal change from the center of the circle of radius b to P is

$b\cos\left(\frac{a}{b}t + t + \pi\right) = -b\cos\left(\frac{a+b}{b}t\right)$ and the

vertical change is

$b\sin\left(\frac{a}{b}t + t + \pi\right) = -b\sin\left(\frac{a+b}{b}t\right)$. Therefore,

$$x = (a+b)\cos t - b\cos\left(\frac{a+b}{b}t\right) \text{ and}$$

$$y = (a+b)\sin t - b\sin\left(\frac{a+b}{b}t\right).$$

$$64. \quad x = 2a\cos t - a\cos 2t = 2a\cos t - 2a\cos^2 t + a$$

$$= 2a\cos t(1 - \cos t) + a$$

$$y = 2a\sin t - a\sin 2t = 2a\sin t - 2a\sin t \cos t$$

$$= 2a\sin t(1 - \cos t)$$

$$x - a = 2a\cos t(1 - \cos t)$$

$$(x - a)^2 + y^2 = 4a^2(1 - \cos t)^2$$

$$(x - a)^2 + y^2 + 2a(x - a) =$$

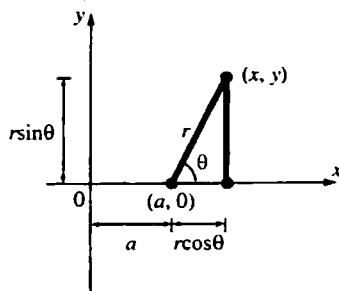
$$4a^2(1 - \cos t)^2 + 4a^2(1 - \cos t)\cos t$$

$$(x-a)^2 + y^2 + 2a(x-a) = 4a^2(1-\cos t)$$

$$[(x-a)^2 + y^2 + 2a(x-a)]^2 = 4a^2 \cdot 4a^2(1-\cos t)^2$$

$$[(x-a)^2 + y^2 + 2a(x-a)]^2 = 4a^2[(x-a)^2 + y^2]$$

If $(a, 0)$ is the pole with the polar axis in the direction of the positive x -axis,
 $x = r \cos \theta + a$ and $y = r \sin \theta$. (See figure below.)



Substitute $x - a = r \cos \theta$ and $y = r \sin \theta$ into the Cartesian equation.

$$[(r \cos \theta)^2 + (r \sin \theta)^2 + 2a(r \cos \theta)]^2$$

$$= 4a^2[(r \cos \theta)^2 + (r \sin \theta)^2]$$

$$(r^2 + 2ar \cos \theta)^2 = 4a^2 r^2$$

$$r^2(r + 2a \cos \theta)^2 = 4a^2 r^2$$

$$r + 2a \cos \theta = 2a$$

$$r = 2a(1 - \cos \theta)$$

65. $\frac{dx}{dt} = \left(\frac{a}{3}\right)(-2 \sin t - 2 \sin 2t),$

$$\frac{dy}{dt} = \left(\frac{a}{3}\right)(2 \cos t - 2 \cos 2t)$$

$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{a}{3}\right)^2 (4 \sin^2 t + 8 \sin t \sin 2t + 4 \sin^2 2t)$$

$$\left(\frac{dy}{dt}\right)^2 = \left(\frac{a}{3}\right)^2 (4 \cos^2 t - 8 \cos t \cos 2t + 4 \cos^2 2t)$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{a}{3}\right)^2 (8 + 8 \sin t \sin 2t - 8 \cos t \cos 2t)$$

$$= \left(\frac{a}{3}\right)^2 (8 + 16 \sin^2 t \cos t - 8 \cos^3 t + 8 \sin^2 t \cos t)$$

$$= \left(\frac{a}{3}\right)^2 (8 + 24 \cos t \sin^2 t - 8 \cos^3 t)$$

$$= \left(\frac{a}{3}\right)^2 (8 + 24 \cos t - 32 \cos^3 t)$$

$$L = 3 \int_0^{2\pi/3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= a \int_0^{2\pi/3} \sqrt{8 + 24 \cos t - 32 \cos^3 t} dt$$

Using a CAS to evaluate the length, $L = \frac{16a}{3}$.

66. a. Let $x = a \cos t$ and $y = b \sin t$.

$$\frac{dx}{dt} = -a \sin t, \frac{dy}{dt} = b \cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2 \sin^2 t + b^2 \cos^2 t$$

$$= a^2 + (b^2 - a^2) \cos^2 t$$

$$= a^2 - c^2 \cos^2 t = a^2 \left(1 - \frac{c^2}{a^2} \cos^2 t\right)$$

$$= a^2 \left(1 - \left(\frac{c}{a}\right)^2 \cos^2 t\right) = a^2 (1 - e^2 \cos^2 t)$$

$$P = 4 \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= 4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2 t} dt$$

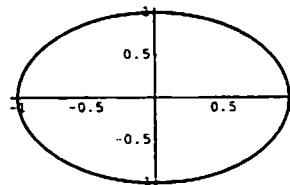
b. $P = 4 \int_0^{\pi/2} \sqrt{1 - \frac{\cos^2 t}{16}} dt$

$$= \int_0^{\pi/2} \sqrt{16 - \cos^2 t} dt \approx 6.1838$$

(The answer is near 2π because it is slightly smaller than a circle of radius 1 whose perimeter is 2π).

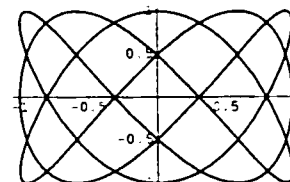
c. $P = \int_0^{\pi/2} \sqrt{16 - \cos^2 t} dt \approx 6.1838$

67. a.

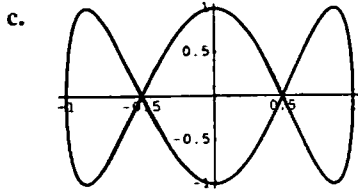


The curve touches a horizontal border once and touches a vertical border once.

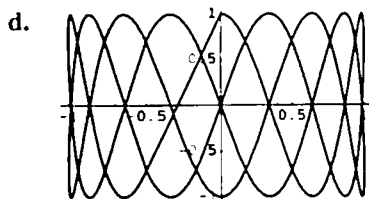
- b.



The curve touches a horizontal border five times and touches a vertical border three times.



The curve touches a horizontal border three times and touches a vertical border once. Note that the curve is traced out three times.



The curve touches a horizontal border nine times and touches a vertical border twice.

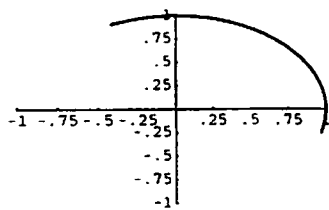
68. a. Figure 15

b. Figure 16

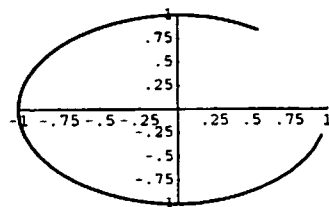
c. Figure 17

d. Figure 14

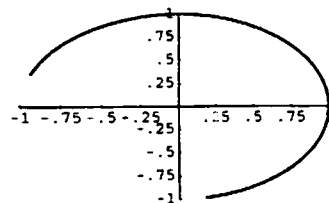
69. a. $0 \leq t \leq 2$



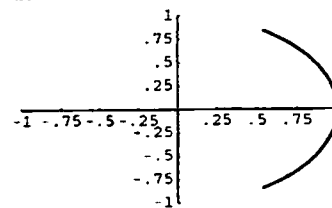
b. $0 \leq t \leq 1$



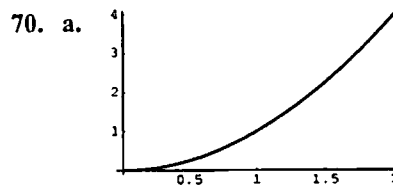
c. $0.25 \leq t \leq 2$



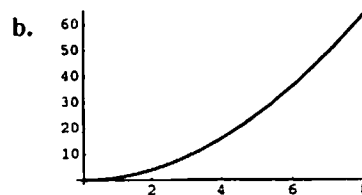
d. $0 \leq t \leq 2\pi$



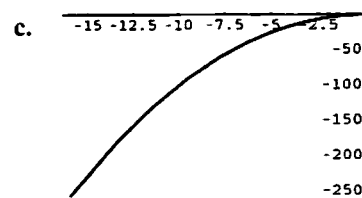
Given a parameterization of the form $x = \cos f(t)$ and $y = \sin f(t)$, the point moves around the curve (which is a circle of radius 1) at a speed of $|f'(t)|$. The point travels clockwise around the circle when $f(t)$ is decreasing and counterclockwise when $f(t)$ is increasing. Note that in part d, only part of the circle will be traced out since the range of $f(t) = \sin t$ is $[-1, 1]$.



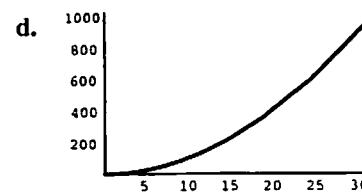
The curve traced out is the graph of $y = x^2$ for $0 \leq x \leq 2$.



The curve traced out is the graph of $y = x^2$ for $0 \leq x \leq 8$.



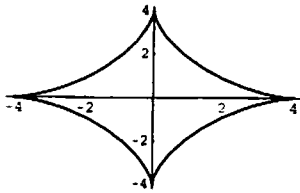
The curve traced out is the graph of $y = -x^2$ for $-16 \leq x \leq 0$.



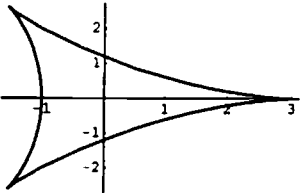
The curve traced out is the graph of $y = x^2$ for $0 \leq x \leq 32$.

All of the curves lie on the graph of $y = \pm x^2$, but trace out different parts because of the parameterization

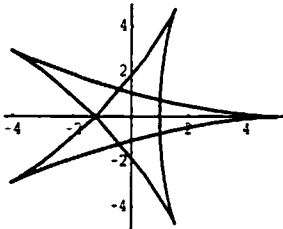
71. a. $0 \leq t \leq 2\pi$



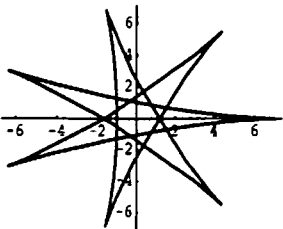
b. $0 \leq t \leq 2\pi$



c. $0 \leq t \leq 4\pi$



d. $0 \leq t \leq 8\pi$



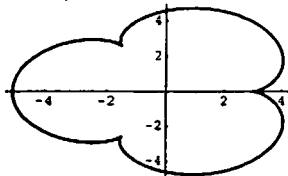
Let $\frac{p}{q} = \frac{a}{b}$ where $\frac{p}{q}$ is the reduced fraction

of $\frac{a}{b}$. The length of the t -interval is $2q\pi$.

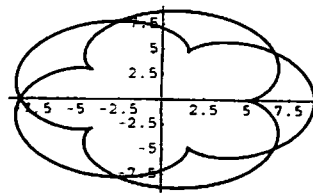
The number of times the graph would touch the circle of radius a during the t -interval is p .

72. Some possible graphs for different a and b are shown below.

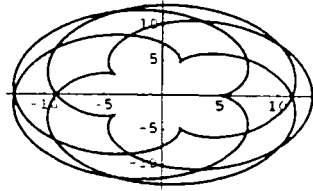
$a = 3, b = 1$



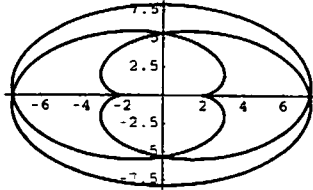
$a = 5, b = 2$



$a = 5, b = 4$



$a = 2, b = 3$

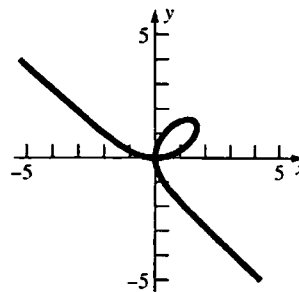


Let $\frac{p}{q} = \frac{a}{b}$ where $\frac{p}{q}$ is the reduced fraction of

$\frac{a}{b}$. The length of the t -interval is $2q\pi$. The

number of times the graph would touch the circle of radius a during the t -interval is p .

73. $x = \frac{3t}{t^3 + 1}, y = \frac{3t^2}{t^3 + 1}$



When $x > 0, t > 0$ or $t < -1$.

When $x < 0, -1 < t < 0$.

When $y > 0, t > -1$. When $y < 0, t < -1$.

Therefore the graph is in quadrant I for $t > 0$,

quadrant II for $-1 < t < 0$,

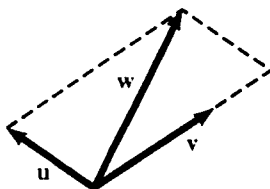
quadrant III for no t , and quadrant IV for $t < -1$.

13.2 Concepts Review

1. magnitude; direction
2. they have the same magnitude and direction.
3. the tail of u ; the head of v
4. force; velocity

Problem Set 13.2

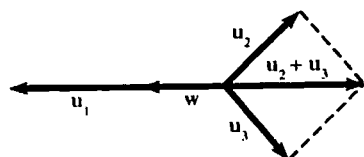
1.



2.



3.



4. 0

$$5. \quad w = \frac{1}{2}(u + v) = \frac{1}{2}u + \frac{1}{2}v$$

$$6. \quad n = \frac{1}{2}(v - u) = \frac{1}{2}v - \frac{1}{2}u$$

$$m = v - n = v - \frac{1}{2}(v - u) = \frac{1}{2}v + \frac{1}{2}u$$

$$7. \quad |w| = |u|\cos 60^\circ + |v|\cos 60^\circ = \frac{1}{2} + \frac{1}{2} = 1$$

$$8. \quad |w| = |u|\cos 45^\circ + |v|\cos 45^\circ = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

9. Let θ be the angle of w measured clockwise from south.

$$|w|\cos \theta = |u|\cos 30^\circ + |v|\cos 45^\circ = 25\sqrt{3} + 25\sqrt{2}$$

$$= 25(\sqrt{3} + \sqrt{2})$$

$$|w|\sin \theta = |v|\sin 45^\circ - |u|\sin 30^\circ = 25\sqrt{2} - 25$$

$$= 25(\sqrt{2} - 1)$$

$$|w|^2 = |w|^2 \cos^2 \theta + |w|^2 \sin^2 \theta$$

$$= 625(\sqrt{3} + \sqrt{2})^2 + 625(\sqrt{2} - 1)^2$$

$$= 625(8 - 2\sqrt{2} + 2\sqrt{6})$$

$$|w| = \sqrt{625(8 - 2\sqrt{2} + 2\sqrt{6})} = 25\sqrt{8 - 2\sqrt{2} + 2\sqrt{6}}$$

$$\approx 79.34$$

$$\tan \theta = \frac{|w|\sin \theta}{|w|\cos \theta} = \frac{\sqrt{2} - 1}{\sqrt{3} + \sqrt{2}}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{2} - 1}{\sqrt{3} + \sqrt{2}}\right) = 7.5^\circ$$

w has magnitude 79.34 lb in the direction S 7.5° W.

10. Let v be the resulting force. Let θ be the angle of v measured clockwise from south.

$$|v|\cos \theta = 60\cos 30^\circ + 80\cos 60^\circ = 30\sqrt{3} + 40$$

$$= 10(3\sqrt{3} + 4)$$

$$|v|\sin \theta = 80\sin 60^\circ - 60\sin 30^\circ = 40\sqrt{3} - 30$$

$$= 10(4\sqrt{3} - 3)$$

$$|v|^2 = |v|^2 \cos^2 \theta + |v|^2 \sin^2 \theta$$

$$= 100(3\sqrt{3} + 4)^2 + 100(4\sqrt{3} - 3)^2$$

$$= 100(100) = 10,000$$

$$|v| = \sqrt{10,000} = 100$$

$$\tan \theta = \frac{|v|\sin \theta}{|v|\cos \theta} = \frac{4\sqrt{3} - 3}{3\sqrt{3} + 4}$$

$$\theta = \tan^{-1}\left(\frac{4\sqrt{3} - 3}{3\sqrt{3} + 4}\right) \approx 23.13^\circ$$

The resultant force has magnitude 100 lb in the direction S 23.13° W.

11. The force of 300 N parallel to the plane has magnitude $300 \sin 30^\circ = 150$ N. Thus, a force of 150 N parallel to the plane will just keep the weight from sliding.

12. Let a be the magnitude of the rope that makes an angle of 27.34° . Let b be the magnitude of the rope that makes an angle of 39.22° .

$$1. \quad a \sin 27.34^\circ = b \sin 39.22^\circ$$

$$2. \quad a \cos 27.34^\circ + b \cos 39.22^\circ = 258.5$$

Solve 1 for b and substitute in 2.

$$a \cos 27.34^\circ + a \frac{\sin 27.34^\circ}{\sin 39.22^\circ} \cos 39.22^\circ = 258.5$$

$$a = \frac{258.5}{\cos 27.34^\circ + \sin 27.34^\circ \cot 39.22^\circ} \approx 178.15$$

$$b = \frac{a \sin 27.34^\circ}{\sin 39.22^\circ} \approx 129.40$$

The magnitudes of the forces exerted by the ropes making angles of 27.34° and 39.22° are 178.15 lb and 129.40 lb, respectively.

13. Let θ be the angle the plane makes from north, measured clockwise.

$$425 \sin \theta = 45 \sin 20^\circ$$

$$\sin \theta = \frac{9}{85} \sin 20^\circ$$

$$\theta = \sin^{-1} \left(\frac{9}{85} \sin 20^\circ \right) \approx 2.08^\circ$$

Let x be the speed of airplane with respect to the ground.

$$x = 45 \cos 20^\circ + 425 \cos \theta \approx 467$$

The plane flies in the direction N 2.08° E, flying 467 mi/h with respect to the ground.

14. Let v be his velocity relative to the surface. Let θ be the angle that his velocity relative to the surface makes with south, measured clockwise.

$$|v| \cos \theta = 20, |v| \sin \theta = 3$$

$$|v|^2 = |v|^2 \cos^2 \theta + |v|^2 \sin^2 \theta = 400 + 9 = 409$$

$$|v| = \sqrt{409} \approx 20.22$$

$$\tan \theta = \frac{|v| \sin \theta}{|v| \cos \theta} = \frac{3}{20}$$

$$\theta = \tan^{-1} \frac{3}{20} \approx 8.53^\circ$$

His velocity has magnitude 20.22 mi/h in the direction S 8.53° W.

15. Let x be the air speed.

$$x \cos 60^\circ = 40$$

$$x = \frac{40}{\cos 60^\circ} = 80$$

The air speed of the plane is 80 mi/hr

16. Let x be the air speed. Let θ be the angle that the plane makes with north measured counter-clockwise.

$$x \cos \theta = 837 + 63 \cos 11.5^\circ$$

$$x \sin \theta = 63 \sin 11.5^\circ$$

$$x^2 = x^2 \cos^2 \theta + x^2 \sin^2 \theta$$

$$= (837 + 63 \cos 11.5^\circ)^2 + (63 \sin 11.5^\circ)^2$$

$$= 704,538 + 105,462 \cos 11.5^\circ$$

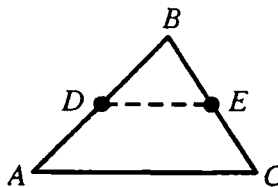
$$x = \sqrt{704,538 + 105,462 \cos 11.5^\circ} \approx 898.82$$

$$\tan \theta = \frac{x \sin \theta}{x \cos \theta} = \frac{63 \sin 11.5^\circ}{837 + 63 \cos 11.5^\circ}$$

$$\theta = \tan^{-1} \left(\frac{63 \sin 11.5^\circ}{837 + 63 \cos 11.5^\circ} \right) \approx 0.80^\circ$$

The plane should fly in the direction N 0.80° W at an air speed of 898.82 mi/h.

17. Given triangle ABC , let D be the midpoint of AB and E be the midpoint of BC . $u = \overrightarrow{AB}$, $v = \overrightarrow{BC}$, $w = \overrightarrow{AC}$, $z = \overrightarrow{DE}$



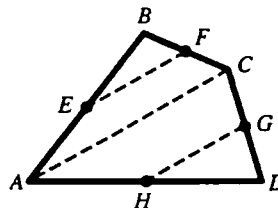
$$u + v = w$$

$$z = \frac{1}{2}u + \frac{1}{2}v = \frac{1}{2}(u + v) = \frac{1}{2}w$$

Thus, DE is parallel to AC .

18. Given quadrilateral $ABCD$, let E be the midpoint of AB , F the midpoint of BC , G the midpoint of CD , and H the midpoint of AD .

ABC and ACD are triangles. From Problem 17, EF and HG are parallel to AC . Thus, EF is parallel to HG . By similar reasoning using triangles ABD and BCD , EH is parallel to FG . Therefore, $EFGH$ is parallelogram.

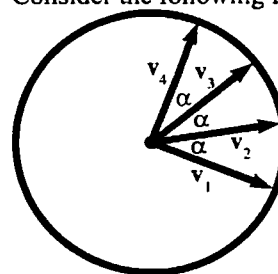


19. Let P_i be the tail of v_i . Then

$$v_1 + v_2 + \dots + v_n = \overrightarrow{P_1 P_2} + \overrightarrow{P_2 P_3} + \dots + \overrightarrow{P_n P_1}$$

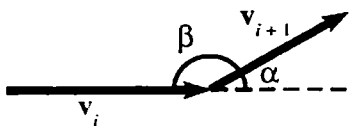
$$= \overrightarrow{P_1 P_1} = \mathbf{0}.$$

20. Consider the following figure of the circle.



$$\alpha = \frac{2\pi}{n}$$

The vectors have the same length. Consider the following figure for adding vectors $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$.



Then $\beta = \pi - \frac{2\pi}{n}$. Note that the interior angle of

a regular n -gon is $\pi - \frac{2\pi}{n}$. Thus the vectors

(placed head to tail from $\mathbf{v}_1$ to $\mathbf{v}_n$) form a regular n -gon. From Problem 19, the sum of the vectors is 0.

21. The components of the forces along the lines containing AP , BP , and CP are in equilibrium; that is,
- $$W = W \cos \alpha + W \cos \beta$$
- $$W = W \cos \beta + W \cos \gamma$$
- $$W = W \cos \alpha + W \cos \gamma$$
- Thus, $\cos \alpha + \cos \beta = 1$, $\cos \beta + \cos \gamma = 1$, and $\cos \alpha + \cos \gamma = 1$. Solving this system of equations results in $\cos \alpha = \cos \beta = \cos \gamma = \frac{1}{2}$.
- Hence $\alpha = \beta = \gamma = 60^\circ$.
Therefore, $\alpha + \beta = \alpha + \gamma = \beta + \gamma = 120^\circ$.
22. Let A' , B' , C' be the points where the weights are attached. The center of gravity is located $\frac{|AA'| + |BB'| + |CC'|}{3}$ units below the plane of the triangle. Then, using the hint, the system is in equilibrium when $|AA'| + |BB'| + |CC'|$ is maximum. Hence, it is in equilibrium when

$|AP| + |BP| + |CP|$ is minimum, because the total length of the string is

$$|AP| + |AA'| + |BP| + |BB'| + |CP| + |CC'|.$$

23. The components of the forces along the lines containing AP , BP , and CP are in equilibrium; that is,
- $$5w \cos \alpha + 4w \cos \beta = 3w$$
- $$3w \cos \beta + 5w \cos \gamma = 4w$$
- $$3w \cos \alpha + 4w \cos \gamma = 5w$$
- Thus, $5 \cos \alpha + 4 \cos \beta = 3$, $3 \cos \beta + 5 \cos \gamma = 4$, and $3 \cos \alpha + 4 \cos \gamma = 5$. Solving this system of equations results in
- $$\cos \alpha = \frac{3}{5}, \cos \beta = 0, \cos \gamma = \frac{4}{5},$$
- from which it follows that $\sin \alpha = \frac{4}{5}$, $\sin \beta = 1$, $\sin \gamma = \frac{3}{5}$.
- Therefore, $\cos(\alpha + \beta) = -\frac{4}{5}$, $\cos(\alpha + \gamma) = 0$,
- $$\cos(\beta + \gamma) = -\frac{3}{5},$$
- so
- $$\alpha + \beta = \cos^{-1}\left(-\frac{4}{5}\right) \approx 143.13^\circ, \alpha + \gamma = 90^\circ,$$
- $$\beta + \gamma = \cos^{-1}\left(-\frac{3}{5}\right) \approx 126.87^\circ.$$
- This problem can be modeled with three strings going through A , four strings through B , and five strings through C , with equal weights attached to the twelve strings. Then the quantity to be minimized is $3|AP| + 4|BP| + 5|CP|$.

24. Written response.

13.3 Concepts Review

- $\langle u_1 + v_1, u_2 + v_2 \rangle; \langle cu_1, cu_2 \rangle; \sqrt{u_1^2 + u_2^2}$
- $u_1v_1 + u_2v_2; |u||v|\cos\theta$
- basis vectors
- $\mathbf{F} \cdot \mathbf{D}$

Problem Set 13.3

- $2\mathbf{a} - 4\mathbf{b} = (-4\mathbf{i} + 6\mathbf{j}) + (-8\mathbf{i} + 12\mathbf{j}) = -12\mathbf{i} + 18\mathbf{j}$
 - $\mathbf{a} \cdot \mathbf{b} = (-2)(2) + (3)(-3) = -13$
 - $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (-2\mathbf{i} + 3\mathbf{j}) \cdot (2\mathbf{i} - 8\mathbf{j}) = (-2)(2) + (3)(-8) = -28$
 - $(-2\mathbf{a} + 3\mathbf{b}) \cdot 5\mathbf{c} = 5[(10\mathbf{i} - 15\mathbf{j}) \cdot (-5\mathbf{j})] = 5[(10)(0) + (-15)(-5)] = 375$
 - $|\mathbf{a}| \mathbf{c} \cdot \mathbf{a} = \sqrt{4+9}[(0)(-2) + (-5)(3)] = -15\sqrt{13}$

$$\begin{aligned} \text{f. } \mathbf{b} \cdot \mathbf{b} - |\mathbf{b}| &= (2)(2) + (-3)(-3) - \sqrt{4+9} \\ &= 13 - \sqrt{13} \end{aligned}$$

$$2. \text{ a. } -4\mathbf{a} + 3\mathbf{b} = \langle -12, 4 \rangle + \langle 3, -3 \rangle = \langle -9, 1 \rangle$$

$$\text{b. } \mathbf{b} \cdot \mathbf{c} = (1)(0) + (-1)(5) = -5$$

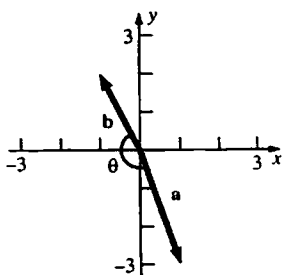
$$\begin{aligned} \text{c. } (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} &= \langle 4, -2 \rangle \cdot \langle 0, 5 \rangle \\ &= (4)(0) + (-2)(5) = -10 \end{aligned}$$

$$\begin{aligned} \text{d. } 2\mathbf{c} \cdot (3\mathbf{a} + 4\mathbf{b}) &= 2\langle 0, 5 \rangle \cdot (\langle 9, -3 \rangle + \langle 4, -4 \rangle) \\ &= 2\langle 0, 5 \rangle \cdot \langle 13, -7 \rangle = 2[(0)(13) + (5)(-7)] \\ &= -70 \end{aligned}$$

$$\text{e. } |\mathbf{b}| \mathbf{b} \cdot \mathbf{a} = \sqrt{1+1}[(1)(3) + (-1)(-1)] = 4\sqrt{2}$$

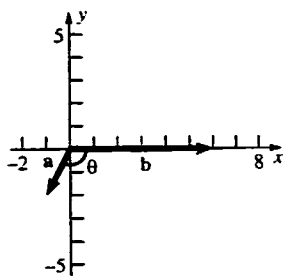
$$\begin{aligned} \text{f. } |\mathbf{c}|^2 - \mathbf{c} \cdot \mathbf{c} &= (\sqrt{0+25})^2 - [(0)(0) + (5)(5)] \\ &= 0 \end{aligned}$$

$$\begin{aligned} 3. \text{ a. } \cos \theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(1)(-1) + (-3)(2)}{(\sqrt{10})(\sqrt{5})} = -\frac{7}{\sqrt{50}} \\ &= -\frac{7}{5\sqrt{2}} \approx -0.9899 \end{aligned}$$

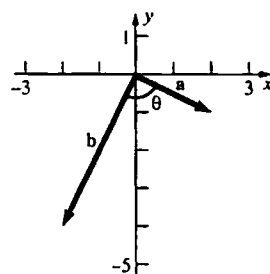


$$\text{b. } \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(-1)(6) + (-2)(0)}{(\sqrt{5})(6)} = -\frac{6}{6\sqrt{5}}$$

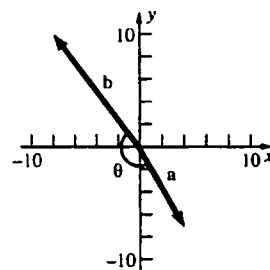
$$= -\frac{1}{\sqrt{5}} \approx -0.4472$$



$$\text{c. } \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(2)(-2) + (-1)(-4)}{(\sqrt{5})(2\sqrt{5})} = \frac{0}{10} = 0$$

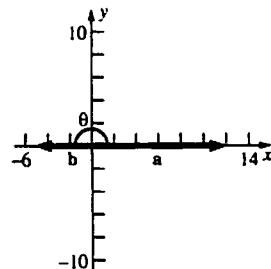


$$\begin{aligned} \text{d. } \cos \theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(4)(-8) + (-7)(10)}{(\sqrt{65})(2\sqrt{41})} \\ &= \frac{-102}{2\sqrt{2665}} = -\frac{51}{\sqrt{2665}} \approx -0.9879 \end{aligned}$$



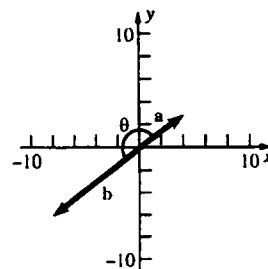
$$4. \text{ a. } \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(12)(-5) + (0)(0)}{(12)(5)} = \frac{-60}{60} = -1$$

$$\theta = \cos^{-1}(-1) = 180^\circ$$



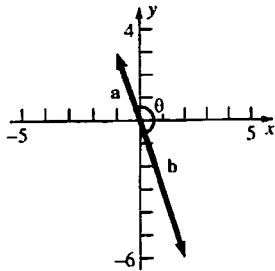
$$\text{b. } \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(4)(-8) + (3)(-6)}{(5)(10)} = \frac{-50}{50} = -1$$

$$\theta = \cos^{-1}(-1) = 180^\circ$$



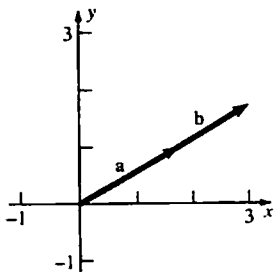
$$c. \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(-1)(2) + (3)(-6)}{(\sqrt{10})(2\sqrt{10})} = \frac{-20}{20} = -1$$

$$\theta = \cos^{-1}(-1) = 180^\circ$$



$$d. \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(\sqrt{3})(3) + (1)(\sqrt{3})}{(2)(2\sqrt{3})} = \frac{4\sqrt{3}}{4\sqrt{3}} = 1$$

$$\theta = \cos^{-1} 1 = 0^\circ$$



$$5. a. \mathbf{a} = (-3 - 2)\mathbf{i} + (4 - 2)\mathbf{j} = -5\mathbf{i} + 2\mathbf{j}$$

$$b. \mathbf{a} = (-6 - 0)\mathbf{i} + (0 - 4)\mathbf{j} = -6\mathbf{i} - 4\mathbf{j}$$

$$c. \mathbf{a} = (0 - \sqrt{2})\mathbf{i} + (0 + \pi)\mathbf{j} = -\sqrt{2}\mathbf{i} + \pi\mathbf{j}$$

$$d. \mathbf{a} = (-4 + 7)\mathbf{i} + \left(\frac{1}{3} - e\right)\mathbf{j} = 3\mathbf{i} + \left(\frac{1}{3} - e\right)\mathbf{j}$$

$$6. a. \mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{-3\mathbf{i} - 4\mathbf{j}}{5} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$b. \mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{i} - 7\mathbf{j}}{5\sqrt{2}} = \frac{1}{5\sqrt{2}}\mathbf{i} - \frac{7}{5\sqrt{2}}\mathbf{j}$$

$$c. \mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{-4\mathbf{j}}{4} = -\mathbf{j}$$

$$d. \mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{-5\mathbf{i} - 12\mathbf{j}}{13} = -\frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}$$

$$7. (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} \\ = |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$$

$$\text{Thus, } |\mathbf{u}|^2 = |\mathbf{v}|^2 \text{ or } |\mathbf{u}| = |\mathbf{v}|.$$

$$8. (\mathbf{u} + \mathbf{v}) \cdot (3\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot (3\mathbf{u}) - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot (3\mathbf{u}) - \mathbf{v} \cdot \mathbf{v} \\ = 3(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \mathbf{v} + 3(\mathbf{u} \cdot \mathbf{v}) - \mathbf{v} \cdot \mathbf{v} \\ = 3|\mathbf{u}|^2 - |\mathbf{v}|^2 + 2\mathbf{u} \cdot \mathbf{v}$$

$$9. \mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{u} = a\langle \mathbf{u}_1, \mathbf{u}_2 \rangle + b\langle \mathbf{u}_1, \mathbf{u}_2 \rangle \\ = \langle a\mathbf{u}_1, a\mathbf{u}_2 \rangle + \langle b\mathbf{u}_1, b\mathbf{u}_2 \rangle = \langle a\mathbf{u}_1 + b\mathbf{u}_1, a\mathbf{u}_2 + b\mathbf{u}_2 \rangle \\ \langle (a+b)\mathbf{u}_1, (a+b)\mathbf{u}_2 \rangle = (a+b)\langle \mathbf{u}_1, \mathbf{u}_2 \rangle \\ = (a+b)\mathbf{u}$$

$$10. \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle \cdot (\langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \langle \mathbf{w}_1, \mathbf{w}_2 \rangle) \\ = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle \cdot \langle \mathbf{v}_1 + \mathbf{w}_1, \mathbf{v}_2 + \mathbf{w}_2 \rangle \\ = u_1(v_1 + w_1) + u_2(v_2 + w_2) \\ = (u_1v_1 + u_2v_2) + (u_1w_1 + u_2w_2) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

$$11. c(\mathbf{u} \cdot \mathbf{v}) = c(\langle \mathbf{u}_1, \mathbf{u}_2 \rangle \cdot \langle \mathbf{v}_1, \mathbf{v}_2 \rangle) \\ = c(u_1v_1 + u_2v_2) = c(u_1v_1) + c(u_2v_2) \\ = (cu_1)v_1 + (cu_2)v_2 = \langle cu_1, cu_2 \rangle \cdot \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \\ = (c\langle \mathbf{u}_1, \mathbf{u}_2 \rangle) \cdot \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = (c\mathbf{u}) \cdot \mathbf{v}$$

$$12. \mathbf{u} + \mathbf{v} = \mathbf{u} \Rightarrow (-\mathbf{u}) + (\mathbf{u} + \mathbf{v}) = (-\mathbf{u}) + \mathbf{u} \\ \Rightarrow [(-\mathbf{u}) + \mathbf{u}] + \mathbf{v} = \mathbf{u} + (-\mathbf{u}) \\ \text{Theorem A, (1) and (2)} \\ \Rightarrow [\mathbf{u} + (-\mathbf{u})] + \mathbf{v} = \mathbf{0} \text{ Theorem A, (1) and (4)} \\ \Rightarrow \mathbf{0} + \mathbf{v} = \mathbf{0} \text{ Theorem A, (4)} \\ \Rightarrow \mathbf{v} = \mathbf{0} \text{ Theorem A, (3)}$$

$$13. 3(6\mathbf{i} - 8\mathbf{j}) = 18\mathbf{i} - 24\mathbf{j}$$

$$14. \frac{-(-5\mathbf{i} + 12\mathbf{j})}{|-5\mathbf{i} + 12\mathbf{j}|} = \frac{5\mathbf{i} - 12\mathbf{j}}{13} = \frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}$$

$$15. \langle 6, 3 \rangle \cdot \langle -1, 2 \rangle = (6)(-1) + (3)(2) = -6 + 6 = 0 \\ \text{By Theorem C, they are perpendicular.}$$

$$16. \langle -5, \sqrt{3} \rangle \cdot \langle \sqrt{27}, 15 \rangle = (-5)(\sqrt{27}) + (\sqrt{3})(15) \\ = -15\sqrt{3} + 15\sqrt{3} = 0 \\ \text{By Theorem C, they are perpendicular.}$$

$$17. \langle c, 6 \rangle \cdot \langle c, -4 \rangle = (c)(c) + (6)(-4) = c^2 - 24 = 0 \\ c^2 = 24 \\ c = \pm 2\sqrt{6}$$

$$18. (2c\mathbf{i} - 8\mathbf{j}) \cdot (3\mathbf{i} + c\mathbf{j}) = (2c)(3) + (-8)(c) \\ = 6c - 8c = -2c = 0 \\ c = 0$$

$$19. \mathbf{r} = k\mathbf{a} + m\mathbf{b} \Rightarrow 7 = k(3) + m(-3) \text{ and} \\ -8 = k(-2) + m(4) \\ 3k - 3m = 7 \\ -2k + 4m = -8$$

Solve the system of equations to get

$$k = \frac{2}{3}, m = -\frac{5}{3}.$$

20. $\mathbf{r} = k\mathbf{a} + m\mathbf{b} \Rightarrow 6 = k(-4) + m(2)$
 $-7 = k(3) + m(-1)$
 $-4k + 2m = 6$
 $3k - m = -7$
 Solve the system of equations to get
 $k = -4, m = -5.$

21. $r_1\mathbf{i} + r_2\mathbf{j} = k(a_1\mathbf{i} + a_2\mathbf{j}) + m(b_1\mathbf{i} + b_2\mathbf{j})$
 $= (ka_1 + mb_1)\mathbf{i} + (ka_2 + mb_2)\mathbf{j}$
 $\Leftrightarrow r_1 = ka_1 + mb_1$ and $r_2 = ka_2 + mb_2.$
 Solve these two equations simultaneously (noting that $a_1b_2 - a_2b_1 \neq 0$ since $\mathbf{a}$ and $\mathbf{b}$ are noncollinear) and obtain

$$k = \frac{b_2r_1 - b_1r_2}{a_1b_2 - a_2b_1} \text{ and } m = \frac{a_1r_2 - a_2r_1}{a_1b_2 - a_2b_1}.$$

22. a and b cannot both be zero. If $a = 0$, then the line $ax + by = c$ is horizontal and $\mathbf{n} = b\mathbf{j}$ is vertical, so $\mathbf{n}$ is perpendicular to the line. Use a similar argument if $b = 0$. If $d \neq 0$ and $b \neq 0$, then

$$P_1\left(\frac{c}{a}, 0\right) \text{ and } P_2\left(0, \frac{c}{b}\right) \text{ are points on the line.}$$

$$\mathbf{n} \cdot \overrightarrow{P_1P_2} = (a\mathbf{i} + b\mathbf{j}) \cdot \left(-\frac{c}{a}\mathbf{i} + \frac{c}{b}\mathbf{j}\right) = -c + c = 0$$

23. $\text{Work} = \mathbf{F} \cdot \mathbf{D} = (3\mathbf{i} + 10\mathbf{j}) \cdot (10\mathbf{j})$
 $= 0 + 100 = 100 \text{ joules}$

24. $\mathbf{F} = 100 \sin 70^\circ \mathbf{i} - 100 \cos 70^\circ \mathbf{j}$
 $\mathbf{D} = 30\mathbf{i}$
 $\text{Work} = \mathbf{F} \cdot \mathbf{D}$
 $= (100 \sin 70^\circ)(30) + (-100 \cos 70^\circ)(0)$
 $= 3000 \sin 70^\circ \approx 2819 \text{ joules}$

25. $\mathbf{D} = 5\mathbf{i} + 8\mathbf{j}$
 $\text{Work} = \mathbf{F} \cdot \mathbf{D} = (6)(5) + (8)(8) = 94 \text{ ft}\cdot\text{lb}$

26. $\mathbf{D} = 12\mathbf{j}$
 $\text{Work} = \mathbf{F} \cdot \mathbf{D} = (-5)(0) + (8)(12) = 96 \text{ joules}$

27. $|\mathbf{u} \cdot \mathbf{v}| = |\cos \theta| |\mathbf{u}| |\mathbf{v}| \leq |\mathbf{u}| |\mathbf{v}|$ since $|\cos \theta| \leq 1.$

28. $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2 \leq |\mathbf{u}|^2 + 2|\mathbf{u} \cdot \mathbf{v}| + |\mathbf{v}|^2$
 $\leq |\mathbf{u}|^2 + 2|\mathbf{u}| |\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2$
 Therefore, $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|.$

29. a. $|\mathbf{u}|^2 + |\mathbf{v}|^2 = 2\mathbf{u} \cdot \mathbf{v} \Rightarrow \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = 0$
 $\Rightarrow (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0 \Rightarrow |\mathbf{u} - \mathbf{v}|^2 = 0$
 $\Rightarrow \mathbf{u} - \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} = \mathbf{v}$

b. $|\mathbf{u}|^2 = |\mathbf{v}|^2 = 2\mathbf{u} \cdot \mathbf{v} \Rightarrow |\mathbf{u}| = |\mathbf{v}|$ and
 $|\mathbf{u}|^2 = 2|\mathbf{u}| |\mathbf{v}| \cos \theta \Rightarrow |\mathbf{u}|^2 = 2|\mathbf{u}|^2 \cos \theta$
 $\Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

Thus, $\mathbf{u}$ and $\mathbf{v}$ have the same length and the angle between them is $\frac{\pi}{3}.$

30. If $\mathbf{u}$ and $\mathbf{v}$ are adjacent edges, the diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}.$

$$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}$$

$$= |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$$

Therefore, $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are perpendicular.

31. Let $x^2 + y^2 = r^2$ be the equation of the circle and let $A(-r, 0)$, $B(r, 0)$, and $C(x, y)$. Then

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = \langle x+r, y \rangle \cdot \langle x-r, y \rangle$$

$$= (x+r)(x-r) + y^2 = (x^2 + y^2) - r^2$$

$$= r^2 - r^2 = 0.$$

32. $\mathbf{w}$ is between $\mathbf{u}$ and $\mathbf{v}$ since it is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ with positive coefficients. Let α and β be the respective angles between $\mathbf{u}$ and $\mathbf{w}$ and between $\mathbf{v}$ and $\mathbf{w}.$

$$\cos \alpha = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}| |\mathbf{w}|} = \frac{\mathbf{u} \cdot (|\mathbf{v}| \mathbf{u} + |\mathbf{u}| \mathbf{v})}{|\mathbf{u}| |\mathbf{w}|}$$

$$= \frac{|\mathbf{v}| \mathbf{u} \cdot \mathbf{u} + |\mathbf{u}| \mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}| |\mathbf{w}|} = \frac{|\mathbf{v}| |\mathbf{u}|^2 + |\mathbf{u}| \mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}| |\mathbf{w}|}$$

$$= \frac{|\mathbf{v}| |\mathbf{u}| + \mathbf{v} \cdot \mathbf{u}}{|\mathbf{w}|}$$

$$\cos \beta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| |\mathbf{w}|} = \frac{\mathbf{v} \cdot (|\mathbf{v}| \mathbf{u} + |\mathbf{u}| \mathbf{v})}{|\mathbf{v}| |\mathbf{w}|} = \frac{|\mathbf{v}| \mathbf{v} \cdot \mathbf{u} + |\mathbf{u}| \mathbf{v} \cdot \mathbf{v}}{|\mathbf{v}| |\mathbf{w}|}$$

$$= \frac{|\mathbf{v}| \mathbf{v} \cdot \mathbf{u} + |\mathbf{u}| |\mathbf{v}|^2}{|\mathbf{v}| |\mathbf{w}|} = \frac{\mathbf{v} \cdot \mathbf{u} + |\mathbf{u}| |\mathbf{v}|}{|\mathbf{w}|}$$

Therefore, $\cos \alpha = \cos \beta$, so $\alpha = \beta$ since α and β are between 0° and $180^\circ.$

33. a. $\text{pr}_{\mathbf{v}} \mathbf{u} = \left(\frac{\langle 0, 5 \rangle \cdot \langle 3, 4 \rangle}{\langle 3, 4 \rangle \cdot \langle 3, 4 \rangle} \right) \langle 3, 4 \rangle$
 $= \frac{20}{9+16} \langle 3, 4 \rangle = \left\langle \frac{12}{5}, \frac{16}{5} \right\rangle$

b. $\text{pr}_{\mathbf{v}} \mathbf{u} = \left(\frac{\langle -3, 2 \rangle \cdot \langle 3, 4 \rangle}{\langle 3, 4 \rangle \cdot \langle 3, 4 \rangle} \right) \langle 3, 4 \rangle$

$$= \frac{-9+8}{9+16} \langle 3, 4 \rangle = \left\langle -\frac{3}{25}, -\frac{4}{25} \right\rangle$$

$$34. \text{ a. } \text{pr}_{\mathbf{u}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{u}}{|\mathbf{u}|^2} \right) \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \mathbf{u}$$

$$\text{b. } \text{pr}_{-\mathbf{u}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot (-\mathbf{u})}{|-\mathbf{u}|^2} \right) \mathbf{u} = \left(\frac{-\mathbf{u} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = -\mathbf{u}$$

$$\text{c. } \text{pr}_{\mathbf{u}} (-\mathbf{u}) = \left(\frac{(-\mathbf{u}) \cdot \mathbf{u}}{|\mathbf{u}|^2} \right) (-\mathbf{u})$$

$$= \left(\frac{-\mathbf{u} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) (-\mathbf{u}) = \mathbf{u}$$

$$\text{d. } \text{pr}_{-\mathbf{u}} (-\mathbf{u}) = \left(\frac{(-\mathbf{u}) \cdot (-\mathbf{u})}{|-\mathbf{u}|^2} \right) (-\mathbf{u})$$

$$= \left(\frac{\mathbf{u} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) (-\mathbf{u}) = -\mathbf{u}$$

35. a. a and b cannot both be zero. If $a = 0$, then the line $ax + by + c = 0$, is horizontal and $\mathbf{n} = b\mathbf{j}$ is vertical, so $\mathbf{n}$ is perpendicular to the line. Use a similar argument if $b = 0$.

If $a \neq 0$ and $b \neq 0$, then $\left(0, -\frac{c}{b}\right)$ and

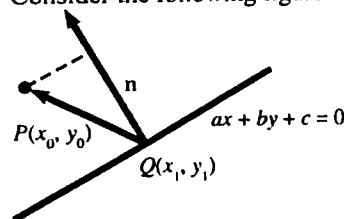
$\left(-\frac{c}{a}, 0\right)$ are on the line. Therefore,

$\left\langle -\frac{c}{a}, \frac{c}{b} \right\rangle$ is a vector in the direction of the

line. $\left\langle -\frac{c}{a}, \frac{c}{b} \right\rangle \cdot \langle a, b \rangle = -c + c = 0$, so

$\left\langle -\frac{c}{a}, \frac{c}{b} \right\rangle$ is perpendicular to $\langle a, b \rangle$. Hence, $\langle a, b \rangle$ is perpendicular to the line.

- b. Consider the following figure.



Observe that $ax_1 + by_1 = -c$.

$$\text{distance} = \left| \text{pr}_{\mathbf{n}} \overrightarrow{QP} \right| = \left| \frac{\overrightarrow{QP} \cdot \mathbf{n}}{|\mathbf{n}|^2} \mathbf{n} \right|$$

$$= \frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{|\mathbf{n}|^2} |\mathbf{n}| = \frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

$$= \frac{|\langle x_0 - x_1, y_0 - y_1 \rangle \cdot \langle a, b \rangle|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|(ax_0 + by_0) - (ax_1 + by_1)|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

13.4 Concepts Review

- a vector-valued function of a real variable
- f and g are continuous at c ; $f'(t)\mathbf{i} + g'(t)\mathbf{j}$
- position
- $\mathbf{r}'(t)$; $\mathbf{r}''(t)$; tangent; concave

Problem Set 13.4

- $\lim_{t \rightarrow 1} [2t\mathbf{i} - t^2\mathbf{j}] = \lim_{t \rightarrow 1} (2t)\mathbf{i} - \lim_{t \rightarrow 1} (t^2)\mathbf{j} = 2\mathbf{i} - \mathbf{j}$
- $\lim_{t \rightarrow 3} [2(t-3)^2\mathbf{i} - 7t^3\mathbf{j}]$
 $= \lim_{t \rightarrow 3} [2(t-3)^2]\mathbf{i} - \lim_{t \rightarrow 3} (7t^3)\mathbf{j} = -189\mathbf{j}$

$$3. \lim_{t \rightarrow 1} \left[\frac{t-1}{t^2-1} \mathbf{i} - \frac{t^2+2t-3}{t-1} \mathbf{j} \right]$$

$$= \lim_{t \rightarrow 1} \left(\frac{t-1}{t^2-1} \right) \mathbf{i} - \lim_{t \rightarrow 1} \left(\frac{t^2+2t-3}{t-1} \right) \mathbf{j}$$

$$= \lim_{t \rightarrow 1} \left(\frac{1}{t+1} \right) \mathbf{i} - \lim_{t \rightarrow 1} (t+3) \mathbf{j} = \frac{1}{2} \mathbf{i} - 4 \mathbf{j}$$

$$4. \lim_{t \rightarrow -2} \left[\frac{2t^2 - 10t - 28}{t+2} \mathbf{i} - \frac{7t^3}{t-3} \mathbf{j} \right]$$

$$= \lim_{t \rightarrow -2} \left(\frac{2t^2 - 10t - 28}{t+2} \right) \mathbf{i} - \lim_{t \rightarrow -2} \left(\frac{7t^3}{t-3} \right) \mathbf{j}$$

$$= \lim_{t \rightarrow -2} (2t - 14) \mathbf{i} - \frac{56}{5} \mathbf{j} = -18\mathbf{i} - \frac{56}{5} \mathbf{j}$$

$$5. \quad \lim_{t \rightarrow 0} \left[\frac{\sin t \cos t}{t} \mathbf{i} - \frac{7t^3}{e^t} \mathbf{j} \right]$$

$$= \lim_{t \rightarrow 0} \left(\frac{\sin t \cos t}{t} \right) \mathbf{i} - \lim_{t \rightarrow 0} \left(\frac{7t^3}{e^t} \right) \mathbf{j} = \mathbf{i}$$

$$6. \quad \lim_{t \rightarrow \infty} \left[\frac{t \sin t}{t^2} \mathbf{i} - \frac{7t^3}{t^3 - 3t} \mathbf{j} \right]$$

$$= \lim_{t \rightarrow \infty} \left(\frac{t \sin t}{t^2} \right) \mathbf{i} - \lim_{t \rightarrow \infty} \left(\frac{7t^3}{t^3 - 3t} \right) \mathbf{j}$$

$$= \lim_{t \rightarrow \infty} \left(\frac{\sin t}{t} \right) \mathbf{i} - 7 \mathbf{j} = -7 \mathbf{j}$$

$$7. \quad \lim_{t \rightarrow 0^+} \langle \ln(t^3), t^2 \ln t \rangle \text{ does not exist because}$$

$$\lim_{t \rightarrow 0^+} \ln(t^3) = -\infty.$$

$$8. \quad \lim_{t \rightarrow 0^-} \left\langle e^{-1/t^2}, \frac{t}{|t|} \right\rangle = \left\langle \lim_{t \rightarrow 0^-} e^{-1/t^2}, \lim_{t \rightarrow 0^-} \frac{t}{|t|} \right\rangle$$

$$= \langle 0, -1 \rangle$$

$$9. \quad \text{a. The domain of } f(t) = \frac{2}{t-4} \text{ is}$$

$$(-\infty, 4) \cup (4, \infty). \text{ The domain of}$$

$$g(t) = \sqrt{3-t} \text{ is } (-\infty, 3]. \text{ Thus, the domain}$$

$$\text{of } \mathbf{r} \text{ is } (-\infty, 3] \text{ or } \{t \in \mathbb{R} : t \leq 3\}.$$

$$\text{b. The domain of } f(t) = \lfloor t^2 \rfloor \text{ is } (-\infty, \infty). \text{ The}$$

$$\text{domain of } g(t) = \sqrt{20-t} \text{ is } (-\infty, 20]. \text{ Thus,}$$

$$\text{the domain of } \mathbf{r} \text{ is } (-\infty, 20] \text{ or}$$

$$\{t \in \mathbb{R} : t \leq 20\}.$$

$$10. \quad \text{a. The domain of } f(t) = \ln(t-1) \text{ is } (1, \infty). \text{ The}$$

$$\text{domain of } g(t) = \sqrt{20-t} \text{ is } (-\infty, 20]. \text{ Thus,}$$

$$\text{the domain of } \mathbf{r} \text{ is } (1, 20] \text{ or}$$

$$\{t \in \mathbb{R} : 1 < t \leq 20\}.$$

$$\text{b. The domain of } f(t) = \ln(t^{-1}) \text{ is } (0, \infty). \text{ The}$$

$$\text{domain of } g(t) = \tan^{-1} t \text{ is } (-\infty, \infty). \text{ Thus,}$$

$$\text{the domain of } \mathbf{r} \text{ is } (0, \infty) \text{ or } \{t \in \mathbb{R} : t > 0\}.$$

$$11. \quad \text{a. } f(t) = \frac{2}{t-4} \text{ is continuous on}$$

$$(-\infty, 4) \cup (4, \infty). \quad g(t) = \sqrt{3-t} \text{ is continuous}$$

$$\text{on } (-\infty, 3]. \text{ Thus, } \mathbf{r} \text{ is continuous on}$$

$$(-\infty, 3) \text{ or } \{t \in \mathbb{R} : t < 3\}.$$

$$\text{b. } f(t) = \lfloor t^2 \rfloor \text{ is continuous on}$$

$$(-\sqrt{n+1}, -\sqrt{n}) \cup (\sqrt{n}, \sqrt{n+1}) \text{ where } n \text{ is a}$$

$$\text{non-negative integer. } g(t) = \sqrt{20-t} \text{ is}$$

$$\text{continuous on } (-\infty, 20) \text{ or } \{t \in \mathbb{R} : t < 20\}.$$

$$\text{Thus, } \mathbf{r} \text{ is continuous on}$$

$$(-\sqrt{n+1}, -\sqrt{n}) \cup (\sqrt{n}, \sqrt{n+1}) \text{ where } n \text{ and}$$

$$k \text{ are non-negative integers and } k < 400 \text{ or}$$

$$\{t \in \mathbb{R} : t < 20, t^2 \text{ not an integer}\}.$$

$$12. \quad \text{a. } f(t) = \ln(t-1) \text{ is continuous on } (1, \infty).$$

$$g(t) = \sqrt{20-t} \text{ is continuous on } (-\infty, 20).$$

$$\text{Thus, } \mathbf{r} \text{ is continuous on } (1, 20) \text{ or}$$

$$\{t \in \mathbb{R} : 1 < t < 20\}.$$

$$\text{b. } f(t) = \ln(t^{-1}) \text{ is continuous on } (0, \infty).$$

$$g(t) = \tan^{-1} t \text{ is continuous on } (-\infty, \infty).$$

$$\text{Thus, } \mathbf{r} \text{ is continuous on } (0, \infty) \text{ or}$$

$$\{t \in \mathbb{R} : t > 0\}.$$

$$13. \quad \text{a. } D_t \mathbf{r}(t) = 9(3t+4)^2 \mathbf{i} + 2te^{t^2} \mathbf{j}$$

$$D_t^2 \mathbf{r}(t) = 54(3t+4) \mathbf{i} + 2(2t^2+1)e^{t^2} \mathbf{j}$$

$$\text{b. } D_t \mathbf{r}(t) = \sin 2t \mathbf{i} - 3 \sin 3t \mathbf{j}$$

$$D_t^2 \mathbf{r}(t) = 2 \cos 2t \mathbf{i} - 9 \cos 3t \mathbf{j}$$

$$14. \quad \text{a. } \mathbf{r}'(t) = (e^t - 2te^{-t^2}) \mathbf{i} + (\ln 2)2^t \mathbf{j}$$

$$\mathbf{r}''(t) = (e^t + 4t^2e^{-t^2} - 2e^{-t^2}) \mathbf{i} + (\ln 2)^2 2^t \mathbf{j}$$

$$\text{b. } \mathbf{r}'(t) = 2 \sec^2 2t \mathbf{i} + \frac{1}{1+t^2} \mathbf{j}$$

$$\mathbf{r}''(t) = 8 \tan 2t \sec^2 2t \mathbf{i} - \frac{2t}{(1+t^2)^2} \mathbf{j}$$

$$15. \quad \mathbf{r}'(t) = -e^{-t} \mathbf{i} - \frac{2}{t} \mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = -e^{-2t} + \frac{2}{t} \ln(t^2)$$

$$D_t[\mathbf{r}(t) \cdot \mathbf{r}'(t)] = 2e^{-2t} + \frac{4}{t^2} - \frac{2}{t^2} \ln(t^2)$$

$$16. \quad \mathbf{r}'(t) = 3 \cos 3t \mathbf{i} + 3 \sin 3t \mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

$$D_t[\mathbf{r}(t) \cdot \mathbf{r}'(t)] = 0$$

$$17. \quad h(t)\mathbf{r}(t) = e^{-3t}\sqrt{t-1}\mathbf{i} + e^{-3t}\ln(2t^2)\mathbf{j}$$

$$D_t[h(t)\mathbf{r}(t)] = -\frac{e^{-3t}}{2}\left(\frac{6t-7}{\sqrt{t-1}}\right)\mathbf{i}$$

$$+ e^{-3t}\left(\frac{2}{t} - 3\ln(2t^2)\right)\mathbf{j}$$

$$18. \quad h(t)\mathbf{r}(t) = \ln(3t-2)\sin 2t\mathbf{i} + \ln(3t-2)\cosh t\mathbf{j}$$

$$D_t[h(t)\mathbf{r}(t)] = \left[2\ln(3t-2)\cos 2t + \frac{3\sin 2t}{3t-2}\right]\mathbf{i}$$

$$+ \left[\ln(3t-2)\sinh t + \frac{3\cosh t}{3t-2}\right]\mathbf{j}$$

$$19. \quad \mathbf{f}'(u) = -\sin u\mathbf{i} + 3e^{3u}\mathbf{j}; \quad \mathbf{g}'(t) = 6t$$

$$\mathbf{F}'(t) = \mathbf{f}'(\mathbf{g}(t))\mathbf{g}'(t)$$

$$= -6t\sin(3t^2-4)\mathbf{i} + 18te^{9t^2-12}\mathbf{j}$$

$$20. \quad \mathbf{f}'(u) = 2u\mathbf{i} + \sin 2u\mathbf{j}; \quad \mathbf{g}'(t) = \sec^2 t$$

$$\mathbf{F}'(t) = \mathbf{f}'(\mathbf{g}(t))\mathbf{g}'(t)$$

$$= 2\tan t \sec^2 t\mathbf{i} + \sec^2 t \sin(2\tan t)\mathbf{j}$$

$$21. \quad \int_0^1 (e^t\mathbf{i} + e^{-t}\mathbf{j})dt = [e^t\mathbf{i} - e^{-t}\mathbf{j}]_0^1$$

$$= (e-1)\mathbf{i} + (1-e^{-1})\mathbf{j}$$

$$22. \quad \int_{-1}^1 [(1+t)^{3/2}\mathbf{i} + (1-t)^{3/2}\mathbf{j}]dt$$

$$= \left[\frac{2}{5}(1+t)^{5/2}\mathbf{i} - \frac{2}{5}(1-t)^{5/2}\mathbf{j}\right]_{-1}^1$$

$$= \frac{8\sqrt{2}}{5}\mathbf{i} + \frac{8\sqrt{2}}{5}\mathbf{j}$$

$$23. \quad \mathbf{r}'(t) = \mathbf{v}(t) = -e^{-t}\mathbf{i} + e^t\mathbf{j}$$

$$\mathbf{r}''(t) = \mathbf{a}(t) = e^{-t}\mathbf{i} + e^t\mathbf{j}$$

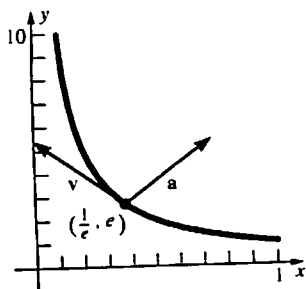
$$|\mathbf{v}(t)| = \sqrt{e^{-2t} + e^{2t}}$$

$$\mathbf{r}(1) = e^{-1}\mathbf{i} + e\mathbf{j}$$

$$\mathbf{v}(1) = -e^{-1}\mathbf{i} + e\mathbf{j}$$

$$\mathbf{a}(1) = e^{-1}\mathbf{i} + e\mathbf{j}$$

$$|\mathbf{v}(1)| = \sqrt{e^{-2} + e^2}$$



$$24. \quad \mathbf{r}'(t) = \mathbf{v}(t) = 6t\mathbf{i} + \mathbf{j}$$

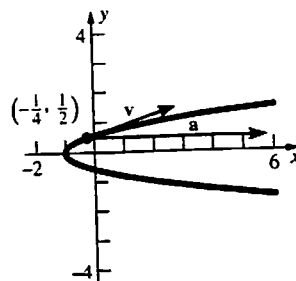
$$\mathbf{r}''(t) = \mathbf{a}(t) = 6\mathbf{i}$$

$$\mathbf{r}\left(\frac{1}{2}\right) = -\frac{1}{4}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\mathbf{v}\left(\frac{1}{2}\right) = 3\mathbf{i} + \mathbf{j}$$

$$\mathbf{a}\left(\frac{1}{2}\right) = 6\mathbf{i}$$

$$\left|\mathbf{v}\left(\frac{1}{2}\right)\right| = \sqrt{9+1} = \sqrt{10}$$



$$25. \quad \mathbf{r}'(t) = \mathbf{v}(t) = -2\sin t\mathbf{i} - 3\sin 2t\mathbf{j}$$

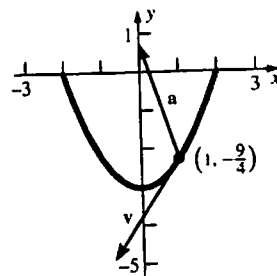
$$\mathbf{r}''(t) = \mathbf{a}(t) = -2\cos t\mathbf{i} - 6\cos 2t\mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{3}\right) = \mathbf{i} - \frac{9}{4}\mathbf{j}$$

$$\mathbf{v}\left(\frac{\pi}{3}\right) = -\sqrt{3}\mathbf{i} - \frac{3\sqrt{3}}{2}\mathbf{j}$$

$$\mathbf{a}\left(\frac{\pi}{3}\right) = -\mathbf{i} + 3\mathbf{j}$$

$$\left|\mathbf{v}\left(\frac{\pi}{3}\right)\right| = \sqrt{3 + \frac{27}{4}} = \frac{\sqrt{39}}{2}$$



$$26. \quad \mathbf{r}'(t) = \mathbf{v}(t) = \sec^2 t\mathbf{i} + \cos t\mathbf{j}$$

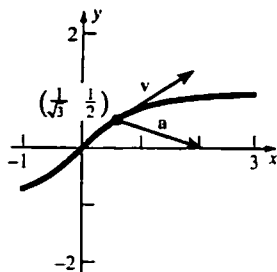
$$\mathbf{r}''(t) = \mathbf{a}(t) = 2\sec^2 t \tan t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\mathbf{v}\left(\frac{\pi}{6}\right) = \frac{4}{3}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\mathbf{a}\left(\frac{\pi}{6}\right) = \frac{8}{3\sqrt{3}}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\left| \mathbf{v}\left(\frac{\pi}{6}\right) \right| = \sqrt{\frac{16}{9} + \frac{3}{4}} = \frac{\sqrt{91}}{6}$$



27. $\mathbf{r}'(t) = \mathbf{v}(t) = 6t\mathbf{i} + 3t^2\mathbf{j}$

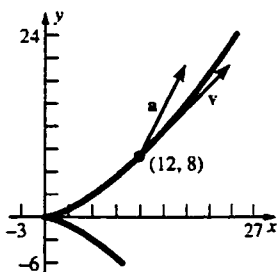
$$\mathbf{r}''(t) = \mathbf{a}(t) = 6\mathbf{i} + 6t\mathbf{j}$$

$$\mathbf{r}(2) = 12\mathbf{i} + 8\mathbf{j}$$

$$\mathbf{v}(2) = 12\mathbf{i} + 12\mathbf{j}$$

$$\mathbf{a}(2) = 6\mathbf{i} + 12\mathbf{j}$$

$$|\mathbf{v}(2)| = \sqrt{144 + 144} = 12\sqrt{2}$$



28. $\mathbf{r}'(t) = \mathbf{v}(t) = a \cos t \mathbf{i} - 2a \sin t \mathbf{j}$

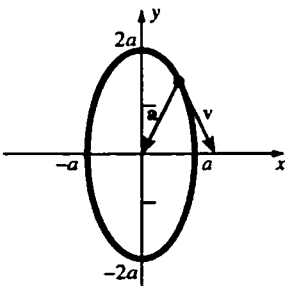
$$\mathbf{r}''(t) = \mathbf{a}(t) = -a \sin t \mathbf{i} - 2a \cos t \mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{a}{\sqrt{2}}\mathbf{i} + \sqrt{2}a\mathbf{j}$$

$$\mathbf{v}\left(\frac{\pi}{4}\right) = \frac{a}{\sqrt{2}}\mathbf{i} - \sqrt{2}a\mathbf{j}$$

$$\mathbf{a}\left(\frac{\pi}{4}\right) = -\frac{a}{\sqrt{2}}\mathbf{i} - \sqrt{2}a\mathbf{j}$$

$$|\mathbf{v}(2)| = \sqrt{\frac{a^2}{2} + 2a^2} = \sqrt{\frac{5}{2}}a$$



29. $\mathbf{r}'(t) = \mathbf{v}(t) = -\sin t \mathbf{i} - 2\sec^2 t \mathbf{j}$

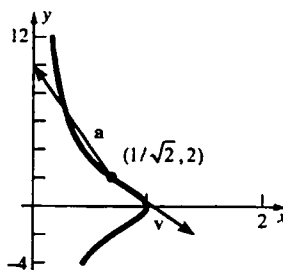
$$\mathbf{r}''(t) = \mathbf{a}(t) = -\cos t \mathbf{i} - 4\sec^2 t \tan t \mathbf{j}$$

$$\mathbf{r}\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{v}\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\mathbf{i} - 4\mathbf{j}$$

$$\mathbf{a}\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\mathbf{i} + 8\mathbf{j}$$

$$|\mathbf{v}\left(-\frac{\pi}{4}\right)| = \sqrt{\frac{1}{2} + 16} = \sqrt{\frac{33}{2}}$$



30. $\mathbf{r}'(t) = \mathbf{v}(t) = \frac{1}{2}e^{t/2}\mathbf{i} - e^{-t}\mathbf{j}$

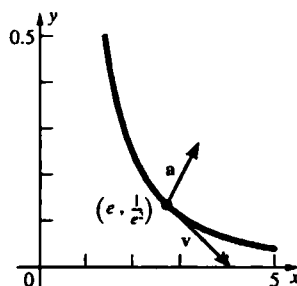
$$\mathbf{r}''(t) = \mathbf{a}(t) = \frac{1}{4}e^{t/2}\mathbf{i} + e^{-t}\mathbf{j}$$

$$\mathbf{r}(2) = e\mathbf{i} + e^{-2}\mathbf{j}$$

$$\mathbf{v}(2) = \frac{1}{2}e\mathbf{i} - e^{-2}\mathbf{j}$$

$$\mathbf{a}(2) = \frac{1}{4}e\mathbf{i} + e^{-2}\mathbf{j}$$

$$|\mathbf{v}(2)| = \sqrt{\frac{1}{4}e^2 + e^{-4}} = \frac{\sqrt{e^6 + 4}}{2e^2}$$



31. $\mathbf{v}(t) = \int \mathbf{a}(t) dt = C_1\mathbf{i} + (-32t + C_2)\mathbf{j}$

$$\mathbf{v}(0) = \mathbf{0} = C_1\mathbf{i} + C_2\mathbf{j}$$

$$C_1 = C_2 = 0$$

$$\mathbf{v}(t) = -32t\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = C_1\mathbf{i} + (-16t^2 + C_2)\mathbf{j}$$

$$\mathbf{r}(0) = \mathbf{0} = C_1\mathbf{i} + C_2\mathbf{j}$$

$$C_1 = C_2 = 0$$

$$\mathbf{r}(t) = -16t^2\mathbf{j}$$

32. $\mathbf{v}(t) = \int \mathbf{a}(t) dt = C_1\mathbf{i} + \left(\frac{1}{2}t^2 + C_2\right)\mathbf{j}$

$$\mathbf{v}(0) = \mathbf{i} + 2\mathbf{j} = C_1\mathbf{i} + C_2\mathbf{j}$$

$$C_1 = 1, C_2 = 2$$

$$\mathbf{v}(t) = \mathbf{i} + \left(\frac{1}{2}t^2 + 2\right)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = (t + C_1)\mathbf{i} + \left(\frac{1}{6}t^3 + 2t + C_2\right)\mathbf{j}$$

$$\mathbf{r}(0) = \mathbf{0} = C_1\mathbf{i} + C_2\mathbf{j}$$

$$C_1 = C_2 = 0$$

$$\mathbf{v}(t) = t\mathbf{i} + \left(\frac{1}{6}t^3 + 2t\right)\mathbf{j}$$

$$33. \mathbf{v}(t) = \int \mathbf{a}(t) dt = (t + C_1)\mathbf{i} + (-e^{-t} + C_2)\mathbf{j}$$

$$\mathbf{v}(0) = 2\mathbf{i} + \mathbf{j} = C_1\mathbf{i} + (-1 + C_2)\mathbf{j}$$

$$C_1 = C_2 = 2$$

$$\mathbf{v}(t) = (t + 2)\mathbf{i} + (-e^{-t} + 2)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \left(\frac{1}{2}t^2 + 2t + C_1\right)\mathbf{i} + (e^{-t} + 2t + C_2)\mathbf{j}$$

$$\mathbf{r}(0) = \mathbf{i} + \mathbf{j} = C_1\mathbf{i} + (1 + C_2)\mathbf{j}$$

$$C_1 = 1, C_2 = 0$$

$$\mathbf{r}(t) = \left(\frac{1}{2}t^2 + 2t + 1\right)\mathbf{i} + (e^{-t} + 2t)\mathbf{j}$$

$$34. \mathbf{v}(t) = \int \mathbf{a}(t) dt = (-\sin t + C_1)\mathbf{i} + (-\cos t + C_2)\mathbf{j}$$

$$\mathbf{v}(0) = \mathbf{i} = C_1\mathbf{i} + (-1 + C_2)\mathbf{j}$$

$$C_1 = C_2 = 1$$

$$\mathbf{v}(t) = (-\sin t + 1)\mathbf{i} + (-\cos t + 1)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = (\cos t + t + C_1)\mathbf{i} + (-\sin t + t + C_2)\mathbf{j}$$

$$\mathbf{r}(0) = \mathbf{i} + 3\mathbf{j} = (1 + C_1)\mathbf{i} + C_2\mathbf{j}$$

$$C_1 = 0, C_2 = 3$$

$$\mathbf{v}(t) = (\cos t + 1)\mathbf{i} + (-\sin t + 1)\mathbf{j}$$

$$35. \mathbf{r}(t) = 5 \cos 6t\mathbf{i} + 5 \sin 6t\mathbf{j}$$

$$\mathbf{v}(t) = -30 \sin 6t\mathbf{i} + 30 \cos 6t\mathbf{j}$$

$$|\mathbf{v}(t)| = \sqrt{900 \sin^2 6t + 900 \cos^2 6t} = 30$$

$$\mathbf{a}(t) = -180 \cos 6t\mathbf{i} - 180 \sin 6t\mathbf{j}$$

$$36. D_t[\mathbf{v}(t) \cdot \mathbf{v}(t)] = \mathbf{v}(t) \cdot \mathbf{a}(t) + \mathbf{a}(t) \cdot \mathbf{v}(t)$$

$$= 2\mathbf{v}(t) \cdot \mathbf{a}(t) = D_t[c] = 0, \text{ so } \mathbf{v}(t) \cdot \mathbf{a}(t) = 0.$$

Thus, $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are perpendicular.

$$37. \text{ Substitute } \theta = 30^\circ \text{ and } v_0 = 96 \text{ ft/s into the equations for } \mathbf{r}(t) \text{ and } \mathbf{v}(t) \text{ in Example 5.}$$

$$\mathbf{r}(t) = 48\sqrt{3}t\mathbf{i} + (48t - 16t^2)\mathbf{j},$$

$$\mathbf{v}(t) = 48\sqrt{3}\mathbf{i} + (48 - 32t)\mathbf{j}$$

The projectile is at ground level when

$$48t - 16t^2 = 0, \text{ so } t = 0 \text{ or } t = 3. \text{ Thus, the}$$

projectile hits the ground after 3 seconds. Its speed is $|\mathbf{v}(3)| = |48\sqrt{3}\mathbf{i} - 48\mathbf{j}| = 96 \text{ ft/s}$. It will

land $48\sqrt{3}(3) = 144\sqrt{3} \text{ ft}$ from the origin.

$$38. \mathbf{v}(t) = \int \mathbf{a}(t) dt = bt\mathbf{i} + (at + c)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \left(bt + d\right)\mathbf{i} + \left(\frac{1}{2}at^2 + ct + e\right)\mathbf{j}$$

Thus, $x = bt + d$ and $y = \frac{1}{2}at^2 + ct + e$. If $b = 0$,

then $x = d$, so the graph will be a vertical line. If

$b \neq 0$, $t = \frac{1}{b}(x - d)$, and

$$y = \frac{1}{2}a\left[\frac{1}{b}(x - d)\right]^2 + c\left[\frac{1}{b}(x - d)\right] + e \text{ is a}$$

quadratic equation, so the graph is a parabola.

Note that if $a = 0$, the graph is a line.

$$39. \mathbf{r}'(t) = \mathbf{v}(t) = \omega \sinh \omega t\mathbf{i} + \omega \cosh \omega t\mathbf{j}$$

$$\mathbf{r}''(t) = \mathbf{a}(t) = \omega^2 \cosh \omega t\mathbf{i} + \omega^2 \sinh \omega t\mathbf{j}$$

$$= \omega^2 (\cosh \omega t\mathbf{i} + \sinh \omega t\mathbf{j}) = \omega^2 \mathbf{r}(t)$$

Thus, $\mathbf{a}(t) = c\mathbf{r}(t)$ where $c = \omega^2$.

$$40. \mathbf{r}'(t) = \mathbf{v}(t) = -a\omega \sin \omega t\mathbf{i} + b\omega \cos \omega t\mathbf{j}$$

$$\mathbf{r}''(t) = \mathbf{a}(t) = -a\omega^2 \cos \omega t\mathbf{i} - b\omega^2 \sin \omega t\mathbf{j}$$

$$= -\omega^2 (a \cos \omega t\mathbf{i} + b \sin \omega t\mathbf{j}) = -\omega^2 \mathbf{r}(t)$$

Thus, $\mathbf{a}(t) = c\mathbf{r}(t)$ where $c = -\omega^2$.

$$41. \text{ Substitute } \theta = 45^\circ \text{ into the equation for } \mathbf{r}(t) \text{ in Example 5.}$$

$$\mathbf{r}(t) = \frac{v_0}{\sqrt{2}}t\mathbf{i} + \left(\frac{v_0}{\sqrt{2}}t - 16t^2\right)\mathbf{j},$$

If t_1 is the time that the ball is caught,

$$\frac{v_0}{\sqrt{2}}t_1 = 300 \text{ and } \frac{v_0}{\sqrt{2}}t_1 - 16t_1^2 = 0. \text{ Thus, from}$$

the first equation $t_1 = \frac{300\sqrt{2}}{v_0}$. Substitute into the

second equation and solve for v_0 .

$$v_0 = \sqrt{9600} = 40\sqrt{6} \approx 97.98 \text{ ft/s}$$

$$42. \text{ Consider the motion from when the ball rolls off the table at time } t = 0. \text{ The only acceleration is due to gravity, so } \mathbf{a}(t) = -32\mathbf{j}, \mathbf{v}(0) = 20\mathbf{i},$$

$$\mathbf{r}(0) = 4\mathbf{j}. \mathbf{v}(t) = \int \mathbf{a}(t) dt = C_1\mathbf{i} + (-32t + C_2)\mathbf{j}$$

$$\mathbf{v}(0) = 20\mathbf{i} = C_1\mathbf{i} + C_2\mathbf{j}, \text{ so } C_1 = 20, C_2 = 0, \text{ and}$$

$$\mathbf{v}(t) = 20\mathbf{i} - 32t\mathbf{j}.$$

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = (20t + C_1)\mathbf{i} + (-16t^2 + C_2)\mathbf{j}$$

$$\mathbf{r}(0) = 4\mathbf{j} = C_1\mathbf{i} + C_2\mathbf{j}, \text{ so } C_1 = 0, C_2 = 4, \text{ and}$$

$$\mathbf{r}(t) = 20t\mathbf{i} + (-16t^2 + 4)\mathbf{j}.$$

The ball will hit the floor when $-16t^2 + 4 = 0$ or

$$t = \frac{1}{2}. \mathbf{v}\left(\frac{1}{2}\right) = 20\mathbf{i} - 16\mathbf{j}, \text{ which is in the}$$

direction of the tangent when the ball hits the

ground. If θ is the angle at which the ball hits the ground, θ is also the angle between the vectors $\mathbf{i}$ and $20\mathbf{i} - 16\mathbf{j}$ or $5\mathbf{i} - 4\mathbf{j}$.

$$\cos \theta = \frac{\mathbf{i} \cdot (5\mathbf{i} - 4\mathbf{j})}{|\mathbf{i}| |5\mathbf{i} - 4\mathbf{j}|} = \frac{5}{\sqrt{41}}, \text{ so}$$

$$\theta = \cos^{-1} \frac{5}{\sqrt{41}} \approx 38.66^\circ.$$

The speed at which the ball hits the ground is

$$\left| \mathbf{v} \left(\frac{1}{2} \right) \right| = 4\sqrt{41} \approx 25.61 \text{ ft/s.}$$

43. From Example 3, the acceleration due to rotation is $\mathbf{a}(t) = -60\omega^2 \cos \omega t \mathbf{i} - 60\omega^2 \sin \omega t \mathbf{j}$. Since it must be at least greater than gravity,

$$980 \leq |\mathbf{a}(t)| = 60\omega^2. \text{ Thus, } \omega^2 \geq \frac{49}{3} \text{ or}$$

$$\omega \geq \frac{7}{\sqrt{3}} \approx 4.04 \text{ rad/s. The angular speed must be at least 4.04 rad/s.}$$

44. a. From Example 3, $a = \frac{v^2}{r}$, and from

Newton's Inverse Square Law, $a = \frac{k}{r^2}$, so

$$\frac{v^2}{r} = \frac{k}{r^2}. \text{ Therefore, } v^2 = \frac{k}{r}.$$

- b. $v = \frac{2\pi r}{T}$; substitute into $v^2 = \frac{k}{r}$, so

$$\frac{4\pi^2 r^2}{T^2} = \frac{k}{r}. \text{ Therefore, } T^2 = \left(\frac{4\pi^2}{k} \right) r^3.$$

- c. At the surface of the earth, $r = R$ and $a = g$.

Substitute into $a = \frac{k}{r^2}$, so $g = \frac{k}{R^2}$.

Therefore, $k = gR^2$.

45. a. From Problem 44a and c, $v^2 = \frac{k}{r} = \frac{gR^2}{r}$.

Convert gravity into mi/h^2 so

$$g \approx (32.17) \left(\frac{1}{5280} \right) \left(\frac{3600}{1} \right)^2 \approx 78,963 \text{ mi/h}^2$$

$$v^2 \approx \frac{(78,963)(3960)^2}{(4160)}$$

$$\approx 297,660,140 \text{ mi}^2/\text{h}^2$$

$$v \approx 17,253 \text{ mi/h}$$

- b. From Problem 44b and c,

$$r^3 = \left(\frac{k}{4\pi^2} \right) T^2 = \left(\frac{gR^2}{4\pi^2} \right) T^2. \text{ Since the}$$

satellite is in synchronous orbit, $T = 24 \text{ h}$.

$$r^3 \approx \left[\frac{(78,963)(3960)^2}{4\pi^2} \right] (24)^2$$

$$\approx 1.80666 \times 10^{13} \text{ mi}^3$$

$$r \approx 26,240 \text{ mi}$$

46. From Problem 44b and c,

$$r^3 = \left(\frac{k}{4\pi^2} \right) T^2 = \left(\frac{gR^2}{4\pi^2} \right) T^2.$$

$$T = (27.32)(24) = 655.68 \text{ h}$$

$$r^3 \approx \left[\frac{(78,963)(3960)^2}{4\pi^2} \right] (655.68)^2$$

$$\approx 1.348360 \times 10^{16} \text{ mi}^3$$

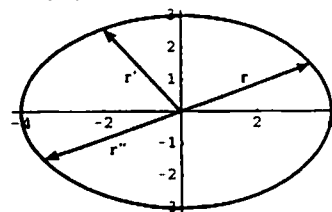
$$r \approx 238,000 \text{ mi}$$

47. $\mathbf{r}'(t) = -4 \sin t \mathbf{i} + 3 \cos t \mathbf{j}$, $\mathbf{r}''(t) = -4 \cos t \mathbf{i} - 3 \sin t \mathbf{j}$

Observe that the graphs of $\mathbf{r}(t)$, $\mathbf{r}'(t)$, and $\mathbf{r}''(t)$ are the same.

a. $\mathbf{r} \left(\frac{\pi}{6} \right) = 2\sqrt{3}\mathbf{i} + \frac{3}{2}\mathbf{j}$, $\mathbf{r}' \left(\frac{\pi}{6} \right) = -2\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}$,

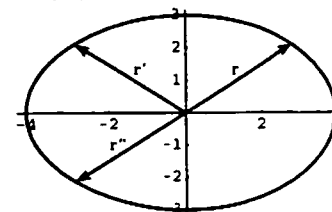
$$\mathbf{r}'' \left(\frac{\pi}{6} \right) = -2\sqrt{3}\mathbf{i} - \frac{3}{2}\mathbf{j}$$



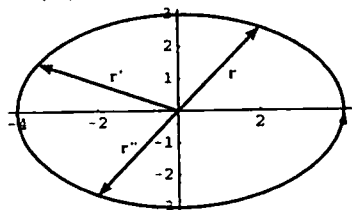
b. $\mathbf{r} \left(\frac{\pi}{4} \right) = 2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$,

$$\mathbf{r}' \left(\frac{\pi}{4} \right) = -2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j},$$

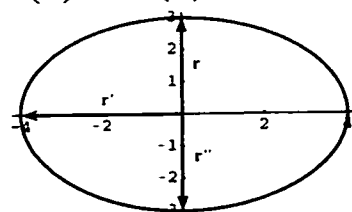
$$\mathbf{r}'' \left(\frac{\pi}{4} \right) = -2\sqrt{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$$



c. $\mathbf{r}\left(\frac{\pi}{3}\right) = 2\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}$, $\mathbf{r}'\left(\frac{\pi}{3}\right) = -2\sqrt{3}\mathbf{i} + \frac{3}{2}\mathbf{j}$,
 $\mathbf{r}''\left(\frac{\pi}{3}\right) = -2\mathbf{i} - \frac{3\sqrt{3}}{2}\mathbf{j}$



d. $\mathbf{r}\left(\frac{\pi}{2}\right) = 3\mathbf{j}$, $\mathbf{r}'\left(\frac{\pi}{2}\right) = -4\mathbf{i}$, $\mathbf{r}''\left(\frac{\pi}{2}\right) = -3\mathbf{j}$



48. The paths of $\mathbf{r}(t)$, $\mathbf{r}'(t)$, and $\mathbf{r}''(t)$ are along the

ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

13.5 Concepts Review

1. $\left|\frac{d\mathbf{T}}{ds}\right|$

2. $\frac{1}{a}$; 0

3. $\kappa = \frac{1}{R}$

4. $\frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\kappa\mathbf{N}$

Problem Set 13.5

1. $\mathbf{r}'(t) = 3\mathbf{i} + 6t\mathbf{j}$

$|\mathbf{r}'(t)| = 3\sqrt{1+4t^2}$

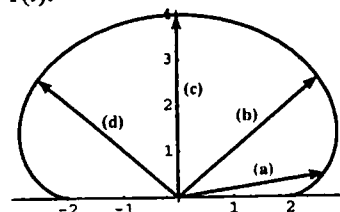
$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+4t^2}}\mathbf{i} + \frac{2t}{\sqrt{1+4t^2}}\mathbf{j}$

$\mathbf{T}\left(\frac{1}{3}\right) = \frac{3}{\sqrt{13}}\mathbf{i} + \frac{2}{\sqrt{13}}\mathbf{j}$

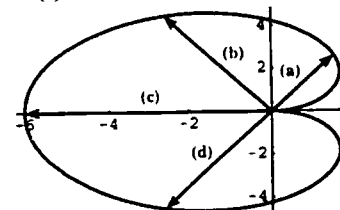
$x'(t) = 3$

$y'(t) = 6t$

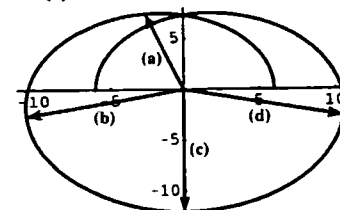
49. $\mathbf{r}(t)$:



$\mathbf{r}'(t)$:



$\mathbf{r}''(t)$:



$x''(t) = 0$

$y''(t) = 6$

$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{18}{27(1+4t^2)^{3/2}}$

$= \frac{2}{3(1+4t^2)^{3/2}}$

$\kappa\left(\frac{1}{3}\right) = \frac{2}{3\left(\frac{13}{9}\right)^{3/2}} = \frac{18}{13\sqrt{13}}$

2. $\mathbf{s}'(t) = 6t\mathbf{i} + 3\mathbf{j}$

$|\mathbf{s}'(t)| = 3\sqrt{4t^2 + 1}$

$\mathbf{T}(t) = \frac{\mathbf{s}'(t)}{|\mathbf{s}'(t)|} = \frac{2t}{\sqrt{4t^2 + 1}}\mathbf{i} + \frac{1}{\sqrt{4t^2 + 1}}\mathbf{j}$

$\mathbf{T}\left(\frac{1}{3}\right) = \frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j}$

$x'(t) = 6t$

$y'(t) = 3$

$x''(t) = 6$

$y''(t) = 0$

$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{18}{27(4t^2 + 1)^{3/2}}$

$$= \frac{2}{3(4t^2 + 1)^{3/2}}$$

$$\kappa\left(\frac{1}{3}\right) = \frac{2}{3\left(\frac{13}{9}\right)^{3/2}} = \frac{18}{13\sqrt{13}}$$

3. $\mathbf{u}'(t) = 8t\mathbf{i} + 4\mathbf{j}$

$$|\mathbf{u}'(t)| = 4\sqrt{4t^2 + 1}$$

$$\mathbf{T}(t) = \frac{\mathbf{u}'(t)}{|\mathbf{u}'(t)|} = \frac{2t}{\sqrt{4t^2 + 1}}\mathbf{i} + \frac{1}{\sqrt{4t^2 + 1}}\mathbf{j}$$

$$\mathbf{T}\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$x'(t) = 8t \quad y'(t) = 4$$

$$x''(t) = 8 \quad y''(t) = 0$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{32}{64(4t^2 + 1)^{3/2}}$$

$$= \frac{1}{2(4t^2 + 1)^{3/2}}$$

$$\kappa\left(\frac{1}{2}\right) = \frac{1}{2(2)^{3/2}} = \frac{1}{4\sqrt{2}}$$

4. $\mathbf{r}'(t) = t^2\mathbf{i} + t\mathbf{j}$

$$|\mathbf{r}'(t)| = t\sqrt{t^2 + 1}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{t}{\sqrt{t^2 + 1}}\mathbf{i} + \frac{1}{\sqrt{t^2 + 1}}\mathbf{j}$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$x'(t) = t^2 \quad y'(t) = t$$

$$x''(t) = 2t \quad y''(t) = 1$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{t^2}{t^3(t^2 + 1)^{3/2}}$$

$$= \frac{1}{t(t^2 + 1)^{3/2}}$$

$$\kappa(1) = \frac{1}{1(2)^{3/2}} = \frac{1}{2\sqrt{2}}$$

5. $\mathbf{z}'(t) = -3\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$$|\mathbf{z}'(t)| = \sqrt{9 + 7\cos^2 t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\frac{3\sin t}{\sqrt{9 + 7\cos^2 t}}\mathbf{i} + \frac{4\cos t}{\sqrt{9 + 7\cos^2 t}}\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$x'(t) = -3\sin t \quad y'(t) = 4\cos t$$

$$x''(t) = -3\cos t \quad y''(t) = -4\sin t$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{12}{(9 + 7\cos^2 t)^{3/2}}$$

$$\kappa\left(\frac{\pi}{4}\right) = \frac{12}{\left(\frac{25}{2}\right)^{3/2}} = \frac{24\sqrt{2}}{125}$$

6. $\mathbf{r}'(t) = e^t\mathbf{i} + e^t\mathbf{j}$

$$|\mathbf{r}'(t)| = e^t\sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{T}(\ln 2) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$x'(t) = e^t \quad y'(t) = e^t$$

$$x''(t) = e^t \quad y''(t) = e^t$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = 0$$

$$\kappa(\ln 2) = 0$$

7. $\mathbf{r}'(t) = (\cos t - \sin t)e^t\mathbf{i} + (\sin t + \cos t)e^t\mathbf{j}$

$$|\mathbf{r}'(t)| = \sqrt{2}e^t$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\cos t - \sin t}{\sqrt{2}}\mathbf{i} + \frac{\sin t + \cos t}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$x'(t) = (\cos t - \sin t)e^t \quad y'(t) = (\sin t + \cos t)e^t$$

$$x''(t) = (-2\sin t)e^t \quad y''(t) = (2\cos t)e^t$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{2e^{2t}}{2\sqrt{2}e^{3t}} = \frac{1}{\sqrt{2}e^t}$$

$$\kappa\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}e^{\pi/2}}$$

8. $\mathbf{r}'(t) = -3e^{-3t}\mathbf{i} + e^t\mathbf{j}$

$$|\mathbf{r}'(t)| = e^{-3t}\sqrt{9 + e^{8t}}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\frac{3}{\sqrt{9 + e^{8t}}}\mathbf{i} + \frac{e^{4t}}{\sqrt{9 + e^{8t}}}\mathbf{j}$$

$$\mathbf{T}(0) = -\frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}$$

$$x'(t) = -3e^{-3t} \quad y'(t) = e^t$$

$$x''(t) = 9e^{-3t} \quad y''(t) = e^t$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{12e^{-2t}}{e^{-9t}(9 + e^{8t})^{3/2}}$$

$$= \frac{12e^{7t}}{(9+e^{8t})^{3/2}}$$

$$\kappa(0) = \frac{12}{10\sqrt{10}} = \frac{6}{5\sqrt{10}}$$

9. $\mathbf{r}'(t) = -2t\mathbf{i} - 3t^2\mathbf{j}$

$$|\mathbf{r}'(t)| = t\sqrt{4+9t^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\frac{2}{\sqrt{4+9t^2}}\mathbf{i} - \frac{3t}{\sqrt{4+9t^2}}\mathbf{j}$$

$$\mathbf{T}(1) = -\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}$$

$$\begin{aligned} x'(t) &= -2t & y'(t) &= -3t^2 \\ x''(t) &= -2 & y''(t) &= -6t \end{aligned}$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{6t^2}{t^3(4+9t^2)^{3/2}}$$

$$= \frac{6}{t(4+9t^2)^{3/2}}$$

$$\kappa(1) = \frac{6}{13\sqrt{13}}$$

10. $\mathbf{r}'(t) = \cosh t\mathbf{i} + \sinh t\mathbf{j}$

$$|\mathbf{r}'(t)| = \sqrt{\sinh^2 t + \cosh^2 t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\cosh t}{\sqrt{\sinh^2 t + \cosh^2 t}}\mathbf{i} + \frac{\sinh t}{\sqrt{\sinh^2 t + \cosh^2 t}}\mathbf{j}$$

$$\mathbf{T}(\ln 3) = \frac{5}{\sqrt{41}}\mathbf{i} + \frac{4}{\sqrt{41}}\mathbf{j}$$

$$x'(t) = \cosh t \quad y'(t) = \sinh t$$

$$x''(t) = \sinh t \quad y''(t) = \cosh t$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{1}{(\sinh^2 t + \cosh^2 t)^{3/2}}; \quad \kappa(\ln 3) = \frac{1}{\left(\frac{41}{9}\right)^{3/2}} = \frac{27}{41\sqrt{41}}$$

11. $\mathbf{r}'(t) = -(\cos t + \sin t)e^{-t}\mathbf{i} + (\cos t - \sin t)e^{-t}\mathbf{j}$

$$|\mathbf{r}'(t)| = \sqrt{2}e^{-t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\frac{\cos t + \sin t}{\sqrt{2}}\mathbf{i} + \frac{\cos t - \sin t}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{T}(0) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$x'(t) = -(\cos t + \sin t)e^{-t} \quad y'(t) = (\cos t - \sin t)e^{-t}$$

$$x''(t) = (2\sin t)e^{-t} \quad y''(t) = (-2\cos t)e^{-t}$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{2e^{-2t}}{2\sqrt{2}e^{-3t}} = \frac{e^t}{\sqrt{2}}$$

$$\kappa(0) = \frac{1}{\sqrt{2}}$$

12. $\mathbf{r}'(t) = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j}$

$$|\mathbf{r}'(t)| = \sqrt{t^2 + 1}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\cos t - t \sin t}{\sqrt{t^2 + 1}}\mathbf{i} + \frac{\sin t + t \cos t}{\sqrt{t^2 + 1}}\mathbf{j}$$

$$\mathbf{T}(1) = \frac{\cos 1 - \sin 1}{\sqrt{2}}\mathbf{i} + \frac{\sin 1 + \cos 1}{\sqrt{2}}\mathbf{j}$$

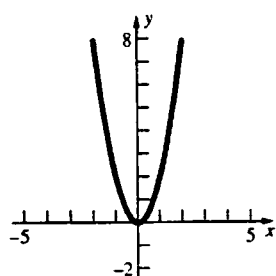
$$x'(t) = \cos t - t \sin t \quad y'(t) = \sin t + t \cos t$$

$$x''(t) = -2 \sin t - t \cos t \quad y''(t) = 2 \cos t - t \sin t$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{t^2 + 2}{(t^2 + 1)^{3/2}}$$

$$\kappa(1) = \frac{3}{2\sqrt{2}}$$

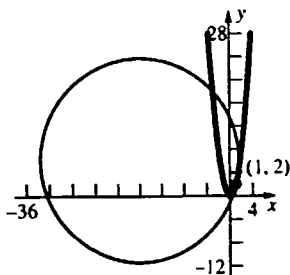
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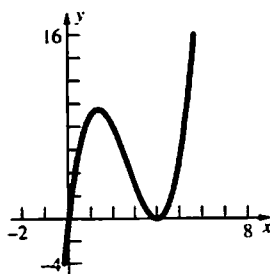
$$y' = 4x, y'' = 4$$

$$\kappa = \frac{4}{(1 + 16x^2)^{3/2}}$$

$$\text{At } (1, 2), \kappa = \frac{4}{17\sqrt{17}} \text{ and } R = \frac{17\sqrt{17}}{4}.$$



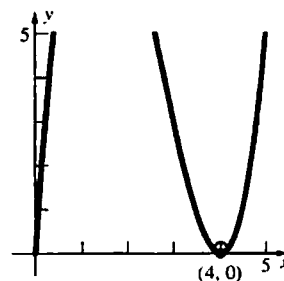
14.



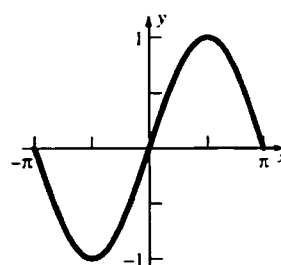
$$y' = 3x^2 - 16x + 16, y'' = 6x - 16$$

$$\kappa = \frac{|6x - 16|}{[1 + (3x^2 - 16x + 16)^2]^{3/2}}$$

$$\text{At } (4, 0), \kappa = \frac{8}{1} = 8 \text{ and } R = \frac{1}{8}.$$



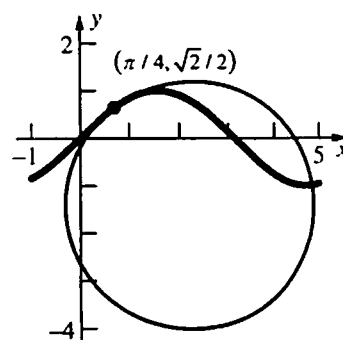
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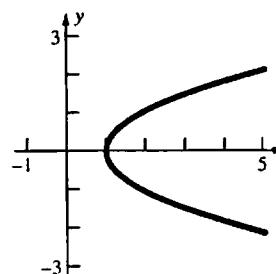
$$y' = -\sin x, y'' = -\cos x$$

$$\kappa = \frac{|\sin x|}{(1 + \cos^2 x)^{3/2}}$$

$$\text{At } \left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right), \kappa = \frac{\frac{1}{\sqrt{2}}}{\left(\frac{3}{2}\right)^{3/2}} = \frac{2}{3\sqrt{3}} \text{ and } R = \frac{3\sqrt{3}}{2}.$$



16.

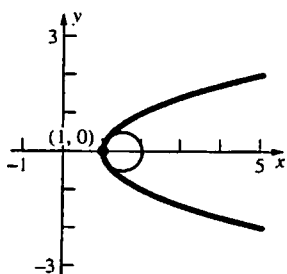


$$2yy' = 1$$

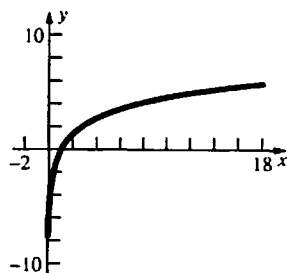
$$y' = \frac{1}{2y}, y'' = -\frac{y'}{2y^2} = -\frac{1}{4y^3}$$

$$\kappa = \frac{\left| \frac{1}{4y^3} \right|}{\left(1 + \frac{1}{4y^2} \right)^{3/2}} = \frac{2}{(4y^2 + 1)^{3/2}}$$

At $(1, 0)$, $\kappa = \frac{2}{1} = 2$ and $R = \frac{1}{2}$.



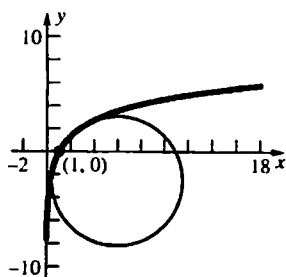
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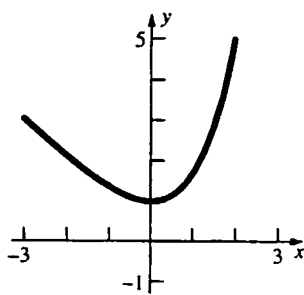
$$y' = \frac{2}{x}, y'' = -\frac{2}{x^2}$$

$$\kappa = \frac{\frac{2}{x^2}}{\left(1 + \frac{4}{x^2} \right)^{3/2}} = \frac{2|x|}{(x^2 + 4)^{3/2}}$$

At $(1, 0)$, $\kappa = \frac{2}{5\sqrt{5}}$ and $R = \frac{5\sqrt{5}}{2}$.



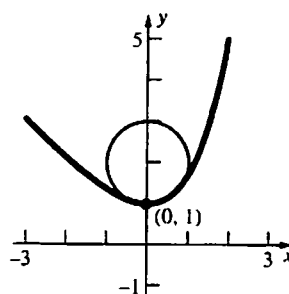
18.



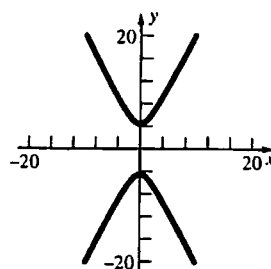
$$y' = e^x - 1, y'' = e^x$$

$$\kappa = \frac{e^x}{(e^{2x} - 2e^x + 2)^{3/2}}$$

At $(0, 1)$, $\kappa = \frac{1}{1} = 1$ and $R = 1$.



19.

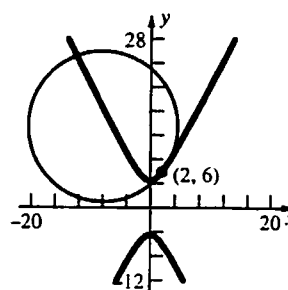


$$2yy' - 8x = 0$$

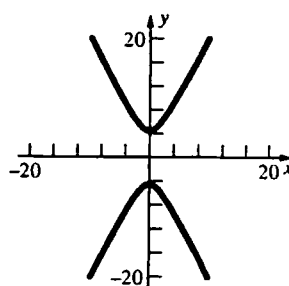
$$y' = \frac{4x}{y}, y'' = \frac{4(y - xy')}{y^2} = \frac{4(y^2 - 4x^2)}{y^3}$$

$$\kappa = \frac{\frac{4y^2 - 16x^2}{y^3}}{\left(1 + \frac{16x^2}{y^2} \right)^{3/2}} = \frac{4(y^2 - 4x^2)}{(y^2 + 16x^2)^{3/2}}$$

At $(2, 6)$, $\kappa = \frac{80}{1000} = \frac{2}{25}$ and $R = \frac{25}{2}$.



20.

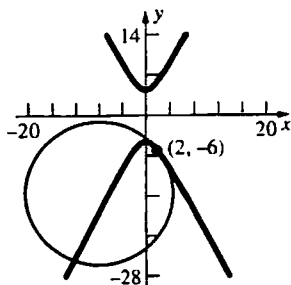


$$2yy' - 8x = 0$$

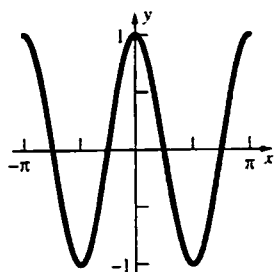
$$y' = \frac{4x}{y}, y'' = \frac{4(y - xy')}{y^2} = \frac{4(y^2 - 4x^2)}{y^3}$$

$$\kappa = \frac{\frac{4(y^2 - 4x^2)}{y^3}}{\left(1 + \frac{16x^2}{y^2}\right)^{3/2}} = \frac{4(y^2 - 4x^2)}{(y^2 + 16x^2)^{3/2}}$$

$$\text{At } (2, -6), \kappa = \frac{80}{1000} = \frac{2}{25} \text{ and } R = \frac{25}{2}.$$



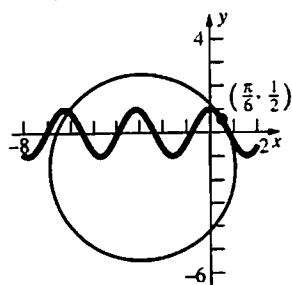
21.



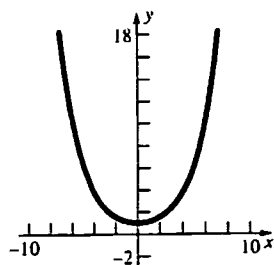
$$y' = -2\sin 2x, y'' = -4\cos 2x$$

$$\kappa = \frac{|4\cos 2x|}{(1 + 4\sin^2 2x)^{3/2}}$$

$$\text{At } \left(\frac{\pi}{6}, \frac{1}{2}\right), \kappa = \frac{2}{8} = \frac{1}{4} \text{ and } R = 4.$$



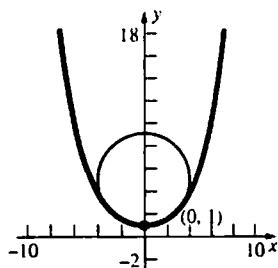
22.



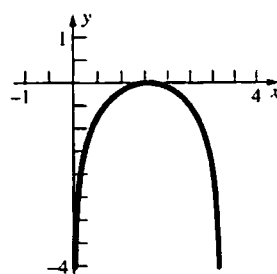
$$y' = \frac{1}{2} \sinh \frac{x}{2}, y'' = \frac{1}{4} \cosh \frac{x}{2}$$

$$\kappa = \frac{\left|\frac{1}{4} \cosh \frac{x}{2}\right|}{\left(1 + \frac{1}{4} \sinh^2 \frac{x}{2}\right)^{3/2}} = \frac{2 \left|\cosh \frac{x}{2}\right|}{\left(4 + \sinh^2 \frac{x}{2}\right)^{3/2}}$$

$$\text{At } (0, 1), \kappa = \frac{2}{8} = \frac{1}{4} \text{ and } R = 4.$$



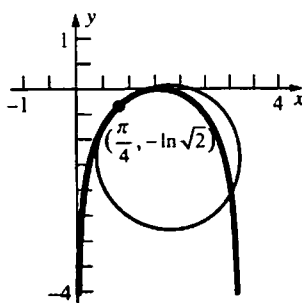
23.



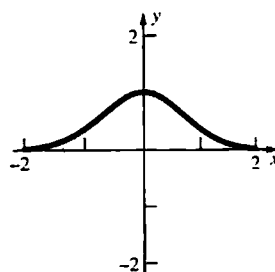
$$y' = \cot x, y'' = -\csc^2 x$$

$$\kappa = \frac{|\csc^2 x|}{(1 + \cot^2 x)^{3/2}} = \frac{1}{|\csc x|}$$

$$\text{At } \left(\frac{\pi}{4}, -\ln \sqrt{2}\right), \kappa = \frac{1}{\sqrt{2}} \text{ and } R = \sqrt{2}.$$



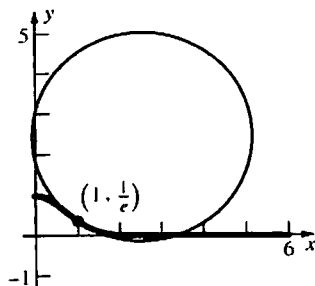
24.



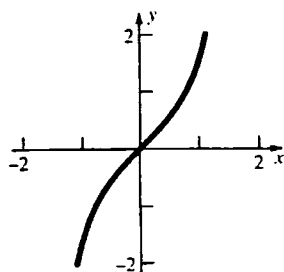
$$y' = -2xe^{-x^2}, y'' = (4x^2 - 2)e^{-x^2}$$

$$\kappa = \frac{|(4x^2 - 2)e^{-x^2}|}{(1 + 4x^2e^{-2x^2})^{3/2}} = \frac{e^{2x^2}|4x^2 - 2|}{(e^{2x^2} + 4x^2)^{3/2}}$$

At $(1, \frac{1}{e})$, $\kappa = \frac{2e^2}{(e^2 + 4)^{3/2}}$ and $R = \frac{(e^2 + 4)^{3/2}}{2e^2}$.



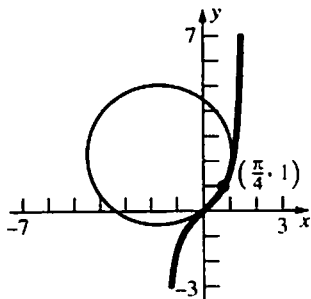
25.



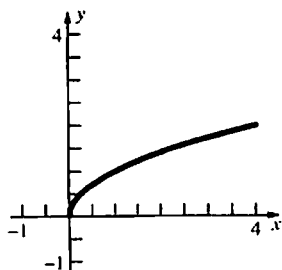
$$y' = \sec^2 x, y'' = 2 \sec^2 x \tan x$$

$$\kappa = \frac{|2 \sec^2 x \tan x|}{(1 + \sec^4 x)^{3/2}}$$

At $(\frac{\pi}{4}, 1)$, $\kappa = \frac{4}{5\sqrt{5}}$ and $R = \frac{5\sqrt{5}}{4}$.



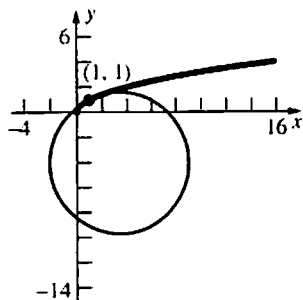
26.



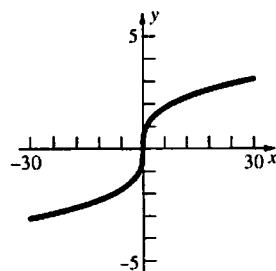
$$y' = \frac{1}{2\sqrt{x}}, y'' = -\frac{1}{4x^{3/2}}$$

$$\kappa = \frac{|\frac{1}{4x^{3/2}}|}{\left(1 + \frac{1}{4x}\right)^{3/2}} = \frac{2}{(4x+1)^{3/2}}$$

At $(1, 1)$, $\kappa = \frac{2}{5\sqrt{5}}$ and $R = \frac{5\sqrt{5}}{2}$.



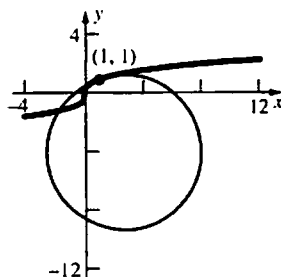
27.



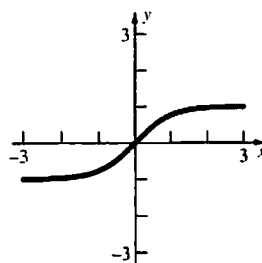
$$y' = \frac{1}{3x^{2/3}}, y'' = -\frac{2}{9x^{5/3}}$$

$$\kappa = \frac{|\frac{2}{9x^{5/3}}|}{\left(1 + \frac{1}{9x^{4/3}}\right)^{3/2}} = \frac{6x^{1/3}}{(9x^{4/3} + 1)^{3/2}}$$

At $(1, 1)$, $\kappa = \frac{6}{10\sqrt{10}} = \frac{3}{5\sqrt{10}}$ and $R = \frac{5\sqrt{10}}{3}$.



28.

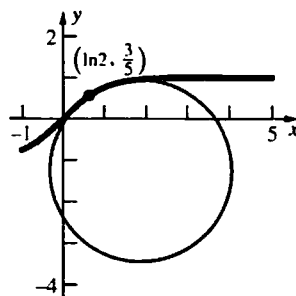


$$y' = \operatorname{sech}^2 x, y'' = -2\operatorname{sech}^2 x \tanh x$$

$$\kappa = \frac{|2\operatorname{sech}^2 x \tanh x|}{(1 + \operatorname{sech}^4 x)^{3/2}}$$

$$\text{At } \left(\ln 2, \frac{3}{5}\right), \kappa = \frac{\frac{96}{125}}{\left(\frac{881}{625}\right)^{3/2}} = \frac{12,000}{881\sqrt{881}} \text{ and}$$

$$R = \frac{881\sqrt{881}}{12,000}.$$



$$29. y' = \frac{1}{x}, y'' = -\frac{1}{x^2}$$

$$\kappa = \frac{\left|\frac{1}{x^2}\right|}{\left(1 + \frac{1}{x^2}\right)^{3/2}} = \frac{|x|}{(x^2 + 1)^{3/2}}$$

$$\text{Since } 0 < x < \infty, \kappa = \frac{x}{(x^2 + 1)^{3/2}}.$$

$$\kappa' = \frac{(x^2 + 1)^{3/2} - 3x^2(x^2 + 1)^{1/2}}{(x^2 + 1)^3} = \frac{-2x^2 + 1}{(x^2 + 1)^{5/2}}$$

$$\kappa' = 0 \text{ when } x = \frac{1}{\sqrt{2}}. \text{ Since } \kappa' > 0 \text{ on } \left(0, \frac{1}{\sqrt{2}}\right) \text{ and } \kappa' < 0 \text{ on } \left(\frac{1}{\sqrt{2}}, \infty\right), \text{ so } \kappa \text{ is maximum when}$$

$$x = \frac{1}{\sqrt{2}}, y = \ln \frac{1}{\sqrt{2}} = -\frac{\ln 2}{2}. \text{ The point of maximum curvature is } \left(\frac{1}{\sqrt{2}}, -\frac{\ln 2}{2}\right).$$

$$30. y' = \cos x, y'' = -\sin x$$

$$\kappa = \frac{|\sin x|}{(1 + \cos^2 x)^{3/2}}$$

$$\kappa' = \frac{\frac{|\sin x|}{\sin x} \cos x (1 + \cos^2 x)^{3/2} + 3|\sin x| \cos x \sin x (1 + \cos^2 x)^{1/2}}{(1 + \cos^2 x)^3}$$

$$= \frac{2|\sin x| \cot x (2 + \cos^2 x)}{(1 + \cos^2 x)^{5/2}}$$

$$\kappa' = 0 \text{ when } x = -\frac{\pi}{2}, \frac{\pi}{2}. \kappa' \text{ is not defined when } x = -\pi, 0. \text{ Since } \kappa' > 0 \text{ on } \left(-\pi, -\frac{\pi}{2}\right) \cup \left(0, \frac{\pi}{2}\right) \text{ and } \kappa' < 0 \text{ on}$$

$$\left(-\frac{\pi}{2}, 0\right) \cup \left(\frac{\pi}{2}, \pi\right), \text{ so } \kappa \text{ has local maxima when } x = -\frac{\pi}{2}, y = -1 \text{ and } x = \frac{\pi}{2}, y = 1.$$

$$\kappa\left(-\frac{\pi}{2}\right) = \kappa\left(\frac{\pi}{2}\right) = 1$$

$$\text{The points of maximum curvature are } (0, 1) \text{ and } \left(\frac{\pi}{2}, 1\right).$$

$$31. y' = \sinh x, y'' = \cosh x$$

$$\kappa = \frac{\cosh x}{(1 + \sinh^2 x)^{3/2}} = \operatorname{sech}^2 x$$

$$\kappa' = -2\operatorname{sech}^2 x \tanh x$$

$\kappa' = 0$ when $x = 0$. Since $\kappa' > 0$ on $(-\infty, 0)$ and $\kappa' < 0$ on $(0, \infty)$, so κ is maximum when $x = 0, y = 1$. The point of maximum curvature is $(0, 1)$.

32. $y' = \cosh x, y'' = \sinh x$

$$\kappa = \frac{|\sinh x|}{(1 + \cosh^2 x)^{3/2}}$$

$$\kappa' = \frac{\frac{|\sinh x|}{\sinh x} \cosh x (1 + \cosh^2 x)^{3/2} - 3|\sinh x| \cosh x \sinh x (1 + \cosh^2 x)^{1/2}}{(1 + \cosh^2 x)^3} = \frac{2|\sinh x| \coth x (2 - \cosh^2 x)}{(1 + \cosh^2 x)^{5/2}}$$

κ' is not defined when $x = 0$ and $\kappa' = 0$ when $\cosh x = \sqrt{2}$ or $x = \pm \ln(\sqrt{2} + 1)$. Since $\kappa' > 0$ on

$(-\infty, -\ln(\sqrt{2} + 1)) \cup (0, \ln(\sqrt{2} + 1))$ and $\kappa' < 0$ on $(-\ln(\sqrt{2} + 1), 0) \cup (\ln(\sqrt{2} + 1), \infty)$. κ has local maxima when $x = -\ln(\sqrt{2} + 1), y = -1$ and $x = \ln(\sqrt{2} + 1), y = 1$.

$$\kappa(-\ln(\sqrt{2} + 1)) = \kappa(\ln(\sqrt{2} + 1)) = \frac{1}{3\sqrt{3}}$$

The points of maximum curvature are $(-\ln(\sqrt{2} + 1), -1)$ and $(\ln(\sqrt{2} + 1), 1)$.

33. $y' = e^x, y'' = e^x$

$$\kappa = \frac{e^x}{(1 + e^{2x})^{3/2}}$$

$$\kappa' = \frac{e^x (1 + e^{2x})^{3/2} - 3e^{3x} (1 + e^{2x})^{1/2}}{(1 + e^{2x})^3} = \frac{e^x (1 - 2e^{2x})}{(1 + e^{2x})^{5/2}}$$

$\kappa' = 0$ when $x = -\frac{1}{2} \ln 2$. Since $\kappa' > 0$ on

$(-\infty, -\frac{1}{2} \ln 2)$ and $\kappa' < 0$ on $(-\frac{1}{2} \ln 2, \infty)$, so

κ is maximum when $x = -\frac{1}{2} \ln 2, y = \frac{1}{\sqrt{2}}$. The

point of maximum curvature is $(-\frac{1}{2} \ln 2, \frac{1}{\sqrt{2}})$.

34. $y' = -\tan x, y'' = -\sec^2 x$

$$\kappa = \frac{|\sec^2 x|}{(1 + \tan^2 x)^{3/2}} = |\cos x|$$

Since $-\frac{\pi}{2} < x < \frac{\pi}{2}, \kappa = \cos x$.

$$\kappa' = -\sin x$$

$\kappa' = 0$ when $x = 0$. Since $\kappa' > 0$ on $(-\frac{\pi}{2}, 0)$

and $\kappa' < 0$ on $(0, \frac{\pi}{2})$, κ is maximum when

$x = 0, y = 0$. The point of maximum curvature is $(0, 0)$.

35. $\mathbf{r}'(t) = 3\mathbf{i} + 6t\mathbf{j}$

$$\mathbf{r}''(t) = 6\mathbf{j}$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = 3\sqrt{1 + 4t^2}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{12t}{\sqrt{1 + 4t^2}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = 36 - \frac{144t^2}{1 + 4t^2} = \frac{36}{1 + 4t^2}$$

$$a_N = \frac{6}{\sqrt{1 + 4t^2}}$$

$$\text{At } t_1 = \frac{1}{3}, a_T = \frac{12}{\sqrt{13}} \text{ and } a_N = \frac{18}{\sqrt{13}}.$$

36. $\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$$\mathbf{r}''(t) = 2\mathbf{i}$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{4t^2 + 1}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{4t}{\sqrt{4t^2 + 1}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = 4 - \frac{16t^2}{4t^2 + 1} = \frac{4}{4t^2 + 1}$$

$$a_N = \frac{2}{\sqrt{4t^2 + 1}}$$

$$\text{At } t_1 = 1, a_T = \frac{4}{\sqrt{5}} \text{ and } a_N = \frac{2}{\sqrt{5}}.$$

$$37. \mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{r}''(t) = 2\mathbf{j}$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = 2\sqrt{1+t^2}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{2t}{\sqrt{1+t^2}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = 4 - \frac{4t^2}{1+t^2} = \frac{4}{1+t^2}$$

$$a_N = \frac{2}{\sqrt{1+t^2}}$$

$$\text{At } t_1 = -1, a_T = -\sqrt{2} \text{ and } a_N = \sqrt{2}.$$

$$38. \mathbf{r}'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j}$$

$$\mathbf{r}''(t) = -a \cos t \mathbf{i} - a \sin t \mathbf{j}$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = a$$

$$a_T = \frac{d^2s}{dt^2} = 0$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = a^2$$

$$a_N = a$$

$$\text{At } t_1 = \frac{\pi}{6}, a_T = 0 \text{ and } a_N = a.$$

$$39. \mathbf{r}'(t) = a \sinh t \mathbf{i} + a \cosh t \mathbf{j}$$

$$\mathbf{r}''(t) = a \cosh t \mathbf{i} + a \sinh t \mathbf{j}$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = a\sqrt{\sinh^2 t + \cosh^2 t}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{2a \cosh t \sinh t}{\sqrt{\sinh^2 t + \cosh^2 t}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = a^2(\cosh^2 t + \sinh^2 t) - \frac{4a^2 \cosh^2 t \sinh^2 t}{\sinh^2 t + \cosh^2 t} = \frac{a^2(\cosh^2 t - \sinh^2 t)^2}{\sinh^2 t + \cosh^2 t}$$

$$a_N = \frac{a(\cosh^2 t - \sinh^2 t)}{\sqrt{\sinh^2 t + \cosh^2 t}}$$

$$\text{At } t_1 = \ln 3, \cosh t = \frac{5}{3}, \sinh t = \frac{4}{3}, \text{ and } \sqrt{\sinh^2 t + \cosh^2 t} = \frac{\sqrt{41}}{3}, \text{ so } a_T = \frac{40a}{3\sqrt{41}} \text{ and } a_N = \frac{3a}{\sqrt{41}}.$$

$$40. \mathbf{r}'(t) = -3 \cos^2 t \sin t \mathbf{i} + 3 \sin^2 t \cos t \mathbf{j}$$

$$\mathbf{r}''(t) = 3 \cos t (2 \sin^2 t - \cos^2 t) \mathbf{i} + 3 \sin t (2 \cos^2 t - \sin^2 t) \mathbf{j}$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = 3 \cos t \sin t = \frac{3}{2} \sin 2t \quad (\text{Note that } t > 0.)$$

$$a_T = \frac{d^2s}{dt^2} = 3 \cos 2t$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = 9(\cos^6 t + \sin^6 t) - 9 \cos^2 2t = 9(\cos^6 t + \sin^6 t - \cos^2 2t)$$

$$a_N = 3\sqrt{\cos^6 t + \sin^6 t - \cos^2 2t}$$

$$\text{At } t_1 = \frac{\pi}{3}, a_T = -\frac{3}{2} \text{ and } a_N = \frac{3\sqrt{3}}{4}.$$

$$41. \mathbf{r}'(t) = e^t \mathbf{i} - e^{-t} \mathbf{j}; \mathbf{r}''(t) = e^t \mathbf{i} + e^{-t} \mathbf{j}$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{e^{2t} + e^{-2t}}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2$$

$$= e^{2t} + e^{-2t} - \frac{e^{4t} - 2 + e^{-4t}}{e^{2t} + e^{-2t}} = \frac{4}{e^{2t} + e^{-2t}}$$

$$a_N = \frac{2}{\sqrt{e^{2t} + e^{-2t}}}$$

At $t_1 = 0$, $a_T = 0$ and $a_N = \sqrt{2}$.

42. $x'(t) = 3$ $y'(t) = -6$

$x''(t) = 0$ $y''(t) = 0$

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = 3\sqrt{5}$$

$$a_T = \frac{d^2s}{dt^2} = 0$$

$$a_N^2 = [x''(t)^2 + y''(t)^2] - a_T^2 = 0; a_N = 0$$

At $t_1 = 0$, $a_T = 0$ and $a_N = 0$.

43. $x'(t) = (\sin t + \cos t)e^t$ $y'(t) = (\cos t - \sin t)e^t$

$x''(t) = (2 \cos t)e^t$ $y''(t) = (-2 \sin t)e^t$

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{2}e^t$$

$$a_T = \frac{d^2s}{dt^2} = \sqrt{2}e^t$$

$$a_N^2 = [x''(t)^2 + y''(t)^2] - a_T^2 = 4e^{2t} - 2e^{2t} = 2e^{2t}$$

$$a_N = \sqrt{2}e^t$$

At $t_1 = \frac{\pi}{3}$, $a_T = \sqrt{2}e^{\pi/3}$ and $a_N = \sqrt{2}e^{\pi/3}$.

44. $\tan \phi = y'$

$$(1 + y'^2)^{3/2} = (1 + \tan^2 \phi)^{3/2} = \sec^3 \phi$$

$$\kappa = \frac{|y''|}{[1 + y'^2]^{3/2}} = \frac{|y''|}{|\sec^3 \phi|} = |y'' \cos^3 \phi|$$

45. See the discussion dealing with s and a parameter t in the text. The important ingredients are that s increases as t increases and t can be expressed as a function of s . Use the Chain Rule.

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left| \frac{d\mathbf{T}}{ds} \right|} = \frac{\frac{\mathbf{T}'(t)}{|\mathbf{v}(t)|}}{\left| \frac{\mathbf{T}'(t)}{|\mathbf{v}(t)|} \right|} = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\frac{d\mathbf{T}}{dt}}{\left| \frac{d\mathbf{T}}{dt} \right|}$$

46. $\mathbf{r}(t) = a \cos \omega t \mathbf{i} + b \sin \omega t \mathbf{j}$

$\mathbf{v}(t) = -a\omega \sin \omega t \mathbf{i} + b\omega \cos \omega t \mathbf{j}$

$\mathbf{a}(t) = \langle -a\omega^2 \cos \omega t, -b\omega^2 \sin \omega t \rangle = -\omega^2 \mathbf{r}(t)$

$$\mathbf{T} = \frac{\mathbf{v}}{(\mathbf{v} \cdot \mathbf{v})^{1/2}};$$

$$\frac{d\mathbf{T}}{dt} = \frac{(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}}{(\mathbf{v} \cdot \mathbf{v})^{3/2}}$$

$$= \frac{-ab\omega}{(a^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{3/2}} (b \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j})$$

$$\left| \frac{d\mathbf{T}}{dt} \right| = \frac{ab\omega(b^2 \cos^2 \omega t + a^2 \sin^2 \omega t)^{1/2}}{(a^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{3/2}}$$

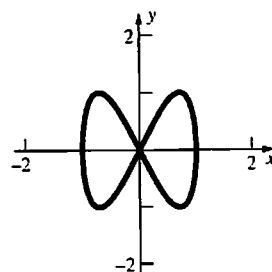
$$= \frac{ab\omega}{a^2 \sin^2 \omega t + b^2 \cos^2 \omega t}$$

Then

$$\frac{\frac{d\mathbf{T}}{dt}}{\left| \frac{d\mathbf{T}}{dt} \right|} = \frac{-1}{(a^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{1/2}} (b \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j})$$

Note that this was done assuming $ab > 0$; if $ab < 0$, drop the negative sign in the numerator.

47.



$\mathbf{v}(t) = \langle \cos t, 2 \cos 2t \rangle$, $\mathbf{a}(t) = \langle -\sin t, -4 \sin 2t \rangle$

$\mathbf{a}(t) = 0$ if and only if $-\sin t = 0$ and $-4 \sin 2t = 0$, which occurs if and only if $t = 0, \pi, 2\pi$, so it occurs only at the origin.

$\mathbf{a}(t)$ points to the origin if and only if $\mathbf{a}(t) = -k\mathbf{r}(t)$ for some k and $\mathbf{r}(t)$ is not $\mathbf{0}$. This occurs if and

only if $t = \frac{\pi}{2}, \frac{3\pi}{2}$, so it occurs only at $(1, 0)$ and $(-1, 0)$.

48. $\mathbf{v}(t) = \langle -\sin t + t \cos t + \sin t, \cos t + t \sin t - \cos t \rangle$

$= t \cos t \mathbf{i} + t \sin t \mathbf{j}$

$\mathbf{a}(t) = \langle -t \sin t + \cos t, t \cos t + \sin t \rangle$

a. $\frac{ds}{dt} = |\mathbf{v}(t)| = |t(\cos^2 t + \sin^2 t)^{1/2}| = t$
(since $t \geq 0$)

b. $a_T = \frac{d^2s}{dt^2} = \left(\frac{d}{dt} \right)(t) = 1$

$$a_N^2 = |\mathbf{a}|^2 - a_T^2$$

$$= [t^2(\sin^2 t + \cos^2 t) + (\cos^2 t + \sin^2 t)] - 1 = t^2$$

Therefore, $a_N = t$.

49. $s''(t) = a_T = 0 \Rightarrow \text{speed} = s'(t) = c$ (a constant)

$$\kappa \left(\frac{ds}{dt} \right)^2 = a_N = 0 \Rightarrow \kappa = 0 \text{ or } \frac{ds}{dt} = 0 \\ \Rightarrow \kappa = 0$$

$$50. \mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left| \frac{d\mathbf{T}}{ds} \right|} = \frac{\langle -\sin \phi, \cos \phi \rangle \left(\frac{d\phi}{ds} \right)}{(\sin^2 \phi + \cos^2 \phi)^{1/2} \left| \frac{d\phi}{ds} \right|} \\ = \frac{\frac{d\phi}{ds} \langle -\sin \phi, \cos \phi \rangle}{\left| \frac{d\phi}{ds} \right|}$$

If $\frac{d\phi}{ds} > 0$, $\mathbf{N} = \langle -\sin \phi, \cos \phi \rangle$ and if

$\frac{d\phi}{ds} < 0$, $\mathbf{N} = \langle \sin \phi, -\cos \phi \rangle$, so $\mathbf{N}$ points to the concave side of the curve in either case.

$$51. \mathbf{T} \cdot \mathbf{T} = 1 \Rightarrow \mathbf{T} \cdot \left(\frac{d\mathbf{T}}{ds} \right) + \mathbf{T} \cdot \left(\frac{d\mathbf{T}}{ds} \right) = 0 \\ \Rightarrow \mathbf{T} \cdot \left(\frac{d\mathbf{T}}{ds} \right) = 0 = \mathbf{T} \cdot \kappa \mathbf{N} = 0$$

Therefore, $\mathbf{T}$ is perpendicular to $\kappa \mathbf{N}$, and hence to $\mathbf{N}$.

52. $a_N = 0$ wherever $\kappa = 0$ or $\frac{ds}{dt} = 0$. κ , the curvature, is 0 at the inflection points, which occur at multiples of $\frac{\pi}{2}$. However, $\frac{ds}{dt} \neq 0$ on this curve. Therefore, $a_N = 0$ at multiples of $\frac{\pi}{2}$.

53. It is given that at $(-12, 16)$, $s'(t) = 10$ ft/s and $s''(t) = 5$ ft/s². From Example 2, $\kappa = \frac{1}{20}$. Therefore, $a_T = 5$ and $a_N = \left(\frac{1}{20} \right) (10)^2 = 5$, so $\mathbf{a} = 5\mathbf{T} + 5\mathbf{N}$. Let $\mathbf{r}(t) = \langle 20 \cos t, 20 \sin t \rangle$

describe the circle.

$$\mathbf{r}(t) = \langle -12, 16 \rangle \Rightarrow \cos t = -\frac{3}{5} \text{ and } \sin t = \frac{4}{5}.$$

$$\mathbf{v}(t) = \langle -20 \sin t, 20 \cos t \rangle, \text{ so } |\mathbf{v}(t)| = 20.$$

$$\text{Then } \mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \langle -\sin t, \cos t \rangle.$$

$$\text{Thus, at } (-12, 16), \mathbf{T} = \left\langle -\frac{4}{5}, -\frac{3}{5} \right\rangle \text{ and}$$

$$\mathbf{N} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle \text{ since } \mathbf{N} \text{ is a unit vector}$$

perpendicular to $\mathbf{T}$ and pointing to the concave side of the curve.

Therefore,

$$\mathbf{a} = 5 \left\langle -\frac{4}{5}, -\frac{3}{5} \right\rangle + 5 \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = -\mathbf{i} - 7\mathbf{j}.$$

54. $s'(t) = 4$ and $s''(t) = 0$.

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

Therefore,

$$\mathbf{a} = (0)\mathbf{T} + (4)^2 \frac{2}{(1 + 4x^2)^{3/2}} \mathbf{N} = \frac{32}{(1 + 4x^2)^{3/2}} \mathbf{N}.$$

55. Let $\mu mg = \frac{mv_R^2}{R}$. Then $v_R = \sqrt{\mu g R}$. At the values given, $v_R = \sqrt{(0.4)(32)(400)} = \sqrt{5120} \approx 71.55$ ft/s (about 49.79 mi/h).

56. a. $\frac{R|\mathbf{F}|\sin \theta}{v_R^2} = \frac{|\mathbf{F}|\cos \theta}{g}$ (from the given equations, equating m in each.) Therefore, $v_R = \sqrt{Rg \tan \theta}$.

- b. For the values given, $v_R = \sqrt{(400)(32)(\tan 10^\circ)} \approx 47.51$ ft/s.

57. Let the polar coordinate equation of the curve be $r = f(\theta)$. Then the curve is parametrized by $x = r \cos \theta$ and $y = r \sin \theta$.

$$x' = -r \sin \theta + r' \cos \theta$$

$$y' = r \cos \theta + r' \sin \theta$$

$$x'' = -r \cos \theta - 2r' \sin \theta + r'' \cos \theta$$

$$y'' = -r \sin \theta + 2r' \cos \theta + r'' \sin \theta$$

By Theorem A, the curvature is

$$\kappa = \frac{|x'y'' - y'x''|}{|x'^2 + y'^2|^{3/2}}$$

$$66. \quad \begin{aligned} r(t) &= f(t)i + g(t)j, \text{ where } x = f(t) \text{ and } y = g(t); \quad v(t) = r'(t) \\ T(t) &= \frac{v(t)}{|v(t)|} = \frac{x' \sqrt{x'^2 + y'^2} i + y' \sqrt{x'^2 + y'^2} j}{\sqrt{x'^2 + y'^2}} = \frac{x'}{\sqrt{x'^2 + y'^2}} i + \frac{y'}{\sqrt{x'^2 + y'^2}} j \\ N(t) &= \frac{T'(t)}{|T'(t)|} = \frac{\left(\frac{x''}{\sqrt{x'^2 + y'^2}} - \frac{x' y''}{(x'^2 + y'^2)^{3/2}} \right) i + \left(\frac{y''}{\sqrt{x'^2 + y'^2}} - \frac{y' x''}{(x'^2 + y'^2)^{3/2}} \right) j}{\sqrt{\left(\frac{x''}{\sqrt{x'^2 + y'^2}} - \frac{x' y''}{(x'^2 + y'^2)^{3/2}} \right)^2 + \left(\frac{y''}{\sqrt{x'^2 + y'^2}} - \frac{y' x''}{(x'^2 + y'^2)^{3/2}} \right)^2}} \\ \kappa &= \frac{|T'(t)|}{|r(t)|} = \frac{\sqrt{\left(\frac{x''}{\sqrt{x'^2 + y'^2}} - \frac{x' y''}{(x'^2 + y'^2)^{3/2}} \right)^2 + \left(\frac{y''}{\sqrt{x'^2 + y'^2}} - \frac{y' x''}{(x'^2 + y'^2)^{3/2}} \right)^2}}{\sqrt{x'^2 + y'^2}} \end{aligned}$$

$$67. \quad \begin{aligned} \kappa &= \frac{\left| \frac{e^6 + 9e^6}{10\sqrt{10}e^9} \right|}{\frac{1}{10\sqrt{10}e^9}} = \frac{10\sqrt{10}e^9}{10\sqrt{10}e^9} = 1 \\ \text{At } \theta = 1, \quad r' &= e^3, \quad r'' = 3e^3, \quad \text{and } r''' = 9e^3. \end{aligned}$$

$$68. \quad \begin{aligned} \kappa &= \frac{\left| \frac{16 + 32 - 0}{48} \right|}{\frac{128\sqrt{2}}{3}} = \frac{8\sqrt{2}}{128\sqrt{2}} = \frac{1}{16} \\ \text{At } \theta = \frac{\pi}{2}, \quad r' &= 4, \quad r'' = -4, \quad \text{and } r''' = 0. \\ 69. \quad r' &= -4\sin\theta, \quad r'' = -4\cos\theta \\ \kappa &= \frac{\left| \frac{1 + 1}{3} \right|}{\frac{2\sqrt{2}}{3}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \\ \text{At } \theta = 1, \quad r' &= 1, \quad r'' = 1, \quad \text{and } r''' = 0. \\ 70. \quad r' &= 1, \quad r'' = 0 \end{aligned}$$

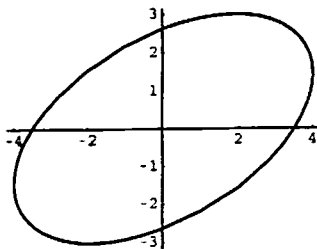
$$58. \quad \begin{aligned} \kappa &= \frac{\left| \frac{16\cos^2\theta + 32\sin^2\theta + 16\sin^2\theta}{32} \right|}{\frac{64}{1}} = \frac{16\cos^2\theta + 32\sin^2\theta + 16\sin^2\theta}{32} \\ r' &= -4\sin\theta, \quad r'' = -4\cos\theta \\ 59. \quad r' &= -\sin\theta, \quad r'' = -\cos\theta \\ \text{At } \theta = 0, \quad r' &= 0, \quad r'' = 2, \quad \text{and } r''' = -1. \\ \kappa &= \frac{\left| \frac{4 + 0 + 2}{6} \right|}{\frac{8}{3}} = \frac{4}{6} = \frac{2}{3} \\ \text{At } \theta = 0, \quad r' &= 1, \quad r'' = 1, \quad \text{and } r''' = 0. \end{aligned}$$

$$64. \quad \begin{aligned} \kappa &= \frac{\left| \frac{e^{12\theta} + 36e^{12\theta}}{37e^{12\theta}} \right|}{\frac{37\sqrt{37}e^{18\theta}}{37e^{12\theta}}} = \frac{1}{37\sqrt{37}} \\ \text{At } \theta = 0, \quad r' &= 6e^{6\theta}, \quad r'' = 36e^{6\theta} \\ \kappa &= \frac{\left| \frac{e^{6\theta}}{1} \right|}{\frac{\sqrt{37}}{1}} = \frac{1}{\sqrt{37}} \\ \text{At } \theta = 0, \quad r' &= 1, \quad r'' = 1, \quad \text{and } r''' = 0. \end{aligned}$$

$$63. \quad \begin{aligned} \kappa &= \frac{\left| \frac{64 + 0 + 32}{96} \right|}{\frac{512}{3}} = \frac{16}{512} = \frac{1}{32} \\ \text{At } \theta = \frac{\pi}{2}, \quad r' &= 8, \quad r'' = 0, \quad r''' = -4. \\ r' &= 4\cos\theta, \quad r'' = -4\sin\theta \end{aligned}$$

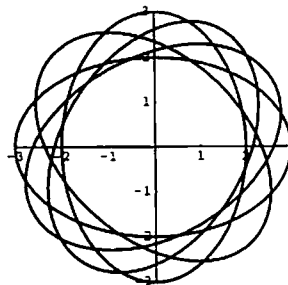
$$\begin{aligned} &= \frac{\left| \frac{r'^2 + 2r'r''}{r'^2 + r''^2} \right|}{\left| \frac{(-r'\sin\theta + r''\cos\theta)(-r'\sin\theta + r''\cos\theta) - (r'\cos\theta + r''\sin\theta)(r'\cos\theta + r''\sin\theta)}{r'^2 + r''^2} \right|} \\ &= \frac{\left| \frac{r'^2 + 2r'r''}{r'^2 + r''^2} \right|}{\left| \frac{(-r'\sin\theta + r''\cos\theta)^2 + (r'\cos\theta + r''\sin\theta)^2}{r'^2 + r''^2} \right|} \end{aligned}$$

67.



Maximum curvature ≈ 0.7606 ,
minimum curvature ≈ 0.1248

68. The following graph was plotted using Mathematica. The "AspectRatio" option was set to "Automatic."



The curves are ellipses with major axis length = 6 and minor axis length = 4. θ determines the angle of the major axis with the positive x -axis. From Example 4, the maximum curvature of such curves is $\frac{3}{4}$ and the minimum curvature is $\frac{2}{9}$.

69. If $a > b$, then the graph is an ellipse centered at the origin with the length of the major axis $2a$ and the length of the minor axis $2b$. The angle of the major axis with the positive x -axis is θ .

13.6 Chapter Review

Concepts Test

- False: For example, $x = 0$, $y = t$, and $x = 0$, $y = -t$ both represent the line $x = 0$.
- True: Eliminating the parameter gives $x = 2y$.
- False: For example, the graph of $x = t^2$, $y = t$ does not represent y as a function of x . $y = \pm\sqrt{x}$, but $h(x) = \pm\sqrt{x}$ is not a function.
- True: When $t = 1$, $x = 0$, and $y = 0$.
- False: For example, if $x = t^3$, $y = t^3$ then $y = x$ so $\frac{d^2y}{dx^2} = 0$, but $\frac{g''(t)}{f''(t)} = 1$.
- True: For example, the graph of the four-leaved rose has two tangent lines at the origin.
- True: $(2\mathbf{i} - 3\mathbf{j}) \cdot (6\mathbf{i} + 4\mathbf{j}) = 0$ if and only if the vectors are perpendicular.
- True: Since $\mathbf{u}$ and $\mathbf{v}$ are unit vectors,
$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \mathbf{u} \cdot \mathbf{v}.$$

- False: The dot product for three vectors $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ is not defined.
- True: $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos \theta| \leq |\mathbf{u}||\mathbf{v}|$ since $|\cos \theta| \leq 1$.
- True: If $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$, then $|\cos \theta| = 1$ since $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos \theta|$. Thus, $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. If $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$,
$$|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u} \cdot k\mathbf{v}| = k|\mathbf{u}|^2 = |\mathbf{u}||k\mathbf{v}| = |\mathbf{u}||\mathbf{v}|.$$
- False: If $\mathbf{u} = -\mathbf{i}$ and $\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$, then
$$\mathbf{u} + \mathbf{v} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} \text{ and } |\mathbf{u}| = |\mathbf{v}| = |\mathbf{u} + \mathbf{v}| = 1.$$
- True: $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$, so $|\mathbf{u}|^2 = |\mathbf{v}|^2$ or $|\mathbf{u}| = |\mathbf{v}|$.
- True:
$$|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u} \cdot \mathbf{v}$$
- True: Theorem 13.4A

16. True: $D_t[\mathbf{F}(t) \cdot \mathbf{F}(t)]$
 $= \mathbf{F}(t) \cdot \mathbf{F}'(t) + \mathbf{F}'(t) \cdot \mathbf{F}(t)$
 $= 2\mathbf{F}(t) \cdot \mathbf{F}'(t)$

17. True: $x' = 3$ $y' = 2$
 $x'' = 0$ $y'' = 0$
 Thus, $\kappa = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = 0$.

18. False: $x' = -2\sin t$ $y' = 2\cos t$
 $x'' = -2\cos t$ $y'' = -2\sin t$
 Thus, $\kappa = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{4}{8} = \frac{1}{2}$.

19. True: $\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|} \mathbf{r}'(t)$
 $\mathbf{T}'(t) = -\frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|^3} \mathbf{r}'(t) + \frac{1}{|\mathbf{r}'(t)|} \mathbf{r}''(t)$
 $\mathbf{T}(t) \cdot \mathbf{T}'(t)$
 $= \frac{1}{|\mathbf{r}'(t)|} \mathbf{r}'(t) \cdot$
 $\left[-\frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|^3} \mathbf{r}'(t) + \frac{1}{|\mathbf{r}'(t)|} \mathbf{r}''(t) \right]$
 $= -\frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|^2} + \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|^2} = 0$

20. False: See Example 3 of Section 13.4.
 $|\mathbf{v}| = 0$ but $|\mathbf{a}| = r\omega^2$.

21. True: $\kappa = \frac{|y''|}{[1 + y'^2]^{3/2}} = 0$

22. False: If $y'' = k$ then $y' = kx + C$ and
 $\kappa = \frac{|y''|}{[1 + y'^2]^{3/2}} = \frac{k}{[1 + (kx + C)^2]^{3/2}}$
 is not constant.

23. False: For example, if $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$, then
 $\mathbf{u} \cdot \mathbf{v} = 0$.

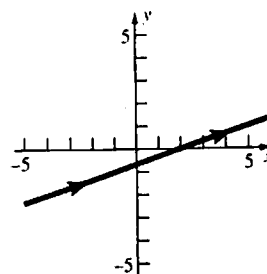
24. False: For example, if $\mathbf{r}(t) = \cos t^2 \mathbf{i} + \sin t^2 \mathbf{j}$,
 then $\mathbf{r}'(t) = -2t \sin t^2 \mathbf{i} + 2t \cos t^2 \mathbf{j}$, so
 $|\mathbf{r}(t)| = 1$ but $|\mathbf{r}'(t)| = 2t$.

25. True: If $\mathbf{v} \cdot \mathbf{v} = \text{constant}$, differentiate both
 sides to get $\mathbf{v} \cdot \mathbf{v}' + \mathbf{v} \cdot \mathbf{v}' = 2\mathbf{v} \cdot \mathbf{v}' = 0$,
 so $\mathbf{v} \cdot \mathbf{v}' = 0$.

Sample Test Problems

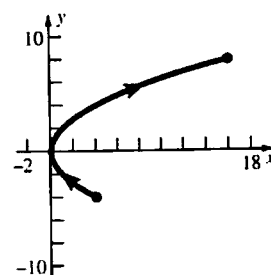
1. $t = \frac{1}{6}(x - 2)$

$y = \frac{1}{3}(x - 2)$



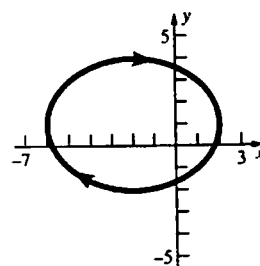
2. $t = \frac{y}{4}$

$x = \frac{y^2}{4}$ or $y^2 = 4x$



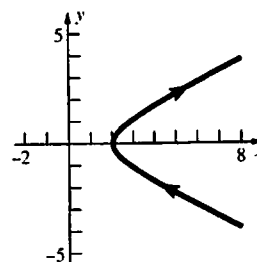
3. $\sin t = \frac{x+2}{4}$, $\cos t = \frac{y-1}{3}$

$\frac{(x+2)^2}{16} + \frac{(y-1)^2}{9} = 1$



4. $\frac{x^2}{4} = \sec^2 t$, $y^2 = \tan^2 t$

$\frac{x^2}{4} - y^2 = 1$



$$5. \frac{dx}{dt} = 6t^2 - 4, \frac{dy}{dt} = 1 + \frac{1}{t+1} = \frac{t+2}{t+1}$$

$$\frac{dy}{dx} = \frac{\frac{t+2}{t+1}}{6t^2 - 4} = \frac{t+2}{(t+1)(6t^2 - 4)}$$

$$\text{At } t = 0, x = 7, y = 0, \text{ and } \frac{dy}{dx} = -\frac{1}{2}.$$

$$\text{Tangent line: } y = -\frac{1}{2}(x - 7) \text{ or } x + 2y - 7 = 0$$

$$\text{Normal line: } y = 2(x - 7) \text{ or } 2x - y - 14 = 0.$$

$$6. \frac{dx}{dt} = -3e^{-t}, \frac{dy}{dt} = \frac{1}{2}e^t$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}e^t}{-3e^{-t}} = -\frac{1}{6}e^{2t}$$

$$\text{At } t = 0, x = 3, y = \frac{1}{2}, \text{ and } \frac{dy}{dx} = -\frac{1}{6}.$$

$$\text{Tangent line: } y - \frac{1}{2} = -\frac{1}{6}(x - 3) \text{ or } x + 6y - 6 = 0$$

$$\text{Normal line: } y - \frac{1}{2} = 6(x - 3) \text{ or}$$

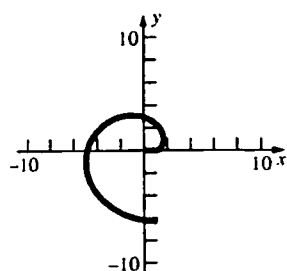
$$12x - 2y - 35 = 0.$$

$$7. \frac{dx}{dt} = -\sin t + \sin t + t \cos t = t \cos t$$

$$\frac{dy}{dt} = \cos t - \cos t + t \sin t = t \sin t$$

$$L = \int_0^{2\pi} \sqrt{(t \cos t)^2 + (t \sin t)^2} dt = \int_0^{2\pi} t dt$$

$$= \left[\frac{1}{2}t^2 \right]_0^{2\pi} = 2\pi^2$$



$$8. \text{ a. } \mathbf{u} = \langle -3, 1 \rangle, |\mathbf{u}| = \sqrt{10} \approx 3.1623$$

$$\text{ b. } \mathbf{u} = \langle -1, 5 \rangle, |\mathbf{u}| = \sqrt{26} \approx 5.0990$$

$$9. \text{ a. } 3\langle 2, -5 \rangle - 2\langle 1, 1 \rangle = \langle 6, -15 \rangle - \langle 2, 2 \rangle = \langle 4, -17 \rangle$$

$$\text{ b. } \langle 2, -5 \rangle \cdot \langle 1, 1 \rangle = 2 + (-5) = -3$$

$$\text{ c. } \langle 2, -5 \rangle \cdot (\langle 1, 1 \rangle + \langle -6, 0 \rangle) = \langle 2, -5 \rangle \cdot \langle -5, 1 \rangle \\ = -10 + (-5) = -15$$

$$\text{ d. } (4\langle 2, -5 \rangle + 5\langle 1, 1 \rangle) \cdot 3\langle -6, 0 \rangle$$

$$= \langle 13, -15 \rangle \cdot \langle -18, 0 \rangle$$

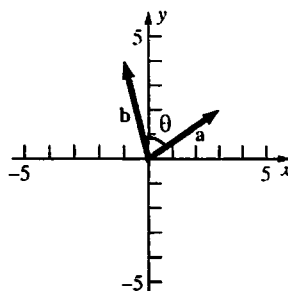
$$= -234 + 0 = -234$$

$$\text{ e. } \sqrt{36+0} \langle -6, 0 \rangle \cdot \langle 1, -1 \rangle = 6(-6+0) = -36$$

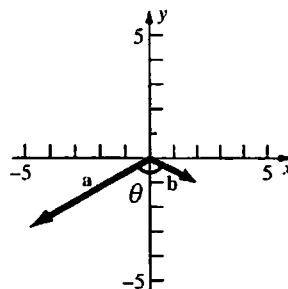
$$\text{ f. } \langle -6, 0 \rangle \cdot \langle -6, 0 \rangle - \sqrt{36+0}$$

$$= (36+0) - 6 = 30$$

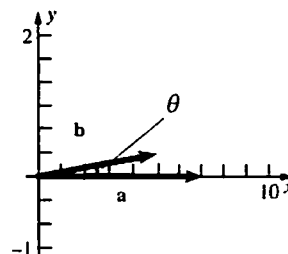
$$10. \text{ a. } \cos \theta = \frac{(3)(-1) + (2)(4)}{\sqrt{9+4}\sqrt{1+16}} = \frac{5}{\sqrt{221}} \approx 0.3363$$



$$\text{ b. } \cos \theta = \frac{(-5)(2) + (-3)(-1)}{\sqrt{25+9}\sqrt{4+1}} = -\frac{7}{\sqrt{170}} \\ \approx -0.5369$$



$$\text{ c. } \cos \theta = \frac{(7)(5) + (0)(1)}{\sqrt{49+0}\sqrt{25+1}} = \frac{5}{\sqrt{26}} \approx 0.9806$$



$$11. 5\mathbf{i} - 4\mathbf{j} = k(-2\mathbf{i}) + m(3\mathbf{i} - 2\mathbf{j}) \\ = (-2k + 3m)\mathbf{i} - (2m)\mathbf{j}, \text{ so } 5 = -2k + 3m \text{ and}$$

$$4 = 2m. \text{ Therefore, } m = 2 \text{ and } k = \frac{1}{2}.$$

$$12. y = x^2; y' = 2x, \text{ so the slope at } (-1, 1) \text{ is } -2 \\ (\text{y changes by } -2 \text{ as } x \text{ changes by } 1). \text{ Therefore, a}$$

vector parallel to the tangent line is $\langle 1, -2 \rangle$. To obtain a vector of length 3, first divide $\langle 1, -2 \rangle$ by its length and then multiply that result by 3.

$$\frac{\langle 1, -2 \rangle}{\|\langle 1, -2 \rangle\|} (3) = \frac{3\langle 1, -2 \rangle}{\sqrt{1+4}} = \left\langle \frac{3}{\sqrt{5}}, -\frac{6}{\sqrt{5}} \right\rangle$$

13. One vector that makes such an angle is $\langle \cos 150^\circ, \sin 150^\circ \rangle$. The required vector is then

$$10 \frac{\langle \cos 150^\circ, \sin 150^\circ \rangle}{\|\langle \cos 150^\circ, \sin 150^\circ \rangle\|} = 5\langle -\sqrt{3}, 1 \rangle.$$

14. $-(\mathbf{F}_1 + \mathbf{F}_2) = -5\mathbf{i} - 9\mathbf{j}$

15. Let the wind vector be

$$\mathbf{w} = \langle 100 \cos 30^\circ, 100 \sin 30^\circ \rangle = \langle 50\sqrt{3}, 50 \rangle.$$

Let $\mathbf{p} = \langle p_1, p_2 \rangle$ be the plane's air velocity vector.

$$\text{We want } \mathbf{w} + \mathbf{p} = 450\mathbf{j} = \langle 0, 450 \rangle.$$

$$\langle 50\sqrt{3}, 50 \rangle + \langle p_1, p_2 \rangle = \langle 0, 450 \rangle$$

$$\Rightarrow 50\sqrt{3} + p_1 = 0, 50 + p_2 = 450$$

$$\Rightarrow p_1 = -50\sqrt{3}, p_2 = 400$$

Therefore, $\mathbf{p} = \langle -50\sqrt{3}, 400 \rangle$. The angle β

formed with the vertical satisfies

$$\cos \beta = \frac{\mathbf{p} \cdot \mathbf{j}}{\|\mathbf{p}\| \|\mathbf{j}\|} = \frac{400}{\sqrt{167,500}}; \beta \approx 12.22^\circ. \text{ Thus, the}$$

heading is N12.22°W. The air speed is

$$\|\mathbf{p}\| = \sqrt{167,500} \approx 409.27 \text{ mi/h.}$$

16. $\mathbf{r}(t) = \langle e^{2t}, e^{-t} \rangle; \mathbf{r}'(t) = \langle 2e^{2t}, -e^{-t} \rangle$

a. $\lim_{t \rightarrow 0} \langle e^{2t}, e^{-t} \rangle = \left\langle \lim_{t \rightarrow 0} e^{2t}, \lim_{t \rightarrow 0} e^{-t} \right\rangle = \langle 1, 1 \rangle$

b. $\lim_{h \rightarrow 0} \frac{\mathbf{r}(0+h) - \mathbf{r}(0)}{h} = \mathbf{r}'(0) = \langle 2, -1 \rangle$

c. $\int_0^{\ln 2} \langle e^{2t}, e^{-t} \rangle dt = \left[\left\langle \frac{1}{2} e^{2t}, -e^{-t} \right\rangle \right]_0^{\ln 2}$
 $= \left\langle 2, -\frac{1}{2} \right\rangle - \left\langle \frac{1}{2}, -1 \right\rangle = \left\langle \frac{3}{2}, \frac{1}{2} \right\rangle$

d. $D_t[\mathbf{r}(t)] = t\mathbf{r}'(t) + \mathbf{r}(t)$
 $= t\langle 2e^{2t}, -e^{-t} \rangle + \langle e^{2t}, e^{-t} \rangle$
 $= \langle e^{2t}(2t+1), e^{-t}(1-t) \rangle$

e. $D_t[\mathbf{r}(3t+10)] = [\mathbf{r}'(3t+10)](3)$
 $= 3\langle 2e^{6t+20}, -e^{-3t-10} \rangle$
 $= \langle 6e^{6t+20}, -3e^{-3t-10} \rangle$

f. $D_t[\mathbf{r}(t) \cdot \mathbf{r}'(t)] = D_t[2e^{4t} - e^{-2t}]$
 $= 8e^{4t} + 2e^{-2t}$

17. a. $\mathbf{r}'(t) = \left\langle \frac{1}{t}, -6t \right\rangle; \mathbf{r}''(t) = \langle -t^{-2}, -6 \rangle$

b. $\mathbf{r}'(t) = \langle \cos t, -2 \sin 2t \rangle;$
 $\mathbf{r}''(t) = \langle -\sin t, -4 \cos 2t \rangle$

c. $\mathbf{r}'(t) = \langle \sec^2 t, -4t^3 \rangle;$
 $\mathbf{r}''(t) = \langle 2 \sec^2 t \tan t, -12t^2 \rangle$

18. $L = \int_0^2 |\mathbf{r}'(t)| dt = \int_0^2 3(4t+1)^{1/2} dt$
 $= \left[\frac{(4t+1)^{3/2}}{2} \right]_0^2 = \frac{27}{2} - \frac{1}{2} = 13$

19. $\mathbf{v}(t) = \langle 4t, 4 \rangle; \mathbf{a}(t) = \langle 4, 0 \rangle; \mathbf{v}(-1) = \langle -4, 4 \rangle;$
 $|\mathbf{v}(-1)| = 4\sqrt{2}; \mathbf{a}(-1) = \langle 4, 0 \rangle$

20. $\mathbf{v}(t) = \langle -4 \cos t, 4(1 - \sin t) \rangle;$
 $\mathbf{a}(t) = \langle 4 \sin t, -4 \cos t \rangle;$
 $\mathbf{v}\left(\frac{2\pi}{3}\right) = 2\mathbf{i} + (4 - 2\sqrt{3})\mathbf{j};$
 $\left| \mathbf{v}\left(\frac{2\pi}{3}\right) \right| = (32 - 16\sqrt{3})^{1/2} \approx 2.0706;$
 $\mathbf{a}\left(\frac{2\pi}{3}\right) = 2\sqrt{3}\mathbf{i} + 2\mathbf{j}$

21. a. $y = x^2 - x; y' = 2x - 1; y'' = 2;$
 $y'(1) = 1; y''(1) = 2;$
 $\kappa(1) = \frac{|2|}{(1+1)^{3/2}} = \frac{1}{\sqrt{2}} \approx 0.7071$

b. Let $x = t + t^3; x' = 1 + 3t^2; x'' = 6t;$
 $y = t + t^2; y' = 1 + 2t; y'' = 2.$
 At $(2, 2), t = 1$, so $x' = 4, x'' = 6,$
 $y' = 3, y'' = 2.$

Therefore, $\kappa = \frac{|(4)(2) - (3)(6)|}{(16+9)^{3/2}} = 0.08.$

$$\text{c. } y = a \cosh\left(\frac{x}{a}\right); y' = \sinh\left(\frac{x}{a}\right);$$

$$y'' = \left(\frac{1}{a}\right) \cosh\left(\frac{x}{a}\right).$$

At $(a, a \cosh 1)$,

$$y' = \sinh 1, y'' = \left(\frac{1}{a}\right) (\cosh 1).$$

Therefore,

$$\begin{aligned} \kappa &= \frac{\left(\frac{1}{a}\right) (\cosh 1)}{(1 + \sinh^2 1)^{3/2}} = \frac{\cosh 1}{a (\cosh^2 1)^{3/2}} \\ &= \frac{1}{a \cosh^2 1} \approx \frac{0.4120}{a}. \end{aligned}$$

$$22. \ x(t) = t, \ x'(t) = 1, \ x''(t) = 0; \ y(t) = \frac{1}{3}t^3,$$

$$y'(t) = t^2, \ y''(t) = 2t$$

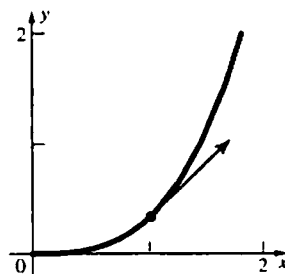
$$x'(1) = 1, \ x''(1) = 0$$

$$y'(1) = 1, \ y''(1) = 2$$

$$\mathbf{T}(1) = \frac{\mathbf{i} + \mathbf{j}}{[(1)^2 + (1)^2]} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\kappa(1) = \frac{2-0}{[1+1]^{3/2}} = \frac{1}{\sqrt{2}}$$

The Cartesian equation is $y = \left(\frac{1}{3}\right)x^3$.



$$23. \ \mathbf{v}(t) = -2t\mathbf{i} + 2\mathbf{j}; \ |\mathbf{v}(t)| = 2(t^2 + 1)^{1/2}; \ \mathbf{a}(t) = -2\mathbf{i};$$

$$|\mathbf{a}(t)| = 2$$

$$a_T = \frac{d^2s}{dt^2} = \left(\frac{d}{dt}\right)[2(t^2 + 1)^{1/2}] = 2t(t^2 + 1)^{-1/2}$$

At $(0, 2)$:

$$t = 1$$

$$a_T = \sqrt{2}$$

$$a_N = |\mathbf{a}|^2 - a_T^2 = 4 - 2 = 2, \text{ so } a_N = \sqrt{2}.$$

14.1 Concepts Review

1. coordinates

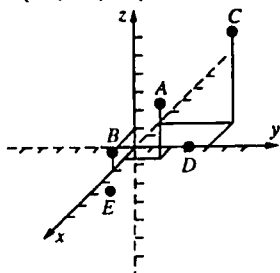
$$2. \sqrt{(x+1)^2 + (y-3)^2 + (z-5)^2}$$

3. $(-1, 3, 5); 4$

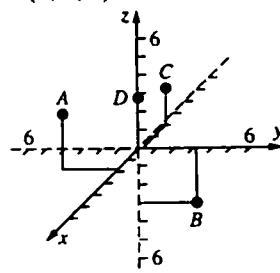
4. plane; 4; $-6; 3$

Problem Set 14.1

1. $A(1, 2, 3), B(2, 0, 1), C(-2, 4, 5), D(0, 3, 0), E(-1, -2, -3)$



2. $A(\sqrt{3}, -3, 3), B(0, \pi, -3), C(-2, \frac{1}{3}, 2), D(0, 0, e)$



3. $x = 0$ in the yz -plane. $x = 0$ and $y = 0$ on the z -axis.

4. $y = 0$ in the xz -plane. $x = 0$ and $z = 0$ on the y -axis.

5. a.
$$\sqrt{(6-1)^2 + (-1-2)^2 + (0-3)^2}$$
$$= \sqrt{25+9+9} = \sqrt{43}$$

b.
$$\sqrt{(-2-2)^2 + (-2+2)^2 + (0+3)^2}$$
$$= \sqrt{16+0+9} = 5$$

c.
$$\sqrt{(e+\pi)^2 + (\pi+4)^2 + (0-\sqrt{3})^2}$$
$$\sqrt{(e+\pi)^2 + (\pi+4)^2} + 3 \approx 9.399$$

6. $P(4, 5, 3), Q(1, 7, 4), R(2, 4, 6)$

$$|PQ| = \sqrt{(4-1)^2 + (5-7)^2 + (3-4)^2}$$
$$= \sqrt{9+4+1} = \sqrt{14}$$

$$|PR| = \sqrt{(4-2)^2 + (5-4)^2 + (3-6)^2}$$
$$= \sqrt{4+1+9} = \sqrt{14}$$

$$|QR| = \sqrt{(1-2)^2 + (7-4)^2 + (4-6)^2}$$
$$= \sqrt{1+9+4} = \sqrt{14}$$

Since the distances are equal, the triangle formed by joining P, Q , and R is equilateral.

7. $P(2, 1, 6), Q(4, 7, 9), R(8, 5, -6)$

$$|PQ| = \sqrt{(2-4)^2 + (1-7)^2 + (6-9)^2}$$
$$= \sqrt{4+36+9} = 7$$

$$|PR| = \sqrt{(2-8)^2 + (1-5)^2 + (6+6)^2}$$
$$= \sqrt{36+16+144} = 14$$

$$|QR| = \sqrt{(4-8)^2 + (7-5)^2 + (9+6)^2}$$
$$= \sqrt{16+4+225} = \sqrt{245}$$

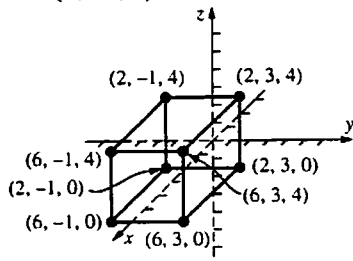
$|PQ|^2 + |PR|^2 = 49 + 196 = 245 = |QR|^2$, so the triangle formed by joining P, Q , and R is a right triangle, since it satisfies the Pythagorean Theorem.

8. a. The distance to the xy -plane is 1 since the point is 1 unit below the plane.

- b. The distance is
$$\sqrt{(2-0)^2 + (3-3)^2 + (-1-0)^2} = \sqrt{5}$$
 since the distance from a point to a line is the length of the shortest segment joining the point and the line. Using the point $(0, 3, 0)$ on the y -axis clearly minimizes the length.

c. $\sqrt{(2-0)^2 + (3-0)^2 + (-1-0)^2}$
 $= \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$

9. Since the faces are parallel to the coordinate planes, the sides of the box are in the planes $x = 2$, $y = 3$, $z = 4$, $x = 6$, $y = -1$, and $z = 0$ and the vertices are at the points where 3 of these planes intersect. Thus, the vertices are $(2, 3, 4)$, $(2, 3, 0)$, $(2, -1, 4)$, $(2, -1, 0)$, $(6, 3, 4)$, $(6, 3, 0)$, $(6, -1, 4)$, and $(6, -1, 0)$



10. It is parallel to the y -axis; $x = 2$ and $z = 3$. (If it were parallel to the x -axis, the y -coordinate could not change, similarly for the z -axis.)

11. a. $(x-1)^2 + (y-2)^2 + (z-3)^2 = 25$

b. $(x+2)^2 + (y+3)^2 + (z+6)^2 = 5$

c. $(x-\pi)^2 + (y-e)^2 + (z-\sqrt{2})^2 = \pi$

12. Since the sphere is tangent to the xy -plane, the point $(2, 4, 0)$ is on the surface of the sphere. Hence, the radius of the sphere is 5 so the equation is $(x-2)^2 + (y-4)^2 + (z-5)^2 = 25$.

13. $(x^2 - 12x + 36) + (y^2 + 14y + 49) + (z^2 - 8z + 16) = -1 + 36 + 49 + 16$

$$(x-6)^2 + (y+7)^2 + (z-4)^2 = 100$$

Center: $(6, -7, 4)$; radius 10

14. $(x^2 + 2x + 1) + (y^2 - 6y + 9) + (z^2 - 10z + 25) = -34 + 1 + 9 + 25$

$$(x+1)^2 + (y-3)^2 + (z-5)^2 = 1$$

Center: $(-1, 3, 5)$; radius 1

15. $x^2 + y^2 + z^2 - x + 2y + 4z = \frac{13}{4}$

$$\left(x^2 - x + \frac{1}{4}\right) + (y^2 + 2y + 1) + (z^2 + 4z + 4) = \frac{13}{4} + \frac{1}{4} + 1 + 4$$

$$\left(x - \frac{1}{2}\right)^2 + (y+1)^2 + (z+2)^2 = \frac{17}{2}$$

Center: $\left(\frac{1}{2}, -1, -2\right)$; radius $\sqrt{\frac{17}{2}} \approx 2.92$

16. $(x^2 + 8x + 16) + (y^2 - 4y + 4) + (z^2 - 22z + 121) = -77 + 16 + 4 + 121$

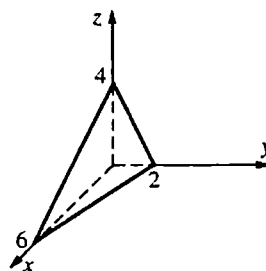
$$(x+4)^2 + (y-2)^2 + (z-11)^2 = 64$$

Center: $(-4, 2, 11)$; radius 8

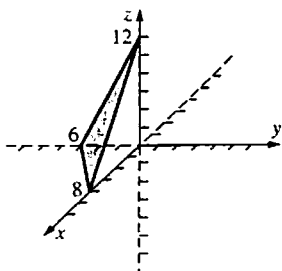
17. x -intercept: $y = z = 0 \Rightarrow 2x = 12, x = 6$

y -intercept: $x = z = 0 \Rightarrow 6y = 12, y = 2$

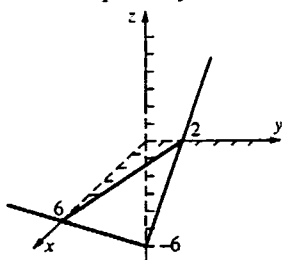
z -intercept: $x = y = 0 \Rightarrow 3z = 12, z = 4$



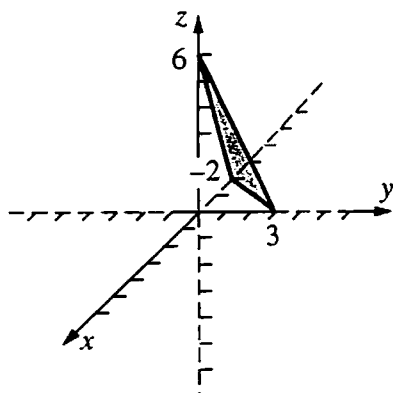
18. x -intercept: $y = z = 0 \Rightarrow 3x = 24, x = 8$
 y -intercept: $y = z = 0 \Rightarrow -4y = 24, y = -6$
 z -intercept: $x = y = 0 \Rightarrow 2z = 24, z = 12$



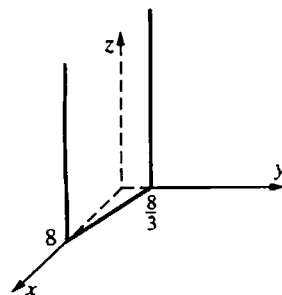
19. x -intercept: $y = z = 0 \Rightarrow x = 6$
 y -intercept: $x = z = 0 \Rightarrow 3y = 6, y = 2$
 z -intercept: $x = y = 0 \Rightarrow -z = 6, z = -6$



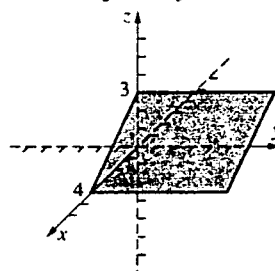
20. x -intercept: $y = z = 0 \Rightarrow -3x = 6, x = -2$
 y -intercept: $x = z = 0 \Rightarrow 2y = 6, y = 3$
 z -intercept: $x = y = 0 \Rightarrow z = 6$



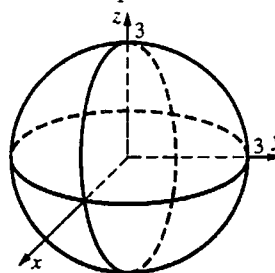
21. x and y cannot both be zero, so the plane is parallel to the z -axis.
 x -intercept: $y = z = 0 \Rightarrow x = 8$
 y -intercept: $x = z = 0 \Rightarrow 3y = 8, y = \frac{8}{3}$



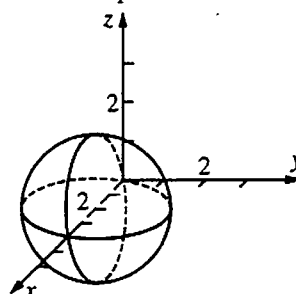
22. x and z cannot both be zero, so the plane is parallel to the y -axis.
 x -intercept: $y = z = 0 \Rightarrow 3x = 12, x = 4$
 z -intercept: $x = y = 0 \Rightarrow 4z = 12, z = 3$



23. This is a sphere with center $(0, 0, 0)$ and radius 3.



24. This is a sphere with center $(2, 0, 0)$ and radius 2.



25. The center of the sphere is the midpoint of the diameter, so it is

$$\left(\frac{-2+4}{2}, \frac{3-1}{2}, \frac{6+5}{2} \right) = \left(1, 1, \frac{11}{2} \right).$$

The radius is $\frac{1}{2} \sqrt{(-2-4)^2 + (3+1)^2 + (6-5)^2} = \frac{\sqrt{53}}{2}$. The

equation is $(x-1)^2 + (y-1)^2 + \left(z - \frac{11}{2} \right)^2 = \frac{53}{4}$.

26. Since the spheres are tangent and have equal radii, the radius of each sphere is $\frac{1}{2}$ of the distance between the centers.

$$r = \frac{1}{2} \sqrt{(-3-5)^2 + (1+3)^2 + (2-6)^2} = 2\sqrt{6}.$$

The spheres are $(x+3)^2 + (y-1)^2 + (z-2)^2 = 24$ and $(x-5)^2 + (y+3)^2 + (z-6)^2 = 24$.

27. The center must be 6 units from each coordinate plane. Since it is in the first octant, the center is

(6, 6, 6). The equation is
 $(x-6)^2 + (y-6)^2 + (z-6)^2 = 36$.

28. $x + y = 12$ is parallel to the z -axis. The distance from $(1, 1, 4)$ to the plane $x + y = 12$ is the same as the distance in the xy -plane of $(1, 1, 0)$ to the line $x + y - 12 = 0$. That distance is

$$\frac{|1+1-12|}{(1+1)^{1/2}} = 5\sqrt{2}.$$

The equation of the sphere is

$$(x-1)^2 + (y-1)^2 + (z-4)^2 = 50.$$

29. a. Plane parallel to and two units above the xy -plane
 b. Plane perpendicular to the xy -plane whose trace in the xy -plane is the line $x = y$.

31. If $P(x, y, z)$ denotes the moving point,

$$\sqrt{(x-1)^2 + (y-2)^2 + (z+3)^2} = 2\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2},$$

which simplifies to $(x-1)^2 + (y-2)^2 + (z-5)^2 = 16$, is a sphere with radius 4 and center $(1, 2, 5)$.

32. If $P(x, y, z)$ denotes the moving point,

$$\sqrt{(x-1)^2 + (y-2)^2 + (z+3)^2} = \sqrt{(x-2)^2 + (y-3)^2 + (z-2)^2},$$

which simplifies to $x + y + 5z = 3/2$, a plane.

33. Note that the volume of a segment of height h in a hemisphere of radius r is $\pi h^2 \left[r - \left(\frac{h}{3} \right) \right]$.

The resulting solid is the union of two segments, one for each ball. Since the two balls have the same radius, each segment will have the same value for h . h is the radius minus half the distance between the centers of the two balls.

$$h = 2 - \frac{1}{2}\sqrt{(2-1)^2 + (4-2)^2 + (3-1)^2} = 2 - \frac{3}{2} = \frac{1}{2}$$

$$V = 2 \left[\pi \left(\frac{1}{2} \right)^2 \left(2 - \frac{1}{6} \right) \right] = \frac{11\pi}{12}$$

34. As in Problem 33, the resulting solid is the union of two segments. Since the radii are not the same, the segments will have different heights. Let h_1 be the height of the segment from the first ball and let h_2 be the height from the second ball. $r_1 = 2$ is the radius of the first ball and $r_2 = 3$ is the radius of the second ball.

Solving for the equation of the plane containing the intersection of the spheres $(x-1)^2 + (y-2)^2 + (z-1)^2 - 4 = 0$ and $(x-2)^2 + (y-4)^2 + (z-3)^2 - 9 = 0$, we get $x + 2y + 2z - 9 = 0$.

The distance from $(1, 2, 1)$ to the plane is $\frac{2}{3}$, and the distance from $(2, 4, 3)$ to the plane is $\frac{7}{3}$. (Use the formula in

Example 6 of Section 14.2.)

$$h_1 = 2 - \frac{2}{3} = \frac{4}{3}; h_2 = 3 - \frac{7}{3} = \frac{2}{3}$$

$$V = \pi \left(\frac{4}{3} \right)^2 \left(2 - \frac{4}{9} \right) + \pi \left(\frac{2}{3} \right)^2 \left(3 - \frac{2}{9} \right) = 4\pi$$

- c. Union of the yz -plane ($x = 0$) and the xz -plane ($y = 0$)
 d. Union of the three coordinate planes
 e. Cylinder of radius 2, parallel to the z -axis
 f. Top half of the sphere with center $(0, 0, 0)$ and radius 3

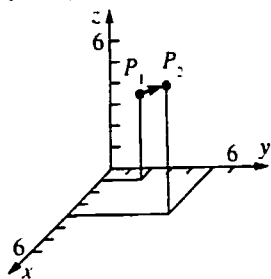
30. The points of the intersection satisfy both
 $(x-1)^2 + (y+2)^2 + (z+1)^2 = 10$ and $z = 2$, so
 $(x-1)^2 + (y+2)^2 + (2+1)^2 = 10$. This simplifies to $(x-1)^2 + (y+2)^2 = 1$, the equation of a circle of radius 1. The center is $(1, -2, 2)$.

14.2 Concepts Review

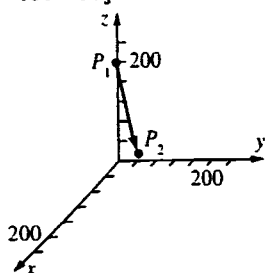
- $\sqrt{2^2 + (-3)^2 + (\sqrt{3})^2} = \sqrt{4+9+3} = 4;$
 $2 \cdot 3 + (-3) \cdot 2 + \sqrt{3} \cdot (-2\sqrt{3}) = 6 - 6 - 6 = -6$
- dot product; 0
- $(2, -1, 0); \langle 3, -2, 4 \rangle$
- $|u||v|\cos\theta; \cos^{-1} \frac{u \cdot v}{|u||v|}$

Problem Set 14.2

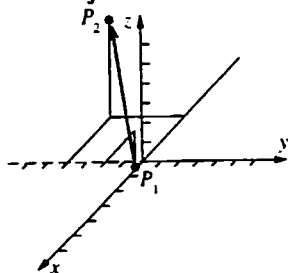
- a. $(4-1)\mathbf{i} + (5-2)\mathbf{j} + (6-4)\mathbf{k} = 3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$



- b. $(-14+1)\mathbf{i} + (52+3)\mathbf{j} + (26-204)\mathbf{k}$
 $= -13\mathbf{i} + 55\mathbf{j} - 178\mathbf{k}$

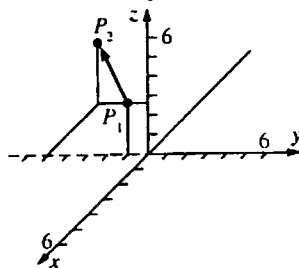


- a. $(-3+2)\mathbf{i} + (-4+2)\mathbf{j} + (5+2)\mathbf{k}$
 $= -\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$



- b. $(-\sqrt{14}-0)\mathbf{i} + (-5+1)\mathbf{j} + (\pi-e)\mathbf{k}$
 $= -\sqrt{14}\mathbf{i} - 4\mathbf{j} + (\pi-e)\mathbf{k}$

$$\approx -3.74\mathbf{i} - 4\mathbf{j} + 0.42\mathbf{k}$$



- length $= \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21} \approx 4.58$
 $\cos\alpha = \frac{4}{\sqrt{21}} \approx 0.87, \cos\beta = \frac{1}{\sqrt{21}} \approx 0.22,$
 $\cos\gamma = \frac{2}{\sqrt{21}} \approx 0.44$
 - length $= \sqrt{(-2)^2 + (-3)^2 + 7^2} = \sqrt{62} \approx 7.87$
 $\cos\alpha = -\frac{2}{\sqrt{62}} \approx -0.25,$
 $\cos\beta = -\frac{3}{\sqrt{62}} \approx -0.38, \cos\gamma = \frac{7}{\sqrt{62}} \approx 0.89$
- length $= \sqrt{2^2 + (-1)^2 + (-2)^2} = 3$
 $\cos\alpha = \frac{2}{3}, \cos\beta = -\frac{1}{3}, \cos\gamma = -\frac{2}{3}$
 - length $= \sqrt{(-1)^2 + 2^2 + (-2)^2} = 3$
 $\cos\alpha = -\frac{1}{3}, \cos\beta = \frac{2}{3}, \cos\gamma = -\frac{2}{3}$
- $|\langle 3, -4, 5 \rangle| = \sqrt{3^2 + (-4)^2 + 5^2} = 5\sqrt{2}$
 $\frac{1}{5\sqrt{2}} \langle 3, -4, 5 \rangle = \left\langle \frac{3}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ is the unit vector with the same direction as $\langle 3, -4, 5 \rangle$.
 $-5 \left\langle \frac{3}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -\frac{3}{\sqrt{2}}, \frac{4}{\sqrt{2}}, -\frac{5}{\sqrt{2}} \right\rangle$ is a vector of length 5 oriented in the opposite direction.
- $|-4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}| = \sqrt{(-4)^2 + 3^2 + (-2)^2} = \sqrt{29}$
 $-\frac{10}{\sqrt{29}}(-4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = \frac{40}{\sqrt{29}}\mathbf{i} - \frac{30}{\sqrt{29}}\mathbf{j} + \frac{20}{\sqrt{29}}\mathbf{k}$ is a vector of length of 10 with direction opposite $-4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

$$7. \cos \theta = \frac{\langle 4, -3, -1 \rangle \cdot \langle -2, -3, 5 \rangle}{\|\langle 4, -3, -1 \rangle\| \|\langle -2, -3, 5 \rangle\|}$$

$$= \frac{(4)(-2) + (-3)(-3) + (-1)(5)}{\sqrt{16+9+1}\sqrt{4+9+25}} = \frac{-4}{\sqrt{26}\sqrt{38}}$$

$$= -\frac{2}{\sqrt{247}}$$

$$\theta = \cos^{-1}\left(-\frac{2}{\sqrt{247}}\right) \approx 97^\circ$$

$$8. \cos \theta = \frac{(-4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + 5\mathbf{k})}{\| -4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \| \| 2\mathbf{i} + \mathbf{j} + 5\mathbf{k} \|}$$

$$= \frac{9}{\sqrt{29}\sqrt{30}} = \frac{9}{\sqrt{870}}$$

$$\theta = \cos^{-1} \frac{9}{\sqrt{870}} \approx 72^\circ$$

9. If $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is perpendicular to $-4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + \mathbf{j}$, then $-4x + 5y + z = 0$ and $4x + y = 0$ since the dot product of perpendicular vectors is 0. Solving these equations yields $y = -4x$ and $z = 24x$. Hence, for any x , $x\mathbf{i} - 4x\mathbf{j} + 24x\mathbf{k}$ is perpendicular to the given vectors.

$$\|x\mathbf{i} - 4x\mathbf{j} + 24x\mathbf{k}\| = \sqrt{x^2 + 16x^2 + 576x^2}$$

$$= |x|\sqrt{593}$$

This length is 10 when $x = \pm \frac{10}{\sqrt{593}}$. The vectors

$$\text{are } \frac{10}{\sqrt{593}}\mathbf{i} - \frac{40}{\sqrt{593}}\mathbf{j} + \frac{240}{\sqrt{593}}\mathbf{k} \text{ and } -\frac{10}{\sqrt{593}}\mathbf{i} + \frac{40}{\sqrt{593}}\mathbf{j} - \frac{240}{\sqrt{593}}\mathbf{k}.$$

10. If $\langle x, y, z \rangle$ is perpendicular to both $\langle 1, -2, -3 \rangle$ and $\langle -3, 2, 0 \rangle$, then $x - 2y - 3z = 0$ and $-3x + 2y = 0$.

Solving these equations for y and z in terms of x yields $y = \frac{3}{2}x$, $z = -\frac{2}{3}x$. Thus, all the vectors

have the form $\left\langle x, \frac{3}{2}x, -\frac{2}{3}x \right\rangle$ where x is a real number.

11. A vector equivalent to $\overline{BA}$ is $\mathbf{u} = \langle 1 + 4, 2 - 5, 3 - 6 \rangle = \langle 5, -3, -3 \rangle$.

A vector equivalent to $\overline{BC}$ is $\mathbf{v} = \langle 1 + 4, 0 - 5, 1 - 6 \rangle = \langle 5, -5, -5 \rangle$.

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{5 \cdot 5 + (-3)(-5) + (-3)(-5)}{\sqrt{25+9+9}\sqrt{25+25+25}}$$

$$= \frac{55}{\sqrt{43}\sqrt{75}} = \frac{11}{\sqrt{129}}, \text{ so } \theta = \cos^{-1} \frac{11}{\sqrt{129}} \approx 14.4^\circ.$$

12. A vector equivalent to $\overline{BA}$ is $\mathbf{u} = \langle 6 - 3, 3 - 1, 3 + 1 \rangle = \langle 3, 2, 4 \rangle$.

A vector equivalent to $\overline{BC}$ is

$$\mathbf{v} = \langle -1 - 3, 10 - 1, -2.5 + 1 \rangle = \langle -4, 9, -1.5 \rangle.$$

$\mathbf{u} \cdot \mathbf{v} = 3(-4) + 2 \cdot 9 + 4 \cdot (-1.5) = 0$ so $\mathbf{u}$ is perpendicular to $\mathbf{v}$. Thus the angle at B is a right angle.

13. $\mathbf{u} \cdot \mathbf{v} = (-1)(-1) + 5 \cdot 1 + 3(-1) = 3$

$$\|\mathbf{v}\| = \sqrt{1+1+1} = \sqrt{3}$$

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

14. $\mathbf{u} \cdot \mathbf{v} = 5(-\sqrt{5}) + 5(\sqrt{5}) + 2(1) = 2$

$$\|\mathbf{v}\| = \sqrt{5+5+1} = \sqrt{11}$$

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{\sqrt{11}}$$

15. $\mathbf{m} = \text{pr}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}$

$$= \frac{(-3)(-3) + 2 \cdot 5 + 1 \cdot (-3)}{9 + 25 + 9} (-3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$$

$$= \frac{16}{43} (-3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) = -\frac{48}{43}\mathbf{i} + \frac{80}{43}\mathbf{j} - \frac{48}{43}\mathbf{k}$$

$$\mathbf{n} = \mathbf{u} - \mathbf{m} = \left(-3 + \frac{48}{43}\right)\mathbf{i} + \left(2 - \frac{80}{43}\right)\mathbf{j} + \left(1 + \frac{48}{43}\right)\mathbf{k}$$

$$= -\frac{81}{43}\mathbf{i} + \frac{6}{43}\mathbf{j} + \frac{91}{43}\mathbf{k}$$

16. $\mathbf{m} = \text{pr}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}$

$$= \frac{e \cdot 1 + \pi \cdot 1 + 1 \cdot 0}{1+1} (\mathbf{i} + \mathbf{j}) = \frac{e+\pi}{2} \mathbf{i} + \frac{e+\pi}{2} \mathbf{j}$$

$$\mathbf{n} = \mathbf{u} - \mathbf{m} = \left(e - \frac{e+\pi}{2}\right)\mathbf{i} + \left(\pi - \frac{e+\pi}{2}\right)\mathbf{j} + (1-0)\mathbf{k}$$

$$= \frac{e-\pi}{2}\mathbf{i} + \frac{\pi-e}{2}\mathbf{j} + \mathbf{k}$$

17. a. $\|\mathbf{u}\| = \sqrt{9+4+1} = \sqrt{14}$

$$\cos \alpha = -\frac{3}{\sqrt{14}}, \alpha = \cos^{-1}\left(-\frac{3}{\sqrt{14}}\right) \approx 143^\circ$$

$$\cos \beta = \frac{2}{\sqrt{14}}, \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ$$

$$\cos \gamma = \frac{1}{\sqrt{14}}, \gamma = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ$$

- b. $|u| = \sqrt{9+36+1} = \sqrt{46}$
- $$\cos \alpha = \frac{3}{\sqrt{46}}, \alpha = \cos^{-1}\left(\frac{3}{\sqrt{46}}\right) \approx 64^\circ$$
- $$\cos \beta = \frac{6}{\sqrt{46}}, \beta = \cos^{-1}\left(\frac{6}{\sqrt{46}}\right) \approx 28^\circ$$
- $$\cos \gamma = -\frac{1}{\sqrt{46}}, \gamma = \cos^{-1}\left(-\frac{1}{\sqrt{46}}\right) \approx 98^\circ$$
18. $\cos^2(46^\circ) + \cos^2(108^\circ) + \cos^2 \gamma = 1$
 $\Rightarrow \cos \gamma \approx \pm 0.6496$
 $\Rightarrow \gamma \approx 49.49^\circ$ or $\gamma \approx 130.51^\circ$
19. $|u| = (4+9+z^2)^{1/2} = 5$ and $z > 0$, so
 $z = 2\sqrt{3} \approx 3.4641$.
20. u and v perpendicular, so $u \cdot v = 0$, thus
 $4 + 18 - 3z = 0, z = \frac{22}{3}$.
21. There are infinitely many such pairs. Note that
 $\langle -4, 2, 5 \rangle \cdot \langle 1, 2, 0 \rangle = -4 + 4 + 0 = 0$, so
 $u = \langle 1, 2, 0 \rangle$ is perpendicular to $\langle -4, 2, 5 \rangle$. For
any c , $\langle -2, 1, c \rangle \cdot \langle 1, 2, 0 \rangle = -2 + 2 + 0 = 0$ so
 $v = \langle -2, 1, c \rangle$ is a candidate.
 $\langle -4, 2, 5 \rangle \cdot \langle -2, 1, c \rangle = 8 + 2 + 5c$
 $8 + 2 + 5c = 0 \Rightarrow c = -2$, so one pair is
 $u = \langle 1, 2, 0 \rangle, v = \langle -2, 1, -2 \rangle$.
22. The midpoint is
 $\left(\frac{3+5}{2}, \frac{2-7}{2}, \frac{-1+2}{2}\right) = \left(4, -\frac{5}{2}, \frac{1}{2}\right)$, so the
vector is $\left\langle 4, -\frac{5}{2}, \frac{1}{2} \right\rangle$.
23. The following do not make sense.
- $v \cdot w$ is not a vector.
 - $u \cdot w$ is not a vector.
 - $|u|$ is not a vector.
24. Pairs of opposite sides are parallel and of equal length in a parallelogram. Thus, side AB must be parallel to side CD and side BC must be parallel to side AD . In addition, the four points must not be colinear. Thus, $b - a = c - d, c - b = d - a$, and $b - a \neq k(c - b)$ for any number k .
25. a. $2(x-1) - 4(y-2) + 3(z+3) = 0$ or
 $2x - 4y + 3z = -15$.
- b. $3(x+2) - 2(y+3) - 1(z-4) = 0$ or
 $3x - 2y - z = -4$.
26. Normals to the planes are $\langle 3, -2, 5 \rangle$ and
 $\langle 4, -2, -3 \rangle$, so the cosine of the angle is
 $\frac{12+4-15}{\sqrt{38}\sqrt{29}} = \frac{1}{\sqrt{1102}}$. The angle is $\approx 88.27^\circ$.
27. The planes are $2x - 4y + 3z = -15$ and
 $3x - 2y - z = 4$. The normals to the planes are
 $u = \langle 2, -4, 3 \rangle$ and $v = \langle 3, -2, -1 \rangle$. If θ is the angle
between the planes,
 $\cos \theta = \frac{u \cdot v}{|u||v|} = \frac{6+8-3}{\sqrt{29}\sqrt{14}} = \frac{11}{\sqrt{406}}$, so
 $\theta = \cos^{-1} \frac{11}{\sqrt{406}} \approx 56.91^\circ$.
28. An equation of the plane has the form
 $2x + 4y - z = D$. And this equation must satisfy
 $2(-1) + 4(2) - (-3) = D$, so $D = 9$. Thus an
equation of the plane is $2x + 4y - z = 9$.
29. a. Planes parallel to the xy -plane may be
expressed as $z = D$, so $z = 2$ is an equation of
the plane.
- b. An equation of the plane is
 $2(x+4) - 3(y+1) - 4(z-2) = 0$ or
 $2x - 3y - 4z = -13$.
30. Distance = $\frac{|(1)+3(-1)+(2)-7|}{\sqrt{1+9+1}} = \frac{7}{\sqrt{11}} \approx 2.1106$
31. The distance is 0 since the point is in the plane.
 $(|(-3)(2) + 2(6) + (3) - 9| = 0)$
32. $(0, 0, 9)$ is on $-3x + 2y + z = 9$. The distance from
 $(0, 0, 9)$ to $6x - 4y - 2z = 19$ is
 $\frac{|6(0) - 4(0) - 2(9) - 19|}{\sqrt{36+16+4}} = \frac{37}{\sqrt{56}} \approx 4.9443$ is the
distance between the planes.
33. $(1, 0, 0)$ is on $5x - 3y - 2z = 5$. The distance from
 $(1, 0, 0)$ to $-5x + 3y + 2z = 7$ is
 $\frac{|-5(1) + 3(0) + 2(0) - 7|}{\sqrt{25+9+4}} = \frac{12}{\sqrt{38}} \approx 1.9467$.
34. The line segment between the points is
perpendicular to the plane and its midpoint,
 $(2, 1, 1)$, is in the plane. Then
 $\langle 6 - (-2), 1 - 1, -2 - 4 \rangle = \langle 8, 0, -6 \rangle$ is
perpendicular to the plane. The equation of the
plane is
 $8(x-2) + 0(y-1) - 6(z-1) = 0$ or $8x - 6z = 10$.

$$35. |u+v|^2 + |u-v|^2 = (u+v) \cdot (u+v) + (u-v) \cdot (u-v) = [u \cdot u + 2(u \cdot v) + v \cdot v] + [u \cdot u - 2(u \cdot v) + v \cdot v] \\ = 2(u \cdot u) + 2(v \cdot v) = 2|u|^2 + 2|v|^2$$

$$36. |u+v|^2 - |u-v|^2 = (u+v) \cdot (u+v) - (u-v) \cdot (u-v) = [u \cdot u + 2(u \cdot v) + v \cdot v] - [u \cdot u - 2(u \cdot v) + v \cdot v] \\ = 4(u \cdot v) \text{ so } u \cdot v = \frac{1}{4}|u+v|^2 - \frac{1}{4}|u-v|^2.$$

37. Orient and scale the axes so that the cube has one end of a main diagonal at (0, 0, 0) and the other end at (1, 1, 1). Thus $\langle 1, 1, 1 \rangle$ is along the diagonal. The angle between the main diagonal and one of its faces is the angle between $\langle 1, 1, 1 \rangle$ and $\langle 1, 1, 0 \rangle$. Let θ be the angle.

$$\cos \theta = \frac{\langle 1, 1, 1 \rangle \cdot \langle 1, 1, 0 \rangle}{\sqrt{3} \cdot \sqrt{2}} = \frac{2}{\sqrt{6}},$$

$$\theta = \cos^{-1} \frac{2}{\sqrt{6}} \approx 35.26^\circ$$

38. Let θ be the common direction angle. Then $3\cos^2 \theta = 1$, so $\cos \theta = \pm \frac{1}{\sqrt{3}}$. There are two unit vectors, $\pm \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$.

39. Place the box so that its corners are at the points (0, 0, 0), (4, 0, 0), (0, 6, 0), (4, 6, 0), (0, 0, 10), (4, 0, 10), (0, 6, 10), and (4, 6, 10). The main diagonals are (0, 0, 0) to (4, 6, 10), (4, 0, 0) to (0, 6, 10), (0, 6, 0) to (4, 0, 10), and (0, 0, 10) to (4, 6, 0). The corresponding vectors are $\langle 4, 6, 10 \rangle$, $\langle -4, 6, 10 \rangle$, $\langle 4, -6, 10 \rangle$, and $\langle 4, 6, -10 \rangle$.

All of these vectors have length $\sqrt{16+36+100} = \sqrt{152}$. Thus, the smallest angle θ between any pair, u and v of the diagonals is found from the largest value of $u \cdot v$, since

$$\cos \theta = \frac{u \cdot v}{|u||v|} = \frac{u \cdot v}{152}.$$

There are six ways of pairing the four vectors. The largest value of $u \cdot v$ is 120 which occurs with

$u = \langle 4, 6, 10 \rangle$ and $v = \langle -4, 6, 10 \rangle$. Thus,

$$\cos \theta = \frac{120}{152} = \frac{15}{19} \text{ so } \theta = \cos^{-1} \frac{15}{19} \approx 37.86^\circ.$$

40. Place the box so that its corners are at the points (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1), and (1, 1, 1). The main diagonals are (0, 0, 0) to (1, 1, 1), (1, 0, 0) to

(0, 1, 1), (0, 1, 0) to (1, 0, 1), and (0, 0, 1) to (1, 1, 0). The corresponding vectors are $\langle 1, 1, 1 \rangle$, $\langle -1, 1, 1 \rangle$, $\langle 1, -1, 1 \rangle$, and $\langle 1, 1, -1 \rangle$.

All of these vectors have length $\sqrt{1+1+1} = \sqrt{3}$.

Thus, the angle θ between any pair, u and v of

the diagonals is found from $\cos \theta = \frac{u \cdot v}{|u||v|} = \frac{u \cdot v}{3}$.

There are six ways of pairing the four vectors, but due to symmetry, there are only two cases we need to consider. In these cases, $u \cdot v = 1$ or

$u \cdot v = -1$. Thus we get that $\cos \theta = \frac{1}{3}$ or

$\cos \theta = -\frac{1}{3}$. Solving for θ gives

$\theta \approx 70.53^\circ$ or $\theta \approx 109.47^\circ$.

41. $D = (4-0)\mathbf{i} + (4-0)\mathbf{j} + (0-8)\mathbf{k} = 4\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$
Thus, $W = \mathbf{F} \cdot \mathbf{D} = 0(4) + 0(4) - 4(-8) = 32$ joules.

42. $D = (9-2)\mathbf{i} + (4-1)\mathbf{j} + (6-3)\mathbf{k} = 7\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$
Thus, $W = \mathbf{F} \cdot \mathbf{D} = 3(7) - 6(3) + 7(3) = 24$ ft-lbs.

43. $D = (3-0)\mathbf{i} + (5-1)\mathbf{j} + (7-2)\mathbf{k} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$
 $\mathbf{F} = 5 \frac{(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})}{\sqrt{4+4+1}} = \frac{10}{3}\mathbf{i} + \frac{10}{3}\mathbf{j} - \frac{5}{3}\mathbf{k}$
 $W = \mathbf{F} \cdot \mathbf{D} = \frac{10}{3}(3) + \frac{10}{3}(4) - \frac{5}{3}(5) = 15$ joules.

44. The 3 wires must offset the weight of the object, thus
 $(3\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}) + (-8\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}) + (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 0\mathbf{i} + 0\mathbf{j} + 30\mathbf{k}$
Thus, $3 - 8 + a = 0$, so $a = 5$;
 $4 - 2 + b = 0$, so $b = -2$;
 $15 + 10 + c = 30$, so $c = 5$.

45. $\langle x, y, z \rangle = \langle 2, 3, -1 \rangle + \frac{1}{5} \langle 7-2, -2-3, 9+1 \rangle$
 $= \langle 2+1, 3-1, -1+2 \rangle = \langle 3, 2, 1 \rangle$,
so (3, 2, 1) is the point.

46. After reflecting from the xy -plane, the ray has direction $a\mathbf{i} + b\mathbf{j} - c\mathbf{k}$. After reflecting from the

xz -plane, the ray now has direction $a\mathbf{i} - b\mathbf{j} - c\mathbf{k}$. After reflecting from the yz -plane, the ray now has direction $-a\mathbf{i} - b\mathbf{j} - c\mathbf{k}$, the opposite of its original direction. ($a < 0, b < 0, c < 0$)

$$\frac{|3(-1) + 4(-3) + 1(4) - 15|}{\sqrt{9+16+1}} - \sqrt{26} = \sqrt{26} - \sqrt{26} = 0,$$

so the sphere is tangent to the plane.

47. The equation of the sphere in standard form is $(x+1)^2 + (y+3)^2 + (z-4)^2 = 26$, so its center is $(-1, -3, 4)$ and radius is $\sqrt{26}$. The distance from the sphere to the plane is the distance from the center to the plane minus the radius of the sphere or

48. Let $P(x_0, y_0, z_0)$ be any point on $Ax + By + Cz = D$, so $Ax_0 + By_0 + Cz_0 = D$. The distance between the planes is the distance from $P(x_0, y_0, z_0)$ to $Ax + By + Cz = E$, which is
$$\frac{|Ax_0 + By_0 + Cz_0 - E|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|D - E|}{\sqrt{A^2 + B^2 + C^2}}.$$

49. Let $\mathbf{x} = \langle x, y, z \rangle$, so

$$\begin{aligned} (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{b}) &= \langle x - a_1, y - a_2, z - a_3 \rangle \cdot \langle x - b_1, y - b_2, z - b_3 \rangle \\ &= x^2 - (a_1 + b_1)x + a_1b_1 + y^2 - (a_2 + b_2)y + a_2b_2 + z^2 - (a_3 + b_3)z + a_3b_3 \end{aligned}$$

Setting this equal to 0 and completing the squares yields

$$\left(x - \frac{a_1 + b_1}{2}\right)^2 + \left(y - \frac{a_2 + b_2}{2}\right)^2 + \left(z - \frac{a_3 + b_3}{2}\right)^2 = \frac{1}{4}[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2].$$

A sphere with center $\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}, \frac{a_3 + b_3}{2}\right)$ and radius $\frac{1}{4}[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2] = \frac{1}{4}|\mathbf{a} - \mathbf{b}|^2$.

50. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ represent the sides of the polygon connected tail to head in succession around the polygon. Then $\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_n = \mathbf{0}$ since the polygon is closed, so $\mathbf{F} \cdot \mathbf{v}_1 + \mathbf{F} \cdot \mathbf{v}_2 + \dots + \mathbf{F} \cdot \mathbf{v}_n = \mathbf{F} \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_n) = \mathbf{F} \cdot \mathbf{0} = 0$.

51. If $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are the position vectors of the vertices labeled A, B , and C , respectively, then the side BC is represented by the vector $\mathbf{c} - \mathbf{b}$. The position vector of the midpoint of BC is $\mathbf{b} + \frac{1}{2}(\mathbf{c} - \mathbf{b}) = \frac{1}{2}(\mathbf{b} + \mathbf{c})$. The segment from A to the midpoint of BC is $\frac{1}{2}(\mathbf{b} + \mathbf{c}) - \mathbf{a}$. Thus, the position vector of P is $\mathbf{a} + \frac{2}{3}\left[\frac{1}{2}(\mathbf{b} + \mathbf{c}) - \mathbf{a}\right] = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$. If the vertices are $(2, 6, 5)$, $(4, -1, 2)$, and $(6, 1, 2)$, the corresponding position vectors are $\langle 2, 6, 5 \rangle$, $\langle 4, -1, 2 \rangle$, and $\langle 6, 1, 2 \rangle$. The position vector of P is $\frac{1}{3}\langle 2 + 4 + 6, 6 - 1 + 1, 5 + 2 + 2 \rangle = \frac{1}{3}\langle 12, 6, 9 \rangle = \langle 4, 2, 3 \rangle$. Thus P is $(4, 2, 3)$.

52. Let A, B, C , and D be the vertices of the tetrahedron with corresponding position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$. The vector representing the segment from A to the centroid of the opposite face, triangle BCD , is $\frac{1}{3}(\mathbf{b} + \mathbf{c} + \mathbf{d}) - \mathbf{a}$ by Problem

49. Similarly, the segments from B, C , and D to the opposite faces are $\frac{1}{3}(\mathbf{a} + \mathbf{c} + \mathbf{d}) - \mathbf{b}$, $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{d}) - \mathbf{c}$, and

$\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) - \mathbf{d}$, respectively. If these segments meet in one point which has a nice formulation as some fraction of the way from a vertex to the centroid of the opposite face, then there is some number k , such that

$$\mathbf{a} + k\left[\frac{1}{3}(\mathbf{b} + \mathbf{c} + \mathbf{d}) - \mathbf{a}\right] = \mathbf{b} + k\left[\frac{1}{3}(\mathbf{a} + \mathbf{c} + \mathbf{d}) - \mathbf{b}\right] = \mathbf{c} + k\left[\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{d}) - \mathbf{c}\right] = \mathbf{d} + k\left[\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) - \mathbf{d}\right].$$

Thus, $\mathbf{a}(1 - k) = \frac{k}{3}\mathbf{a}$, so $k = \frac{3}{4}$. Hence the segments joining the vertices to the centroids of the opposite faces meet

in a common point which is $\frac{3}{4}$ of the way from a vertex to the corresponding centroid. With $k = \frac{3}{4}$, all of the above formulas simplify to $\frac{1}{4}(a + b + c + d)$, the position vector of the point.

14.3 Concepts Review

$$1. \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{k}$$

$$= (-2 - 1)\mathbf{i} - (1 - 3)\mathbf{j} + (-1 - 6)\mathbf{k}$$

$$= -3\mathbf{i} + 2\mathbf{j} - 7\mathbf{k} = \langle -3, 2, -7 \rangle$$

$$2. |\mathbf{u}||\mathbf{v}|\sin\theta$$

$$3. -(\mathbf{v} \times \mathbf{u})$$

$$4. \text{parallel}$$

Problem Set 14.3

$$1. \text{ a. } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -2 \\ -1 & 2 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 \\ 2 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & -2 \\ -1 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{k}$$

$$= (-8 + 4)\mathbf{i} - (12 - 2)\mathbf{j} + (-6 + 2)\mathbf{k}$$

$$= -4\mathbf{i} - 10\mathbf{j} - 4\mathbf{k}$$

$$\text{b. } \mathbf{b} + \mathbf{c} = 6\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}, \text{ so}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -2 \\ 6 & 5 & -8 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 \\ 5 & -8 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & -2 \\ 6 & -8 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 2 \\ 6 & 5 \end{vmatrix} \mathbf{k}$$

$$= (-16 + 10)\mathbf{i} - (24 + 12)\mathbf{j} + (-15 - 12)\mathbf{k}$$

$$= -6\mathbf{i} - 36\mathbf{j} - 27\mathbf{k}$$

$$\text{c. } \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = -3(6) + 2(5) - 2(-8) = 8$$

$$\text{d. } \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -4 \\ 7 & 3 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -4 \\ 3 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -4 \\ 7 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 7 & 3 \end{vmatrix} \mathbf{k}$$

$$= (-8 + 12)\mathbf{i} - (4 + 28)\mathbf{j} + (-3 - 14)\mathbf{k}$$

$$= 4\mathbf{i} - 32\mathbf{j} - 17\mathbf{k}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -2 \\ 4 & -32 & -17 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 \\ -32 & -17 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & -2 \\ 4 & -17 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 2 \\ 4 & -32 \end{vmatrix} \mathbf{k}$$

$$= (-34 - 64)\mathbf{i} - (51 + 8)\mathbf{j} + (96 - 8)\mathbf{k}$$

$$= -98\mathbf{i} - 59\mathbf{j} + 88\mathbf{k}$$

$$2. \text{ a. } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ -2 & -1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 1 \\ -2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 3 \\ -2 & -1 \end{vmatrix} \mathbf{k}$$

$$= (0 + 1)\mathbf{i} - (0 + 2)\mathbf{j} + (-3 + 6)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

$$= \langle 1, -2, 3 \rangle$$

$$\text{b. } \mathbf{b} + \mathbf{c} = \langle -2 - 2, -1 - 3, 0 - 1 \rangle$$

$$= \langle -4, -4, -1 \rangle$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ -4 & -4 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ -4 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 1 \\ -4 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 3 \\ -4 & -4 \end{vmatrix} \mathbf{k}$$

$$= (-3 + 4)\mathbf{i} - (-3 + 4)\mathbf{j} + (-12 + 12)\mathbf{k}$$

$$= \mathbf{i} - \mathbf{j} = \langle 1, -1, 0 \rangle$$

$$\text{c. } \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 0 \\ -2 & -3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 \\ -3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 0 \\ -2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -1 \\ -2 & -3 \end{vmatrix} \mathbf{k}$$

$$= (1 - 0)\mathbf{i} - (2 - 0)\mathbf{j} + (6 - 2)\mathbf{k} = \langle 1, -2, 4 \rangle$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 3(1) + 3(-2) + 1(4) = 1$$

$$\text{d. } \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ -2 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 3 \\ 1 & -2 \end{vmatrix} \mathbf{k}$$

$$= (12 + 2)\mathbf{i} - (12 - 1)\mathbf{j} + (-6 - 3)\mathbf{k}$$

$$= \langle 14, -11, -9 \rangle$$

3. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 2 & -4 \end{vmatrix}$

$$= \begin{vmatrix} 2 & 3 \\ 2 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k}$$

$$= (-8 - 6)\mathbf{i} - (-4 + 6)\mathbf{j} + (2 + 4)\mathbf{k} = -14\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. All perpendicular vectors will have the form $c(-14\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ where c is a real number.

4. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 5 & -2 \\ 3 & -2 & 4 \end{vmatrix}$

$$= \begin{vmatrix} 5 & -2 \\ -2 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & -2 \\ 3 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 5 \\ 3 & -2 \end{vmatrix} \mathbf{k}$$

$$= (20 - 4)\mathbf{i} - (-8 + 6)\mathbf{j} + (4 - 15)\mathbf{k}$$

$$= 16\mathbf{i} + 2\mathbf{j} - 11\mathbf{k}$$

All vectors perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ will have the form $c(16\mathbf{i} + 2\mathbf{j} - 11\mathbf{k})$ where c is a real number.

5. $\mathbf{u} = \langle 3 - 1, -1 - 3, 2 - 5 \rangle = \langle 2, -4, -3 \rangle$ and $\mathbf{v} = \langle 4 - 1, 0 - 3, 1 - 5 \rangle = \langle 3, -3, -4 \rangle$ are in the plane.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 3 & -3 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} -4 & -3 \\ -3 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -3 \\ 3 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -4 \\ 3 & -3 \end{vmatrix} \mathbf{k}$$

$$= (16 - 9)\mathbf{i} - (-8 + 9)\mathbf{j} + (-6 + 12)\mathbf{k}$$

$$= \langle 7, -1, 6 \rangle \text{ is perpendicular to the plane.}$$

$$\pm \frac{1}{\sqrt{49 + 1 + 36}} \langle 7, -1, 6 \rangle = \pm \left\langle \frac{7}{\sqrt{86}}, -\frac{1}{\sqrt{86}}, \frac{6}{\sqrt{86}} \right\rangle$$

are the vectors perpendicular to the plane.

6. $\mathbf{u} = \langle 5 + 1, 1 - 3, 2 - 0 \rangle = \langle 6, -2, 2 \rangle$ and $\mathbf{v} = \langle 4 + 1, -3 - 3, -1 - 0 \rangle = \langle 5, -6, -1 \rangle$ are in the plane.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -2 & 2 \\ 5 & -6 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 2 \\ -6 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 6 & 2 \\ 5 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 6 & -2 \\ 5 & -6 \end{vmatrix} \mathbf{k}$$

$$= (2 + 12)\mathbf{i} - (-6 - 10)\mathbf{j} + (-36 + 10)\mathbf{k}$$

$$= \langle 14, 16, -26 \rangle$$

is perpendicular to the plane.

$$\pm \frac{1}{\sqrt{196 + 256 + 676}} \langle 14, 16, -26 \rangle$$

$$= \pm \left\langle \frac{7}{\sqrt{282}}, \frac{8}{\sqrt{282}}, -\frac{13}{\sqrt{282}} \right\rangle$$

are the unit vectors perpendicular to the plane.

7. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -3 \\ 4 & 2 & -4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -3 \\ 4 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 4 & 2 \end{vmatrix} \mathbf{k}$$

$$= (-4 + 6)\mathbf{i} - (4 + 12)\mathbf{j} + (-2 - 4)\mathbf{k}$$

$$= 2\mathbf{i} - 16\mathbf{j} - 6\mathbf{k}$$

Area of parallelogram = $|\mathbf{a} \times \mathbf{b}|$

$$= \sqrt{4 + 256 + 36} = 2\sqrt{74}$$

8. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 1 & -4 \end{vmatrix}$

$$= \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} \mathbf{k}$$

$$= (-8 + 1)\mathbf{i} - (-8 - 1)\mathbf{j} + (2 + 2)\mathbf{k}$$

$$= -7\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$$

Area of parallelogram = $|\mathbf{a} \times \mathbf{b}| = \sqrt{49 + 81 + 16}$

$$= \sqrt{146}$$

9. $\mathbf{a} = \langle 2 - 3, 4 - 2, 6 - 1 \rangle = \langle -1, 2, 5 \rangle$ and $\mathbf{b} = \langle -1 - 3, 2 - 2, 5 - 1 \rangle = \langle -4, 0, 4 \rangle$ are adjacent sides of the triangle. The area of the triangle is half the area of the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as adjacent sides.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 5 \\ -4 & 0 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 5 \\ 0 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 5 \\ -4 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ -4 & 0 \end{vmatrix} \mathbf{k}$$

$$= (8 - 0)\mathbf{i} - (-4 + 20)\mathbf{j} + (0 + 8)\mathbf{k} = \langle 8, -16, 8 \rangle$$

Area of triangle = $\frac{1}{2} \sqrt{64 + 256 + 64} = \frac{1}{2} (8\sqrt{6})$

$$= 4\sqrt{6}$$

10. $\mathbf{a} = \langle 3 - 1, 1 - 2, 5 - 3 \rangle = \langle 2, -1, 2 \rangle$ and $\mathbf{b} = \langle 4 - 1, 5 - 2, 6 - 3 \rangle = \langle 3, 3, 3 \rangle$ are adjacent sides of the triangle.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 3 & 3 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix} \mathbf{k}$$

$$= (-3 - 6)\mathbf{i} - (6 - 6)\mathbf{j} + (6 + 3)\mathbf{k} = \langle -9, 0, 9 \rangle$$

$$\text{Area of triangle} = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} \sqrt{81 + 81} = \frac{9\sqrt{2}}{2}$$

11. $\mathbf{u} = \langle 0 - 1, 3 - 3, 0 - 2 \rangle = \langle -1, 0, -2 \rangle$ and
 $\mathbf{v} = \langle 2 - 1, 4 - 3, 3 - 2 \rangle = \langle 1, 1, 1 \rangle$ are in the plane.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{k}$$

$$= (0 + 2)\mathbf{i} - (-1 + 2)\mathbf{j} + (-1 - 0)\mathbf{k} = \langle 2, -1, -1 \rangle$$

The plane through $(1, 3, 2)$ with normal $\langle 2, -1, -1 \rangle$ has equation

$$2(x - 1) - 1(y - 3) - 1(z - 2) = 0 \text{ or } 2x - y - z = -3.$$

12. $\mathbf{u} = \langle 0 - 1, 0 - 1, 1 - 2 \rangle = \langle -1, -1, -1 \rangle$ and
 $\mathbf{v} = \langle -2 - 1, -3 - 1, 0 - 2 \rangle = \langle -3, -4, -2 \rangle$ are in the plane.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -1 \\ -3 & -4 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 \\ -3 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -1 \\ -3 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & -1 \\ -3 & -4 \end{vmatrix} \mathbf{k}$$

17. Volume = $|\langle 2, 3, 4 \rangle \cdot (\langle 0, 4, -1 \rangle \times \langle 5, 1, 3 \rangle)| = |\langle 2, 3, 4 \rangle \cdot \langle 13, -5, -20 \rangle| = |-69| = 69$

18. Volume = $|(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot [(-\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \times (3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})]| = |(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot (12\mathbf{i} + 8\mathbf{j} - 4\mathbf{k})| = |-4| = 4$

19. a. Volume = $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\langle 3, 2, 1 \rangle \cdot \langle -3, -1, 2 \rangle| = |-9| = 9$

b. Area = $|\mathbf{u} \times \mathbf{v}| = |\langle 3, -5, 1 \rangle| = \sqrt{9 + 25 + 1} = \sqrt{35}$

c. Let θ be the angle. Then θ is the complement of the smaller angle between $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$.

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{|\mathbf{u}| |\mathbf{v} \times \mathbf{w}|} = \frac{9}{\sqrt{9 + 4 + 1} \sqrt{9 + 1 + 4}} = \frac{9}{14}, \theta \approx 40.01^\circ$$

$$= (2 - 4)\mathbf{i} - (2 - 3)\mathbf{j} + (4 - 3)\mathbf{k} = \langle -2, 1, 1 \rangle$$

The plane through $(0, 0, 1)$ with normal $\langle -2, 1, 1 \rangle$ has equation $-2(x - 0) + 1(y - 0) + 1(z - 1) = 0$ or $-2x + y + z = 1$.

13. The plane's normals will be perpendicular to the normals of the other two planes. Then a normal is $\langle 1, -3, 2 \rangle \times \langle 2, -2, -1 \rangle = \langle 7, 5, 4 \rangle$. An equation of the plane is $7(x + 1) + 5(y + 2) + 4(z - 3) = 0$ or $7x + 5y + 4z = -5$.

14. $(4\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}) = 13\mathbf{i} - 26\mathbf{j} - 26\mathbf{k} = 13(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$ is normal to the plane. An equation of the plane is $1(x - 2) - 2(y + 3) - 2(z - 2) = 0$ or $x - 2y - 2z = 4$.

15. Each vector normal to the plane we seek is parallel to the line of intersection of the given planes. Also, the cross product of vectors normal to the given planes will be parallel to both planes, hence parallel to the line of intersection. Thus, $\langle 4, -3, 2 \rangle \times \langle 3, 2, -1 \rangle = \langle -1, 10, 17 \rangle$ is normal to the plane we seek. An equation of the plane is $-1(x - 6) + 10(y - 2) + 17(z + 1) = 0$ or $-x + 10y + 17z = -3$.

16. $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$, hence $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is perpendicular to $\mathbf{a} \times \mathbf{b}$ hence it is parallel to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

20. From Theorem C, $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$, which leads to $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$. Again from Theorem C, $|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| = |(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}| = |-(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}|$, which leads to $|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| = |\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})|$. Therefore, we have that $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})| = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$.
21. Choice (c) does not make sense because $(\mathbf{a} \cdot \mathbf{b})$ is a scalar and can't be crossed with a vector. Choice (d) does not make sense because $(\mathbf{a} \times \mathbf{b})$ is a vector and can't be added to a constant.
22. $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ will both be perpendicular to the common plane. Hence $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are parallel so $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$.
23. Let $\mathbf{b}$ and $\mathbf{c}$ determine the (triangular) base of the tetrahedron. Then the area of the base is $\frac{1}{2}|\mathbf{b} \times \mathbf{c}|$ which is half of the area of the parallelogram determined by $\mathbf{b}$ and $\mathbf{c}$. Thus,
- $$\begin{aligned} \frac{1}{3}(\text{area of base})(\text{height}) &= \frac{1}{3} \left[\frac{1}{2}(\text{area of corresponding parallelogram})(\text{height}) \right] \\ &= \frac{1}{6}(\text{area of corresponding parallelepiped}) = \frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \end{aligned}$$
24. $\mathbf{a} = \langle 4+1, -1-2, 2-3 \rangle = \langle 5, -3, -1 \rangle$,
 $\mathbf{b} = \langle 5+1, 6-2, 3-3 \rangle = \langle 6, 4, 0 \rangle$,
 $\mathbf{c} = \langle 1+1, 1-2, -2-3 \rangle = \langle 2, -1, -5 \rangle$
 $\text{Volume} = \frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \frac{1}{6}|\langle 5, -3, -1 \rangle \cdot \langle -20, 30, -14 \rangle| = \frac{1}{6}|-176| = \frac{88}{3}$
25. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ then
 $\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$
 $|\mathbf{u} \times \mathbf{v}|^2 = u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2$
 $= u_1^2(v_1^2 + v_2^2 + v_3^2) - u_1^2v_1^2 + u_2^2(v_1^2 + v_2^2 + v_3^2) - u_2^2v_2^2 + u_3^2(v_1^2 + v_2^2 + v_3^2)$
 $\quad - u_3^2v_3^2 - 2u_2u_3v_2v_3 - 2u_1u_3v_1v_3 - 2u_1u_2v_1v_2$
 $= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2) + 2u_2u_3v_2v_3 + 2u_1u_3v_1v_3 + 2u_1u_2v_1v_2$
 $= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$
26. $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$
 $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle (u_2v_3 - u_3v_2) + (u_2w_3 - u_3w_2), (u_3v_1 - u_1v_3) + (u_3w_1 - u_1w_3), (u_1v_2 - u_2v_1) + (u_1w_2 - u_2w_1) \rangle$
 $= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
27. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = -[\mathbf{u} \times (\mathbf{v} + \mathbf{w})] = -[(\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})] = -(\mathbf{u} \times \mathbf{v}) - (\mathbf{u} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$
28. $\mathbf{u} \times \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel. $\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are perpendicular. Thus, either $\mathbf{u}$ or $\mathbf{v}$ is $\mathbf{0}$.
29. $\overrightarrow{PQ} = \langle -a, b, 0 \rangle$, $\overrightarrow{PR} = \langle -a, 0, c \rangle$,
The area of the triangle is half the area of the parallelogram with $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as adjacent sides, so area
 $(\Delta PQR) = \frac{1}{2}|\langle -a, b, 0 \rangle \times \langle -a, 0, c \rangle| = \frac{1}{2}|\langle bc, ac, ab \rangle| = \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$.
30. The area of the triangle is
 $\frac{1}{2}|\langle x_2 - x_1, y_2 - y_1, 0 \rangle \times \langle x_3 - x_1, y_3 - y_1, 0 \rangle| = \frac{1}{2}|\langle 0, 0, (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1) \rangle|$

$$= \frac{1}{2} |(x_2 y_3 - x_3 y_2) - (x_1 y_3 - x_3 y_1) + (x_1 y_2 - x_2 y_1)|$$

which is half of the absolute value of the determinant given. (Expand the determinant along the third column to see the equality.)

$$31. \text{ From Problem 29, } D^2 = \frac{1}{4}(b^2 c^2 + a^2 c^2 + a^2 b^2) = \left(\frac{1}{2}bc\right)^2 + \left(\frac{1}{2}ac\right)^2 + \left(\frac{1}{2}ab\right)^2 = A^2 + B^2 + C^2.$$

$$32. \text{ Note that the area of the face determined by } \mathbf{a} \text{ and } \mathbf{b} \text{ is } \frac{1}{2}|\mathbf{a} \times \mathbf{b}|.$$

Label the tetrahedron so that $\mathbf{m} = \frac{1}{2}(\mathbf{a} \times \mathbf{b})$, $\mathbf{n} = \frac{1}{2}(\mathbf{b} \times \mathbf{c})$, and $\mathbf{p} = \frac{1}{2}(\mathbf{c} \times \mathbf{a})$ point outward.

The fourth face is determined by $\mathbf{a} - \mathbf{c}$ and $\mathbf{b} - \mathbf{c}$, so

$$\mathbf{q} = \frac{1}{2}[(\mathbf{b} - \mathbf{c}) \times (\mathbf{a} - \mathbf{c})] = \frac{1}{2}[(\mathbf{b} \times \mathbf{a}) - (\mathbf{b} \times \mathbf{c}) - (\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{c})] = \frac{1}{2}[-(\mathbf{a} \times \mathbf{b}) - (\mathbf{b} \times \mathbf{c}) - (\mathbf{c} \times \mathbf{a})].$$

$$\mathbf{m} + \mathbf{n} + \mathbf{p} + \mathbf{q} = \frac{1}{2}[(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a}) - (\mathbf{a} \times \mathbf{b}) - (\mathbf{b} \times \mathbf{c}) - (\mathbf{c} \times \mathbf{a})] = \mathbf{0}$$

$$33. \text{ The area of the triangle is } A = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|. \text{ Thus,}$$

$$\begin{aligned} A^2 &= \frac{1}{4}|\mathbf{a} \times \mathbf{b}|^2 = \frac{1}{4}(|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2) = \frac{1}{4} \left[|\mathbf{a}|^2 |\mathbf{b}|^2 - \frac{1}{4}(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2)^2 \right] \\ &= \frac{1}{16} [4a^2 b^2 - (a^2 + b^2 - c^2)^2] = \frac{1}{16} (2a^2 b^2 - a^4 + 2a^2 c^2 - b^4 + 2b^2 c^2 - c^4). \end{aligned}$$

$$\text{Note that } s - a = \frac{1}{2}(b + c - a),$$

$$s - b = \frac{1}{2}(a + c - b), \text{ and } s - c = \frac{1}{2}(a + b - c).$$

$$s(s - a)(s - b)(s - c) = \frac{1}{16}(a + b + c)(b + c - a)(a + c - b)(a + b - c) = \frac{1}{16}(2a^2 b^2 - a^4 + 2a^2 c^2 - b^4 + 2b^2 c^2 - c^4)$$

which is the same as was obtained above.

$$\begin{aligned} 34. \quad \mathbf{u} \times \mathbf{v} &= (u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \\ &= (u_1 v_1)(\mathbf{i} \times \mathbf{i}) + (u_1 v_2)(\mathbf{i} \times \mathbf{j}) + (u_1 v_3)(\mathbf{i} \times \mathbf{k}) + (u_2 v_1)(\mathbf{j} \times \mathbf{i}) + (u_2 v_2)(\mathbf{j} \times \mathbf{j}) + (u_2 v_3)(\mathbf{j} \times \mathbf{k}) \\ &\quad + (u_3 v_1)(\mathbf{k} \times \mathbf{i}) + (u_3 v_2)(\mathbf{k} \times \mathbf{j}) + (u_3 v_3)(\mathbf{k} \times \mathbf{k}) \\ &= (u_1 v_1)(0) + (u_1 v_2)(\mathbf{k}) + (u_1 v_3)(-\mathbf{j}) + (u_2 v_1)(-\mathbf{k}) + (u_2 v_2)(0) + (u_2 v_3)(\mathbf{i}) \\ &\quad + (u_3 v_1)(\mathbf{j}) + (u_3 v_2)(-\mathbf{i}) + (u_3 v_3)(0) \\ &= (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k} \end{aligned}$$

14.4 Concepts Review

$$1. \quad 1 + 4t; -3 - 2t; 2 - t$$

$$2. \quad \frac{x-1}{4} = \frac{y+3}{-2} = \frac{z-2}{-1}$$

$$3. \quad 2t\mathbf{i} - 3\mathbf{j} + 3t^2\mathbf{k}$$

$$4. \quad \langle 2, -3, 3 \rangle; \frac{x-1}{2} = \frac{y+3}{-3} = \frac{z-1}{3}$$

Problem Set 14.4

1. A parallel vector is

$$\mathbf{v} = \langle 4 - 1, 5 + 2, 6 - 3 \rangle = \langle 3, 7, 3 \rangle.$$

$$x = 1 + 3t, y = -2 + 7t, z = 3 + 3t$$

2. A parallel vector is

$$\mathbf{v} = \langle 7 - 2, -2 + 1, 3 + 5 \rangle = \langle 5, -1, 8 \rangle$$

$$x = 2 + 5t, y = -1 - t, z = -5 + 8t$$

3. A parallel vector is
 $\mathbf{v} = \langle 6 - 4, 2 - 2, -1 - 3 \rangle = \langle 2, 0, -4 \rangle$ or $\langle 1, 0, -2 \rangle$.
 $x = 4 + t, y = 2, z = 3 - 2t$
4. A parallel vector is
 $\mathbf{v} = \langle 5 - 5, 4 + 3, 2 + 3 \rangle = \langle 0, 7, 5 \rangle$
 $x = 5, y = -3 + 7t, z = -3 + 5t$
5. $x = 4 + 3t, y = 5 + 2t, z = 6 + t$
 $\frac{x-4}{3} = \frac{y-5}{2} = \frac{z-6}{1}$
6. $x = -1 - 2t, y = 3, z = -6 + 5t$
 Since the second direction number is 0, the line does not have symmetric equations.
7. Another parallel vector is $\langle 1, 10, 100 \rangle$.
 $x = 1 + t, y = 1 + 10t, z = 1 + 100t$
 $\frac{x-1}{1} = \frac{y-1}{10} = \frac{z-1}{100}$
8. $x = -2 + 7t, y = 2 - 6t, z = -2 + 3t$
 $\frac{x+2}{7} = \frac{y-2}{-6} = \frac{z+2}{3}$
9. Set $z = 0$. Solving $4x + 3y = 1$ and $10x + 6y = 10$ yields $x = 4, y = -5$. Thus $P_1(4, -5, 0)$ is on the line. Set $y = 0$. Solving $4x - 7z = 1$ and $10x - 5z = 10$ yields $x = \frac{13}{10}, z = \frac{3}{5}$. Thus $P_2(\frac{13}{10}, 0, \frac{3}{5})$ is on the line.
 $\overrightarrow{P_1P_2} = \langle \frac{13}{10} - 4, 0 - (-5), \frac{3}{5} - 0 \rangle = \langle -\frac{27}{10}, 5, \frac{3}{5} \rangle$ is a direction vector. Thus $\langle 27, -50, -6 \rangle = -10\overrightarrow{P_1P_2}$ is also a direction vector. The symmetric equations are thus
 $\frac{x-4}{27} = \frac{y+5}{-50} = \frac{z}{-6}$
10. With $x = 0, y - z = 2$ and $-2y + z = 3$ yield $(0, -5, -7)$.
 With $y = 0, x - z = 2$ and $3x + z = 3$ yield $(\frac{5}{4}, 0, -\frac{3}{4})$.
 A vector parallel to the line is
 $\langle \frac{5}{4}, 5, -\frac{3}{4} + 7 \rangle = \langle \frac{5}{4}, 5, \frac{25}{4} \rangle$ or $\langle 1, 4, 5 \rangle$.
 $\frac{x}{1} = \frac{y+5}{4} = \frac{z+7}{5}$
11. $\mathbf{u} = \langle 1, 4, -2 \rangle$ and $\mathbf{v} = \langle 2, -1, -2 \rangle$ are both perpendicular to the line, so $\mathbf{u} \times \mathbf{v} = \langle -10, -2, -9 \rangle$, and hence $\langle 10, 2, 9 \rangle$ is parallel to

the line.

With $y = 0, x - 2z = 13$ and $2x - 2z = 5$ yield $(-8, 0, -\frac{21}{2})$. The symmetric equations are

$$\frac{x+8}{10} = \frac{y}{2} = \frac{z+\frac{21}{2}}{9}$$

12. $\mathbf{u} = \langle 1, -3, 1 \rangle$ and $\mathbf{v} = \langle 6, -5, 4 \rangle$ are both perpendicular to the line, so $\mathbf{u} \times \mathbf{v} = \langle -7, 2, 13 \rangle$ is parallel to the line.
 With $x = 0, -3y + z = -1$ and $-5y + 4z = 9$ yield $(0, \frac{13}{7}, \frac{32}{7})$.
 $\frac{x}{-7} = \frac{y-\frac{13}{7}}{2} = \frac{z-\frac{32}{7}}{13}$
13. $\langle 1, -5, 2 \rangle$ is a vector in the direction of the line.
 $\frac{x-4}{1} = \frac{y}{-5} = \frac{z-6}{2}$
14. $\langle 2, 1, -3 \rangle \times \langle 5, 4, -1 \rangle = \langle 11, -13, 3 \rangle$ is in the direction of the line.
 $\frac{x+5}{11} = \frac{y-7}{-13} = \frac{z+2}{3}$
15. The point of intersection on the z -axis is $(0, 0, 4)$. A vector in the direction of the line is $\langle 5 - 0, -3 - 0, 4 - 4 \rangle = \langle 5, -3, 0 \rangle$. Parametric equations are $x = 5t, y = -3t, z = 4$.
16. $\langle 3, 1, -2 \rangle \times \langle 2, 3, -1 \rangle = \langle 5, -1, 7 \rangle$ is in the direction of the line since the line is perpendicular to $\langle 3, 1, -2 \rangle$ and $\langle 2, 3, -1 \rangle$.
 $\frac{x-2}{5} = \frac{y+4}{-1} = \frac{z-5}{7}$
17. Using $t = 0$ and $t = 1$, two points on the first line are $(-2, 1, 2)$ and $(0, 5, 1)$. Using $t = 0$, a point on the second line is $(2, 3, 1)$. Thus, two nonparallel vectors in the plane are $\langle 0 + 2, 5 - 1, 1 - 2 \rangle = \langle 2, 4, 1 \rangle$ and $\langle 2 + 2, 3 - 1, 1 - 2 \rangle = \langle 4, 2, -1 \rangle$.
 Hence, $\langle 2, 4, 1 \rangle \times \langle 4, 2, -1 \rangle = \langle -2, -2, -12 \rangle$ is a normal to the plane, and so is $\langle 1, 1, 6 \rangle$. An equation of the plane is
 $1(x + 2) + 1(y - 1) + 6(z - 2) = 0$ or
 $x + y + 6z = 11$.

18. Solve $\frac{x-1}{-4} = \frac{y-2}{3}$ and $\frac{x-2}{-1} = \frac{y-1}{1}$ simultaneously to get $x = 1, y = 2$. From the first line $\frac{1-1}{-4} = \frac{z-4}{-2}$, so $z = 4$ and $(1, 2, 4)$ is on the first line. This point also satisfies the equations of the second line, so the lines intersect. $\langle -4, 3, -2 \rangle$ and $\langle -1, 1, 6 \rangle$ are parallel to the plane determined by the lines, so $\langle -4, 3, -2 \rangle \times \langle -1, 1, 6 \rangle = \langle 20, 26, -1 \rangle$ is a normal to the plane. An equation of the plane is $20(x-1) + 26(y-2) - 1(z-4) = 0$ or $20x + 26y - z = 68$.

19. Using $t = 0$, another point in the plane is $(1, -1, 4)$ and $\langle 2, 3, 1 \rangle$ is parallel to the plane. Another parallel vector is $\langle 1-1, -1+1, 5-4 \rangle = \langle 0, 0, 1 \rangle$. Thus, $\langle 2, 3, 1 \rangle \times \langle 0, 0, 1 \rangle = \langle 3, -2, 0 \rangle$ is a normal to the plane. An equation of the plane is $3(x-1) - 2(y+1) + 0(z-5) = 0$ or $3x - 2y = 5$.
20. Using $t = 0$, one point of the plane is $(0, 1, 0)$. $\langle 2, -1, 1 \rangle \times \langle 0, 1, 1 \rangle = \langle -2, -2, 2 \rangle = -2\langle 1, 1, -1 \rangle$ is perpendicular to the normals of both planes, hence parallel to their line of intersection. $\langle 3, 1, 2 \rangle$ is parallel to the line in the plane we seek, thus $\langle 3, 1, 2 \rangle \times \langle 1, 1, -1 \rangle = \langle -3, 5, 2 \rangle$ is a normal to the plane. An equation of the plane is $-3(x-0) + 5(y-1) + 2(z-0) = 0$ or $-3x + 5y + 2z = 5$.

21. a. With $t = 0$ in the first line, $x = 2 - 0 = 2$, $y = 3 + 4 \cdot 0 = 3$, $z = 2 \cdot 0 = 0$, so $(2, 3, 0)$ is on the first line.
- b. $\langle -1, 4, 2 \rangle$ is parallel to the first line, while $\langle 1, 0, 2 \rangle$ is parallel to the second line, so $\langle -1, 4, 2 \rangle \times \langle 1, 0, 2 \rangle = \langle 8, 4, -4 \rangle = 4\langle 2, 1, -1 \rangle$ is normal to both. Thus, π has equation $2(x-2) + 1(y-3) - 1(z-0) = 0$ or $2x + y - z = 7$, and contains the first line.
- c. With $t = 0$ in the second line, $x = -1 + 0 = -1$, $y = 2$, $z = -1 + 2 \cdot 0 = -1$, so $Q(-1, 2, -1)$ is on the second line.
- d. From Example 6 of Section 14.2, the distance from Q to π is $\frac{|2(-1) + (2) - (-1) - 7|}{\sqrt{4+1+1}} = \frac{6}{\sqrt{6}} = \sqrt{6} \approx 2.449$.

22. With $t = 0$, $(1, -3, -1)$ is on the first line. $\langle 2, 4, -1 \rangle \times \langle -2, 3, 2 \rangle = \langle 11, -2, 14 \rangle$ is perpendicular to both lines, so $11(x-1) - 2(y+3) + 14(z+1) = 0$ or $11x - 2y + 14z = 3$ is parallel to both lines and contains the first line. With $t = 0$, $(4, 1, 0)$ is on the second line. The distance from $(4, 1, 0)$ to $11x - 2y + 14z = 3$ is $\frac{|11(4) - 2(1) + 14(0) - 3|}{\sqrt{121+4+196}} = \frac{39}{\sqrt{321}} \approx 2.1768$.

23. $\mathbf{r}\left(\frac{\pi}{3}\right) = \mathbf{i} + 3\sqrt{3}\mathbf{j} + \frac{\pi}{3}\mathbf{k}$, so $\left(1, 3\sqrt{3}, \frac{\pi}{3}\right)$ is on the tangent line. $\mathbf{r}'(t) = -2\sin t\mathbf{i} + 6\cos t\mathbf{j} + \mathbf{k}$, so $\mathbf{r}'\left(\frac{\pi}{3}\right) = -\sqrt{3}\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ is parallel to the tangent line at $t = \frac{\pi}{3}$. The symmetric equations of the line are $\frac{x-1}{-\sqrt{3}} = \frac{y-3\sqrt{3}}{3} = \frac{z-\frac{\pi}{3}}{1}$.

24. The curve is given by $\mathbf{r}(t) = 2t^2\mathbf{i} + 4t\mathbf{j} + t^3\mathbf{k}$. $\mathbf{r}(1) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, so $(2, 4, 1)$ is on the tangent line. $\mathbf{r}'(t) = 4t\mathbf{i} + 4\mathbf{j} + 3t^2\mathbf{k}$, so $\mathbf{r}'(1) = 4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ is parallel to the tangent line. The parametric equations of the line are $x = 2 + 4t$, $y = 4 + 4t$, $z = 1 + 3t$.

25. The curve is given by $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j} + t^5\mathbf{k}$. $\mathbf{r}(-1) = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, so $(-3, 2, -1)$ is on the plane. $\mathbf{r}'(t) = 3\mathbf{i} + 4t\mathbf{j} + 5t^4\mathbf{k}$, so $\mathbf{r}'(-1) = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ is in the direction of the curve at $t = -1$, hence normal to the plane. An equation of the plane is $3(x+3) - 4(y-2) + 5(z+1) = 0$ or $3x - 4y + 5z = -22$.

26. $\mathbf{r}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\mathbf{i} + \frac{3\pi}{2}\mathbf{j}$, so $\left(\frac{\pi}{2}, \frac{3\pi}{2}, 0\right)$ is on the plane. $\mathbf{r}'(t) = (t\cos t + \sin t)\mathbf{i} + 3\mathbf{j} + (2\cos t - 2t\sin t)\mathbf{k}$ so $\mathbf{r}'\left(\frac{\pi}{2}\right) = \mathbf{i} + 3\mathbf{j} - \pi\mathbf{k}$ is in the direction of the curve at $\frac{\pi}{2}$, hence normal to the plane. An equation of the plane is $1\left(x - \frac{\pi}{2}\right) + 3\left(y - \frac{3\pi}{2}\right) - \pi(z-0) = 0$ or $x + 3y - \pi z = 5\pi$.

27. a. $[x(t)]^2 + [y(t)]^2 + [z(t)]^2 = (\sin t \cos t)^2 + (\sin^2 t)^2 + \cos^2 t = \sin^2 t \cos^2 t + \sin^4 t + \cos^2 t$
 $= \sin^2 t (\cos^2 t + \sin^2 t) + \cos^2 t = \sin^2 t + \cos^2 t = 1$

Thus, the curve lies on the sphere $x^2 + y^2 + z^2 = 1$
 whose center is at the origin.

b. $\mathbf{r}\left(\frac{\pi}{6}\right) = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k} = \frac{\sqrt{3}}{4}\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$, so $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ is on the tangent line.

$\mathbf{r}'(t) = (\cos^2 t - \sin^2 t)\mathbf{i} + 2 \cos t \sin t \mathbf{j} - \sin t \mathbf{k}$ so $\mathbf{r}'\left(\frac{\pi}{6}\right) = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$ is parallel to the line.

The line has equations $x = \frac{\sqrt{3}}{4} + t$, $y = \frac{1}{4} + \sqrt{3}t$, $z = \frac{\sqrt{3}}{2} - t$.

The line intersects the xy -plane when $z = 0$, so $t = \frac{\sqrt{3}}{2}$, hence $x = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$, $y = \frac{1}{4} + \frac{3}{2} = \frac{7}{4}$.

The point is $\left(\frac{3\sqrt{3}}{4}, \frac{7}{4}, 0\right)$.

28. In Figure 7, d is the magnitude of the scalar projection of $\overrightarrow{PQ}$ on $\mathbf{n}$. $\text{pr}_{\mathbf{n}} \overrightarrow{PQ} = \frac{\overrightarrow{PQ} \cdot \mathbf{n}}{|\mathbf{n}|^2} \mathbf{n}$, so

$$\left| \text{pr}_{\mathbf{n}} \overrightarrow{PQ} \right| = \left| \frac{\overrightarrow{PQ} \cdot \mathbf{n}}{|\mathbf{n}|^2} \mathbf{n} \right| = \frac{\left| \overrightarrow{PQ} \cdot \mathbf{n} \right|}{|\mathbf{n}|^2} |\mathbf{n}| = \frac{\left| \overrightarrow{PQ} \cdot \mathbf{n} \right|}{|\mathbf{n}|}$$

The point $(0, 0, 1)$ is on the plane $4x - 4y + 2z = 2$. With $P(0, 0, 1)$, $Q(4, -2, 3)$, and $\mathbf{n} = \langle 2, -2, 1 \rangle = \frac{1}{2} \langle 4, -4, 2 \rangle$,

$$\overrightarrow{PQ} = \langle 4 - 0, -2 - 0, 3 - 1 \rangle = \langle 4, -2, 2 \rangle \text{ and } d = \frac{|\langle 4, -2, 2 \rangle \cdot \langle 2, -2, 1 \rangle|}{\sqrt{4 + 4 + 1}} = \frac{14}{3}.$$

From Example 6 of Section 14.2, $d = \frac{|4(4) - 4(-2) + 2(3) - 2|}{\sqrt{16 + 16 + 4}} = \frac{28}{6} = \frac{14}{3}$.

29. Let $\overrightarrow{PR}$ be the scalar projection of $\overrightarrow{PQ}$ on $\mathbf{n}$. Then $\left| \overrightarrow{PQ} \right|^2 = \left| \overrightarrow{PR} \right|^2 + d^2$ so

$$d^2 = \left| \overrightarrow{PQ} \right|^2 - \left| \overrightarrow{PR} \right|^2 = \left| \overrightarrow{PQ} \right|^2 - \frac{\left| \overrightarrow{PQ} \cdot \mathbf{n} \right|^2}{|\mathbf{n}|^2}$$

$$= \frac{|\vec{PQ}|^2 |\mathbf{n}|^2 - (\vec{PQ} \cdot \mathbf{n})^2}{|\mathbf{n}|^2} = \frac{|\vec{PQ} \times \mathbf{n}|^2}{|\mathbf{n}|^2}$$

by Lagrange's Identity. Thus, $d = \frac{|\vec{PQ} \times \mathbf{n}|}{|\mathbf{n}|}$.

- a. $P(3, -2, 1)$ is on the line, so $\vec{PQ} = \langle 1-3, 0+2, -4-1 \rangle = \langle -2, 2, -5 \rangle$ while $\mathbf{n} = \langle 2, -2, 1 \rangle$, so

$$d = \frac{|\langle -2, 2, -5 \rangle \times \langle 2, -2, 1 \rangle|}{\sqrt{4+4+1}} = \frac{|\langle -8, -8, 0 \rangle|}{3}$$

$$= \frac{8\sqrt{2}}{3} \approx 3.771$$

- b. $P(1, -1, 0)$ is on the line, so $\vec{PQ} = \langle 2-1, -1+1, 3-0 \rangle = \langle 1, 0, 3 \rangle$ while $\mathbf{n} = \langle 2, 3, -6 \rangle$.

$$d = \frac{|\langle 1, 0, 3 \rangle \times \langle 2, 3, -6 \rangle|}{\sqrt{4+9+36}} = \frac{|\langle -9, 12, 3 \rangle|}{7}$$

$$= \frac{3\sqrt{26}}{7} \approx 2.185$$

30. d is the distance between the parallel planes containing the lines. Since $\mathbf{n}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$, it is normal to the planes. Thus, d is the magnitude of the scalar projection of $\vec{PQ}$

on $\mathbf{n}$, which is $\frac{|\vec{PQ} \cdot \mathbf{n}|}{|\mathbf{n}|}$.

- a. $P(3, -2, 1)$ is on the first line, $Q(-4, -5, 0)$ is on the second line, $\mathbf{n}_1 = \langle 1, 1, 2 \rangle$, and $\mathbf{n}_2 = \langle 3, 4, 5 \rangle$.
 $\vec{PQ} = \langle -4-3, -5+2, 0-1 \rangle = \langle -7, -3, -1 \rangle$

$$\mathbf{n} = \langle 1, 1, 2 \rangle \times \langle 3, 4, 5 \rangle = \langle -3, 1, 1 \rangle$$

$$d = \frac{|\langle -7, -3, -1 \rangle \cdot \langle -3, 1, 1 \rangle|}{\sqrt{9+1+1}} = \frac{17}{\sqrt{11}} \approx 5.126$$

- b. $P(1, -2, 0)$ is on the first line, $Q(0, 1, 0)$ is on the second line, $\mathbf{n}_1 = \langle 2, 3, -4 \rangle$, and

$$\mathbf{n}_2 = \langle 3, 1, -5 \rangle.$$

$$\vec{PQ} = \langle 0-1, 1+2, 0-0 \rangle = \langle -1, 3, 0 \rangle$$

$$\mathbf{n} = \langle 2, 3, -4 \rangle \times \langle 3, 1, -5 \rangle = \langle -11, -2, -7 \rangle$$

$$d = \frac{|\langle -1, 3, 0 \rangle \cdot \langle -11, -2, -7 \rangle|}{\sqrt{121+4+49}} = \frac{5}{\sqrt{174}} \approx 0.379$$

14.5 Concepts Review

- $\mathbf{r}'(t); \mathbf{r}''(t)$
- $\int_a^b |\mathbf{r}'(t)| dt$
- parallel; concave
- $\left| \frac{dT}{ds} \right|$

Problem Set 14.5

- $\mathbf{v}(t) = \mathbf{r}'(t) = 4\mathbf{i} + 10t\mathbf{j} + 2\mathbf{k}$
 $\mathbf{a}(t) = \mathbf{r}''(t) = 10\mathbf{j}$
 $\mathbf{v}(1) = 4\mathbf{i} + 10\mathbf{j} + 2\mathbf{k}; \mathbf{a}(1) = 10\mathbf{j};$
 $s(1) = \sqrt{16+100+4} = 2\sqrt{30} \approx 10.954$
- $\mathbf{v}(t) = \mathbf{i} + 2(t-1)\mathbf{j} + 3(t-3)^2\mathbf{k}$
 $\mathbf{a}(t) = 2\mathbf{j} + 6(t-3)\mathbf{k}$
 $\mathbf{v}(0) = \mathbf{i} - 2\mathbf{j} + 27\mathbf{k}; \mathbf{a}(0) = 2\mathbf{j} - 18\mathbf{k};$
 $s(0) = \sqrt{1+4+729} = \sqrt{734} \approx 27.092$

$$3. \mathbf{v}(t) = -\frac{1}{t^2}\mathbf{i} - \frac{2t}{(t^2-1)^2}\mathbf{j} + 5t^4\mathbf{k}$$

$$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{i} + \frac{2+6t^2}{(t^2-1)^3}\mathbf{j} + 20t^3\mathbf{k}$$

$$\mathbf{v}(2) = -\frac{1}{4}\mathbf{i} - \frac{4}{9}\mathbf{j} + 80\mathbf{k}; \mathbf{a}(2) = \frac{1}{4}\mathbf{i} + \frac{26}{27}\mathbf{j} + 160\mathbf{k};$$

$$s(2) = \sqrt{\frac{1}{16} + \frac{16}{81} + 6400} = \frac{\sqrt{8,294,737}}{36} \approx 80.002$$

$$4. \mathbf{v}(t) = 6t^5\mathbf{i} + 72t(6t^2-5)^5\mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = 30t^4\mathbf{i} + 72(66t^2-5)(6t^2-5)^4\mathbf{j}$$

$$\mathbf{v}(1) = 6\mathbf{i} + 72\mathbf{j} + \mathbf{k}; \mathbf{a}(1) = 30\mathbf{i} + 4392\mathbf{j};$$

$$s(1) = \sqrt{36 + 5184 + 1} = \sqrt{5221} \approx 72.256$$

$$5. \mathbf{v}(t) = t^2\mathbf{j} + \frac{2}{3}t^{-1/3}\mathbf{k}; \mathbf{a}(t) = 2t\mathbf{j} - \frac{2}{9}t^{-4/3}\mathbf{k}$$

$$\mathbf{v}(2) = 4\mathbf{j} + \frac{2^{2/3}}{3}\mathbf{k}; \mathbf{a}(2) = 4\mathbf{j} - \frac{2^{-1/3}}{9}\mathbf{k}$$

$$s(2) = \sqrt{16 + \frac{2^{4/3}}{9}} \approx 4.035$$

$$\mathbf{v}(2) = 4\mathbf{j} + \frac{2^{2/3}}{3}\mathbf{k}; \mathbf{a}(2) = 4\mathbf{j} - \frac{1}{9\sqrt[3]{2}}\mathbf{k};$$

$$s(2) = \sqrt{16 + \frac{2^{4/3}}{9}} \approx 4.035$$

$$6. \mathbf{v}(t) = t^2\mathbf{i} + 5(t-1)^3\mathbf{j} + \sin \pi t\mathbf{k}$$

$$\mathbf{a}(t) = 2t\mathbf{i} + 15(t-1)^2\mathbf{j} + \pi \cos \pi t\mathbf{k}$$

$$11. \mathbf{v}(t) = (\pi t \cos \pi t + \sin \pi t)\mathbf{i} + (\cos \pi t - \pi t \sin \pi t)\mathbf{j} - e^{-t}\mathbf{k}$$

$$\mathbf{a}(t) = (2\pi \cos \pi t - \pi^2 t \sin \pi t)\mathbf{i} + (-2\pi \sin \pi t - \pi^2 t \cos \pi t)\mathbf{j} + e^{-t}\mathbf{k}$$

$$\mathbf{v}(2) = 2\pi\mathbf{i} + \mathbf{j} - e^{-2}\mathbf{k}; \mathbf{a}(2) = 2\pi\mathbf{i} - 2\pi^2\mathbf{j} + e^{-2}\mathbf{k};$$

$$s(2) = \sqrt{4\pi^2 + 1 + e^{-4}} \approx 6.364$$

$$12. \mathbf{v}(t) = \frac{1}{t}\mathbf{i} + \frac{2}{t}\mathbf{j} + \frac{3}{t}\mathbf{k}$$

$$\mathbf{a}(t) = -\frac{1}{t^2}\mathbf{i} - \frac{2}{t^2}\mathbf{j} - \frac{3}{t^2}\mathbf{k}$$

$$\mathbf{v}(2) = \frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{3}{2}\mathbf{k}; \mathbf{a}(2) = -\frac{1}{4}\mathbf{i} - \frac{1}{2}\mathbf{j} - \frac{3}{4}\mathbf{k};$$

$$s(2) = \sqrt{\frac{1}{4} + 1 + \frac{9}{4}} = \frac{\sqrt{14}}{2} \approx 1.871$$

$$\mathbf{v}(2) = 4\mathbf{i} + 5\mathbf{j}; \mathbf{a}(2) = 4\mathbf{i} + 15\mathbf{j} + \pi\mathbf{k};$$

$$s(2) = \sqrt{16 + 25} = \sqrt{41} \approx 6.403$$

$$7. \mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{v}(\pi) = -\mathbf{j} + \mathbf{k}; \mathbf{a}(\pi) = \mathbf{i};$$

$$s(\pi) = \sqrt{1+1} = \sqrt{2} \approx 1.414$$

$$8. \mathbf{v}(t) = 2 \cos 2t\mathbf{i} - 3 \sin 3t\mathbf{j} - 4 \sin 4t\mathbf{k}$$

$$\mathbf{a}(t) = -4 \sin 2t\mathbf{i} - 9 \cos 3t\mathbf{j} - 16 \cos 4t\mathbf{k}$$

$$\mathbf{v}\left(\frac{\pi}{2}\right) = -2\mathbf{i} + 3\mathbf{j}; \mathbf{a}\left(\frac{\pi}{2}\right) = -16\mathbf{k};$$

$$s\left(\frac{\pi}{2}\right) = \sqrt{4+9} = \sqrt{13} \approx 3.606$$

$$9. \mathbf{v}(t) = \sec^2 t\mathbf{i} + 3e^t\mathbf{j} - 4 \sin 4t\mathbf{k}$$

$$\mathbf{a}(t) = 2 \sec^2 t \tan t\mathbf{i} + 3e^t\mathbf{j} - 16 \cos 4t\mathbf{k}$$

$$\mathbf{v}\left(\frac{\pi}{4}\right) = 2\mathbf{i} + 3e^{\pi/4}\mathbf{j}; \mathbf{a}\left(\frac{\pi}{4}\right) = 4\mathbf{i} + 3e^{\pi/4}\mathbf{j} + 16\mathbf{k};$$

$$s\left(\frac{\pi}{4}\right) = \sqrt{4+9e^{\pi/2}} \approx 6.877$$

$$10. \mathbf{v}(t) = -e^t\mathbf{i} - \sin \pi t\mathbf{j} + \frac{2}{3}t^{-1/3}\mathbf{k}$$

$$\mathbf{a}(t) = -e^t\mathbf{i} - \pi \cos \pi t\mathbf{j} - \frac{2}{9}t^{-4/3}\mathbf{k}$$

$$\mathbf{v}(2) = -e^2\mathbf{i} + \frac{2^{2/3}}{3}\mathbf{k}; \mathbf{a}(2) = -e^2\mathbf{i} - \pi\mathbf{j} - \frac{1}{9\sqrt[3]{2}}\mathbf{k};$$

$$s(0) = \sqrt{e^4 + \frac{2^{4/3}}{9}} \approx 7.408$$

13. If $|\mathbf{v}| = C$, then $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = C^2$. Differentiate implicitly to get $D_t(\mathbf{v} \cdot \mathbf{v}) = 2\mathbf{v} \cdot \mathbf{v}' = 0$. Thus, $\mathbf{v} \cdot \mathbf{v}' = \mathbf{v} \cdot \mathbf{a} = 0$, so $\mathbf{a}$ is perpendicular to $\mathbf{v}$.

14. If $|\mathbf{r}(t)| = C$, similar to Problem 13, $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$. Conversely, if $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$, then $2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$. But since $2\mathbf{r}(t) \cdot \mathbf{r}'(t) = D_t[\mathbf{r}(t) \cdot \mathbf{r}(t)]$, this means that $\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2$ is a constant, so $|\mathbf{r}(t)|$ is constant.

$$15. s = \int_0^2 \sqrt{1^2 + \cos^2 t + (-\sin t)^2} dt = \int_0^2 \sqrt{1 + \cos^2 t + \sin^2 t} dt = \int_0^2 \sqrt{1+1} dt = \sqrt{2} \int_0^2 dt = \sqrt{2}(2-0) = 2\sqrt{2}$$

$$16. s = \int_0^2 \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 2} dt = \int_0^2 \sqrt{t^2 + 3} dt = \left[\frac{t}{2} \sqrt{t^2 + 3} + \frac{3}{2} \ln \left| t + \sqrt{t^2 + 3} \right| \right]_0^2$$

$$= \sqrt{7} + \frac{3}{2} \ln(2 + \sqrt{7}) - \frac{3}{2} \ln \sqrt{3} \approx 4.126$$

Use Formula 44 with $u = t$ and $a = \sqrt{3}$ for $\int \sqrt{t^2 + 3} dt$.

$$17. s = \int_3^6 \sqrt{24t^2 + 4t^4 + 36} dt = \int_3^6 2\sqrt{t^4 + 6t^2 + 9} dt$$

$$= \int_3^6 2(t^2 + 3) dt = 2 \left[\frac{t^3}{3} + 3t \right]_3^6 = 2[72 + 18 - (9 + 9)] = 144$$

$$18. s = \int_0^1 \sqrt{4t^2 + 36t^4 + 324t^6} dt = \int_0^1 2t\sqrt{1 + 90t^2} dt = \left[\frac{1}{135} (1 + 90t^2)^{3/2} \right]_0^1 = \frac{1}{135} (91^{3/2} - 1) \approx 6.423$$

$$19. s = \int_0^1 \sqrt{9t^4 + 36t^4 + 324t^4} dt = \int_0^1 3\sqrt{41}t^2 dt = \left[\sqrt{41}t^3 \right]_0^1 = \sqrt{41} \approx 6.403$$

$$20. s = \int_0^1 \sqrt{343t^{12} + 98t^{12} + 1764t^{12}} dt = \int_0^1 21\sqrt{5}t^6 dt = \left[3\sqrt{5}t^7 \right]_0^1 = 3\sqrt{5} \approx 6.708$$

$$21. s = \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} dt = \int_0^{\pi/2} 3\cos t \sin t \sqrt{\cos^2 t + \sin^2 t} dt$$

$$= \int_0^{\pi/2} 3\cos t \sin t dt = \left[\frac{3}{2} \sin^2 t \right]_0^{\pi/2} = \frac{3}{2}$$

$$22. s = \int_0^{\pi} \sqrt{e^{4t} (2\cos t - \sin t)^2 + e^{4t} (2\sin t + \cos t)^2 + 4e^{4t}} dt = \int_0^{\pi} e^{2t} \sqrt{4\cos^2 t + \sin^2 t + 4\sin^2 t + \cos^2 t + 4} dt$$

$$= \int_0^{\pi} 3e^{2t} dt = \left[\frac{3}{2} e^{2t} \right]_0^{\pi} = \frac{3}{2} (e^{2\pi} - 1) \approx 801.737$$

$$23. s = \int_0^{\pi} \sqrt{\cosh^2 t + \sinh^2 t + 1} dt = \int_0^{\pi} \sqrt{2\cosh^2 t} dt = \int_0^{\pi} \sqrt{2} \cosh t dt = \left[\sqrt{2} \sinh t \right]_0^{\pi} = \sqrt{2} \sinh \pi \approx 16.332$$

$$24. s = \int_0^{\pi} \sqrt{9\cosh^2 3t + 9\sinh^2 3t + 9} dt = \int_0^{\pi} 3\sqrt{2\cosh^2 3t} dt = \int_0^{\pi} 3\sqrt{2} \cosh 3t dt = \left[\sqrt{2} \sinh 3t \right]_0^{\pi} = \sqrt{2} \sinh 3\pi \approx 8762$$

$$25. \mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j} + (2t - 4)\mathbf{k}$$

$$\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{k}$$

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \frac{4\mathbf{i} + 2\mathbf{j}}{\sqrt{16 + 4}} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

$$a_T(2) = \frac{\mathbf{r}'(2) \cdot \mathbf{r}''(2)}{|\mathbf{r}'(2)|} = \frac{(4\mathbf{i} + 2\mathbf{j}) \cdot (2\mathbf{i} + 2\mathbf{k})}{2\sqrt{5}} = \frac{8}{2\sqrt{5}} = \frac{4}{\sqrt{5}}$$

$$a_N(2) = \frac{|\mathbf{r}'(2) \times \mathbf{r}''(2)|}{|\mathbf{r}'(2)|} = \frac{|(4\mathbf{i} + 2\mathbf{j}) \times (2\mathbf{i} + 2\mathbf{k})|}{2\sqrt{5}} = \frac{|4\mathbf{i} - 8\mathbf{j} - 4\mathbf{k}|}{2\sqrt{5}} = \frac{4\sqrt{6}}{2\sqrt{5}} = 2\sqrt{\frac{6}{5}}$$

$$\mathbf{N}(2) = \frac{\mathbf{r}''(2) - a_T(2)\mathbf{T}(2)}{a_N(2)} = \frac{(2\mathbf{i} + 2\mathbf{k}) - \frac{4}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}\right)}{2\sqrt{\frac{6}{5}}} = \frac{\sqrt{5}}{2\sqrt{6}}\left(\frac{2}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} + 2\mathbf{k}\right) = \frac{1}{\sqrt{30}}\mathbf{i} - \frac{2}{\sqrt{30}}\mathbf{j} + \frac{5}{\sqrt{30}}\mathbf{k}$$

$$\kappa(2) = \frac{|\mathbf{r}'(2) \times \mathbf{r}''(2)|}{|\mathbf{r}'(2)|^3} = \frac{4\sqrt{6}}{(2\sqrt{5})^3} = \frac{\sqrt{6}}{10\sqrt{5}}$$

$$\mathbf{B}(2) = \mathbf{T}(2) \times \mathbf{N}(2) = \left(\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}\right) \times \left(\frac{1}{\sqrt{30}}\mathbf{i} - \frac{2}{\sqrt{30}}\mathbf{j} + \frac{5}{\sqrt{30}}\mathbf{k}\right) = \frac{1}{\sqrt{6}}\mathbf{i} - \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

26. $\mathbf{r}'(t) = t^2\mathbf{i} + t\mathbf{k}$

$$\mathbf{r}''(t) = 2t\mathbf{i} + \mathbf{k}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{\mathbf{i} + \mathbf{k}}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}$$

$$a_T(1) = \frac{\mathbf{r}'(1) \cdot \mathbf{r}''(1)}{|\mathbf{r}'(1)|} = \frac{(\mathbf{i} + \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{k})}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$a_N(1) = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|} = \frac{|(\mathbf{i} + \mathbf{k}) \times (2\mathbf{i} + \mathbf{k})|}{\sqrt{2}} = \frac{1}{\sqrt{2}}|\mathbf{j}| = \frac{1}{\sqrt{2}}$$

$$\mathbf{N}(1) = \frac{\mathbf{r}''(1) - a_T(1)\mathbf{T}(1)}{a_N(1)} = \frac{(2\mathbf{i} + \mathbf{k}) - \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}\right)}{\frac{1}{\sqrt{2}}} = \sqrt{2}\left(\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{k}\right) = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}$$

$$\kappa(1) = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{1}{(\sqrt{2})^3} = 2^{-3/2}$$

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}\right) \times \left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}\right) = \mathbf{j}$$

27. $\mathbf{r}'(t) = t\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$

$$\mathbf{r}''(t) = \mathbf{i} + 2t\mathbf{k}$$

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \frac{2\mathbf{i} + \mathbf{j} + 4\mathbf{k}}{\sqrt{4+1+16}} = \frac{2}{\sqrt{21}}\mathbf{i} + \frac{1}{\sqrt{21}}\mathbf{j} + \frac{4}{\sqrt{21}}\mathbf{k}$$

$$a_T(2) = \frac{\mathbf{r}'(2) \cdot \mathbf{r}''(2)}{|\mathbf{r}'(2)|} = \frac{(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + 4\mathbf{k})}{\sqrt{21}} = \frac{18}{\sqrt{21}}$$

$$a_N(2) = \frac{|\mathbf{r}'(2) \times \mathbf{r}''(2)|}{|\mathbf{r}'(2)|} = \frac{|(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} + 4\mathbf{k})|}{\sqrt{21}} = \frac{1}{\sqrt{21}}|4\mathbf{i} - 4\mathbf{j} - \mathbf{k}| = \frac{\sqrt{33}}{\sqrt{21}} = \sqrt{\frac{11}{7}}$$

$$\mathbf{N}(2) = \frac{\mathbf{r}''(2) - a_T(2)\mathbf{T}(2)}{a_N(2)} = \frac{(\mathbf{i} + 4\mathbf{k}) - \frac{18}{\sqrt{21}}\left(\frac{2}{\sqrt{21}}\mathbf{i} + \frac{1}{\sqrt{21}}\mathbf{j} + \frac{4}{\sqrt{21}}\mathbf{k}\right)}{\sqrt{\frac{11}{7}}} = \sqrt{\frac{7}{11}}\left(-\frac{15}{21}\mathbf{i} - \frac{18}{21}\mathbf{j} + \frac{12}{21}\mathbf{k}\right)$$

$$= \sqrt{\frac{7}{11}}\left(-\frac{5}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{4}{7}\mathbf{k}\right) = -\frac{5}{\sqrt{77}}\mathbf{i} - \frac{6}{\sqrt{77}}\mathbf{j} + \frac{4}{\sqrt{77}}\mathbf{k}$$

$$\kappa(2) = \frac{|\mathbf{r}'(2) \times \mathbf{r}''(2)|}{|\mathbf{r}'(2)|^3} = \frac{\sqrt{33}}{(\sqrt{21})^3} = \sqrt{\frac{33}{9261}} = \sqrt{\frac{11}{3087}} = \frac{\sqrt{11}}{21\sqrt{7}}$$

$$\mathbf{B}(2) = \mathbf{T}(2) \times \mathbf{N}(2) = \left(\frac{2}{\sqrt{21}}\mathbf{i} + \frac{1}{\sqrt{21}}\mathbf{j} + \frac{4}{\sqrt{21}}\mathbf{k} \right) \times \left(-\frac{5}{\sqrt{77}}\mathbf{i} - \frac{6}{\sqrt{77}}\mathbf{j} + \frac{4}{\sqrt{77}}\mathbf{k} \right) = \frac{4}{\sqrt{33}}\mathbf{i} - \frac{4}{\sqrt{33}}\mathbf{j} - \frac{1}{\sqrt{33}}\mathbf{k}$$

28. $\mathbf{r}(t) = \langle \sin 3t, \cos 3t, t \rangle$

$$\mathbf{r}'(t) = \langle 3 \cos 3t, -3 \sin 3t, 1 \rangle$$

$$\mathbf{r}''(t) = \langle -9 \sin 3t, -9 \cos 3t, 0 \rangle$$

$$\mathbf{T}\left(\frac{\pi}{9}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{9}\right)}{|\mathbf{r}'\left(\frac{\pi}{9}\right)|} = \frac{\left\langle \frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1 \right\rangle}{\sqrt{\frac{9}{4} + \frac{27}{4} + 1}} = \left\langle \frac{3}{2\sqrt{10}}, -\frac{3\sqrt{3}}{2\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$$

$$a_T\left(\frac{\pi}{9}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{9}\right) \cdot \mathbf{r}''\left(\frac{\pi}{9}\right)}{|\mathbf{r}'\left(\frac{\pi}{9}\right)|} = \frac{\left\langle \frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1 \right\rangle \cdot \left\langle -\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0 \right\rangle}{\sqrt{10}} = 0$$

$$a_N\left(\frac{\pi}{9}\right) = \frac{|\mathbf{r}'\left(\frac{\pi}{9}\right) \times \mathbf{r}''\left(\frac{\pi}{9}\right)|}{|\mathbf{r}'\left(\frac{\pi}{9}\right)|} = \frac{\left| \left\langle \frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1 \right\rangle \times \left\langle -\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0 \right\rangle \right|}{\sqrt{10}} = \frac{1}{\sqrt{10}} \left| \left\langle \frac{9}{2}, -\frac{9\sqrt{3}}{2}, -27 \right\rangle \right| = \frac{1}{\sqrt{10}} (9\sqrt{10}) = 9$$

$$\mathbf{N}\left(\frac{\pi}{9}\right) = \frac{\mathbf{r}''\left(\frac{\pi}{9}\right) - a_N\left(\frac{\pi}{9}\right)\mathbf{T}\left(\frac{\pi}{9}\right)}{a_N} = \frac{1}{9} \left\langle -\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0 \right\rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0 \right\rangle$$

$$\kappa\left(\frac{\pi}{9}\right) = \frac{|\mathbf{r}'\left(\frac{\pi}{9}\right) \times \mathbf{r}''\left(\frac{\pi}{9}\right)|}{|\mathbf{r}'\left(\frac{\pi}{9}\right)|^3} = \frac{9\sqrt{10}}{(\sqrt{10})^3} = \frac{9}{10}$$

$$\mathbf{B}\left(\frac{\pi}{9}\right) = \mathbf{T}\left(\frac{\pi}{9}\right) \times \mathbf{N}\left(\frac{\pi}{9}\right) = \left\langle \frac{3}{2\sqrt{10}}, -\frac{3\sqrt{3}}{2\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle \times \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0 \right\rangle = \left\langle \frac{1}{2\sqrt{10}}, -\frac{\sqrt{3}}{2\sqrt{10}}, -\frac{3}{\sqrt{10}} \right\rangle$$

29. $\mathbf{r}(t) = \langle 7 \sin 3t, 7 \cos 3t, 14t \rangle$

$$\mathbf{r}'(t) = \langle 21 \cos 3t, -21 \sin 3t, 14 \rangle$$

$$\mathbf{r}''(t) = \langle -63 \sin 3t, -63 \cos 3t, 0 \rangle$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{|\mathbf{r}'\left(\frac{\pi}{3}\right)|} = \frac{\langle -21, 0, 14 \rangle}{\sqrt{441 + 196}} = \frac{1}{7\sqrt{13}} \langle -21, 0, 14 \rangle = \left\langle -\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}} \right\rangle$$

$$a_T\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right) \cdot \mathbf{r}''\left(\frac{\pi}{3}\right)}{|\mathbf{r}'\left(\frac{\pi}{3}\right)|} = \frac{\langle -21, 0, 14 \rangle \cdot \langle 0, -63, 0 \rangle}{7\sqrt{13}} = 0$$

$$a_N\left(\frac{\pi}{3}\right) = \frac{|\mathbf{r}'\left(\frac{\pi}{3}\right) \times \mathbf{r}''\left(\frac{\pi}{3}\right)|}{|\mathbf{r}'\left(\frac{\pi}{3}\right)|} = \frac{|\langle -21, 0, 14 \rangle \times \langle 0, -63, 0 \rangle|}{7\sqrt{13}} = \frac{|\langle 882, 0, -1323 \rangle|}{7\sqrt{13}} = \frac{441\sqrt{13}}{7\sqrt{13}} = 63$$

$$\mathbf{N}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}''\left(\frac{\pi}{3}\right) - a_N\left(\frac{\pi}{3}\right)\mathbf{T}\left(\frac{\pi}{3}\right)}{a_N} = \frac{1}{63} \langle 0, 63, 0 \rangle = \langle 0, 1, 0 \rangle$$

$$\kappa\left(\frac{\pi}{3}\right) = \frac{\left|\mathbf{r}'\left(\frac{\pi}{3}\right) \times \mathbf{r}''\left(\frac{\pi}{3}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|^3} = \frac{441\sqrt{13}}{(7\sqrt{13})^3} = \frac{9}{91}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right) \times \mathbf{N}\left(\frac{\pi}{3}\right) = \left\langle -\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}} \right\rangle \times \langle 0, 1, 0 \rangle = \left\langle -\frac{2}{\sqrt{13}}, 0, -\frac{3}{\sqrt{13}} \right\rangle$$

30. $\mathbf{r}'(t) = -3\cos^2 t \sin t \mathbf{i} + 3\sin^2 t \cos t \mathbf{k}$

$$\mathbf{r}''(t) = (6\cos t \sin^2 t - 3\cos^3 t) \mathbf{i} + (6\cos^2 t \sin t - 3\sin^3 t) \mathbf{k}$$

$$\mathbf{r}'\left(\frac{\pi}{2}\right) = \mathbf{0} \text{ so the object is motionless at } t_1 = \frac{\pi}{2}.$$

κ , $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ do not exist.

31. $\mathbf{r}'(t) = \sinh \frac{t}{3} \mathbf{i} + \mathbf{j}$

$$\mathbf{r}''(t) = \frac{1}{3} \cosh \frac{t}{3} \mathbf{i}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{\sinh \frac{1}{3} \mathbf{i} + \mathbf{j}}{\sqrt{\sinh^2 \frac{1}{3} + 1}} = \frac{\sinh \frac{1}{3} \mathbf{i} + \mathbf{j}}{\cosh \frac{1}{3}} = \tanh \frac{1}{3} \mathbf{i} + \operatorname{sech} \frac{1}{3} \mathbf{j}$$

$$a_T(1) = \frac{\mathbf{r}'(1) \cdot \mathbf{r}''(1)}{|\mathbf{r}'(1)|} = \frac{(\sinh \frac{1}{3} \mathbf{i} + \mathbf{j}) \cdot (\frac{1}{3} \cosh \frac{1}{3} \mathbf{i})}{\cosh \frac{1}{3}} = \frac{1}{3} \sinh \frac{1}{3}$$

$$a_N(1) = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|} = \frac{|(\sinh \frac{1}{3} \mathbf{i} + \mathbf{j}) \times (\frac{1}{3} \cosh \frac{1}{3} \mathbf{i})|}{\cosh \frac{1}{3}} = \frac{|\frac{1}{3} \cos \frac{1}{3} \mathbf{k}|}{\cosh \frac{1}{3}} = \frac{1}{3}$$

$$\mathbf{N}(1) = \frac{\mathbf{r}''(1) - a_T(1)\mathbf{T}(1)}{a_N(1)} = 3 \left[\frac{1}{3} \cosh \frac{1}{3} \mathbf{i} - \frac{1}{3} \sinh \frac{1}{3} \left(\tanh \frac{1}{3} \mathbf{i} + \operatorname{sech} \frac{1}{3} \mathbf{j} \right) \right] = \left(\cosh \frac{1}{3} - \frac{\sinh^2 \frac{1}{3}}{\cosh \frac{1}{3}} \right) \mathbf{i} - \tanh \frac{1}{3} \mathbf{j}$$

$$= \operatorname{sech} \frac{1}{3} \mathbf{i} - \tanh \frac{1}{3} \mathbf{j}$$

$$\kappa(1) = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{\frac{1}{3} \cosh \frac{1}{3}}{\cosh^3 \frac{1}{3}} = \frac{1}{3} \operatorname{sech}^2 \frac{1}{3}$$

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left(\tanh \frac{1}{3} \mathbf{i} + \operatorname{sech} \frac{1}{3} \mathbf{j} \right) \times \left(\operatorname{sech} \frac{1}{3} \mathbf{i} - \tanh \frac{1}{3} \mathbf{j} \right) = \left(-\operatorname{sech}^2 \frac{1}{3} - \tanh^2 \frac{1}{3} \right) \mathbf{k} = -\mathbf{k}$$

32. $\mathbf{r}'(t) = e^{7t} (7 \cos 2t - 2 \sin 2t) \mathbf{i} + e^{7t} (7 \sin 2t + 2 \cos 2t) \mathbf{j} + 7e^{7t} \mathbf{k}$

$$\mathbf{r}''(t) = e^{7t} (45 \cos 2t - 28 \sin 2t) \mathbf{i} + e^{7t} (45 \sin 2t + 28 \cos 2t) \mathbf{j} + 49e^{7t} \mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{|\mathbf{r}'\left(\frac{\pi}{3}\right)|} = \frac{e^{7\pi/3} \left(-\frac{7}{2} - \sqrt{3}\right) \mathbf{i} + e^{7\pi/3} \left(\frac{7\sqrt{3}}{2} - 1\right) \mathbf{j} + 7e^{7\pi/3} \mathbf{k}}{e^{7\pi/3} \sqrt{\left(-\frac{7}{2} - \sqrt{3}\right)^2 + \left(\frac{7\sqrt{3}}{2} - 1\right)^2 + 49}} = \left(-\frac{7+2\sqrt{3}}{2\sqrt{102}}\right) \mathbf{i} + \left(\frac{7\sqrt{3}-2}{2\sqrt{102}}\right) \mathbf{j} + \frac{7}{\sqrt{102}} \mathbf{k}$$

$$a_T\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right) \cdot \mathbf{r}''\left(\frac{\pi}{3}\right)}{|\mathbf{r}'\left(\frac{\pi}{3}\right)|} = \frac{714e^{14\pi/3}}{e^{7\pi/3} \sqrt{102}} = 7\sqrt{102}e^{7\pi/3}$$

$$a_N\left(\frac{\pi}{3}\right) = \frac{\left|\mathbf{r}'\left(\frac{\pi}{3}\right) \times \mathbf{r}''\left(\frac{\pi}{3}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|} = \frac{\left|e^{14\pi/3}(49+14\sqrt{3})\mathbf{i} + e^{14\pi/3}(14-49\sqrt{3})\mathbf{j} + 106e^{14\pi/3}\mathbf{k}\right|}{e^{7\pi/3}\sqrt{102}} = 2\sqrt{53}e^{7\pi/3}$$

$$\mathbf{N}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}''\left(\frac{\pi}{3}\right) - a_T\left(\frac{\pi}{3}\right)\mathbf{T}\left(\frac{\pi}{3}\right)}{a_N} = \frac{2-7\sqrt{3}}{2\sqrt{53}}\mathbf{i} - \frac{7+2\sqrt{3}}{2\sqrt{53}}\mathbf{j}$$

$$\kappa\left(\frac{\pi}{3}\right) = \frac{\left|\mathbf{r}'\left(\frac{\pi}{3}\right) \times \mathbf{r}''\left(\frac{\pi}{3}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|^3} = \frac{2\sqrt{5406}e^{14\pi/3}}{\sqrt{102^3}e^{7\pi}} = \frac{\sqrt{53}}{51}e^{-7\pi/3}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right) \times \mathbf{N}\left(\frac{\pi}{3}\right) = \frac{42+49\sqrt{3}}{6\sqrt{1802}}\mathbf{i} + \frac{14\sqrt{3}-147}{6\sqrt{1802}}\mathbf{j} + \sqrt{\frac{53}{102}}\mathbf{k}$$

33. $\mathbf{r}'(t) = -2e^{-2t}\mathbf{i} + 2e^{2t}\mathbf{j} + 2\sqrt{2}\mathbf{k}$

$$\mathbf{r}''(t) = 4e^{-2t}\mathbf{i} + 4e^{2t}\mathbf{j}$$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{-2\mathbf{i} + 2\mathbf{j} + 2\sqrt{2}\mathbf{k}}{\sqrt{4+4+8}} = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

$$a_T(0) = \frac{\mathbf{r}'(0) \cdot \mathbf{r}''(0)}{|\mathbf{r}'(0)|} = \frac{(-2\mathbf{i} + 2\mathbf{j} + 2\sqrt{2}\mathbf{k}) \cdot (4\mathbf{i} + 4\mathbf{j})}{4} = 0$$

$$a_N(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|} = \frac{|-8\sqrt{2}\mathbf{i} + 8\sqrt{2}\mathbf{j} - 16\mathbf{k}|}{4} = \frac{16\sqrt{2}}{4} = 4\sqrt{2}$$

$$\mathbf{N}(0) = \frac{\mathbf{r}''(0) - a_T(0)\mathbf{T}(0)}{a_N(0)} = \frac{4\mathbf{i} + 4\mathbf{j}}{4\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{16\sqrt{2}}{64} = \frac{\sqrt{2}}{4}$$

$$\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \left(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}\right) \times \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$$

34. $\mathbf{r}(t) = \langle \ln t, 3t, t^2 \rangle$

$$\mathbf{r}'(t) = \left\langle \frac{1}{t}, 3, 2t \right\rangle$$

$$\mathbf{r}''(t) = \left\langle -\frac{1}{t^2}, 0, 2 \right\rangle$$

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \frac{\left\langle \frac{1}{2}, 3, 4 \right\rangle}{\sqrt{\frac{1}{4} + 9 + 16}} = \frac{\left\langle \frac{1}{2}, 3, 4 \right\rangle}{\frac{\sqrt{101}}{2}} = \left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}} \right\rangle$$

$$a_T(2) = \frac{\mathbf{r}'(2) \cdot \mathbf{r}''(2)}{|\mathbf{r}'(2)|} = \frac{2}{\sqrt{101}} \left(\left\langle \frac{1}{2}, 3, 4 \right\rangle \cdot \left\langle -\frac{1}{4}, 0, 2 \right\rangle \right) = \frac{2}{\sqrt{101}} \left(\frac{63}{8} \right) = \frac{63}{4\sqrt{101}}$$

$$a_N(2) = \frac{|\mathbf{r}'(2) \times \mathbf{r}''(2)|}{|\mathbf{r}'(2)|} = \frac{2}{\sqrt{101}} \left| \left\langle 6, -2, \frac{3}{4} \right\rangle \right| = \frac{2}{\sqrt{101}} \left(\frac{\sqrt{649}}{4} \right) = \frac{\sqrt{649}}{2\sqrt{101}}$$

$$\mathbf{N}(2) = \frac{\mathbf{r}''(2) - a_T(2)\mathbf{T}(2)}{a_N(2)} = \frac{2\sqrt{101}}{\sqrt{649}} \left(\left\langle -\frac{1}{4}, 0, 2 \right\rangle - \frac{63}{4\sqrt{101}} \left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}} \right\rangle \right)$$

$$= \frac{2\sqrt{101}}{\sqrt{649}} \left\langle -\frac{41}{101}, -\frac{189}{202}, \frac{76}{101} \right\rangle = \left\langle -\frac{82}{\sqrt{65,549}}, -\frac{189}{\sqrt{65,549}}, \frac{152}{\sqrt{65,549}} \right\rangle$$

$$\kappa(2) = \frac{|\mathbf{r}'(2) \times \mathbf{r}''(2)|}{|\mathbf{r}'(2)|^3} = \left(\frac{\sqrt{649}}{4} \right) \left(\frac{2}{\sqrt{101}} \right)^3 = \frac{2\sqrt{649}}{101\sqrt{101}}$$

$$\mathbf{B}(2) = \mathbf{T}(2) \times \mathbf{N}(2) = \left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}} \right\rangle \times \left\langle -\frac{82}{\sqrt{65,549}}, -\frac{189}{\sqrt{65,549}}, \frac{152}{\sqrt{65,549}} \right\rangle = \left\langle \frac{24}{\sqrt{649}}, -\frac{8}{\sqrt{649}}, \frac{3}{\sqrt{649}} \right\rangle$$

35. $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

$$\mathbf{r}''(t) = 2\mathbf{k}$$

$$a_T(t) = \frac{(\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}) \cdot (2\mathbf{k})}{\sqrt{1+9+4t^2}} = \frac{4t}{\sqrt{10+4t^2}}$$

$$a_N(t) = \frac{|(\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}) \times (2\mathbf{k})|}{\sqrt{10+4t^2}} = \frac{|6\mathbf{i} - 2\mathbf{j}|}{\sqrt{10+4t^2}} = \frac{\sqrt{36+4}}{\sqrt{10+4t^2}} = 2\sqrt{\frac{10}{10+4t^2}} = 2\sqrt{\frac{5}{5+2t^2}}$$

36. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\mathbf{r}''(t) = \langle 0, 2, 6t \rangle$$

$$a_T(t) = \frac{\langle 1, 2t, 3t^2 \rangle \cdot \langle 0, 2, 6t \rangle}{\sqrt{1+4t^2+9t^4}} = \frac{4t+18t^3}{\sqrt{1+4t^2+9t^4}}$$

$$a_N(t) = \frac{|\langle 1, 2t, 3t^2 \rangle \times \langle 0, 2, 6t \rangle|}{\sqrt{1+4t^2+9t^4}} = \frac{|\langle 6t^2, -6t, 2 \rangle|}{\sqrt{1+4t^2+9t^4}} = \frac{\sqrt{36t^4+36t^2+4}}{\sqrt{1+4t^2+9t^4}} = 2\sqrt{\frac{9t^4+9t^2+1}{1+4t^2+9t^4}}$$

37. $\mathbf{r}(t) = \langle e^{-t}, 2t, e^t \rangle$

$$\mathbf{r}'(t) = \langle -e^{-t}, 2, e^t \rangle$$

$$\mathbf{r}''(t) = \langle e^{-t}, 0, e^t \rangle$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -e^{-2t} + e^{2t}$$

$$|\mathbf{r}'(t)| = \sqrt{e^{-2t} + 4 + e^{2t}}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \left| \langle 2e^t, 2, -2e^{-t} \rangle \right| = \sqrt{4e^{2t} + 4 + 4e^{-2t}} = 2\sqrt{e^{2t} + 1 + e^{-2t}}$$

$$a_T(t) = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + 4 + e^{-2t}}}$$

$$a_N(t) = 2\sqrt{\frac{e^{2t} + 1 + e^{-2t}}{e^{2t} + 4 + e^{-2t}}}$$

38. $\mathbf{r}'(t) = 2(t-2)\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$

$$\mathbf{r}''(t) = 2\mathbf{i} - 2\mathbf{j}$$

$$a_T(t) = \frac{[2(t-2)\mathbf{i} - 2t\mathbf{j} + \mathbf{k}] \cdot (2\mathbf{i} - 2\mathbf{j})}{\sqrt{4(t-2)^2 + 4t^2 + 1}} = \frac{4(t-2) + 4t}{\sqrt{8t^2 - 16t + 17}} = \frac{8t - 8}{\sqrt{8t^2 - 16t + 17}}$$

$$a_N(t) = \frac{|[2(t-2)\mathbf{i} - 2t\mathbf{j} + \mathbf{k}] \times (2\mathbf{i} - 2\mathbf{j})|}{\sqrt{8t^2 - 16t + 17}} = \frac{|2\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}|}{\sqrt{8t^2 - 16t + 17}} = \frac{6\sqrt{2}}{\sqrt{8t^2 - 16t + 17}}$$

39. $\mathbf{r}'(t) = (1-t^2)\mathbf{i} - (1+t^2)\mathbf{j} + \mathbf{k}$

$$\mathbf{r}''(t) = -2t\mathbf{i} - 2t\mathbf{j}$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -2t(1-t^2) + 2t(1+t^2) = 4t^3$$

$$|\mathbf{r}'(t)| = \sqrt{(1-t^2)^2 + (1+t^2)^2 + 1} = \sqrt{2t^4 + 3}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |2t\mathbf{i} - 2t\mathbf{j} - 4t\mathbf{k}| = \sqrt{4t^2 + 4t^2 + 16t^2} = 2\sqrt{6}|t|$$

$$a_T(t) = \frac{4t^3}{\sqrt{2t^4 + 3}}$$

$$a_N(t) = \frac{2\sqrt{6}|t|}{\sqrt{2t^4 + 3}}$$

40. $\mathbf{r}'(t) = \mathbf{i} + t^2\mathbf{j} - \frac{1}{t^2}\mathbf{k}, t > 0$

$$\mathbf{r}''(t) = 2t\mathbf{j} + \frac{2}{t^3}\mathbf{k}$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 2t^3 - \frac{2}{t^5} = \frac{2}{t^5}(t^8 - 1)$$

$$|\mathbf{r}'(t)| = \sqrt{1 + t^4 + \frac{1}{t^4}} = \frac{1}{t^2}\sqrt{t^4 + t^8 + 1}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \left| \frac{4}{t}\mathbf{i} - \frac{2}{t^3}\mathbf{j} + 2t\mathbf{k} \right| = \sqrt{\frac{4}{t^6} + \frac{16}{t^2} + 4t^2} = \frac{2}{t^3}\sqrt{1 + 4t^4 + t^8}$$

$$a_T(t) = \frac{\frac{2}{t^5}(t^8 - 1)}{\frac{1}{t^2}\sqrt{t^4 + t^8 + 1}} = \frac{2(t^8 - 1)}{t^3\sqrt{t^8 + t^4 + 1}}$$

$$a_N(t) = \frac{\frac{2}{t^3}\sqrt{1 + 4t^4 + t^8}}{\frac{1}{t^2}\sqrt{t^4 + t^8 + 1}} = \frac{2}{t}\sqrt{\frac{t^8 + 4t^4 + 1}{t^8 + t^4 + 1}}$$

41. $\mathbf{r}'(t) = \cot t\mathbf{i} - \tan t\mathbf{j} + \mathbf{k}$

$$\mathbf{r}''(t) = -\csc^2 t\mathbf{i} - \sec^2 t\mathbf{j}$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -\frac{\cos t}{\sin^3 t} + \frac{\sin t}{\cos^3 t} = \frac{-\cos^4 t + \sin^4 t}{\sin^3 t \cos^3 t} = \tan t \sec^2 t - \cot t \csc^2 t$$

$$|\mathbf{r}'(t)| = \sqrt{\cot^2 t + \tan^2 t + 1}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\sec^2 t\mathbf{i} - \csc^2 t\mathbf{j} - 2\csc t \sec t\mathbf{k}| = \sqrt{\sec^4 t + \csc^4 t + 4\csc^2 t \sec^2 t}$$

$$a_T(t) = \frac{\tan t \sec^2 t - \cot t \csc^2 t}{\sqrt{\cot^2 t + \tan^2 t + 1}}$$

$$a_N(t) = \frac{\sqrt{\sec^4 t + \csc^4 t + 4\csc^2 t \sec^2 t}}{\sqrt{\cot^2 t + \tan^2 t + 1}}$$

$$42. \mathbf{r}'(t) = (t \cos t + \sin t)\mathbf{i} + (\cos t - t \sin t)\mathbf{j} + 2t\mathbf{k}$$

$$\mathbf{r}''(t) = (2 \cos t - t \sin t)\mathbf{i} + (-t \cos t - 2 \sin t)\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (t \cos t + \sin t)(2 \cos t - t \sin t) + (\cos t - t \sin t)(-t \cos t - 2 \sin t) + 4t = 5t$$

$$|\mathbf{r}'(t)| = \sqrt{(t \cos t + \sin t)^2 + (\cos t - t \sin t)^2 + 4t^2} = \sqrt{t^2 + 1 + 4t^2} = \sqrt{5t^2 + 1}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |(2 \cos t + 2t^2 \cos t + 2t \sin t)\mathbf{i} + (2t \cos t - 2 \sin t - 2t^2 \sin t)\mathbf{j} + (-2 - t^2)\mathbf{k}| = \sqrt{5t^4 + 16t^2 + 8}$$

$$a_T(t) = \frac{5t}{\sqrt{5t^2 + 1}}$$

$$a_N(t) = \sqrt{\frac{5t^4 + 16t^2 + 8}{5t^2 + 1}}$$

$$43. \mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3 \right\rangle$$

$$\mathbf{r}'(t) = \langle 1, 2t, 2t^2 \rangle$$

$$\mathbf{r}''(t) = \langle 0, 2, 4t \rangle$$

$$\mathbf{r}'(1) = \langle 1, 2, 2 \rangle$$

$$\mathbf{r}''(1) = \langle 0, 2, 4 \rangle$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{1+4+4}} \langle 1, 2, 2 \rangle = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

$$a_T(1) = \frac{1}{3}(0+4+8) = 4$$

$$a_N(1) = \frac{1}{3}|\langle 4, -4, 2 \rangle| = \frac{1}{3}\sqrt{16+16+4} = \frac{6}{3} = 2$$

$$\mathbf{N}(1) = \frac{1}{2} \left(\langle 0, 2, 4 \rangle - 4 \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \right) = \frac{1}{2} \left\langle -\frac{4}{3}, -\frac{2}{3}, \frac{4}{3} \right\rangle = \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$$

$$\mathbf{B}(1) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \times \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

$$44. \mathbf{r}'(t) = -3 \cos^2 t \sin t \mathbf{i} + 3 \sin^2 t \cos t \mathbf{k}$$

$$\mathbf{r}''(t) = (6 \cos t \sin^2 t - 3 \cos^3 t)\mathbf{i} + (6 \cos^2 t \sin t - 3 \sin^3 t)\mathbf{k}$$

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = -\frac{9}{8}\mathbf{i} + \frac{3\sqrt{3}}{8}\mathbf{k}$$

$$\mathbf{r}''\left(\frac{\pi}{6}\right) = -\frac{3\sqrt{3}}{8}\mathbf{i} + \frac{15}{8}\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{-\frac{9}{8}\mathbf{i} + \frac{3\sqrt{3}}{8}\mathbf{k}}{\sqrt{\frac{81}{64} + \frac{27}{64}}} = \frac{4}{3\sqrt{3}} \left(-\frac{9}{8}\mathbf{i} + \frac{3\sqrt{3}}{8}\mathbf{k} \right) = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{k}$$

$$a_T\left(\frac{\pi}{6}\right) = \frac{4}{3\sqrt{3}} \left(\frac{27\sqrt{3}}{64} + \frac{45\sqrt{3}}{64} \right) = \frac{3}{2}$$

$$a_N\left(\frac{\pi}{6}\right) = \frac{4}{3\sqrt{3}} \left| \frac{27}{16}\mathbf{j} \right| = \frac{9}{4\sqrt{3}} = \frac{3\sqrt{3}}{4}$$

$$\mathbf{N}\left(\frac{\pi}{6}\right) = \frac{4}{3\sqrt{3}} \left[\left(-\frac{3\sqrt{3}}{8} \mathbf{i} + \frac{15}{8} \mathbf{k} \right) - \frac{3}{2} \left(-\frac{\sqrt{3}}{2} \mathbf{i} + \frac{1}{2} \mathbf{k} \right) \right] = \frac{1}{2} \mathbf{i} + \frac{\sqrt{3}}{2} \mathbf{k}$$

$$\mathbf{B}\left(\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2} \mathbf{i} + \frac{1}{2} \mathbf{k} \right) \times \left(\frac{1}{2} \mathbf{i} + \frac{\sqrt{3}}{2} \mathbf{k} \right) = \mathbf{j}$$

45. $\mathbf{r}'(t) = \cosh \frac{t}{c} \mathbf{i} + \mathbf{k}$

$$\mathbf{r}''(t) = \frac{1}{c} \sinh \frac{t}{c} \mathbf{i}$$

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = \cosh \frac{\pi}{6c} \mathbf{i} + \mathbf{k}$$

$$\mathbf{r}''\left(\frac{\pi}{6}\right) = \frac{1}{c} \sinh \frac{\pi}{6c} \mathbf{i}$$

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{\cosh^2 \frac{\pi}{6c} + 1}} \left(\cosh \frac{\pi}{6c} \mathbf{i} + \mathbf{k} \right)$$

$$a_T\left(\frac{\pi}{6}\right) = \frac{\frac{1}{c} \cosh \frac{\pi}{6c} \sinh \frac{\pi}{6c}}{\sqrt{\cosh^2 \frac{\pi}{6c} + 1}}$$

$$a_N\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{\cosh^2 \frac{\pi}{6c} + 1}} \left| \frac{1}{c} \sinh \frac{\pi}{6c} \mathbf{j} \right| = \frac{\frac{1}{c} \sinh \frac{\pi}{6c}}{\sqrt{\cosh^2 \frac{\pi}{6c} + 1}}$$

$$\mathbf{N}\left(\frac{\pi}{6}\right) = \frac{\sqrt{\cosh^2 \frac{\pi}{6c} + 1}}{\frac{1}{c} \sinh \frac{\pi}{6c}} \left[\frac{1}{c} \sinh \frac{\pi}{6c} \mathbf{i} - \frac{\frac{1}{c} \cosh \frac{\pi}{6c} \sinh \frac{\pi}{6c}}{\cosh^2 \frac{\pi}{6c} + 1} \left(\cosh \frac{\pi}{6c} \mathbf{i} + \mathbf{k} \right) \right] = \frac{1}{\sqrt{\cosh^2 \frac{\pi}{6c} + 1}} \left(\mathbf{i} - \cosh \frac{\pi}{6c} \mathbf{k} \right)$$

$$\mathbf{B}\left(\frac{\pi}{6}\right) = \mathbf{T}\left(\frac{\pi}{6}\right) \times \mathbf{N}\left(\frac{\pi}{6}\right) = \mathbf{j}$$

46. $\mathbf{r}'(t) = (t \cos t + \sin t) \mathbf{i} + (\cos t - t \sin t) \mathbf{j} + 2t \mathbf{k}$

$$\mathbf{r}''(t) = (2 \cos t - t \sin t) \mathbf{i} + (-t \cos t - 2 \sin t) \mathbf{j} + 2 \mathbf{k}$$

$$\mathbf{r}'\left(\frac{\pi}{2}\right) = \mathbf{i} - \frac{\pi}{2} \mathbf{j} + \pi \mathbf{k}$$

$$\mathbf{r}''\left(\frac{\pi}{2}\right) = -\frac{\pi}{2} \mathbf{i} - 2 \mathbf{j} + 2 \mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{1 + \frac{\pi^2}{4} + \pi^2}} \left(\mathbf{i} - \frac{\pi}{2} \mathbf{j} + \pi \mathbf{k} \right) = \frac{1}{\sqrt{4 + 5\pi^2}} (2\mathbf{i} - \pi \mathbf{j} + 2\pi \mathbf{k})$$

$$a_T\left(\frac{\pi}{2}\right) = \frac{2}{\sqrt{4 + 5\pi^2}} \left(-\frac{\pi}{2} + \pi + 2\pi \right) = \frac{5\pi}{\sqrt{4 + 5\pi^2}}$$

$$a_N\left(\frac{\pi}{2}\right) = \frac{2}{\sqrt{4 + 5\pi^2}} \left| \pi \mathbf{i} + \left(-2 - \frac{\pi^2}{2} \right) \mathbf{j} + \left(-2 - \frac{\pi^2}{4} \right) \mathbf{k} \right| = \frac{2}{\sqrt{4 + 5\pi^2}} \cdot \frac{1}{4} \sqrt{128 + 64\pi^2 + 5\pi^4} = \frac{1}{2} \sqrt{\frac{128 + 64\pi^2 + 5\pi^4}{4 + 5\pi^2}}$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = 2 \sqrt{\frac{4 + 5\pi^2}{128 + 64\pi^2 + 5\pi^4}} \left[\left(-\frac{\pi}{2} \mathbf{i} - 2 \mathbf{j} + 2 \mathbf{k} \right) - \frac{5\pi}{4 + 5\pi^2} (2\mathbf{i} - \pi \mathbf{j} + 2\pi \mathbf{k}) \right]$$

$$= \frac{1}{\sqrt{(128 + 64\pi^2 + 5\pi^4)(4 + 5\pi^2)}} \left[(-5\pi^3 - 24\pi)\mathbf{i} + (-16 - 10\pi^2)\mathbf{j} + 16\mathbf{k} \right]$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \mathbf{T}\left(\frac{\pi}{2}\right) \times \mathbf{N}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{128 + 64\pi^2 + 5\pi^4}} (4\pi\mathbf{i} - (8 + 2\pi^2)\mathbf{j} - (8 + \pi^2)\mathbf{k})$$

47. $\mathbf{r}'(t) = (\cos t - t \sin t)\mathbf{i} + (t \cos t + \sin t)\mathbf{j} + 2t\mathbf{k}$
 $\mathbf{r}''(t) = (-t \cos t - 2 \sin t)\mathbf{i} + (2 \cos t - t \sin t)\mathbf{j} + 2\mathbf{k}$
 $\mathbf{r}'(\pi) = -\mathbf{i} - \pi\mathbf{j} + 2\pi\mathbf{k}$
 $\mathbf{r}''(\pi) = \pi\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

$$\mathbf{T}(\pi) = \frac{1}{\sqrt{1 + 5\pi^2}} (-\mathbf{i} - \pi\mathbf{j} + 2\pi\mathbf{k})$$

$$a_T(\pi) = \frac{1}{\sqrt{1 + 5\pi^2}} (2\pi - \pi + 4\pi) = \frac{5\pi}{\sqrt{1 + 5\pi^2}}$$

$$a_N(\pi) = \frac{1}{\sqrt{1 + 5\pi^2}} |2\pi\mathbf{i} + (2 + 2\pi^2)\mathbf{j} + (2 + \pi^2)\mathbf{k}| = \sqrt{\frac{8 + 16\pi^2 + 5\pi^4}{1 + 5\pi^2}}$$

$$\mathbf{N}(\pi) = \frac{1}{\sqrt{8 + 16\pi^2 + 5\pi^4}} \left[(\pi\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) - \frac{5\pi}{1 + 5\pi^2} (-\mathbf{i} - \pi\mathbf{j} + 2\pi\mathbf{k}) \right]$$

$$= \frac{1}{\sqrt{(8 + 16\pi^2 + 5\pi^4)(1 + 5\pi^2)}} \left[(5\pi^3 + 6\pi)\mathbf{i} + (-2 - 5\pi^2)\mathbf{j} - 2\mathbf{k} \right]$$

$$\mathbf{B}(\pi) = \frac{1}{\sqrt{8 + 16\pi^2 + 5\pi^4}} (2\pi\mathbf{i} + (2 + 2\pi^2)\mathbf{j} + (2 + \pi^2)\mathbf{k})$$

48. $\mathbf{r}'(t) = \mathbf{i} + t^2\mathbf{j} - \frac{1}{t^2}\mathbf{k}$

$$\mathbf{r}''(t) = 2t\mathbf{j} + \frac{2}{t^3}\mathbf{k}$$

$$\mathbf{r}'(1) = \mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\mathbf{r}''(1) = 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{1+1+1}} (\mathbf{i} + \mathbf{j} - \mathbf{k}) = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$$

$$a_T(1) = \frac{1}{\sqrt{3}} (0 + 2 - 2) = 0$$

$$a_N(1) = \frac{1}{\sqrt{3}} |4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}| = \frac{2\sqrt{6}}{\sqrt{3}} = 2\sqrt{2}$$

$$\mathbf{N}(1) = \frac{1}{2\sqrt{2}} (2\mathbf{j} + 2\mathbf{k}) = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$$

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \sqrt{\frac{2}{3}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$$

49. $\mathbf{r}'(t) = e^t\mathbf{i} + e^t(\cos t - \sin t)\mathbf{j} + e^t(\cos t + \sin t)\mathbf{k}$

$$\mathbf{r}''(t) = e^t\mathbf{i} - 2e^t \sin t\mathbf{j} + 2e^t \cos t\mathbf{k}$$

$$\mathbf{r}'\left(\frac{\pi}{3}\right) = e^{\pi/3}\mathbf{i} + \frac{e^{\pi/3}}{2}(1 - \sqrt{3})\mathbf{j} + \frac{e^{\pi/3}}{2}(1 + \sqrt{3})\mathbf{k}$$

$$\mathbf{r}'\left(\frac{\pi}{3}\right) = e^{\pi/3}\mathbf{i} - \sqrt{3}e^{\pi/3}\mathbf{j} + e^{\pi/3}\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}e^{\pi/3}} \left[e^{\pi/3}\mathbf{i} + \frac{e^{\pi/3}}{2}(1-\sqrt{3})\mathbf{j} + \frac{e^{\pi/3}}{2}(1+\sqrt{3})\mathbf{k} \right] = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{2}\left(\frac{1}{\sqrt{3}}-1\right)\mathbf{j} + \frac{1}{2}\left(\frac{1}{\sqrt{3}}+1\right)\mathbf{k}$$

$$a_T\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}e^{\pi/3}} \left[e^{2\pi/3} - \frac{\sqrt{3}}{2}e^{2\pi/3}(1-\sqrt{3}) + \frac{e^{2\pi/3}}{2}(1+\sqrt{3}) \right] = \frac{1}{\sqrt{3}e^{\pi/3}}(3e^{2\pi/3}) = \sqrt{3}e^{\pi/3}$$

$$a_N\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}e^{\pi/3}} \left| 2e^{2\pi/3}\mathbf{i} + \frac{e^{2\pi/3}}{2}(\sqrt{3}-1)\mathbf{j} + \frac{e^{2\pi/3}}{2}(-\sqrt{3}-1)\mathbf{k} \right| = \frac{1}{\sqrt{3}e^{\pi/3}}(\sqrt{6}e^{2\pi/3}) = \sqrt{2}e^{\pi/3}$$

$$\begin{aligned} \mathbf{N}\left(\frac{\pi}{3}\right) &= \frac{1}{\sqrt{2}e^{\pi/3}} \left[e^{\pi/3}(\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k}) - \sqrt{3}e^{\pi/3} \left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{2}\left(\frac{1}{\sqrt{3}}-1\right)\mathbf{j} + \frac{1}{2}\left(\frac{1}{\sqrt{3}}+1\right)\mathbf{k} \right) \right] \\ &= \frac{1}{\sqrt{2}} \left(-\frac{1+\sqrt{3}}{2}\mathbf{j} + \frac{1-\sqrt{3}}{2}\mathbf{k} \right) = -\frac{1+\sqrt{3}}{2\sqrt{2}}\mathbf{j} + \frac{1-\sqrt{3}}{2\sqrt{2}}\mathbf{k} \end{aligned}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right) \times \mathbf{N}\left(\frac{\pi}{3}\right) = \sqrt{\frac{2}{3}}\mathbf{i} + \frac{1}{12}(3\sqrt{2}-\sqrt{6})\mathbf{j} - \frac{1}{12}(3\sqrt{2}+\sqrt{6})\mathbf{k}$$

50. $\mathbf{r}'(t) = e^t\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j} + e^t(\cos t - \sin t)\mathbf{k}$

$$\mathbf{r}''(t) = e^t\mathbf{i} + 2e^t \cos t \mathbf{j} - 2e^t \sin t \mathbf{k}$$

$$\mathbf{r}'\left(\frac{\pi}{3}\right) = e^{\pi/3}\mathbf{i} + \frac{e^{\pi/3}}{2}(1+\sqrt{3})\mathbf{j} + \frac{e^{\pi/3}}{2}(1-\sqrt{3})\mathbf{k}$$

$$\mathbf{r}''\left(\frac{\pi}{3}\right) = e^{\pi/3}\mathbf{i} + e^{\pi/3}\mathbf{j} - \sqrt{3}e^{\pi/3}\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{2}\left(\frac{1}{\sqrt{3}}+1\right)\mathbf{j} + \frac{1}{2}\left(\frac{1}{\sqrt{3}}-1\right)\mathbf{k}$$

$$a_T\left(\frac{\pi}{3}\right) = \sqrt{3}e^{\pi/3}$$

$$a_N\left(\frac{\pi}{3}\right) = \sqrt{2}e^{\pi/3}$$

$$\mathbf{N}\left(\frac{\pi}{3}\right) = \frac{1-\sqrt{3}}{2\sqrt{2}}\mathbf{j} - \frac{1+\sqrt{3}}{2\sqrt{2}}\mathbf{k}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = -\sqrt{\frac{2}{3}}\mathbf{i} + \frac{1}{12}(3\sqrt{2}+\sqrt{6})\mathbf{j} + \frac{1}{12}(\sqrt{6}-3\sqrt{2})\mathbf{k}$$

Note that this is the $\mathbf{r}(t)$ from Problem 49 with the $\mathbf{j}$ and $\mathbf{k}$ components switched, thus the values computed in Problem 49 can be used.

51. $\mathbf{r}'(t) = \cosh t \mathbf{i} + \mathbf{j} + \sinh t \mathbf{k}$

$$\mathbf{r}''(t) = \sinh t \mathbf{i} + \cosh t \mathbf{k}$$

$$\mathbf{r}'\left(\frac{\pi}{3}\right) = \cosh \frac{\pi}{3} \mathbf{i} + \mathbf{j} + \sinh \frac{\pi}{3} \mathbf{k}$$

$$\mathbf{r}''\left(\frac{\pi}{3}\right) = \sinh \frac{\pi}{3} \mathbf{i} + \cosh \frac{\pi}{3} \mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{\cosh^2 \frac{\pi}{3} + 1 + \sinh^2 \frac{\pi}{3}}} \left(\cosh \frac{\pi}{3} \mathbf{i} + \mathbf{j} + \sinh \frac{\pi}{3} \mathbf{k} \right) = \frac{1}{\sqrt{2} \cosh \frac{\pi}{3}} \left(\cosh \frac{\pi}{3} \mathbf{i} + \mathbf{j} + \sinh \frac{\pi}{3} \mathbf{k} \right)$$

$$= \frac{1}{\sqrt{2}} \mathbf{i} + \frac{\operatorname{sech} \frac{\pi}{3}}{\sqrt{2}} \mathbf{j} + \frac{\tanh \frac{\pi}{3}}{\sqrt{2}} \mathbf{k}$$

$$a_T\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{2} \cosh \frac{\pi}{3}} \left(2 \cosh \frac{\pi}{3} \sinh \frac{\pi}{3} \right) = \sqrt{2} \sinh \frac{\pi}{3}$$

$$a_N\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{2} \cosh \frac{\pi}{3}} \left| \cosh \frac{\pi}{3} \mathbf{i} - \mathbf{j} - \sinh \frac{\pi}{3} \mathbf{k} \right| = 1$$

$$\mathbf{N}\left(\frac{\pi}{3}\right) = \left(\sinh \frac{\pi}{3} \mathbf{i} + \cosh \frac{\pi}{3} \mathbf{k} \right) - \sqrt{2} \sinh \frac{\pi}{3} \left(\frac{1}{\sqrt{2}} \mathbf{i} + \frac{\operatorname{sech} \frac{\pi}{3}}{\sqrt{2}} \mathbf{j} + \frac{\tanh \frac{\pi}{3}}{\sqrt{2}} \mathbf{k} \right) = -\tanh \frac{\pi}{3} \mathbf{j} + \operatorname{sech} \frac{\pi}{3} \mathbf{k}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right) \times \mathbf{N}\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{2}} \mathbf{i} - \frac{\operatorname{sech} \frac{\pi}{3}}{\sqrt{2}} \mathbf{j} - \frac{\tanh \frac{\pi}{3}}{\sqrt{2}} \mathbf{k}$$

52. a. $\mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + (2t - 3) \mathbf{k}$

$2t - 3 < 0$ for $t < \frac{3}{2}$, so the particle moves downward for $0 \leq t < \frac{3}{2}$.

b. $|\mathbf{r}'(t)| = \sqrt{\cos^2 t + \sin^2 t + (2t - 3)^2} = \sqrt{4t^2 - 12t + 10}$

$4t^2 - 12t + 10 = 0$ has no real-number solutions, so the particle never has speed 0, i.e., it never stops moving.

c. $t^2 - 3t + 2 = 12$ when $t^2 - 3t - 10 = (t + 2)(t - 5) = 0$, $t = -2, 5$. Since $t \geq 0$, the particle is 12 meters above the ground when $t = 5$.

d. $\mathbf{v}(5) = \cos 5 \mathbf{i} - \sin 5 \mathbf{j} + 7 \mathbf{k}$

e. $\mathbf{v}(5)$ is tangent to the helix at the point where the particle is 12 meters above the ground. Its path is described by
 $\mathbf{v}(5) = \cos 5 \mathbf{i} - \sin 5 \mathbf{j} + 7 \mathbf{k}$.

53. a. The motion of the planet with respect to the sun can be given by $x = R_p \cos t$, $y = R_p \sin t$.

Assume that when $t = 0$, both the planet and the moon are on the x -axis.

Since the moon orbits the planet 10 times for every time the planet orbits the sun, the motion of the moon with respect to the planet can be given by $x = R_m \cos 10t$, $y = R_m \sin 10t$.

Combining these equations, the motion of the moon with respect to the sun is given by

$$x = R_p \cos t + R_m \cos 10t, \quad y = R_p \sin t + R_m \sin 10t.$$

b. $x'(t) = -R_p \sin t - 10R_m \sin 10t$

$$y'(t) = R_p \cos t + 10R_m \cos 10t$$

The moon is motionless with respect to the sun when $x'(t)$ and $y'(t)$ are both 0.

Solve $x'(t) = 0$ for $\sin t$ and $y'(t) = 0$ for $\cos t$ to get $\sin t = -\frac{10R_m}{R_p} \sin 10t$, $\cos t = -\frac{10R_m}{R_p} \cos 10t$.

Since $\sin^2 t + \cos^2 t = 1$

$$1 = \frac{100R_m^2}{R_p^2} \sin^2 10t + \frac{100R_m^2}{R_p^2} \cos^2 10t$$

$$= \frac{100R_m^2}{R_p^2}. \text{ Thus, } R_p^2 = 100R_m^2 \text{ or } R_p = 10R_m. \text{ Substitute this into } x'(t) = 0 \text{ and } y'(t) = 0 \text{ to get}$$

$$-R_p(\sin t + \sin 10t) = 0 \text{ and } R_p(\cos t + \cos 10t) = 0.$$

If $0 \leq t \leq \frac{\pi}{2}$, then to have $\sin t + \sin 10t = 0$ and $\cos t + \cos 10t = 0$ it must be that

$$10t = \pi + t \text{ or } t = \frac{\pi}{9}.$$

Thus, when the radius of the planet's orbit around the sun is ten times the radius of the moon's orbit around the planet and $t = \frac{\pi}{9}$, the moon is motionless with respect to the sun.

54. a. Years

b. The sun orbits the earth once each year while the moon orbits the earth roughly 13 times each year.

c. $|\mathbf{r}(0)| = |93.24\mathbf{i}| = 93.24$ million mi, the sum of the orbital radii.

d. $93 - 0.24 = 92.76$ million mi

e. No; since the moon orbits the earth 13 times for each time the earth orbits the sun, the moon could not be stationary with respect to the sun unless the radius of its orbit around the earth were $\frac{1}{13}$ the the radius of the earth's orbit around the sun.

f. $\mathbf{v}(t) = [-186\pi \sin(2\pi t) - 6.24\pi \sin(26\pi t)]\mathbf{i} + [186\pi \cos(2\pi t) + 6.24\pi \cos(26\pi t)]\mathbf{j}$

$\mathbf{a}(t) = [-372\pi^2 \cos(2\pi t) - 162.24\pi^2 \cos(26\pi t)]\mathbf{i} + [-372\pi^2 \sin(2\pi t) - 162.24\pi^2 \sin(26\pi t)]\mathbf{j}$

$\mathbf{v}\left(\frac{1}{2}\right) = 0\mathbf{i} + (-186\pi - 6.24\pi)\mathbf{j} = -192.24\pi\mathbf{j}$

$s\left(\frac{1}{2}\right) = 192.24\pi$ million mi/yr

$\mathbf{a}\left(\frac{1}{2}\right) = (372\pi^2 + 162.24\pi^2)\mathbf{i} + 0\mathbf{j} = 534.24\pi^2\mathbf{i}$

55. a. Winding upward around the right circular cylinder $x = \sin t$, $y = \cos t$ as t increases.

b. Same as part a, but winding faster/slower by a factor of $3t^2$.

c. With standard orientation of the axes, the motion is winding to the right around the right circular cylinder $x = \sin t$, $z = \cos t$.

d. Spiraling upward, with increasing radius, along the spiral $x = t \sin t$, $y = t \cos t$.

e. Spiraling upward, with decreasing radius, along the spiral $x = \frac{1}{t^2} \sin t$, $y = \frac{1}{t^2} \cos t$.

f. Spiraling to the right, with increasing radius, along the spiral $x = t^2 \sin(\ln t)$,
 $z = t^2 \cos(\ln t)$.

56. Since $\lim_{x \rightarrow 0^-} y = \lim_{x \rightarrow 0^+} y = 0 = y(0)$, y is continuous.

$$y'(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 3x^2 & \text{if } x > 0 \end{cases}$$

is continuous since

$$\lim_{x \rightarrow 0^-} y' = \lim_{x \rightarrow 0^+} y' = 0 = y'(0).$$

$$y''(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 6x & \text{if } x > 0 \end{cases}$$

is continuous since

$$\lim_{x \rightarrow 0^-} y'' = \lim_{x \rightarrow 0^+} y'' = 0 = y''(0).$$

Thus, $\kappa = \frac{|y''|}{(1 + y'^2)^{3/2}}$ is continuous also. If

$x \neq 0$ then y' and κ are continuous as elementary functions.

57. Let

$$P_5(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5.$$

$$P_5(0) = 0 \Rightarrow a_0 = 0$$

$$P_5'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4, \text{ so}$$

$$P_5'(0) = 0 \Rightarrow a_1 = 0.$$

$$P_5''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3, \text{ so}$$

$$P_5''(0) = 0 \Rightarrow a_2 = 0.$$

$$\text{Thus, } P_5(x) = a_3x^3 + a_4x^4 + a_5x^5,$$

$$P_5'(x) = 3a_3x^2 + 4a_4x^3 + 5a_5x^4, \text{ and}$$

$$P_5''(x) = 6a_3x + 12a_4x^2 + 20a_5x^3.$$

$$P_3(1) = 1, P_3'(1) = 0, \text{ and } P_3''(1) = 0 \Rightarrow$$

$$a_3 + a_4 + a_5 = 1$$

$$3a_3 + 4a_4 + 5a_5 = 0$$

$$6a_3 + 12a_4 + 20a_5 = 0$$

The simultaneous solution to these equations is

$$a_3 = 10, a_4 = -15, a_5 = 6, \text{ so}$$

$$P_3(x) = 10x^3 - 15x^4 + 6x^5.$$

$$58. \mathbf{B} \cdot \mathbf{B} = 1$$

$$\frac{d}{ds}(\mathbf{B} \cdot \mathbf{B}) = 2\mathbf{B} \cdot \frac{d\mathbf{B}}{ds} = 0$$

Thus, $\frac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{B}$.

$$59. \frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

$$\text{Since } \mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|}, \frac{d\mathbf{T}}{ds} \times \mathbf{N} = 0, \text{ so } \frac{d\mathbf{B}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}.$$

$$\text{Thus } \mathbf{T} \cdot \frac{d\mathbf{B}}{ds} = \mathbf{T} \cdot \left(\mathbf{T} \times \frac{d\mathbf{N}}{ds} \right) = (\mathbf{T} \times \mathbf{T}) \cdot \frac{d\mathbf{N}}{ds} = 0, \text{ so}$$

$$\frac{d\mathbf{B}}{ds} \text{ is perpendicular to } \mathbf{T}.$$

$$60. \mathbf{N} \text{ is perpendicular to } \mathbf{T}, \text{ and } \mathbf{B} = \mathbf{T} \times \mathbf{N} \text{ is}$$

perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. Thus, since $\frac{d\mathbf{B}}{ds}$

is perpendicular to both $\mathbf{T}$ and $\mathbf{B}$, it is parallel to

$$63. \mathbf{r}(t) = 6 \cos \pi t \mathbf{i} + 6 \sin \pi t \mathbf{j} + 2t \mathbf{k}, t > 0$$

$$\text{Let } (6 \cos \pi t)^2 + (6 \sin \pi t)^2 + (2t)^2 = 100.$$

$$\text{Then } 36(\cos^2 \pi t + \sin^2 \pi t) + 4t^2 = 100; 4t^2 = 64; t = 4.$$

$$\mathbf{r}(4) = 6\mathbf{i} + 8\mathbf{k}, \text{ so the fly will hit the sphere at the point } (6, 0, 8).$$

$$\mathbf{r}'(t) = -6\pi \sin \pi t \mathbf{i} + 6\pi \cos \pi t \mathbf{j} + 2\mathbf{k}, \text{ so the fly will have traveled}$$

$$\int_0^4 \sqrt{(-6\pi \sin \pi t)^2 + (6\pi \cos \pi t)^2 + (2)^2} dt = \int_0^4 \sqrt{36\pi^2 + 4} dt = \sqrt{36\pi^2 + 4}(4 - 0)$$

$$= 8\sqrt{9\pi^2 + 1} \approx 75.8214$$

$$64. \mathbf{r}(t) = \left\langle 10 \cos t, 10 \sin t, \left(\frac{34}{2\pi}\right)t \right\rangle$$

Using the result of Example 1 with $a = 10$ and $c = \frac{34}{2\pi}$, the length of one complete turn is

$$2\pi \sqrt{(10)^2 + \left(\frac{34}{2\pi}\right)^2} \text{ angstroms} = 10^{-8} \sqrt{400\pi^2 + 34^2} \text{ cm. Therefore, the total length of the helix is}$$

$$(2.9)(10^8)(10^{-8})\sqrt{400\pi^2 + 34^2} \approx 207.1794 \text{ cm.}$$

$$65. \text{ a. } \mathbf{r}(t) = \text{constant, so } \mathbf{r} = \mathbf{r}_0.$$

$\mathbf{N}$, and hence there is some number $\tau(s)$ such that

$$\frac{d\mathbf{B}}{ds} = -\tau(s)\mathbf{N}.$$

61. Let $ax + by + cz + d = 0$ be the equation of the plane containing the curve. Since $\mathbf{T}$ and $\mathbf{N}$ lie in the plane $\mathbf{B} = \pm \frac{a\mathbf{i} + b\mathbf{j} + c\mathbf{k}}{\sqrt{a^2 + b^2 + c^2}}$. Thus, $\mathbf{B}$ is a

constant vector and $\frac{d\mathbf{B}}{ds} = 0$, so $\tau(s) = 0$, since $\mathbf{N}$ will not necessarily be 0 everywhere.

$$62. \mathbf{r}'(t) = a_0\mathbf{i} + b_0\mathbf{j} + c_0\mathbf{k}$$

$$\mathbf{r}''(t) = \mathbf{0}$$

Thus, $\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{0}$ and since

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}, \kappa = 0.$$

To show that $\tau = 0$, note that the curve is confined to a plane. This means that the curve is two-dimensional and thus $\tau = 0$.

b. $r(t) = ct + C$

$$r(0) = r_0 \Rightarrow r(t) = ct + r_0$$

c. $r'(t) = ct + C_1$

$$r'(0) = v_0 \Rightarrow r'(t) = ct + v_0$$

$$r(t) = \frac{1}{2}ct^2 + v_0t + C_2$$

$$r(0) = r_0 \Rightarrow r(t) = \frac{1}{2}ct^2 + v_0t + r_0$$

d. Say $r(t) = x(t)i + y(t)j + z(t)k$.

Then $\frac{dr}{dt} = cr$ is equivalent to $\frac{dx}{dt} = cx$, $\frac{dy}{dt} = cy$, $\frac{dz}{dt} = cz$.

These equations have the solutions $x(t) = x_0e^{ct}$, $y(t) = y_0e^{ct}$, $z(t) = z_0e^{ct}$.

Hence, $r(t) = r_0e^{ct}$.

66. $r(t) = \langle \cos t, \sin t, 16t \rangle$

Death occurred at $r(12) = \langle \cos 12, \sin 12, 192 \rangle$.

$$r'(t) = \langle -\sin t, \cos t, 16 \rangle$$

$$r'(12) = \langle -\sin 12, \cos 12, 16 \rangle$$

Now let $t = 0$ when the bee died, and let $s(t)$ describe the path of the bee from the instant of death.

Then $s_0 = s(0) = \langle \cos 12, \sin 12, 192 \rangle$ and $v_0 = v(0) = \langle -\sin 12, \cos 12, 16 \rangle$.

We know that $\frac{d^2s}{dt^2} = \langle 0, 0, -32 \rangle$, so using the result of Problem 65 part c,

$$s(t) = \langle 0, 0, -32 \rangle \left[\left(\frac{1}{2} \right) t^2 \right] + \langle -\sin 12, \cos 12, 16 \rangle t + \langle \cos 12, \sin 12, 192 \rangle.$$

It landed when $z = 0$; $-32 \left[\left(\frac{1}{2} \right) t^2 \right] + 16t + 192 = 0$; $0 = -16(t^2 - t - 12) = -16(t - 4)(t + 3)$ so $t = 4$.

$s(4) = \langle -4\sin 12 + \cos 12, 4\cos 12 + \sin 12, 0 \rangle$, so it landed at approximately $(2.99, 2.84, 0)$.

67. It left the path at the point $r(4\pi) = \langle 4\pi, 0, 4\pi \rangle$; and $r'(4\pi) = \langle 1, 4\pi, 1 \rangle$.

Hence, parametric equations of the tangent line are:

$$x = 4\pi + t, y = 4\pi t, z = 4\pi + t.$$

Then the point of intersection of the tangent line and the plane $x + y = 30$

occurs where $(4\pi + t) + (4\pi t) = 30$; $t = \frac{30 - 4\pi}{4\pi + 1}$.

The bee hits the plane at the point with coordinates:

$$x = 4\pi + \frac{30 - 4\pi}{4\pi + 1} = \frac{2(8\pi^2 + 15)}{4\pi + 1} \approx 13.85,$$

$$y = 4\pi \left[\frac{30 - 4\pi}{4\pi + 1} \right] = \frac{8\pi(15 - 2\pi)}{4\pi + 1} \approx 16.15, z \approx 13.85 \text{ (same as } x).$$

68. $(F \times G)' = \left[\langle f_2g_3 - f_3g_2, f_3g_1 - f_1g_3, f_1g_2 - f_2g_1 \rangle \right]'$

$$= \langle (f_2g_3' + f_2'g_3) - (f_3g_2' + f_3'g_2), (f_3g_1' + f_3'g_1) - (f_1g_3' + f_1'g_3), (f_1g_2' + f_1'g_2) - (f_2g_1' + f_2'g_1) \rangle$$

$$F \times G' + F' \times G = \langle f_2g_3' - f_3g_2', f_3g_1' - f_1g_3', f_1g_2' - f_2g_1' \rangle + \langle f_2'g_3 - f_3'g_2, f_3'g_1 - f_1'g_3, f_1'g_2 - f_2'g_1 \rangle$$

$$= \langle (f_2 g_3' + f_2' g_3) - (f_3 g_2' + f_3' g_2), (f_3 g_1' + f_3' g_1) - (f_1 g_3' + f_1' g_3), (f_1 g_2' + f_1' g_2) - (f_2 g_1' + f_2' g_1) \rangle = (\mathbf{F} \times \mathbf{G})'$$

$$\text{Thus, } \frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t) + \mathbf{r}'(t) \times \mathbf{r}'(t) = \mathbf{r}(t) \times \mathbf{r}''(t) + 0 = \mathbf{r}(t) \times \mathbf{r}''(t)$$

$$69. \frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t) = \mathbf{r}(t) \times c\mathbf{r}(t) = c[\mathbf{r}(t) \times \mathbf{r}(t)] = 0$$

Hence, $\mathbf{r}(t) \times \mathbf{r}'(t) = \mathbf{c}$, a constant vector, with $\mathbf{r}'(t)$ perpendicular to $\mathbf{c}$ for all t . Thus, the motion is in a plane with $\mathbf{c}$ as a normal.

$$70. \text{ a. } \mathbf{L}'(t) = m\mathbf{r}(t) \times \mathbf{v}'(t) + m\mathbf{r}'(t) \times \mathbf{v}(t) = m\mathbf{r}(t) \times \mathbf{a}(t) + m[\mathbf{v}(t) \times \mathbf{v}(t)] = m\mathbf{r}(t) \times \mathbf{a}(t) = \boldsymbol{\tau}(t)$$

b. If $\boldsymbol{\tau}(t) = 0$, then, by part a, $\mathbf{L}'(t) = 0$, so $\mathbf{L}(t)$ is a constant vector.

c. From Problem 69, a particle moving under a central force satisfies $\mathbf{a}(t) = \mathbf{r}''(t) = c\mathbf{r}(t)$.

Thus, for such a particle,

$$\boldsymbol{\tau}(t) = m\mathbf{r}(t) \times c\mathbf{r}(t) = mc[\mathbf{r}(t) \times \mathbf{r}(t)] = 0$$

and angular momentum is conserved.

71. $\mathbf{r} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}$ where $r = r(t)$ and $\theta = \theta(t)$ are functions of t .

$$\mathbf{v}(t) = \mathbf{r}'(t) = (r' \cos \theta - r \sin \theta \theta') \mathbf{i} + (r' \sin \theta + r \cos \theta \theta') \mathbf{j}$$

$$\begin{aligned} \text{Thus } \mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{v}(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ mr \cos \theta & mr \sin \theta & 0 \\ r' \cos \theta - r \sin \theta \theta' & r' \sin \theta + r \cos \theta \theta' & 0 \end{vmatrix} \\ &= 0\mathbf{i} - 0\mathbf{j} + [mr \cos \theta (r' \sin \theta + r \cos \theta \theta') - mr \sin \theta (r' \cos \theta - r \sin \theta \theta')] \mathbf{k} \\ &= (mr^2 \cos^2 \theta \theta' + mr^2 \sin^2 \theta \theta') \mathbf{k} = mr^2 \theta' \mathbf{k} = mr^2 \frac{d\theta}{dt} \mathbf{k} \end{aligned}$$

72. From Example 5, at $P\left(1, 1, \frac{1}{3}\right)$,

$$\begin{aligned} \mathbf{B} = \mathbf{T} \times \mathbf{N} &= \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \times \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k}) \\ &= \frac{1}{2\sqrt{3}}(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k}) \end{aligned}$$

So, $\mathbf{i} - \mathbf{j} + \mathbf{k}$ is a normal to the osculating plane.

$$\text{An equation of the plane is } 1(x-1) - 1(y-1) + 1\left(z - \frac{1}{3}\right) = 0 \text{ or } x - y + z = \frac{1}{3}.$$

$$73. \mathbf{B} \times \mathbf{T} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{T} = -[\mathbf{T} \times (\mathbf{T} \times \mathbf{N})] = -[(\mathbf{T} \cdot \mathbf{N})\mathbf{T} - (\mathbf{T} \cdot \mathbf{T})\mathbf{N}] = \mathbf{N}$$

since $\mathbf{T}$ and $\mathbf{N}$ are perpendicular unit vectors, hence $\mathbf{T} \cdot \mathbf{N} = 0$ and $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1^2 = 1$.

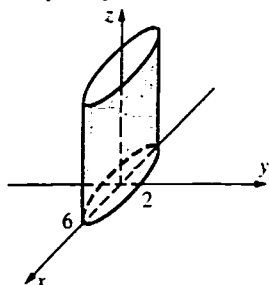
14.6 Concepts Review

1. traces; cross sections
2. cylinders; z -axis
3. ellipsoid
4. elliptic paraboloid

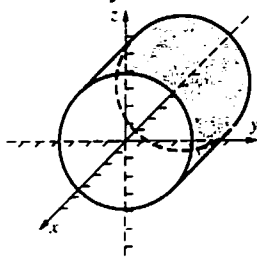
Problem Set 14.6

1. $\frac{x^2}{36} + \frac{y^2}{4} = 1$

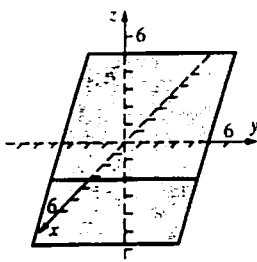
Elliptic cylinder



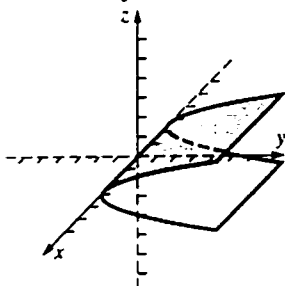
2. Circular cylinder



3. Plane

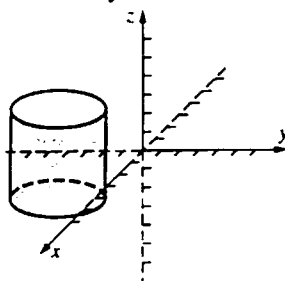


4. Parabolic cylinder

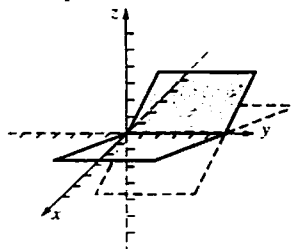


5. $(x-4)^2 + (y+2)^2 = 7$

Circular cylinder

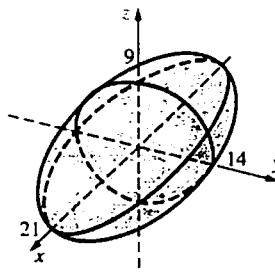


6. Two planes



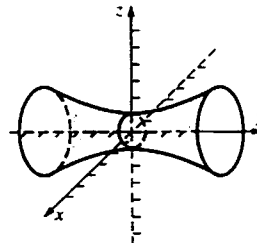
7. $\frac{x^2}{441} + \frac{y^2}{196} + \frac{z^2}{36} = 1$

Ellipsoid



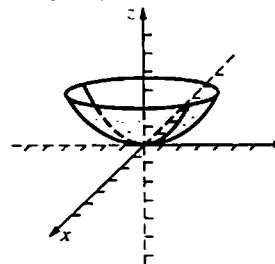
8. $\frac{x^2}{1} - \frac{y^2}{9} + \frac{z^2}{1} = 1$

Hyperboloid of one sheet

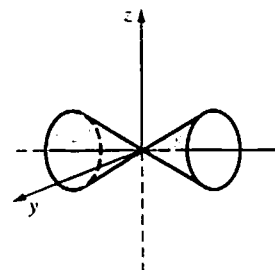


9. $z = \frac{x^2}{8} + \frac{y^2}{2}$

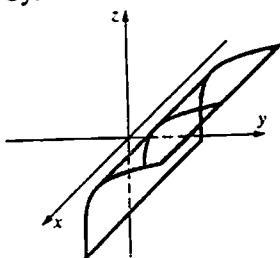
Elliptic paraboloid



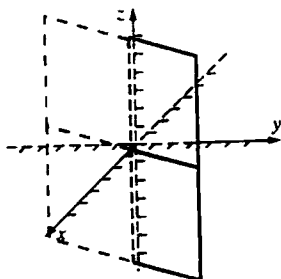
10. Circular cone



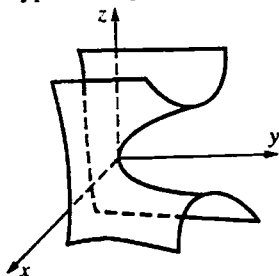
11. Cylinder



12. Plane

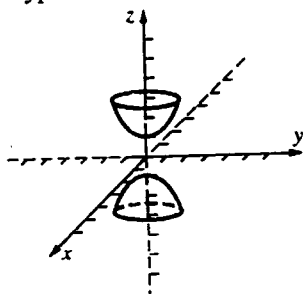


13. Hyperbolic paraboloid



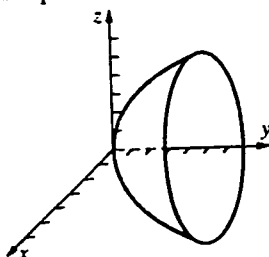
$$14. -\frac{x^2}{4} - \frac{y^2}{4} + \frac{z^2}{1} = 1$$

Hyperboloid of two sheets



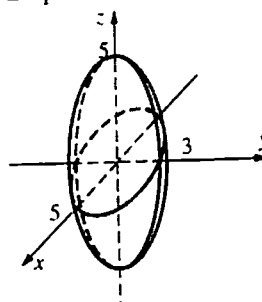
$$15. y = \frac{x^2}{4} + \frac{z^2}{9}$$

Elliptic Paraboloid

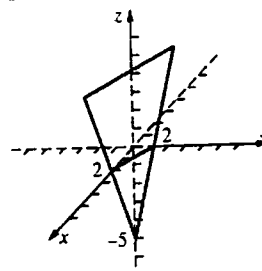


$$16. \frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{25} = 1$$

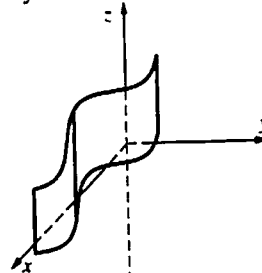
Ellipsoid



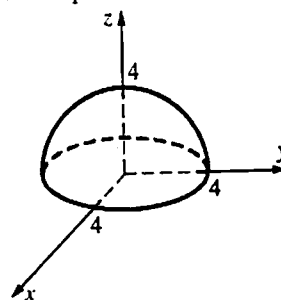
17. Plane



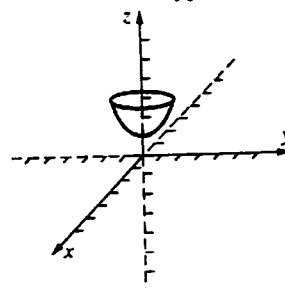
18. Cylinder



19. Hemisphere



20. One sheet of a hyperboloid of two sheets.

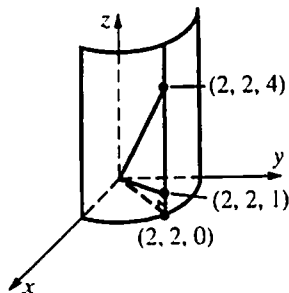


21. a. Replacing x by $-x$ results in an equivalent equation.
 b. Replacing x by $-x$ and y by $-y$ results in an equivalent equation.
 c. Replacing y by $-y$ and z by $-z$ results in an equivalent equation.
 d. Replacing x by $-x$, y by $-y$, and z by $-z$, results in an equivalent equation.
22. a. 1, 2, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 18
 b. 1, 2, 6, 7, 8, 9, 10, 14, 16, 19, 20
23. At $y = x$, the revolution generates a circle of radius $x = \sqrt{\frac{y}{2}} = \sqrt{\frac{k}{2}}$. Thus, the cross section in the plane $y = k$ is the circle $x^2 + z^2 = \frac{k}{2}$ or $2x^2 + 2z^2 = k$. The equation of the surface is $y = 2x^2 + 2z^2$.
24. At $z = k$, the revolution generates a circle of radius $y = \frac{z}{2} = \frac{k}{2}$. Thus, the cross section in the plane $z = k$ is the circle $x^2 + y^2 = \frac{k^2}{4}$ or $4x^2 + 4y^2 = k^2$. The equation of the surface is $z^2 = 4x^2 + 4y^2$.
25. At $y = k$, the revolution generates a circle of radius $x = \sqrt{3 - \frac{3}{4}y^2} = \sqrt{3 - \frac{3}{4}k^2}$. Thus, the cross section in the plane $y = k$ is the circle $x^2 + z^2 = 3 - \frac{3}{4}k^2$ or $12 - 4x^2 - 4z^2 = 3k^2$. The equation of the surface is $4x^2 + 3y^2 + 4z^2 = 12$.
26. At $x = k$, the revolution generates a circle of radius $y = \sqrt{\frac{4}{3}x^2 - 4} = \sqrt{\frac{4}{3}k^2 - 4}$. Thus, the cross section in the plane $x = k$ is the circle $y^2 + z^2 = \frac{4}{3}k^2 - 4$ or $12 + 3y^2 + 3z^2 = 4k^2$. The equation of the surface is $4x^2 = 12 + 3y^2 + 3z^2$.
27. When $z = 4$ the equation is $4 = \frac{x^2}{4} + \frac{y^2}{9}$ or $1 = \frac{x^2}{16} + \frac{y^2}{36}$, so $a^2 = 36, b^2 = 16$, and

$c^2 = a^2 - b^2 = 20$, hence $c = \pm 2\sqrt{5}$. The major axis of the ellipse is on the y -axis so the foci are at $(0, \pm 2\sqrt{5}, 4)$.

28. When $x = 4$, the equation is $z = \frac{16}{4} + \frac{y^2}{9}$ or $y^2 = 9(z - 4) = 4 \cdot \frac{9}{4}(z - 4)$, hence $p = \frac{9}{4}$. The vertex is at $(4, 0, 4)$ so the focus is $(4, 0, 4 + \frac{9}{4}) = (4, 0, \frac{25}{4})$.
29. When $z = h$, the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{h^2}{c^2} = 1$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{c^2 - h^2}{c^2}$ which is equivalent to $\frac{\frac{x^2}{a^2(c^2 - h^2)}}{\frac{c^2}{c^2}} + \frac{\frac{y^2}{b^2(c^2 - h^2)}}{\frac{c^2}{c^2}} = 1$, which is $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ with $A = \frac{a}{c}\sqrt{c^2 - h^2}$ and $B = \frac{b}{c}\sqrt{c^2 - h^2}$. Thus, the area is $\pi\left(\frac{a}{c}\sqrt{c^2 - h^2}\right)\left(\frac{b}{c}\sqrt{c^2 - h^2}\right) = \frac{\pi ab(c^2 - h^2)}{c^2}$.
30. The equation of the elliptical cross section is $\frac{x^2}{a^2(h - z)} + \frac{y^2}{b^2(h - z)} = 1$, for each z in $[0, h)$. Therefore, $\Delta V \approx \pi(a\sqrt{h - z})(b\sqrt{h - z})\Delta z = \pi ab(h - z)\Delta z$, using the area formula mentioned in Problem 29. Therefore, $V = \int_0^h \pi ab(h - z)dz = \pi ab\left[hz - \frac{z^2}{2}\right]_0^h = \pi ab\left[h^2 - \frac{h^2}{2}\right] = \frac{\pi abh^2}{2}$, which is the height times one half the area of the base ($z = 0$), $\pi(a\sqrt{h})(b\sqrt{h}) = \pi abh$.
31. Equating the expressions for y , $4 - x^2 = x^2 + z^2$ or $1 = \frac{x^2}{2} + \frac{z^2}{4}$ which is the equation of an ellipse in the xz -plane with major diameter of $2\sqrt{4} = 4$ and minor diameter $2\sqrt{2}$.

32.



$y = x$ intersects the cylinder when $x = y = 2$. Thus, the vertices of the triangle are $(0, 0, 0)$, $(2, 2, 1)$, and $(2, 2, 4)$. The area of the triangle with sides represented by $\langle 2, 2, 1 \rangle$ and $\langle 2, 2, 4 \rangle$ is

$$\begin{aligned} \frac{1}{2} |\langle 2, 2, 1 \rangle \times \langle 2, 2, 4 \rangle| &= \frac{1}{2} |\langle 6, -6, 0 \rangle| \\ &= \frac{1}{2} (6\sqrt{2}) = 3\sqrt{2}. \end{aligned}$$

33. $(t \cos t)^2 + (t \sin t)^2 - t^2 = t^2(\cos^2 t + \sin^2 t) - t^2 = t^2 - t^2 = 0$, hence every point on the spiral is on the cone.
For $\mathbf{r} = 3t \cos t \mathbf{i} + t \sin t \mathbf{j} + t \mathbf{k}$, every point

satisfies $x^2 + 9y^2 - 9z^2 = 0$ so the spiral lies on the elliptical cone.

34. It is clear that $x = y$ at each point on the curve. Thus, the curve lies in the plane $x = y$. Since $z = t^2 = y^2$, the curve is the intersection of the plane $x = y$ with the parabolic cylinder $z = y^2$. Let the line $y = x$ in the xy -plane be the u -axis, then the curve determined by $\mathbf{r}$ is in the uz -plane. The u -coordinate of a point on the curve, (y, y, z) is the signed distance of the point $(y, y, 0)$ from the origin, i.e., $u = \sqrt{2}y$. Thus, $u^2 = 2y^2 = 2z$ or $u^2 = 4\left(\frac{1}{2}\right)z$. This is a parabola in the uz -plane with vertex at $u = 0, z = 0$ and focus at $u = 0, z = \frac{1}{2}$. The focus is at $\left(0, 0, \frac{1}{2}\right)$.

14.7 Concepts Review

1. circular cylinder; sphere
2. plane; cone
3. $\rho^2 = r^2 + z^2$
4. $x^2 + y^2 + z^2 = 4z$, so $x^2 + y^2 + z^2 - 4z + 4 = 4$ or $x^2 + y^2 + (z - 2)^2 = 4$.

Problem Set 14.7

1. Cylindrical to Spherical:

$$\begin{aligned} \rho &= \sqrt{r^2 + z^2} \\ \cos \phi &= \frac{z}{\sqrt{r^2 + z^2}} \end{aligned}$$

$$\theta = \theta$$

Spherical to Cylindrical:

$$r = \rho \sin \phi$$

$$z = \rho \cos \phi$$

$$\theta = \theta$$

2. a. $(\rho, \theta, \phi) = \left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$

b. $(\rho, \theta, \phi) = \left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{4}\right)$

3. a. $x = 6 \cos\left(\frac{\pi}{6}\right) = 3\sqrt{3}$

$$y = 6 \sin\left(\frac{\pi}{6}\right) = 3$$

$$z = -2$$

b. $x = 4 \cos\left(\frac{4\pi}{3}\right) = -2$

$$y = 4 \sin\left(\frac{4\pi}{3}\right) = -2\sqrt{3}$$

$$z = -8$$

4. a. $x = 8 \sin\left(\frac{\pi}{6}\right) \cos\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$$y = 8 \sin\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{4}\right) = 2\sqrt{2}$$

$$z = 8 \cos\left(\frac{\pi}{6}\right) = 4\sqrt{3}$$

b. $x = 4 \sin\left(\frac{3\pi}{4}\right) \cos\left(\frac{\pi}{3}\right) = \sqrt{2}$
 $y = 4 \sin\left(\frac{3\pi}{4}\right) \sin\left(\frac{\pi}{3}\right) = \sqrt{6}$
 $z = 4 \cos\left(\frac{3\pi}{4}\right) = -2\sqrt{2}$

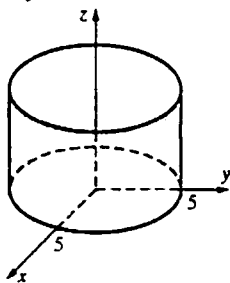
5. a. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 12 + 16} = 4\sqrt{2}$
 $\tan \theta = \frac{y}{x} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$ and (x, y) is in the
 4th quadrant so $\theta = \frac{5\pi}{3}$.
 $\cos \phi = \frac{z}{\rho} = \frac{4}{4\sqrt{2}} = \frac{\sqrt{2}}{2}$ so $\phi = \frac{\pi}{4}$.
 Spherical: $\left(4\sqrt{2}, \frac{5\pi}{3}, \frac{\pi}{4}\right)$

b. $\rho = \sqrt{2 + 2 + 12} = 4$
 $\tan \theta = \frac{\sqrt{2}}{-\sqrt{2}} = -1$ and (x, y) is in the 2nd
 quadrant so $\theta = \frac{3\pi}{4}$.
 $\cos \phi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$ so $\phi = \frac{\pi}{6}$.
 Spherical: $\left(4, \frac{3\pi}{4}, \frac{\pi}{6}\right)$

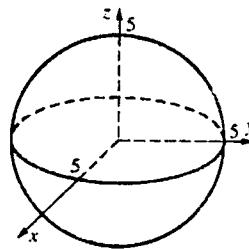
6. a. $r = \sqrt{4 + 4} = 2\sqrt{2}$
 $\tan \theta = \frac{2}{2} = 1$, $x > 0$, $y > 0$, so $\theta = \frac{\pi}{4}$. $z = 3$

b. $r = \sqrt{48 + 16} = 8$
 $\tan \theta = -\frac{4}{4\sqrt{3}} = -\frac{1}{\sqrt{3}}$, $x > 0$, $y < 0$, so
 $\theta = \frac{11\pi}{6}$. $z = 6$

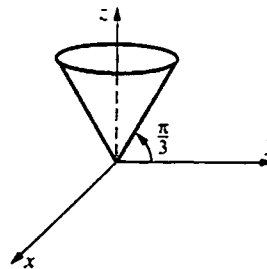
7. $r = 5$
 Cylinder



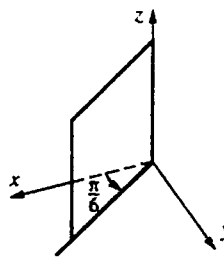
8. $\rho = 5$
 Sphere



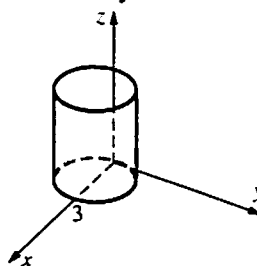
9. $\phi = \frac{\pi}{6}$
 Cone



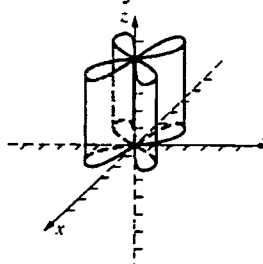
10. $\theta = \frac{\pi}{6}$
 Plane



11. $r = 3 \cos \theta$
 Circular cylinder



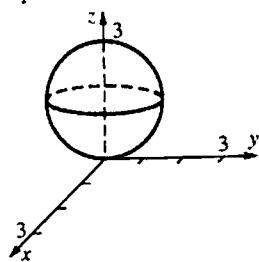
12. $r = 2 \sin 2\theta$
 4-leaved cylinder



13. $\rho = 3 \cos \phi$

$$x^2 + y^2 + \left(z - \frac{3}{2}\right)^2 = \frac{9}{4}$$

Sphere

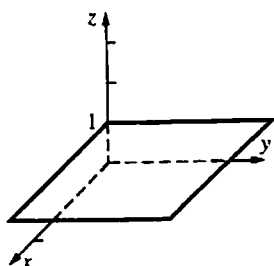


14. $\rho = \sec \phi$

$$\rho \cos \phi = 1$$

$$z = 1$$

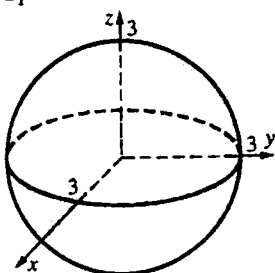
Plane



15. $r^2 + z^2 = 9$

$$x^2 + y^2 + z^2 = 9$$

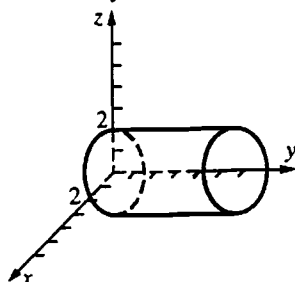
Sphere



16. $r^2 \cos^2 \theta + z^2 = 4$

$$x^2 + z^2 = 4$$

Circular cylinder



17. $x^2 + y^2 = 9; r^2 = 9; r = 3$

18. $r^2 \cos^2 \theta - r^2 \sin^2 \theta = 25; r^2 \cos 2\theta = 25;$

$$r^2 = 25 \sec^2 \theta; r = 5 \sec \theta$$

19. $r^2 + 4z^2 = 10$

20. $(x^2 + y^2 + z^2) + 3z^2 = 10; \rho^2 + 3\rho^2 \cos^2 \phi = 10;$

$$\rho^2 = \frac{10}{1 + 3 \cos^2 \phi}$$

21. $(x^2 + y^2 + z^2) - 3z^2 = 0; \rho^2 - 3\rho^2 \cos^2 \phi = 0;$

$$\cos^2 \phi = \frac{1}{3} \text{ (pole is not lost); } \cos^2 \phi = \frac{1}{3} \text{ (or}$$

$$\sin^2 \phi = \frac{2}{3} \text{ or } \tan^2 \phi = 2)$$

22. $\rho^2 [\sin^2 \phi \cos^2 \theta - \sin^2 \phi \sin^2 \theta - \cos^2 \phi] = 1;$

$$\rho^2 [\sin^2 \phi \cos^2 \theta - \sin^2 \phi \sin^2 \theta - 1 + \sin^2 \phi] = 1;$$

$$\rho^2 [\sin^2 \phi \cos^2 \theta - 1 + \sin^2 \phi (1 - \sin^2 \theta)] = 1;$$

$$\rho^2 [\sin^2 \phi \cos^2 \theta - 1 + \sin^2 \phi \cos^2 \theta] = 1;$$

$$\rho^2 [2 \sin^2 \phi \cos^2 \theta - 1] = 1;$$

$$\rho^2 = \frac{1}{2 \sin^2 \phi \cos^2 \theta - 1}$$

23. $(r^2 + z^2) + z^2 = 4; \rho^2 + \rho^2 \cos^2 \phi = 4;$

$$\rho^2 = \frac{4}{1 + \cos^2 \phi}$$

24. $\rho^2 = 2\rho \cos \phi; r^2 + z^2 = 2z; r^2 = 2z - z^2;$

$$r = \sqrt{2z - z^2}$$

25. $r \cos \theta + r \sin \theta = 4; r = \frac{4}{\sin \theta + \cos \theta}$

26. $\rho \sin \phi \cos \theta + \rho \sin \phi \sin \theta + \rho \cos \phi = 1;$

$$\rho = \frac{1}{\sin \phi (\sin \theta + \cos \theta) + \cos \phi}$$

27. $(x^2 + y^2 + z^2) - z^2 = 9; \rho^2 - \rho^2 \cos^2 \phi = 9;$

$$\rho^2 (1 - \cos^2 \phi) = 9; \rho^2 \sin^2 \phi = 9$$

28. $r^2 = 2r \sin \theta; x^2 + y^2 = 2y; x^2 + (y - 1)^2 = 1$

29. $r^2 \cos 2\theta = z; r^2 (\cos^2 \theta - \sin^2 \theta) = z;$

$$(r \cos \theta)^2 - (r \sin \theta)^2 = z; x^2 - y^2 = z$$

30. $\rho \sin \phi = 1$ (spherical); $r = 1$ (cylindrical);

$$x^2 + y^2 = 1 \text{ (Cartesian)}$$

31. $z = 2x^2 + 2y^2 = 2(x^2 + y^2)$ (Cartesian); $z = 2r^2$ (cylindrical)

32. $2x^2 + 2y^2 - z^2 = 2$ (Cartesian); $2r^2 - z^2 = 2$ (cylindrical)

33. For St. Paul:

$$\rho = 3960,$$

$$\theta = 360^\circ - 93.1^\circ = 266.9^\circ \approx 4.6583 \text{ rad}$$

$$\phi = 90^\circ - 45^\circ = 45^\circ = \frac{\pi}{4} \text{ rad}$$

$$x = 3960 \sin \frac{\pi}{4} \cos 4.6583 \approx -151.4$$

$$y = 3960 \sin \frac{\pi}{4} \sin 4.6583 \approx -2796.0$$

$$z = 3960 \cos \frac{\pi}{4} \approx 2800.1$$

For Oslo:

$$\rho = 3960, \theta = 10.5^\circ \approx 0.1833 \text{ rad.}$$

$$\phi = 90^\circ - 59.6^\circ = 30.4^\circ \approx 0.5306 \text{ rad}$$

$$x = 3960 \sin 0.5306 \cos 0.1833 \approx 1970.4$$

$$y = 3960 \sin 0.5306 \sin 0.1833 \approx 365.3$$

$$z = 3960 \cos 0.5306 \approx 3415.5$$

As in Example 7,

$$\cos \gamma \approx \frac{(-151.4)(1970.4) + (-2796.0)(365.3) + (2800.1)(3415.5)}{3960^2} \approx 0.5257$$

so $\gamma \approx 1.0173$ and the great-circle distance is

$$d \approx 3960(1.0173) \approx 4029 \text{ mi}$$

34. For New York:

$$\rho = 3960, \theta = 360^\circ - 74^\circ = 286^\circ \approx 4.9916 \text{ rad}$$

$$\phi = 90^\circ - 40.4^\circ = 49.6^\circ \approx 0.8657 \text{ rad}$$

$$x = 3960 \sin 0.8657 \cos 4.9916 \approx 831.1$$

$$y = 3960 \sin 0.8657 \sin 4.9916 \approx -2898.9$$

$$z = 3960 \cos 0.8657 \approx 2566.5$$

For Greenwich:

$$\rho = 3960, \theta = 0, \phi = 90^\circ - 51.3^\circ = 38.7^\circ \approx 0.6754 \text{ rad}$$

$$x = 3960 \sin 0.6754 \cos 0 \approx 2475.8$$

$$y = 0$$

$$z = 3960 \cos 0.6754 \approx 3090.6$$

$$\cos \gamma = \frac{(831.1)(2475.8) + (-2898.9)(0) + (2566.5)(3090.6)}{3960^2} \approx 0.6370$$

so $\gamma \approx 0.8802$ and the great-circle distance is

$$d \approx 3960(0.8802) \approx 3485 \text{ mi}$$

35. From Problem 33, the coordinates of St. Paul are $P(-151.4, -2796.0, 2800.1)$.

For Turin:

$$\rho = 3960, \theta = 7.4^\circ \approx 0.1292 \text{ rad, } \phi = \frac{\pi}{4} \text{ rad}$$

$$x = 3960 \sin \frac{\pi}{4} \cos 0.1292 \approx 2776.8$$

$$y = 3960 \sin \frac{\pi}{4} \sin 0.1292 \approx 360.8$$

$$z = 3960 \cos \frac{\pi}{4} \approx 2800.1$$

$$\cos \gamma \approx \frac{(-151.4)(2776.8) + (-2796.0)(360.8) + (2800.1)(2800.1)}{3960^2} \approx 0.4088$$

so $\gamma \approx 1.1497$ and the great-circle distance is $d \approx 3960(1.1497) \approx 4553$ mi

36. The circle inscribed on the earth at 45° parallel ($\phi = 45^\circ$) has radius $3960 \cos \frac{\pi}{4}$. The longitudinal angle between St. Paul and Turin is $93.1^\circ + 7.4^\circ = 100.5^\circ \approx 1.7541$ rad

Thus, the distance along the 45° parallel is $\left(3960 \cos \frac{\pi}{4}\right)(1.7541) \approx 4912$ mi

37. Let St. Paul be at $P_1(-151.4, -2796.0, 2800.1)$ and Turin be at $P_2(2776.8, 360.8, 2800.1)$ and O be the center of the earth. Let β be the angle between the z -axis and the plane determined by O , P_1 , and P_2 . $\overrightarrow{OP_1} \times \overrightarrow{OP_2}$ is normal to the plane. The angle between the z -axis and $\overrightarrow{OP_1} \times \overrightarrow{OP_2}$ is complementary to β . Hence

$$\beta = \frac{\pi}{2} - \cos^{-1} \left(\frac{\left(\overrightarrow{OP_1} \times \overrightarrow{OP_2} \right) \cdot \mathbf{k}}{\left| \overrightarrow{OP_1} \times \overrightarrow{OP_2} \right| |\mathbf{k}|} \right) \approx \frac{\pi}{2} - \cos^{-1} \left(\frac{7.709 \times 10^6}{1.431 \times 10^7} \right) \approx 0.5689.$$

The distance between the North Pole and the St. Paul-Turin great-circle is $3960(0.5689) \approx 2253$ mi

38. $x_i = \rho_i \sin \phi_i \cos \theta_i$, $y_i = \rho_i \sin \phi_i \sin \theta_i$, $z_i = \rho_i \cos \phi_i$ for $i = 1, 2$.

$$\begin{aligned} d^2 &= (\rho_2 \sin \phi_2 \cos \theta_2 - \rho_1 \sin \phi_1 \cos \theta_1)^2 + (\rho_2 \sin \phi_2 \sin \theta_2 - \rho_1 \sin \phi_1 \sin \theta_1)^2 + (\rho_2 \cos \phi_2 - \rho_1 \cos \phi_1)^2 \\ &= \rho_2^2 \sin^2 \phi_2 (\cos^2 \theta_2 + \sin^2 \theta_2) + \rho_1^2 \sin^2 \phi_1 (\cos^2 \theta_1 + \sin^2 \theta_1) + \rho_2^2 \cos^2 \phi_2 + \rho_1^2 \cos^2 \phi_1 \\ &\quad - 2\rho_1 \rho_2 \sin \phi_1 \sin \phi_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) - 2\rho_1 \rho_2 \cos \phi_1 \cos \phi_2 \\ &= \rho_2^2 \sin^2 \phi_2 + \rho_1^2 \sin^2 \phi_1 + \rho_2^2 \cos^2 \phi_2 + \rho_1^2 \cos^2 \phi_1 - 2\rho_1 \rho_2 \sin \phi_1 \sin \phi_2 [\cos(\theta_1 - \theta_2)] - 2\rho_1 \rho_2 \cos \phi_1 \cos \phi_2 \\ &= \rho_2^2 (\sin^2 \phi_2 + \cos^2 \phi_2) + \rho_1^2 (\sin^2 \phi_1 + \cos^2 \phi_1) + 2\rho_1 \rho_2 [-\cos(\theta_1 - \theta_2) \sin \phi_1 \sin \phi_2 - \cos \phi_1 \cos \phi_2] \\ &= \rho_1^2 + \rho_2^2 + 2\rho_1 \rho_2 [-\cos(\theta_1 - \theta_2) \sin \phi_1 \sin \phi_2 - \cos \phi_1 \cos \phi_2] \\ &= \rho_1^2 - 2\rho_1 \rho_2 + \rho_2^2 + 2\rho_1 \rho_2 [1 - \cos(\theta_1 - \theta_2) \sin \phi_1 \sin \phi_2 - \cos \phi_1 \cos \phi_2] \\ &= (\rho_1 - \rho_2)^2 + 2\rho_1 \rho_2 [1 - \cos(\theta_1 - \theta_2) \sin \phi_1 \sin \phi_2 - \cos \phi_1 \cos \phi_2] \end{aligned}$$

$$\text{Hence, } d = \{(\rho_1 - \rho_2)^2 + 2\rho_1 \rho_2 [1 - \cos(\theta_1 - \theta_2) \sin \phi_1 \sin \phi_2 - \cos \phi_1 \cos \phi_2]\}^{1/2}$$

39. Let P_1 be (a_1, θ_1, ϕ_1) and P_2 be (a_2, θ_2, ϕ_2) . If γ is the angle between $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ then the great-circle distance between P_1 and P_2 is $a\gamma$. $|OP_1| = |OP_2| = a$ while the straight-line distance between P_1 and P_2 is (from

$$\begin{aligned} \text{Problem 38) } d^2 &= (a - a)^2 + 2a^2 [1 - \cos(\theta_1 - \theta_2) \sin \phi_1 \sin \phi_2 - \cos \phi_1 \cos \phi_2] \\ &= 2a^2 \{1 - [\cos(\theta_1 - \theta_2) \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2]\}. \end{aligned}$$

Using the Law Of Cosines on the triangle OP_1P_2 .

$$d^2 = a^2 + a^2 - 2a^2 \cos \gamma = 2a^2 (1 - \cos \gamma).$$

Thus, γ is the central angle and $\cos \gamma = \cos(\theta_1 - \theta_2) \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2$.

40. The longitude/latitude system (α, β) is related to a spherical coordinate system (ρ, θ, ϕ) by the following relations: $\rho = 3960$; the trigonometric function values of α and θ are identical but

$$-\pi \leq \alpha \leq \pi \text{ rather than } 0 \leq \theta \leq 2\pi, \text{ and } \beta = \frac{\pi}{2} - \phi \text{ so } \sin \beta = \cos \phi \text{ and } \cos \beta = \sin \phi.$$

From Problem 39, the great-circle distance between $(3960, \theta_1, \phi_1)$ and $(3960, \theta_2, \phi_2)$ is 3960γ where $0 \leq \gamma \leq \pi$ and $\cos \gamma = \cos(\theta_1 - \theta_2) \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 = \cos(\alpha_1 - \alpha_2) \cos \beta_1 \cos \beta_2 + \sin \beta_1 \sin \beta_2$.

41. a. New York $(-74^\circ, 40.4^\circ)$; Greenwich $(0^\circ, 51.3^\circ)$
 $\cos \gamma = \cos(-74^\circ - 0^\circ) \cos(40.4^\circ) \cos(51.3^\circ) + \sin(40.4^\circ) \sin(51.3^\circ) \approx 0.637$
 Then $\gamma \approx 0.880$ rad, so $d \approx 3960(0.8801) \approx 3485$ mi.
- b. St. Paul $(-93.1^\circ, 45^\circ)$; Turin $(7.4^\circ, 45^\circ)$
 $\cos \gamma = \cos(-93.1^\circ - 7.4^\circ) \cos(45^\circ) \cos(45^\circ) + \sin(45^\circ) \sin(45^\circ) \approx 0.4089$
 Then $\gamma \approx 1.495$ rad, so $d \approx 3960(1.495) \approx 4552$ mi.
- c. South Pole $(7.4^\circ, -90^\circ)$; Turin $(7.4^\circ, 45^\circ)$
 Note that any value of α can be used for the poles.
 $\cos \gamma = \cos 0^\circ \cos(-90^\circ) \cos(45^\circ) + \sin(-90^\circ) \sin(45^\circ) = -\frac{1}{\sqrt{2}}$
 thus $\gamma = 135^\circ = \frac{3\pi}{4}$ rad, so $d = 3960\left(\frac{3\pi}{4}\right) \approx 9331$ mi.
- d. New York $(-74^\circ, 40.4^\circ)$; Cape Town $(18.4^\circ, -33.9^\circ)$
 $\cos \gamma = \cos(-74^\circ - 18.4^\circ) \cos(40.4^\circ) \cos(-33.9^\circ) + \sin(40.4^\circ) \sin(-33.9^\circ) \approx -0.3880$
 Then $\gamma \approx 1.9693$ rad, so $d \approx 3960(1.9693) \approx 7798$ mi.
- e. For these points $\alpha_1 = 100^\circ$ and $\alpha_2 = -80^\circ$ while $\beta_1 = \beta_2 = 0$, hence
 $\cos \gamma = \cos 180^\circ$ and $\gamma = \pi$ rad, so $d = 3960\pi \approx 12,441$ mi.

42. $\rho = 2a \sin \phi$ is independent of θ so the cross section in each half-plane, $\theta = k$, is a circle tangent to the origin and with radius $2a$. Thus, the graph of $\rho = 2a \sin \phi$ is the surface of revolution generated by revolving about the z -axis a circle of radius $2a$ and tangent to the z -axis at the origin.

14.8 Chapter Review

Concepts Test

- True: The coordinates are defined in terms of distances from the coordinate planes in such a way that they are unique.
- False: The equation is $(x-2)^2 + y^2 + z^2 = -5$, so the solution set is the empty set.
- True: See Section 14.2.
- False: See previous problem. It represents a plane if A and B are not both zero.
- False: The distance between $(0, 0, 3)$ and $(0, 0, -3)$ (a point from each plane) is 6, so the distance between the planes is less than or equal to 6 units.
- False: It is normal to the plane.
- True: Let $t = \frac{1}{2}$.
- True: Direction cosines are $\frac{a}{|\mathbf{u}|}, \frac{b}{|\mathbf{u}|}, \frac{c}{|\mathbf{u}|}$ which are a, b, c .
- True: $\|\mathbf{u}\|\mathbf{u} = |\mathbf{u}||\mathbf{u}| = |\mathbf{u}|^2$
- False: The dot product of a scalar and a vector is not defined.
- True: $|\mathbf{u} \times \mathbf{v}| = |-\mathbf{v} \times \mathbf{u}| = |-1||\mathbf{v} \times \mathbf{u}| = |\mathbf{v} \times \mathbf{u}|$
- True: $(k\mathbf{v}) \times \mathbf{v} = k(\mathbf{v} \times \mathbf{v}) = k(\mathbf{0}) = \mathbf{0}$
- False: Obviously not true if $\mathbf{u} = \mathbf{v}$. (More generally, it is only true when $\mathbf{u}$ and $\mathbf{v}$ are also perpendicular.)
- False: It multiplies $\mathbf{v}$ by a ; it multiplies the length of $\mathbf{v}$ by $|a|$.

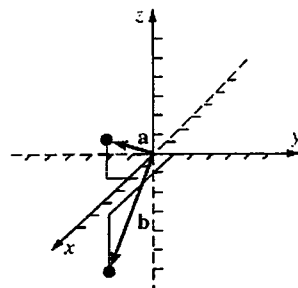
15. True: $\frac{|\mathbf{u} \times \mathbf{v}|}{(\mathbf{u} \cdot \mathbf{v})} = \frac{|\mathbf{u}||\mathbf{v}|\sin\theta}{|\mathbf{u}||\mathbf{v}|\cos\theta} = \tan\theta$
16. True: The vectors are both parallel and perpendicular, so one or both must be 0.
17. True: $|(2\mathbf{i} \times 2\mathbf{j}) \cdot (\mathbf{j} \times \mathbf{i})| = 4|(\mathbf{k}) \cdot (-\mathbf{k})| = 4(\mathbf{k} \cdot \mathbf{k}) = 4$
18. False: Let $\mathbf{u} = \mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$. Then $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0} \times \mathbf{w} = \mathbf{0}$; but $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$.
19. True: Since $\langle b_1, b_2, b_3 \rangle$ is normal to the plane.
20. False: Each line can be represented by parametric equations, but lines with any zero direction number cannot be represented by symmetric equations.
21. True: $\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = 0 \Rightarrow |\mathbf{r}' \times \mathbf{r}''| = |\mathbf{r}'||\mathbf{r}''|\sin\theta = 0$. Thus, either $\mathbf{r}'$ and $\mathbf{r}''$ are parallel or either $\mathbf{r}'$ or $\mathbf{r}''$ is $\mathbf{0}$, which implies that the path is a straight line.
22. True: An ellipse bends the sharpest at points on the major axis.
23. False: κ depends only on the shape of the curve.
24. False: Suppose $\mathbf{v}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$, then $|\mathbf{v}(t)| = 1$ but $\mathbf{a}(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$ which is non-zero.
25. True: $\mathbf{T}$ depends only upon the shape of the curve, hence $\mathbf{N}$ and $\mathbf{B}$ also.
26. True: If $\mathbf{v}$ is perpendicular to $\mathbf{a}$, then $\mathbf{T}$ is also perpendicular to $\mathbf{a}$, so $\frac{d}{dt}\left(\frac{ds}{dt}\right) = a_T = \mathbf{T} \cdot \mathbf{a} = 0$. Thus $\text{speed} = \frac{ds}{dt}$ is a constant.
27. False: If $\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + ct \mathbf{k}$ then $\mathbf{v}$ is perpendicular to $\mathbf{a}$, but the path of motion is a circular helix, not a circle.
28. False: The circular helix (see Problem 27) has constant curvature.

29. True: The curves are identical, although the motion of an object moving along the curves would be different.
30. False: At any time $0 < t < 1$, $r_1(t) \neq r_2(t)$
31. True: The parametrization affects only the rate at which the curve is traced out.
32. True: If a curve lies in a plane, then $\mathbf{T}$ and $\mathbf{N}$ will lie in the plane, so $\mathbf{T} \times \mathbf{N} = \mathbf{B}$ will be a unit vector normal to the plane.
33. False: For $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$, $|\mathbf{r}(t)| = 1$, but $\mathbf{r}'(t) \neq \mathbf{0}$.
34. True: The plane passes through the origin so its intersection with the sphere is a great circle. The radius of the circle is 1, so is curvature is $\frac{1}{1} = 1$.
35. False: It is the part of the z -axis with $z \geq 0$.
36. False: It is a parabolic cylinder.
37. False: The origin, $\rho = 0$, has infinitely many spherical coordinates, since any value of θ and ϕ can be used.

Sample Test Problems

1. The center of the sphere is the midpoint $\left(\frac{-2+4}{2}, \frac{3+1}{2}, \frac{3+5}{2}\right) = (1, 2, 4)$ of the diameter. The radius is $r = \sqrt{(1+2)^2 + (2-3)^2 + (4-3)^2} = \sqrt{9+1+1} = \sqrt{11}$. The equation of the sphere is $(x-1)^2 + (y-2)^2 + (z-4)^2 = 11$
2. $(x^2 - 6x + 9) + (y^2 + 2y + 1) + (z^2 - 8z + 16) = 9 + 1 + 16$; $(x-3)^2 + (y+1)^2 + (z-4)^2 = 26$
Center: $(3, -1, 4)$; radius: $\sqrt{26}$

3.



a. $|\mathbf{a}| = \sqrt{4+1+4} = 3$; $|\mathbf{b}| = \sqrt{25+1+9} = \sqrt{35}$

b. $\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$, direction cosines $\frac{2}{3}$, $-\frac{1}{3}$, and $\frac{2}{3}$.

$\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{5}{\sqrt{35}}\mathbf{i} + \frac{1}{\sqrt{35}}\mathbf{j} - \frac{3}{\sqrt{35}}\mathbf{k}$, direction cosines $\frac{5}{\sqrt{35}}$, $\frac{1}{\sqrt{35}}$, $-\frac{3}{\sqrt{35}}$

c. $\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

d. $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{10-1-6}{3\sqrt{35}} = \frac{3}{3\sqrt{35}} = \frac{1}{\sqrt{35}}$
 $\theta = \cos^{-1} \frac{1}{\sqrt{35}} \approx 1.4010 \approx 80.27^\circ$

4. a. $\langle -5, -5, 5 \rangle = -5\langle 1, 1, -1 \rangle$

b. $\langle 2, -1, 1 \rangle \times \langle 0, 5, 1 \rangle = \langle 6, -2, 10 \rangle$

c. $\langle 2, -1, 1 \rangle \cdot \langle -7, 1, -5 \rangle = -20$

d. $\langle 2, -1, 1 \rangle \times \langle -7, 1, -5 \rangle = \langle 4, 3, -5 \rangle$

5. $c\langle 3, 3, -1 \rangle \times \langle -1, -2, 4 \rangle = c\langle 10, -11, -3 \rangle$ for any c in $\mathbb{R}$.

6. Two vectors determined by the points are $\langle -1, 7, -3 \rangle$ and $\langle 3, -1, -3 \rangle$. Then $\langle -1, 7, -3 \rangle \times \langle 3, -1, -3 \rangle = -4\langle 6, 3, 5 \rangle$ and $\langle 6, 3, 5 \rangle$ are normal to the plane.

$\frac{\pm\langle 6, 3, 5 \rangle}{\sqrt{36+9+25}} = \frac{\pm\langle 6, 3, 5 \rangle}{\sqrt{70}}$ are the unit vectors normal to the plane.

7. a. $y = 7$, since y must be a constant.

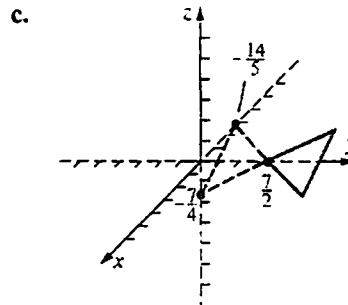
b. $x = -5$, since it is parallel to the yz -plane.

c. $z = -2$, since it is parallel to the xy -plane.

d. $3x - 4y + z = -45$, since it can be expressed as $3x - 4y + z = D$ and D must satisfy $3(-5) - 4(7) + (-2) = D$, so $D = -45$.

8. a. $\langle 4+1, 1-5, 1+7 \rangle = \langle 5, -4, 8 \rangle$ is along the line, hence normal to the plane, which has equation $\langle x-2, y+4, z+5 \rangle \cdot \langle 5, -4, 8 \rangle = 0$.

b. $5(x-2) - 4(y+4) + 8(z+5) = 0$ or $5x - 4y + 8z = -14$



9. If the planes are perpendicular, their normals will also be perpendicular. Thus $0 = \langle 1, 5, C \rangle \cdot \langle 4, -1, 1 \rangle = 4 - 5 + C$, so $C = 1$.

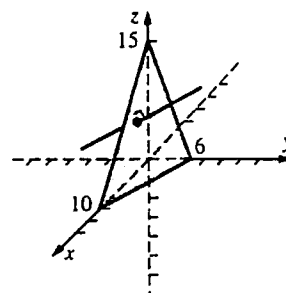
10. Two vectors in the same plane are $\langle 3, -2, -3 \rangle$ and $\langle 3, 7, 0 \rangle$. Their cross product, $3\langle 7, -3, 9 \rangle$, is normal to the plane. An equation of the plane is $7(x-2) - 3(y-3) + 9(z+1) = 0$ or $7x - 3y + 9z = -4$.

11. A vector in the direction of the line is $\langle 8, 1, -8 \rangle$. Parametric equations are $x = -2 + 8t$, $y = 1 + t$, $z = 5 - 8t$.

12. In the yz -plane, $x = 0$. Solve $-2y + 4z = 14$ and $2y - 5z = -30$, obtaining $y = 25$ and $z = 16$. In the xz -plane, $y = 0$. Solve $x + 4z = 14$ and $-x - 5z = -30$, obtaining $x = -50$ and $z = 16$. Therefore, the points are $(0, 25, 16)$ and $(-50, 0, 16)$.

13. $(0, 25, 16)$ and $(-50, 0, 16)$ are on the line, so $\langle 50, 25, 0 \rangle = 25\langle 2, 1, 0 \rangle$ is in the direction of the line. Parametric equations are $x = 0 + 2t$, $y = 25 + 1t$, $z = 16 + 0t$ or $x = 2t$, $y = 25 + t$, $z = 16$.

14. $\langle 3, 5, 2 \rangle$ is normal to the plane, so is in the direction of the line. Symmetric equations of the line are $\frac{x-4}{3} = \frac{y-5}{5} = \frac{z-8}{2}$.



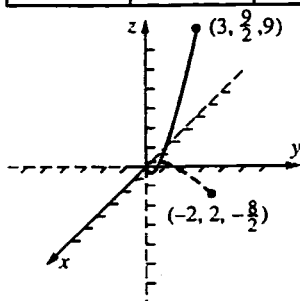
15. $\langle 5, -4, -3 \rangle$ is a vector in the direction of the line, and $\langle 2, -2, 1 \rangle$ is a position vector to the line.

Then a vector equation of the line is

$$\mathbf{r}(t) = \langle 2, -2, 1 \rangle + t \langle 5, -4, -3 \rangle.$$

16.

t	x	y	z
-2	-2	2	-8/3
-1	-1	1/2	-1/3
0	0	0	0
1	1	1/2	1/3
2	2	2	8/3
3	3	9/2	9



17. $\mathbf{r}'(t) = \langle 1, t, t^2 \rangle$, $\mathbf{r}'(2) = \langle 1, 2, 4 \rangle$ and $\mathbf{r}(2) = \langle 2, 2, \frac{8}{3} \rangle$. Symmetric equations for the

tangent line are $\frac{x-2}{1} = \frac{y-2}{2} = \frac{z-\frac{8}{3}}{4}$. Normal plane is $1(x-2) + 2(y-2) + 4(z-\frac{8}{3}) = 0$ or $3x + 6y + 12z = 50$.

18. $\mathbf{r}(t) = \langle t \cos t, t \sin t, 2t \rangle$;
 $\mathbf{r}'(t) = \langle -t \sin t + \cos t, t \cos t + \sin t, 2 \rangle$;
 $\mathbf{r}''(t) = \langle -t \cos t - 2 \sin t, -t \sin t + 2 \cos t, 0 \rangle$;
 $\mathbf{r}'(\frac{\pi}{2}) = \langle -\frac{\pi}{2}, 1, 2 \rangle$; $\mathbf{r}''(\frac{\pi}{2}) = \langle -2, -\frac{\pi}{2}, 0 \rangle$;
 $|\mathbf{r}'(\frac{\pi}{2})| = \frac{\sqrt{\pi^2 + 20}}{2}$;
 $\mathbf{T}(\frac{\pi}{2}) = \frac{\mathbf{r}'(\frac{\pi}{2})}{|\mathbf{r}'(\frac{\pi}{2})|} = \frac{\langle -\pi, 2, 4 \rangle}{\sqrt{\pi^2 + 20}}$

19. $\mathbf{r}'(t) = e^t \langle \cos t + \sin t, -\sin t + \cos t, 1 \rangle$;
 $|\mathbf{r}'(t)| = \sqrt{3}e^t$

Length is

$$\int_1^5 \sqrt{3}e^t dt = \left[\sqrt{3}e^t \right]_1^5 = \sqrt{3}(e^5 - e) \approx 252.3509.$$

20. $\mathbf{v}(t) = \langle e^t, -e^{-t}, 2 \rangle$

$$\mathbf{a}(t) = \langle e^t, e^{-t}, 0 \rangle$$

$$\mathbf{v}(\ln 2) = \langle 2, -\frac{1}{2}, 2 \rangle$$

$$\mathbf{a}(\ln 2) = \langle 2, \frac{1}{2}, 0 \rangle$$

$$\kappa(\ln 2) = \frac{|\mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2)|}{|\mathbf{v}(\ln 2)|^3} = \frac{|\langle -1, 4, 2 \rangle|}{\left(\sqrt{\frac{33}{4}}\right)^3} = \frac{8}{33^{3/2}} \sqrt{21} = 8 \sqrt{\frac{21}{35937}} = \frac{8\sqrt{7}}{\sqrt{11979}} \approx 0.1934$$

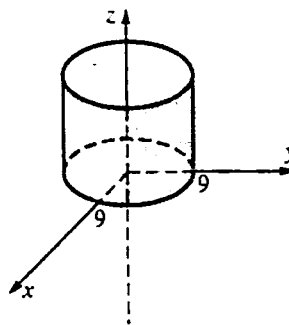
21. $\mathbf{v}(t) = \langle 1, 2t, 3t^2 \rangle$; $\mathbf{a}(t) = \langle 0, 2, 6t \rangle$

$$\mathbf{v}(1) = \langle 1, 2, 3 \rangle$$
; $|\mathbf{v}(1)| = \sqrt{14}$; $\mathbf{a}(1) = \langle 0, 2, 6 \rangle$

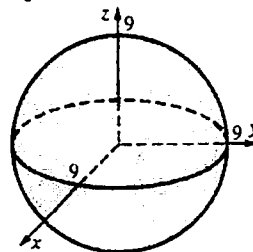
$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = \frac{0 + 4 + 18}{\sqrt{14}} = \frac{22}{\sqrt{14}} \approx 5.880$$

$$a_N = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|} = \frac{|\langle 6, -6, 2 \rangle|}{\sqrt{14}} = \frac{2\sqrt{19}}{\sqrt{14}} \approx 2.330$$

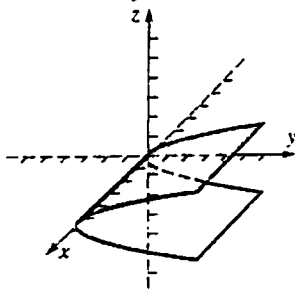
22. Circular cylinder



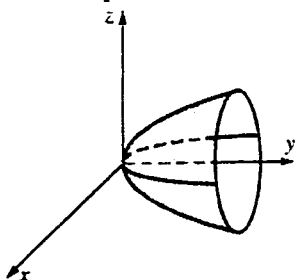
23. Sphere



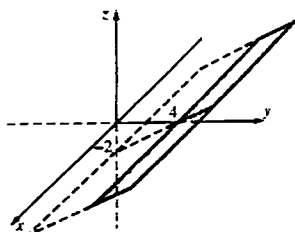
24. Parabolic cylinder



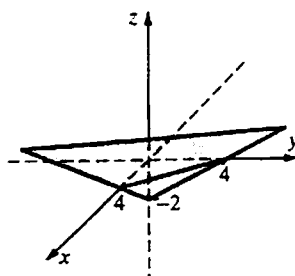
25. Circular paraboloid



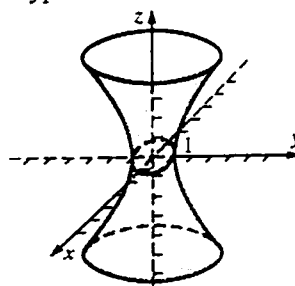
26. Plane



27. Plane

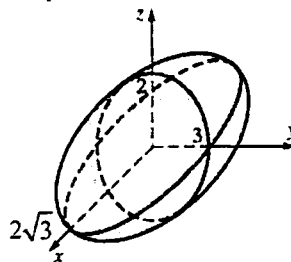


28. Hyperboloid of one sheet



$$29. \frac{x^2}{12} + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

Ellipsoid



30. The graph of $3x^2 + 4y^2 + 9z^2 = -36$ is the empty set.

31. a. $r^2 = 9$; $r = 3$

b. $(x^2 + y^2) + 3y^2 = 16$

$$r^2 + 3r^2 \sin^2 \theta = 16, r^2 = \frac{16}{1 + 3\sin^2 \theta}$$

c. $r^2 = 9z$

d. $r^2 + 4z^2 = 10$

32. a. $x^2 + y^2 + z^2 = 9$

b. $x^2 + z^2 = 4$

c. $r^2(\cos^2 \theta - \sin^2 \theta) + z^2 = 1$;
 $x^2 - y^2 + z^2 = 1$

33. a. $\rho^2 = 4$; $\rho = 2$

b. $x^2 + y^2 + z^2 - 2z^2 = 0$;

$$\rho^2 - 2\rho^2 \cos^2 \phi = 0; \rho^2(1 - 2\cos^2 \phi) = 0;$$

$$1 - 2\cos^2 \phi = 0; \cos^2 \phi = \frac{1}{2}; \phi = \frac{\pi}{4} \text{ or}$$

$$\phi = \frac{3\pi}{4}.$$

Any of the following (as well as others) would be acceptable:

$$\left(\phi - \frac{\pi}{4}\right)\left(\phi - \frac{3\pi}{4}\right) = 0$$

$$\cos^2 \phi = \frac{1}{2}$$

$$\sec^2 \phi = 2$$

$$\tan^2 \phi = 1$$

$$\begin{aligned} \text{c. } 2x^2 - (x^2 + y^2 + z^2) &= 1; \\ 2\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 &= 1; \\ \rho^2 &= \frac{1}{2 \sin^2 \phi \cos^2 \theta - 1} \end{aligned}$$

$$\begin{aligned} \text{d. } x^2 + y^2 &= z; \\ \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta &= \rho \cos \phi \\ \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) &= \rho \cos \phi; \\ \rho \sin^2 \phi &= \cos \phi; \quad \rho = \cot \phi \csc \phi \end{aligned}$$

(Note that when we divided through by ρ in part c and d we did not lose the pole since it is also a solution of the resulting equations.)

34. Cartesian coordinates are $(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3})$ and $(\sqrt{2}, \sqrt{6}, -2\sqrt{2})$. Distance

$$\left[2 + (2\sqrt{2} - \sqrt{6})^2 + (4\sqrt{3} + 2\sqrt{2})^2 \right]^{1/2} \approx 9.8659.$$

35. $(2, 0, 0)$ is a point of the first plane. The distance between the planes is

$$\frac{|2(2) - 3(0) + \sqrt{3}(0) - 9|}{\sqrt{4 + 9 + 3}} = \frac{5}{\sqrt{16}} = 1.25$$

36. $\langle 2, -4, 1 \rangle$ and $\langle 3, 2, -5 \rangle$ are normal to the respective planes. The acute angle between the two planes is the same as the acute angle θ between the normal vectors.

$$\cos \theta = \frac{|6 - 8 - 5|}{\sqrt{21}\sqrt{38}} = \frac{7}{\sqrt{798}}, \text{ so } \theta \approx 1.3204 \text{ rad} \\ \approx 75.65^\circ$$

37. If speed $= \frac{ds}{dt} = c$, a constant, then

$$\mathbf{a} = \frac{d^2s}{dt^2} \mathbf{T} + \left(\frac{ds}{dt} \right)^2 \kappa \mathbf{N} = c^2 \kappa \mathbf{N} \text{ since } \frac{d^2s}{dt^2} = 0. \mathbf{T}$$

is in the direction of $\mathbf{v}$, while $\mathbf{N}$ is perpendicular to $\mathbf{T}$ and hence to $\mathbf{v}$ also. Thus, $\mathbf{a} = c^2 \kappa \mathbf{N}$ is perpendicular to $\mathbf{v}$.

15.1 Concepts Review

1. real-valued function of two real variables
2. level curve; contour map
3. concentric circles
4. parallel lines

Problem Set 15.1

1. a. 5
b. 0
c. 6
d. $a^6 + a^2$
e. $2x^2, x \neq 0$
f. Undefined

The natural domain is the set of all (x, y) such that y is nonnegative.

2. a. 4
b. 17
c. $\frac{17}{16}$
d. $1 + a^2, a \neq 0$
e. $x^3 + x, x \neq 0$
f. Undefined

The natural domain is the set of all (x, y) such that x is nonzero.

3. a. $\sin(2\pi) = 0$
b. $4\sin\left(\frac{\pi}{6}\right) = 2$

c. $16\sin\left(\frac{\pi}{2}\right) = 16$

d. $\pi^2 \sin(\pi^2) \approx -4.2469$

e. $1.44 \sin[(3.1)(4.2)] \approx 0.6311$

4. a. 6

b. 12

c. 2

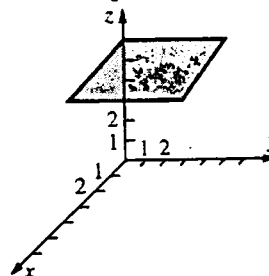
d. $(3\cos 6)^{1/2} + 1.44 \approx 3.1372$

e. $(-2\cos 2)^{1/2} + 9 \approx 9.9123$

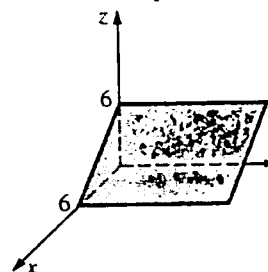
5. $F(t \cos t, \sec^2 t) = t^2 \cos^2 t \sec^2 t = t^2, \cos t \neq 0$

6. $F(f(t), g(t)) = F(\ln t^2, e^{t/2})$
 $= \exp(\ln t^2) + (e^{t/2})^2 = t^2 + e^t, t \neq 0$

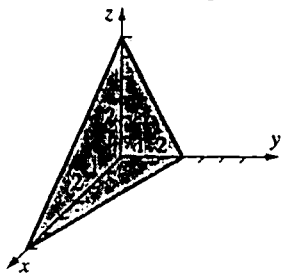
7. $z = 6$ is a plane.



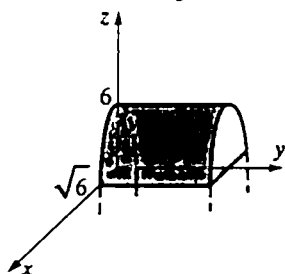
8. $x + z = 6$ is a plane.



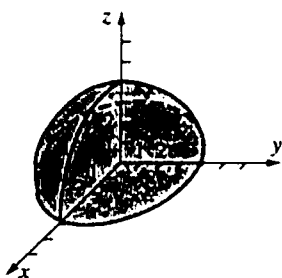
9. $x + 2y + z = 6$ is a plane.



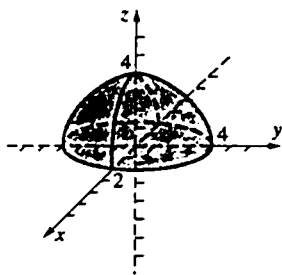
10. $z = 6 - x^2$ is a parabolic cylinder.



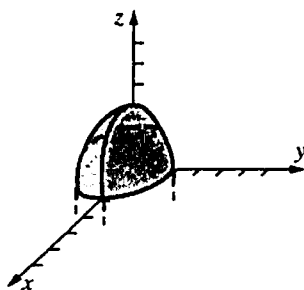
11. $x^2 + y^2 + z^2 = 16, z \geq 0$ is a hemisphere.



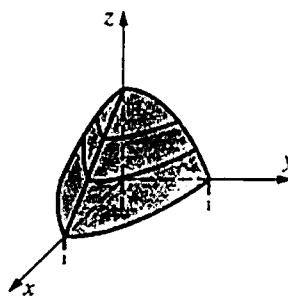
12. $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1, z \geq 0$ is a hemi-ellipsoid.



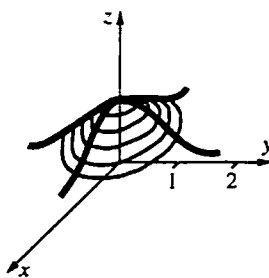
13. $z = 3 - x^2 - y^2$ is a paraboloid.



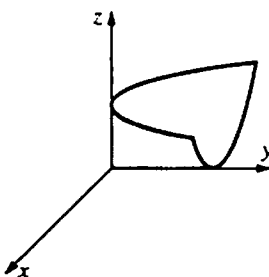
14. $z = 2 - x - y^2$



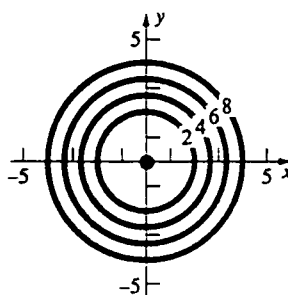
15. $z = \exp[-(x^2 + y^2)]$



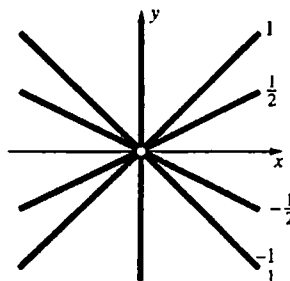
16. $z = \frac{x^2}{y}, y > 0$



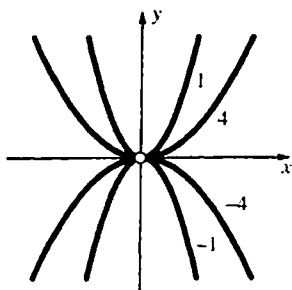
17. $x^2 + y^2 = 2z; x^2 + y^2 = 2k$



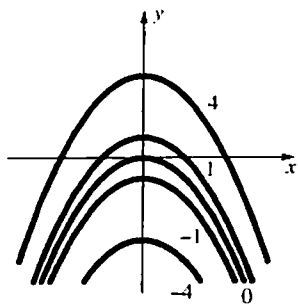
18. $x = zy, y \neq 0; x = ky, y \neq 0$



19. $x^2 = zy, y \neq 0; x^2 = ky, y \neq 0$



20. $x^2 = -(y-z); x^2 = -(y-k)$



21. $z = \frac{x^2 + y}{x + y^2}, x \neq -y^2$

$k = 0: y = -x^2$

Parabola except (0, 0) and (-1, -1)

$k = 1: x^2 + y = x + y^2$

$$\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2 = 0$$

$y = x$ or $y = -x + 1$

Intersecting lines except (0, 0) and (-1, -1)

$k = 2: x^2 + y = 2x + 2y^2$

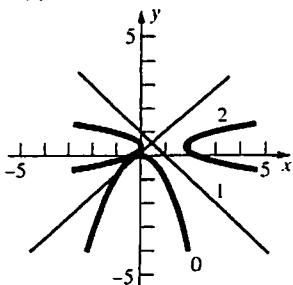
$$\frac{(x-1)^2}{\frac{7}{8}} - \frac{\left(y - \frac{1}{4}\right)^2}{\frac{7}{16}} = 1$$

Hyperbola except (0, 0) and (-1, -1)

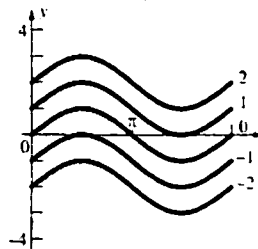
$k = 4: x^2 + y = 4x + 4y^2$

$$\frac{(x-2)^2}{\frac{63}{16}} - \frac{\left(y - \frac{1}{8}\right)^2}{\frac{63}{64}} = 1$$

Hyperbola except (0, 0) and (-1, -1)

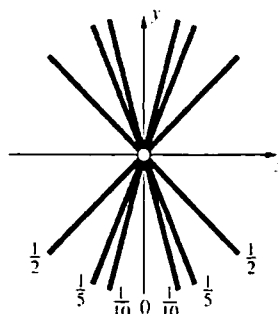


22. $y = \sin x + z; y = \sin x + k$

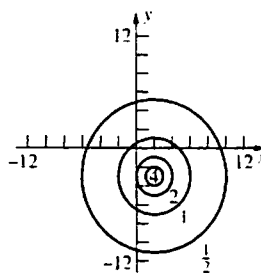


23. $x = 0$, if $T = 0$:

$$y^2 = \left(\frac{1}{T} - 1\right)x^2, \text{ if } y \neq 0.$$



24. $(x-2)^2 + (y+3)^2 = \frac{16}{r^2}$



25. a. San Francisco and St. Louis had a temperature between 70 and 80 degrees Fahrenheit.
- b. Drive northwest to get to cooler temperatures, and drive southeast to get warmer temperatures.
- c. Since the level curve for 70 runs southwest to northeast, you could drive southwest or northeast and stay at about the same temperature.
26. a. The lowest barometric pressure, 1000 millibars and under, occurred in the region of the Great Lakes, specifically near Wisconsin. The highest barometric pressure, 1025 millibars and over, occurred on the east coast, from Massachusetts to South Carolina.
- b. Driving northwest would take you to lower barometric pressure, and driving southeast

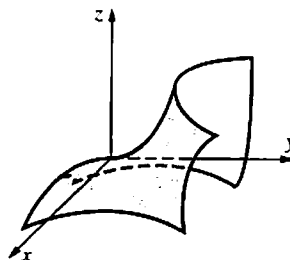
would take you to higher barometric pressure.

- c. Since near St. Louis the level curves run southwest to northeast, you could drive southwest or northeast and stay at about the same barometric pressure.

27. $x^2 + y^2 + z^2 \geq 16$; the set of all points on and outside the sphere of radius 4 that is centered at the origin
28. The set of all points inside (the part containing the z -axis) and on the hyperboloid of one sheet; $\frac{x^2}{9} + \frac{y^2}{9} - \frac{z^2}{9} = 1$.
29. $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{1} \leq 1$; points inside and on the ellipsoid
30. Points inside (the part containing the z -axis) or on the hyperboloid of one sheet, $\frac{x^2}{9} + \frac{y^2}{9} - \frac{z^2}{16} = 1$, excluding points on the coordinate planes
31. Since the argument to the natural logarithm function must be positive, we must have $x^2 + y^2 + z^2 > 0$. This is true for all (x, y, z) except $(x, y, z) = (0, 0, 0)$. The domain consists all points in $\mathbb{R}^3$ except the origin.
32. Since the argument to the natural logarithm function must be positive, we must have $xy > 0$. This occurs when the ordered pair (x, y) is in the first quadrant or the third quadrant of the xy -plane. There is no restriction on z . Thus, the domain consists of all points (x, y, z) such that x and y are both positive or both negative.
33. $x^2 + y^2 + z^2 = k$, $k > 0$; set of all spheres centered at the origin
34. $100x^2 + 16y^2 + 25z^2 = k$, $k > 0$;
 $\frac{x^2}{\frac{k}{100}} + \frac{y^2}{\frac{k}{16}} + \frac{z^2}{\frac{k}{25}} = 1$; set of all ellipsoids centered at origin such that their axes have ratio $\left(\frac{1}{10}\right) : \left(\frac{1}{4}\right) : \left(\frac{1}{5}\right)$ or 2:5:4.

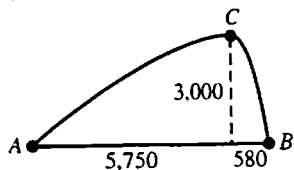
35. $\frac{x^2}{\frac{1}{16}} + \frac{y^2}{\frac{1}{4}} - \frac{z^2}{1} = k$; the elliptic cone
 $\frac{x^2}{9} + \frac{y^2}{9} = \frac{z^2}{16}$ and all hyperboloids (one and two sheets) with z -axis for axis such that $a:b:c$ is $\left(\frac{1}{4}\right) : \left(\frac{1}{4}\right) : \left(\frac{1}{3}\right)$ or 3:3:4.
36. $\frac{x^2}{\frac{1}{9}} - \frac{y^2}{\frac{1}{4}} - \frac{z^2}{1} = k$; the elliptical cone
 $\frac{y^2}{9} + \frac{z^2}{36} = \frac{x^2}{4}$ and all hyperboloids (one and two sheets) with x -axis for axis such that $a:b:c$ is $\left(\frac{1}{3}\right) : \left(\frac{1}{2}\right) : 1$ or 2:3:6
37. $4x^2 - 9y^2 = k$, k in $\mathbb{R}$; $\frac{x^2}{\frac{k}{4}} - \frac{y^2}{\frac{k}{9}} = 1$, if $k \neq 0$;
 planes $y = \pm \frac{2x}{3}$ (for $k = 0$) and all hyperbolic cylinders parallel to the z -axis such that the ratio $a:b$ is $\left(\frac{1}{2}\right) : \left(\frac{1}{3}\right)$ or 3:2 (where a is associated with the x -term)
38. $e^{x^2+y^2+z^2} = k$, $k > 0$
 $x^2 + y^2 + z^2 = \ln k$
 concentric circles centered at the origin.
39. a. All (w, x, y, z) except $(0, 0, 0, 0)$, which would cause division by 0.
 b. All $(x_1, x_2, \dots, x_n)$ in n -space.
 c. All $(x_1, x_2, \dots, x_n)$ that satisfy $x_1^2 + x_2^2 + \dots + x_n^2 \leq 1$; other values of $(x_1, x_2, \dots, x_n)$ would lead to the square root of a negative number.

40. If $z = 0$, then $x = 0$ or $x = \pm\sqrt{3}y$.



41. a. AC is the least steep path and BC is the most steep path between A and C since the level curves are farthest apart along AC and closest together along BC .

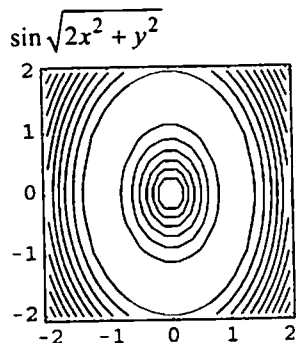
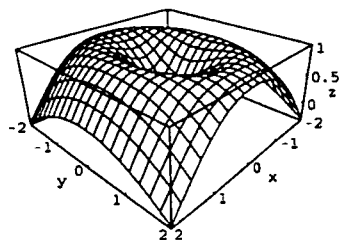
b. $|AC| \approx \sqrt{(5750)^2 + (3000)^2} \approx 6490$ ft
 $|BC| \approx \sqrt{(580)^2 + (3000)^2} \approx 3060$ ft



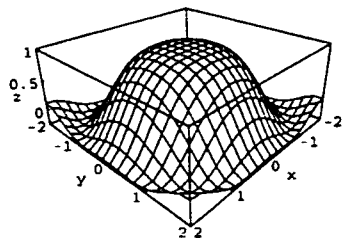
42. Completing the squares on x and y yields the equivalent equation

$$f(x, y) + 25.25 = (x - 0.5)^2 + 3(y + 2)^2, \text{ an elliptic paraboloid.}$$

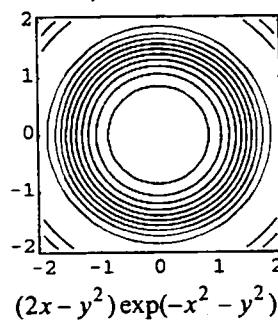
43.



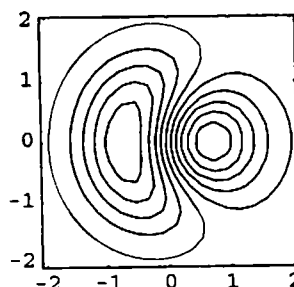
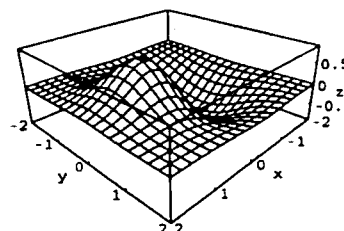
44.



$$\frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

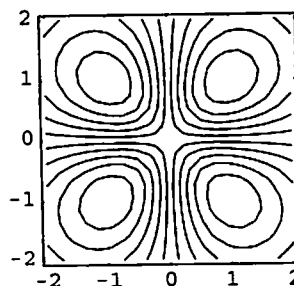
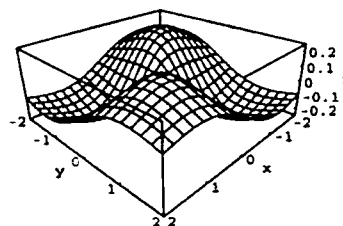


45.



$$\frac{\sin x \sin y}{(1 + x^2 + y^2)}$$

46.



15.2 Concepts Review

- $\lim_{h \rightarrow 0} \frac{[(f(x_0 + h, y_0) - f(x_0, y_0))]}{h}$; partial derivative of f with respect to x
- 5; 1
- $\frac{\partial^2 f}{\partial y \partial x}$
- 0

Problem Set 15.2

- $f_x(x, y) = 8(2x - y)^3$; $f_y(x, y) = -4(2x - y)^3$
- $f_x(x, y) = 6(4x - y^2)^{1/2}$;
 $f_y(x, y) = -3y(4x - y^2)^{1/2}$
- $f_x(x, y) = \frac{(xy)(2x) - (x^2 - y^2)(y)}{(xy)^2} = \frac{x^2 + y^2}{x^2 y}$
 $f_y(x, y) = \frac{(xy)(-2y) - (x^2 - y^2)(x)}{(xy)^2}$
 $= -\frac{(x^2 + y^2)}{xy^2}$
- $f_x(x, y) = e^x \cos y$; $f_y(x, y) = -e^x \sin y$
- $f_x(x, y) = e^y \cos x$; $f_y(x, y) = e^y \sin x$
- $f_x(x, y) = \left(-\frac{1}{3}\right)(3x^2 + y^2)^{-4/3}(6x)$
 $= -2x(3x^2 + y^2)^{-4/3}$;
 $f_y(x, y) = \left(-\frac{1}{3}\right)(3x^2 + y^2)^{-4/3}(2y)$
 $= \left(-\frac{2y}{3}\right)(3x^2 + y^2)^{-4/3}$
- $f_x(x, y) = x(x^2 - y^2)^{-1/2}$;
 $f_y(x, y) = -y(x^2 - y^2)^{-1/2}$
- $f_u(u, v) = ve^{uv}$; $f_v(u, v) = ue^{uv}$
- $g_x(x, y) = -ye^{-xy}$; $g_y(x, y) = -xe^{-xy}$

- $f_s(s, t) = 2s(s^2 - t^2)^{-1}$;
 $f_t(s, t) = -2t(s^2 - t^2)^{-1}$
- $f_x(x, y) = 4[1 + (4x - 7y)^2]^{-1}$;
 $f_y(x, y) = -7[1 + (4x - 7y)^2]^{-1}$
- $F_w(w, z) = w \frac{1}{\sqrt{1 - \left(\frac{w}{z}\right)^2}} \left(\frac{1}{z}\right) + \sin^{-1}\left(\frac{w}{z}\right)$
 $= \frac{\frac{w}{z}}{\sqrt{1 - \left(\frac{w}{z}\right)^2}} + \sin^{-1}\left(\frac{w}{z}\right)$;
 $F_z = (w, z) = w \frac{1}{\sqrt{1 - \left(\frac{w}{z}\right)^2}} \left(-\frac{w}{z^2}\right) = \frac{-\left(\frac{w}{z}\right)^2}{\sqrt{1 - \left(\frac{w}{z}\right)^2}}$
- $f_x(x, y) = -2xy \sin(x^2 + y^2)$;
 $f_y(x, y) = -2y^2 \sin(x^2 + y^2) + \cos(x^2 + y^2)$
- $f_s(s, t) = -2se^{t^2 - s^2}$; $f_t(s, t) = 2te^{t^2 - s^2}$
- $F_x(x, y) = 2 \cos x \cos y$; $F_y(x, y) = -2 \sin x \sin y$
- $f_r(r, \theta) = 9r^2 \cos 2\theta$; $f_\theta(r, \theta) = -6r^3 \sin 2\theta$
- $f_x(x, y) = 4xy^3 - 3x^2y^5$;
 $f_{xy}(x, y) = 12xy^2 - 15x^2y^4$
 $f_y(x, y) = 6x^2y^2 - 5x^3y^4$;
 $f_{yx}(x, y) = 12xy^2 - 15x^2y^4$
- $f_x(x, y) = 5(x^3 + y^2)^4(3x^2)$;
 $f_{xy}(x, y) = 60x^2(x^3 + y^2)^3(2y)$
 $= 120x^2y(x^3 + y^2)^3$
 $f_y(x, y) = 5(x^3 + y^2)^4(2y)$;
 $f_{yx}(x, y) = 40y(x^3 + y^2)^3(3x^2)$
 $= 120x^2y(x^3 + y^2)^3$
- $f_x(x, y) = 6e^{2x} \cos y$; $f_{xy}(x, y) = -6e^{2x} \sin y$
 $f_y(x, y) = -3e^{2x} \sin y$; $f_{yx}(x, y) = -6e^{2x} \sin y$
- $f_x(x, y) = y(1 + x^2y^2)^{-1}$;
 $f_{xy}(x, y) = (1 - x^2y^2)(1 + x^2y^2)^{-2}$

$$f_x(x, y) = x(1 + x^2 y^2)^{-1};$$

$$f_{xy}(x, y) = (1 - x^2 y^2)(1 + x^2 y^2)^{-2}$$

$$21. F_x(x, y) = \frac{(xy)(2) - (2x - y)(y)}{(xy)^2} = \frac{y^2}{x^2 y^2} = \frac{1}{x^2};$$

$$F_x(3, -2) = \frac{1}{9}$$

$$F_y(x, y) = \frac{(xy)(-1) - (2x - y)(x)}{(xy)^2} = \frac{-2x^2}{x^2 y^2} = -\frac{2}{x^2};$$

$$F_y(3, -2) = -\frac{1}{2}$$

$$22. F_x(x, y) = (2x + y)(x^2 + xy + y^2)^{-1};$$

$$F_x(-1, 4) = \frac{2}{13} \approx 0.1538$$

$$F_y(x, y) = (x + 2y)(x^2 + xy + y^2)^{-1};$$

$$F_y(-1, 4) = \frac{7}{13} \approx 0.5385$$

$$23. f_x(x, y) = -y^2(x^2 + y^4)^{-1};$$

$$f_x(\sqrt{5}, -2) = -\frac{4}{21} \approx -0.1905$$

$$f_y(x, y) = 2xy(x^2 + y^4)^{-1};$$

$$f_y(\sqrt{5}, -2) = -\frac{4\sqrt{5}}{21} \approx -0.4259$$

$$24. f_x(x, y) = e^y \sinh x;$$

$$f_x(-1, 1) = e \sinh(-1) \approx -3.1945$$

$$f_y(x, y) = e^y \cosh x;$$

$$f_y(-1, 1) = e \cosh(-1) \approx 4.1945$$

$$25. \text{ Let } z = f(x, y) = \frac{x^2}{9} + \frac{y^2}{4}.$$

$$f_y(x, y) = \frac{y}{2}$$

The slope is $f_y(3, 2) = 1$.

$$26. \text{ Let } z = f(x, y) = (1/3)(36 - 9x^2 - 4y^2)^{1/2}.$$

$$f_y(x, y) = \left(-\frac{4}{3}\right)y(36 - 9x^2 - y^2)^{-1/2}$$

The slope is $f_y(1, -2) = \frac{8}{3\sqrt{11}} \approx 0.8040$.

$$27. z = f(x, y) = \left(\frac{1}{2}\right)(9x^2 + 9y^2 - 36)^{1/2}.$$

$$f_x(x, y) = \frac{9x}{2(9x^2 + 9y^2 - 36)^{1/2}}$$

$$f_x(2, 1) = 3$$

$$28. z = f(x, y) = \left(\frac{5}{4}\right)(16 - x^2)^{1/2}.$$

$$f_x(x, y) = \left(-\frac{5}{4}\right)x(16 - x^2)^{-1/2}$$

$$f_x(2, 3) = -\frac{5}{4\sqrt{3}} \approx -0.7217$$

$$29. V_r(r, h) = 2\pi r h;$$

$$V_r(6, 10) = 120\pi \approx 376.99 \text{ in.}^2$$

$$30. T_y(x, y) = 3y^2; T_y(3, 2) = 12 \text{ degrees per ft}$$

$$31. P(V, T) = \frac{kT}{V}$$

$$P_T(V, T) = \frac{k}{V};$$

$$P_T(100, 300) = \frac{k}{100} \text{ lb/in.}^2 \text{ per degree}$$

$$32. V[P_V(V, T)] + T[P_T(V, T)]$$

$$= V(-kTV^{-2}) + T(kV^{-1}) = 0$$

$$P_V V_T T_P = \left(-\frac{kT}{V^2}\right)\left(\frac{k}{P}\right)\left(\frac{V}{k}\right) = -\frac{kT}{PV} = -\frac{PV}{PV} = -1$$

$$33. f_x(x, y) = 3x^2 y - y^3; f_{xx}(x, y) = 6xy;$$

$$f_y(x, y) = x^3 - 3xy^2; f_{yy}(x, y) = -6xy$$

Therefore, $f_{xx}(x, y) + f_{yy}(x, y) = 0$.

$$34. f_x(x, y) = 2x(x^2 + y^2)^{-1};$$

$$f_{xx}(x, y) = -2(x^2 - y^2)(x^2 + y^2)^{-1}$$

$$f_y(x, y) = 2y(x^2 + y^2)^{-1};$$

$$f_{yy}(x, y) = 2(x^2 - y^2)(x^2 + y^2)^{-1}$$

$$35. F_y(x, y) = 15x^4 y^4 - 6x^2 y^2;$$

$$F_{yy}(x, y) = 60x^4 y^3 - 12x^2 y;$$

$$F_{yyy}(x, y) = 180x^4 y^2 - 12x^2$$

$$36. f_x(x, y) = [-\sin(2x^2 - y^2)](4x)$$

$$\begin{aligned}
 &= -4x \sin(2x^2 - y^2) \\
 f_{xx}(x, y) &= (-4x)[\cos(2x^2 - y^2)](4x) \\
 &\quad + [\sin(2x^2 - y^2)](-4) \\
 f_{xy}(x, y) &= -16x^2[-\sin(2x^2 - y^2)](-2y) \\
 &\quad - 4[\cos(2x^2 - y^2)](-2y) \\
 &= -32x^2y \sin(2x^2 - y^2) + 8y \cos(2x^2 - y^2)
 \end{aligned}$$

37. a. $\frac{\partial^3 f}{\partial y^3}$

b. $\frac{\partial^3 y}{\partial y \partial x^2}$

c. $\frac{\partial^4 y}{\partial y^3 \partial x}$

38. a. f_{yxx}

b. f_{yyxx}

c. f_{yyxxx}

39. a. $f_x(x, y, z) = 6xy - yz$

b. $f_y(x, y, z) = 3x^2 - xz + 2yz^2$;
 $f_y(0, 1, 2) = 8$

c. Using the result in a, $f_{xy}(x, y, z) = 6x - z$.

40. a. $12x^2(x^3 + y^2 + z)^3$

b. $f_y(x, y, z) = 8y(x^3 + y^2 + z)^3$;
 $f_y(0, 1, 1) = 64$

c. $f_z(x, y, z) = 4(x^3 + y^2 + z)^3$;
 $f_{zz}(x, y, z) = 12(x^3 + y^2 + z)^2$

41. $f_x(x, y, z) = -yze^{-xyz} - y(xy - z^2)^{-1}$

42. $f_x(x, y, z) = \left(\frac{1}{2}\right)\left(\frac{xy}{z}\right)^{-1/2}\left(\frac{y}{z}\right)$;
 $f_x(-2, -1, 8) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^{-1/2}\left(-\frac{1}{8}\right) = -\frac{1}{8}$

43. If $f(x, y) = x^4 + xy^3 + 12$, $f_y(x, y) = 3xy^2$;
 $f_y(1, -2) = 12$. Therefore, along the tangent line
 $\Delta y = 1 \Rightarrow \Delta z = 12$, so $\langle 0, 1, 12 \rangle$ is a tangent
vector (since $\Delta x = 0$). Then parametric equations

of the tangent line are $\begin{cases} x = 1 \\ y = -2 + t \\ z = 5 + 12t \end{cases}$. Then the

point of xy -plane at which the bee hits is
 $(1, 0, 29)$ [since $y = 0 \Rightarrow t = 2 \Rightarrow x = 1, z = 29$].

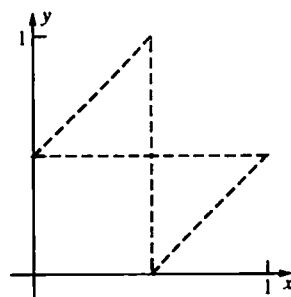
44. The largest rectangle that can be contained in the
circle is a square of diameter length 20. The edge
of such a square has length $10\sqrt{2}$, so its area is
200. Therefore, the domain of A is
 $\{(x, y) : 0 \leq x^2 + y^2 < 400\}$, and the range is
 $(0, 200]$.

45. Domain: (Case $x < y$)
The lengths of the sides are then x , $y - x$, and
 $1 - y$. The sum of the lengths of any two sides
must be greater than the length of the remaining
side, leading to three inequalities:

$$x + (y - x) > 1 - y \Rightarrow y > \frac{1}{2}$$

$$(y - x) + (1 - y) > x \Rightarrow x < \frac{1}{2}$$

$$x + (1 - y) > y - x \Rightarrow y < x + \frac{1}{2}$$



The case for $y < x$ yields similar inequalities
(x and y interchanged). The graph of D_A , the
domain of A is given above. In set notation it is

$$\begin{aligned}
 D_A &= \left\{ (x, y) : x < \frac{1}{2}, y > \frac{1}{2}, y < x + \frac{1}{2} \right\} \\
 &\quad \cup \left\{ (x, y) : x > \frac{1}{2}, y < \frac{1}{2}, x < y + \frac{1}{2} \right\}.
 \end{aligned}$$

Range: The area is greater than zero but can be
arbitrarily close to zero since one side can be
arbitrarily small and the other two sides are
bounded above. It seems that the area would be
largest when the triangle is equilateral. An
equilateral triangle with sides equal to $\frac{1}{3}$ has

area $\frac{\sqrt{3}}{36}$. Hence the range of A is $\left(0, \frac{\sqrt{3}}{36}\right]$. (In

Sections 8 and 9 of this chapter methods will be presented which will make it easy to prove that the largest value of A will occur when the triangle is equilateral.)

46. a. $u = \cos(x) \cos(ct)$:

$$u_x = -\sin(x) \cos(ct)$$

$$u_t = -c \cos(x) \sin(ct)$$

$$u_{xx} = -\cos(x) \cos(ct)$$

$$u_{tt} = -c^2 \cos(x) \cos(ct)$$

$$\text{Therefore, } c^2 u_{xx} = u_{tt}.$$

$$u = e^x \cosh(ct):$$

$$u_x = e^x \cosh(ct), u_t = ce^x \sinh(ct)$$

$$u_{xx} = e^x \cosh(ct), u_{tt} = c^2 e^x \cosh(ct)$$

$$\text{Therefore, } c^2 u_{xx} = u_{tt}.$$

- b. $u = e^{-ct} \sin(x)$:

$$u_x = e^{-ct} \cos x$$

$$u_{xx} = -e^{-ct} \sin x$$

$$u_t = -ce^{ct} \sin x$$

$$\text{Therefore, } cu_{xx} = u_t.$$

$$u = t^{-1/2} e^{-x^2/4ct}:$$

$$u_x = t^{-1/2} e^{-x^2/4ct} \left(-\frac{x}{2ct} \right)$$

$$u_{xx} = \frac{(x^2 - 2ct)}{(4c^2 t^{5/2} e^{x^2/4ct})}$$

$$u_t = \frac{(x^2 - 2ct)}{(4ct^{5/2} e^{x^2/4ct})}$$

$$\text{Therefore, } cu_{xx} = u_t.$$

47. a. Moving parallel to the y -axis from the point $(1, 1)$ to the nearest level curve and

approximating $\frac{\Delta z}{\Delta y}$, we obtain

$$f_y(1, 1) = \frac{4-5}{1.25-1} = -4.$$

- b. Moving parallel to the x -axis from the point $(-4, 2)$ to the nearest level curve and

approximating $\frac{\Delta z}{\Delta x}$, we obtain

$$f_x(-4, 2) \approx \frac{1-0}{-2.5-(-4)} = \frac{2}{3}.$$

- c. Moving parallel to the x -axis from the point $(-5, -2)$ to the nearest level curve and

approximating $\frac{\Delta z}{\Delta x}$, we obtain

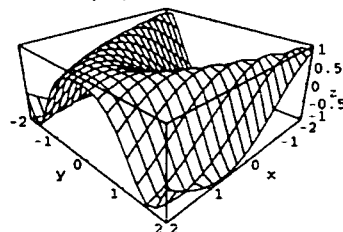
$$f_x(-4, -5) \approx \frac{1-0}{-2.5-(-5)} = \frac{2}{5}.$$

- d. Moving parallel to the y -axis from the point $(0, -2)$ to the nearest level curve and

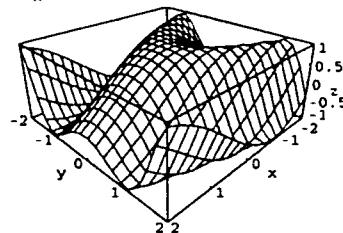
approximating $\frac{\Delta z}{\Delta y}$, we obtain

$$f_y(0, 2) \approx \frac{0-1}{\frac{-19}{8}-(-2)} = \frac{8}{3}.$$

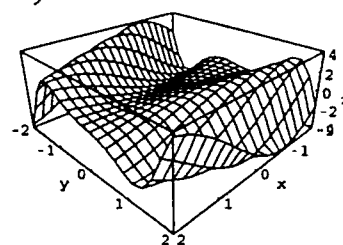
48. a. $\sin(x + y^2)$



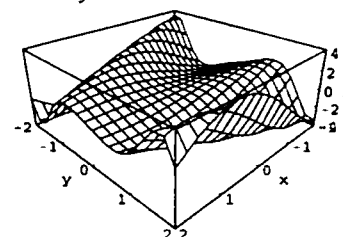
- b. $D_x(\sin(x + y^2))$



- c. $D_y(\sin(x + y^2))$



- d. $D_x(D_y(\sin(x + y^2)))$



49. a. $f_y(x, y, z)$

$$= \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

b. $f_z(x, y, z)$

$$= \lim_{\Delta z \rightarrow 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

c. $G_x(w, x, y, z)$

$$= \lim_{\Delta x \rightarrow 0} \frac{G(w, x + \Delta x, y, z) - G(w, x, y, z)}{\Delta x}$$

d. $\frac{\partial}{\partial z} \lambda(x, y, z, t)$

$$= \lim_{\Delta z \rightarrow 0} \frac{\lambda(x, y, z + \Delta z, t) - \lambda(x, y, z, t)}{\Delta z}$$

e. $\frac{\partial}{\partial b_2} S(b_0, b_1, b_2, \dots, b_n) =$

$$= \lim_{\Delta b_2 \rightarrow 0} \left[\frac{S(b_0, b_1, b_2 + \Delta b_2, \dots, b_n) - S(b_0, b_1, b_2, \dots, b_n)}{\Delta b_2} \right]$$

50. a. $\frac{\partial}{\partial w} (\sin w \sin x \cos y \cos z)$

$$= \cos w \sin x \cos y \cos z$$

b. $\frac{\partial}{\partial x} [x \ln(wxyz)] = x \cdot \frac{wyz}{wxyz} + 1 \cdot \ln(wxyz)$

$$= 1 + \ln(wxyz)$$

c. $\lambda_t(x, y, z, t)$

$$= \frac{(1 + xyz t) \cos x - t (\cos x) xyz}{(1 + xyz t)^2}$$

$$= \frac{\cos x}{(1 + xyz t)^2}$$

15.3 Concepts Review

- 3; (x, y) approaches $(1, 2)$.
- $\lim_{(x, y) \rightarrow (1, 2)} f(x, y) = f(1, 2)$
- contained in S
- an interior point of S ; boundary points

Problem Set 15.3

- 18
- 3
- $\lim_{(x, y) \rightarrow (2, \pi)} \left[x \cos^2 xy - \sin\left(\frac{xy}{3}\right) \right]$

$$= 2 \cos^2 2\pi - \sin\left(\frac{2\pi}{3}\right) = 2 - \frac{\sqrt{3}}{2} \approx 1.1340$$
- $-\frac{5}{2}$
- $\frac{1}{3}$

6. $\lim_{(x, y) \rightarrow (0, 0)} \frac{\tan(x^2 + y^2)}{(x^2 + y^2)}$

$$= \lim_{(x, y) \rightarrow (0, 0)} \frac{\sin(x^2 + y^2)}{(x^2 + y^2)} \cdot \frac{1}{\cos(x^2 + y^2)}$$

$$= (1)(1) = 1$$

7. The limit does not exist since the function is not defined anywhere along the line $y = x$. That is, there is no neighborhood of the origin in which the function is defined everywhere except possibly at the origin.

8. $\lim_{(x, y) \rightarrow (0, 0)} \frac{(x^2 + y^2)(x^2 - y^2)}{x^2 + y^2}$

$$= \lim_{(x, y) \rightarrow (0, 0)} (x^2 - y^2) = 0$$

9. The entire plane since $x^2 + y^2 + 1$ is never zero.
10. Require $1 - x^2 - y^2 > 0$; $x^2 + y^2 < 1$. S is the interior of the unit circle centered at the origin.
11. Require $y - x^2 \neq 0$. S is the entire plane except the parabola $y = x^2$.

12. The only points at which f might be discontinuous occur when $xy = 0$.

$$\lim_{(x,y) \rightarrow (a,0)} \frac{\sin(xy)}{xy} = 1 = f(a, 0) \text{ for all nonzero } a \text{ in } \mathbb{R}, \text{ and then } \lim_{(x,y) \rightarrow (0,b)} \frac{\sin(xy)}{xy} = 1 = f(0, b)$$

$$\text{for all } b \text{ in } \mathbb{R}. \text{ Therefore, } f \text{ is continuous on the entire plane.}$$

13. Require $x - y + 1 \geq 0$; $y \leq x + 1$. S is the region below and on the line $y = x + 1$.

14. Require $4 - x^2 - y^2 > 0$; $x^2 + y^2 < 4$. S is the interior of the circle of radius 2 centered at the origin.

15. Along x -axis ($y = 0$): $\lim_{(x,y) \rightarrow (0,0)} \frac{0}{x^2 + 0} = 0$.

Along $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{2x^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{1}{2} = \frac{1}{2}.$$

Hence, the limit does not exist because for some points near the origin $f(x, y)$ is getting closer to 0, but for others it is getting closer to $\frac{1}{2}$.

16. Along $y = 0$: $\lim_{x \rightarrow 0} \frac{0}{x^2 + 0} = 0$. Along $y = x$:

$$\lim_{x \rightarrow 0} \frac{x^2 + x^3}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{1 + x}{2} = \frac{1}{2}.$$

17. a. $\lim_{x \rightarrow 0} \frac{x^2(mx)}{x^4 + (mx)^2} = \lim_{x \rightarrow 0} \frac{mx^3}{x^4 + m^2x^2} = \lim_{x \rightarrow 0} \frac{mx}{x^2 + m^2} = 0$

b. $\lim_{x \rightarrow 0} \frac{x^2(x^2)}{x^4 + (x^2)^2} = \lim_{x \rightarrow 0} \frac{x^4}{2x^4} = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}$

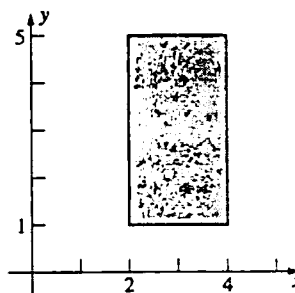
c. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4 + y} \text{ does not exist.}$

18. $\left| \frac{xy^2}{x^2 + y^2} \right| \leq \sqrt{x^2 + y^2} < \epsilon$ in some δ -neighborhood of $(0, 0)$ since

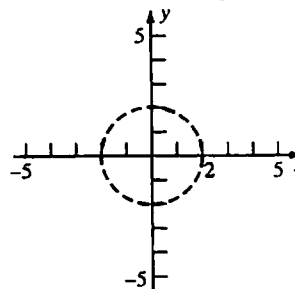
$$\lim_{(x,y) \rightarrow (0,0)} \sqrt{x^2 + y^2} = 0. \text{ Therefore,}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} = 0.$$

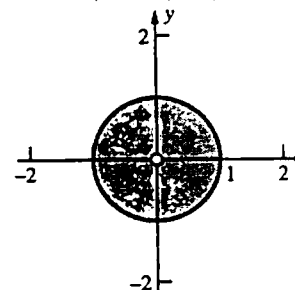
19. The boundary consists of the points that form the outer edge of the rectangle. The set is closed.



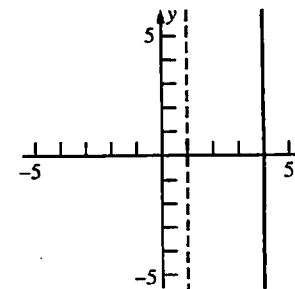
20. The boundary consists of the points of the circle shown. The set is open.



21. The boundary consists of the circle and the origin. The set is neither open (since, for example, $(1, 0)$ is not an interior point), nor closed (since $(0, 0)$ is not in the set).

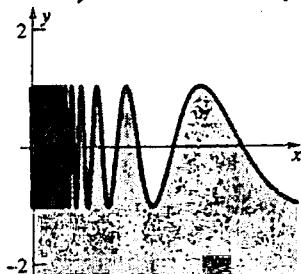


22. The boundary consists of the points on the line $x = 1$ along with the points on the line $x = 4$. The set is neither closed nor open.

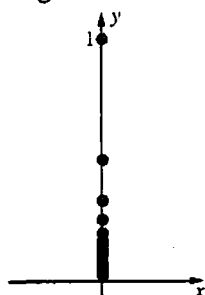


23. The boundary consists of the graph of $y = \sin\left(\frac{1}{x}\right)$ along with the part of the y -axis for

which $y \leq 1$. The set is open.



24. The boundary is the set itself along with the origin. The set is neither open (since none of its points are interior points) nor closed (since the origin is not in the set).



25.
$$\frac{x^2 - 4y^2}{x - 2y} = \frac{(x + 2y)(x - 2y)}{x - 2y} = x + 2y \text{ (if } x \neq 2y\text{)}$$

We want

$$g(x) = x + 2\left(\frac{x}{2}\right) \left(\text{if } x = 2y, \text{ or } y = \frac{x}{2} \right) = 2x$$

26. Let L and M be the latter two limits.

$$|f(x, y) + g(x, y) - [L + M]|$$

$$\leq |f(x, y) - L| + |f(x, y) - M| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

for (x, y) in some δ -neighborhood of $(a, b) = \epsilon$. Therefore,

$$\lim_{(x, y) \rightarrow (a, b)} [f(x, y) + g(x, y)] = L + M.$$

27. Note: $(x, y) \rightarrow (0, 0)$ is equivalent to $r \rightarrow 0$.

a.
$$f(x, y) = \frac{(r \cos \theta)(r \sin \theta)}{\sqrt{r^2}} = |r| \sin \theta \cos \theta$$

$$= \frac{|r| \sin 2\theta}{2} \rightarrow 0 \text{ as } r \rightarrow 0.$$

b.
$$f(x, y) = \frac{(r \cos \theta)(r \sin \theta)}{r^2} = \frac{\sin 2\theta}{2}$$
 which does not approach 0 as $r \rightarrow 0$.

c.
$$f(x, y) = \frac{r^{7/3} \cos^{7/3} \theta}{r^2} = r \cos^{7/3} \theta \rightarrow 0 \text{ as } r \rightarrow 0.$$

d.
$$f(x, y) = (r \cos \theta)(r \sin \theta)(\cos 2\theta)$$
 (See introduction to this problem for third factor.)

$$= \frac{r^2 \sin 2\theta \cos 2\theta}{2} = \frac{r^2 \sin 4\theta}{4} \rightarrow 0 \text{ as } r \rightarrow 0.$$

e.
$$f(x, y) = \frac{r^4 \cos^2 \theta \sin^2 \theta}{r^2 \cos^2 \theta + r^4 \sin^4 \theta}$$

$$= r^2 \left(\frac{\cos^2 \theta \sin^2 \theta}{\cos^2 \theta + r^2 \sin^4 \theta} \right)$$

$$= r^2 \left(\frac{\sin^2 \theta}{1 + r^2 \sin^2 \theta \tan^2 \theta} \right) \text{ if } \theta \neq \pm \frac{\pi}{2}.$$
 This

converges to 0 as $r \rightarrow 0$ since the fraction is bounded (the numerator is in $[0, 1]$ and the denominator is greater than or equal to 1). If

$$\theta = \pm \frac{\pi}{2}, f(x, y) = 0.$$

- f. This one is not easier in polar coordinates. Here is a Cartesian coordinates solution.

Along curve $x = y^2$:

$$\frac{xy^2}{x^2 + y^4} = \frac{(y^2)y^2}{(y^2)^2 + y^4} = \frac{1}{2} \text{ which does not approach 0.}$$

Conclusion: The functions of parts a, c, d, and e are continuous at the origin. Those of parts b and f are discontinuous at the origin.

28. f is discontinuous at each overhang. More interesting, f is discontinuous along the Continental Divide.

29. a. $\{(x, y, z) : x^2 + y^2 = 1, z \in [1, 2]\}$ [For $x^2 + y^2 < 1$, the particle hits the hemisphere and then slides to the origin (or bounds toward the origin); for $x^2 + y^2 = 1$, it bounces up; for $x^2 + y^2 > 1$, it falls straight down.]

- b. $\{(x, y, z) : x^2 + y^2 = 1, z = 1\}$ (As one moves at a level of $z = 1$ from the rim of the bowl toward any position away from the bowl there is a change from seeing all of the interior of the bowl to seeing none of it.)

- c. $\{(x, y, z) : z = 1\}$ [$f(x, y, z)$ is undefined (infinite) at $(x, y, 1)$.]

- d. ϕ (Small changes in points of the domain result in small changes in the shortest path from the points to the origin.)

30. f is continuous on an open set D and P_0 is in D implies that there is neighborhood of P_0 with radius r on which f is continuous. f is continuous at $P_0 \Rightarrow \lim_{P \rightarrow P_0} f(P) = f(P_0)$.

Now let $\varepsilon = f(P_0)$ which is positive. Then there

is a δ such that $0 < \delta < r$ and $|f(P) - f(P_0)| < f(P_0)$ if P is in the δ -neighborhood of P_0 . Therefore, $-f(P_0) < f(P) - f(P_0) < f(P_0)$, so $0 < f(P)$ (using the left-hand inequality) in that δ -neighborhood of P_0 .

31. a. $f(x, y) = \begin{cases} (x^2 + y^2)^{1/2} + 1 & \text{if } y \neq 0 \\ |x - 1| & \text{if } y = 0 \end{cases}$. Check discontinuities where $y = 0$.

As $y = 0$, $(x^2 + y^2)^{1/2} + 1 = |x| + 1$, so f is continuous if $|x| + 1 = |x - 1|$. Squaring each side and simplifying yields $|x| = -x$, so f is continuous for $x \leq 0$. That is, f is discontinuous along the positive x -axis.

- b. Let $P = (u, v)$ and $Q = (x, y)$.

$$f(u, v, x, y) = \begin{cases} |OP| + |OQ| & \text{if } P \text{ and } Q \text{ are not on same ray from the origin and neither is the origin} \\ |PQ| & \text{otherwise} \end{cases}$$

This means that in the first case one travels from P to the origin and then to Q ; in the second case one travels directly from P to Q without passing through the origin, so f is discontinuous on the set

$$\{(u, v, x, y) : \langle u, v \rangle = k \langle x, y \rangle \text{ for some } k > 0, \langle u, v \rangle \neq 0, \langle x, y \rangle \neq 0\}.$$

32. a. $f_x(0, y) = \lim_{h \rightarrow 0} \left(\frac{\frac{hy(h^2 - y^2)}{h^2 + y^2} - 0}{h} \right) = \lim_{h \rightarrow 0} \frac{y(h^2 - y^2)}{h^2 + y^2} = -y$

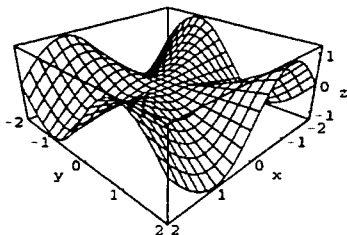
b. $f_y(x, 0) = \lim_{h \rightarrow 0} \left(\frac{\frac{xh(x^2 - h^2)}{x^2 + h^2} - 0}{h} \right) = \lim_{h \rightarrow 0} \frac{y(x^2 - h^2)}{x^2 + y^2} = x$

c. $f_{yx}(0, 0) = \lim_{h \rightarrow 0} \frac{f_y(0 + h, y) - f_y(0, y)}{h} = \lim_{h \rightarrow 0} \frac{h - 0}{h} = 1$

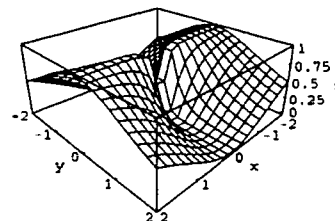
d. $f_{xy}(0, 0) = \lim_{h \rightarrow 0} \frac{f_x(x, 0 + h) - f_x(x, 0)}{h} = \lim_{h \rightarrow 0} \frac{-h - 0}{h} = -1$

Therefore, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

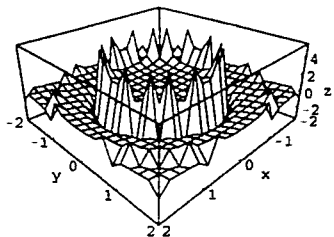
33.



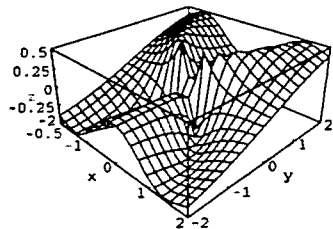
34. a.



b.



35.



36. A function f of three variables is continuous at a point (a, b, c) if $f(a, b, c)$ is defined and equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (a, b, c) . In other words,

$$\lim_{(x,y,z) \rightarrow (a,b,c)} f(x, y, z) = f(a, b, c).$$

A function of three variables is continuous on an open set S if it is continuous at every point in the

interior of the set. The function is continuous at a boundary point P of S if $f(Q)$ approaches $f(P)$ as Q approaches P along any path through points in S in the neighborhood of P .

37. If we approach the point $(0, 0, 0)$ along a straight path from the point (x, x, x) , we have

$$\lim_{(x,x,x) \rightarrow (0,0,0)} \frac{x(x)(x)}{x^3 + x^3 + x^3} = \lim_{(x,x,x) \rightarrow (0,0,0)} \frac{x^3}{3x^3} = \frac{1}{3}$$

Since the limit does not equal to $f(0, 0, 0)$, the function is not continuous at the point $(0, 0, 0)$.

38. If we approach the point $(0, 0, 0)$ along the x -axis, we get

$$\lim_{(x,0,0) \rightarrow (0,0,0)} (0+1) \frac{(x^2 - 0^2)}{(x^2 + 0^2)} = \lim_{(x,0,0) \rightarrow (0,0,0)} \frac{x^2}{x^2} = 1$$

Since the limit does not equal $f(0, 0, 0)$, the function is not continuous at the point $(0, 0, 0)$.

15.4 Concepts Review

1. gradient
2. locally linear
3. $\frac{\partial f}{\partial x}(\mathbf{p})\mathbf{i} + \frac{\partial f}{\partial y}(\mathbf{p})\mathbf{j}; y^2\mathbf{i} + 2xy\mathbf{j}$
4. tangent plane

Problem Set 15.4

1. $\langle 2xy + 3y, x^2 + 3x \rangle$
2. $\langle 3x^2y, x^3 - 3y^2 \rangle$
3. $\nabla f(x, y) = \langle (x)(e^{xy}y) + (e^{xy})(1), xe^{xy}x \rangle$
 $= e^{xy} \langle xy + 1, x^2 \rangle$
4. $\langle 2xy \cos y, x^2(\cos y - y \sin y) \rangle$
5. $x(x+y)^{-2} \langle y(x+2), x^2 \rangle$
6. $\nabla f(x, y) = \langle 3[\sin^2(x^2y)][\cos(x^2y)](2xy), 3[\sin^2(x^2y)][\cos(x^2y)](x^2) \rangle = 3x \sin^2(x^2y) \cos(x^2y) \langle 2y, x \rangle$
7. $(x^2 + y^2 + z^2)^{-1/2} \langle x, y, z \rangle$
8. $\langle 2xy + z^2, x^2 + 2yz, y^2 + 2xz \rangle$

$$9. \nabla f(x, y) = \left\langle (x^2 y)(e^{x-z}) + (e^{x-z})(2xy), x^2 e^{x-z}, x^2 y e^{x-z}(-1) \right\rangle = x e^{x-z} \langle y(x+2), x, -xy \rangle$$

$$10. \left\langle xz(x+y+z)^{-1} + z \ln(x+y+z), xz(x+y+z)^{-1}, xz(x+y+z)^{-1} + x \ln(x+y+z) \right\rangle$$

$$11. \nabla f(x, y) = \langle 2xy - y^2, x^2 - 2xy \rangle; \nabla f(-2, 3) = \langle -21, 16 \rangle$$

$$z = f(-2, 3) + \langle -21, 16 \rangle \cdot \langle x+2, y-3 \rangle = 30 + (-21x - 42 + 16y - 48)$$

$$z = -21x + 16y - 60$$

$$12. \nabla f(x, y) = \langle 3x^2 y + 3y^2, x^3 + 6xy \rangle, \text{ so } \nabla f(2, -2) = \langle -12, -16 \rangle.$$

Tangent plane:

$$z = f(2, -2) + \nabla f(2, -2) \cdot \langle x-2, y+2 \rangle = 8 + \langle -12, -16 \rangle \cdot \langle x-2, y+2 \rangle = 8 + (-12x + 24 - 16y - 32)$$

$$z = -12x - 16y$$

$$13. \nabla f(x, y) = \langle -\pi \sin(\pi x) \sin(\pi y), \pi \cos(\pi x) \cos(\pi y) + 2\pi \cos(2\pi y) \rangle$$

$$\nabla f\left(-1, \frac{1}{2}\right) = \langle 0, -2\pi \rangle$$

$$z = f\left(-1, \frac{1}{2}\right) + \langle 0, -2\pi \rangle \cdot \left\langle x+1, y-\frac{1}{2} \right\rangle = -1 + (0 - 2\pi y + \pi);$$

$$z = -2\pi y + (\pi - 1)$$

$$14. \nabla f(x, y) = \left\langle \frac{2x}{y}, -\frac{x^2}{y^2} \right\rangle; \nabla f(2, -1) = \langle -4, -4 \rangle$$

$$z = f(2, -1) + \langle -4, -4 \rangle \cdot \langle x-2, y+1 \rangle$$

$$= -4 + (-4x + 8 - 4y - 4)$$

$$z = -4x - 4y$$

$$15. \nabla f(x, y, z) = \langle 6x + z^2, -4y, 2xz \rangle, \text{ so } \nabla f(1, 2, -1) = \langle 7, -8, -2 \rangle$$

Tangent hyperplane:

$$w = f(1, 2, -1) + \nabla f(1, 2, -1) \cdot \langle x-1, y-2, z+1 \rangle = -4 + \langle 7, -8, -2 \rangle \cdot \langle x-1, y-2, z+1 \rangle$$

$$= -4 + (7x - 7 - 8y + 16 - 2z - 2)$$

$$w = 7x - 8y - 2z + 3$$

$$16. \nabla f(x, y, z) = \langle yz + 2x, xz, xy \rangle; \nabla f(2, 0, -3) = \langle 4, -6, 0 \rangle$$

$$w = f(2, 0, -3) + \langle 4, -6, 0 \rangle \cdot \langle x-2, y, z+3 \rangle = 4 + (4x - 8 - 6y + 0)$$

$$w = 4x - 6y - 4$$

$$17. \nabla \left(\frac{f}{g} \right) = \frac{\langle gf_x - fg_x, gf_y - fg_y, gf_z - fg_z \rangle}{g^2} = \frac{g \langle f_x, f_y, f_z \rangle - f \langle g_x, g_y, g_z \rangle}{g^2} = \frac{g \nabla f - f \nabla g}{g^2}$$

$$18. \nabla(f^r) = \langle rf^{r-1} f_x, rf^{r-1} f_y, rf^{r-1} f_z \rangle = rf^{r-1} \langle f_x, f_y, f_z \rangle = rf^{r-1} \nabla f$$

$$19. \text{ Let } F(x, y, z) = x^2 - 6x + 2y^2 - 10y + 2xy - z = 0$$

$$\nabla F(x, y, z) = \langle 2x - 6 + 2y, 4y - 10 + 2x, -1 \rangle$$

The tangent plane will be horizontal if

$$\nabla F(x, y, z) = \langle 0, 0, k \rangle, \text{ where } k \neq 0. \text{ Therefore,}$$

we have the following system of equations:

$$2x + 2y - 6 = 0$$

$$2x + 4y - 10 = 0$$

Solving this system yields $x = 1$ and $y = 2$. Thus,

there is a horizontal tangent plane at $(x, y) = (1, 2)$.

20. Let $F(x, y, z) = x^3 - z = 0$

$$\nabla F(x, y, z) = \langle 3x^2, 0, -1 \rangle$$

The tangent plane will be horizontal if

$$\nabla F(x, y, z) = \langle 0, 0, k \rangle, \text{ where } k \neq 0. \text{ Therefore,}$$

we need only solve the equation $3x^2 = 0$. There is a horizontal tangent plane at $(x, y) = (0, y)$.

(Note: there are infinitely many points since y can take on any value).

21. a. The point $(2, 1, 9)$ projects to $(2, 1, 0)$ on the xy plane. The equation of a plane containing this point and parallel to the x -axis is given by $y = 1$. The tangent plane to the surface at the point $(2, 1, 9)$ is given by

$$\begin{aligned} z &= f(2, 1) + \nabla f(2, 1) \cdot \langle x - 2, y - 1 \rangle \\ &= 9 + \langle 12, 10 \rangle \langle x - 2, y - 1 \rangle \\ &= 12x + 10y - 25 \end{aligned}$$

The line of intersection of the two planes is the tangent line to the surface, passing through the point $(2, 1, 9)$, whose projection in the xy plane is parallel to the x -axis. This line of intersection is parallel to the cross product of the normal vectors for the planes. The normal vectors are

$$\langle 12, 10, -1 \rangle \text{ and } \langle 0, 1, 0 \rangle \text{ for the tangent plane and vertical plane respectively. The cross product is given by}$$

$$\langle 12, 10, -1 \rangle \times \langle 0, 1, 0 \rangle = \langle 1, 0, 12 \rangle$$

Thus, parametric equations for the desired tangent line are

$$x = 2 + t$$

$$y = 1$$

$$z = 9 + 12t$$

- b. Using the equation for the tangent plane from the previous part, we now want the vertical plane to be parallel to the y -axis, but still pass through the projected point $(2, 1, 0)$. The vertical plane now has equation $x = 2$. The normal equations are given by $\langle 12, 10, -1 \rangle$ and $\langle 1, 0, 0 \rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:

$$\langle 12, 10, -1 \rangle \times \langle 1, 0, 0 \rangle = \langle 0, 10, 10 \rangle$$

Thus, parametric equations for the desired

tangent line are

$$x = 2$$

$$y = 1 + 10t$$

$$z = 9 + 10t$$

- c. Using the equation for the tangent plane from the first part, we now want the vertical plane to be parallel to the line $y = x$, but still pass through the projected point $(2, 1, 0)$. The vertical plane now has equation $y - x + 1 = 0$. The normal equations are given by $\langle 12, 10, -1 \rangle$ and $\langle 1, -1, 0 \rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:

$$\langle 12, 10, -1 \rangle \times \langle 1, -1, 0 \rangle = \langle -1, -1, -22 \rangle$$

Thus, parametric equations for the desired tangent line are

$$x = 2 - t$$

$$y = 1 - t$$

$$z = 9 - 22t$$

22. a. The point $(3, 2, 72)$ on the surface is the point $(3, 2, 0)$ when projected into the xy plane. The equation of a plane containing this point and parallel to the x -axis is given by $y = 2$. The tangent plane to the surface at the point $(3, 2, 72)$ is given by

$$\begin{aligned} z &= f(3, 2) + \nabla f(3, 2) \cdot \langle x - 3, y - 2 \rangle \\ &= 72 + \langle 48, 108 \rangle \langle x - 3, y - 2 \rangle \\ &= 48x + 108y - 288 \end{aligned}$$

The line of intersection of the two planes is the tangent line to the surface, passing through the point $(3, 2, 72)$, whose

projection in the xy plane is parallel to the x -axis. This line of intersection is parallel to the cross product of the normal vectors for the planes. The normal vectors are

$$\langle 48, 108, -1 \rangle \text{ and } \langle 0, 2, 0 \rangle \text{ for the tangent plane and vertical plane respectively. The cross product is given by}$$

$$\langle 48, 108, -1 \rangle \times \langle 0, 2, 0 \rangle = \langle 2, 0, 96 \rangle$$

Thus, parametric equations for the desired tangent line are

$$x = 3 + 2t$$

$$y = 2$$

$$z = 72 + 96t$$

- b. Using the equation for the tangent plane from the previous part, we now want the vertical plane to be parallel to the y -axis, but still pass through the projected point

$(3, 2, 72)$. The vertical plane now has equation $x = 3$. The normal equations are given by $\langle 48, 108, -1 \rangle$ and $\langle 3, 0, 0 \rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:

$$\langle 48, 108, -1 \rangle \times \langle 3, 0, 0 \rangle = \langle 0, -3, -324 \rangle$$

Thus, parametric equations for the desired tangent line are

$$x = 3$$

$$y = 2 - 3t$$

$$z = 72 - 324t$$

- c. Using the equation for the tangent plane from the first part, we now want the vertical

plane to be parallel to the line $x = -y$, but still pass through the projected point $(3, 2, 72)$. The vertical plane now has equation $y + x - 5 = 0$. The normal

equations are given by $\langle 48, 108, -1 \rangle$ and $\langle 1, 1, 0 \rangle$ for the tangent and vertical planes respectively. Again we find the cross

product of the normal vectors:

$$\langle 48, 108, -1 \rangle \times \langle 1, 1, 0 \rangle = \langle 1, -1, -60 \rangle$$

Thus, parametric equations for the desired tangent line are

$$x = 3 + t$$

$$y = 2 - t$$

$$z = 72 - 60t$$

$$23. \nabla f(x, y) = \left\langle -10 \left(\frac{1}{2\sqrt{|xy|}} \frac{|xy|}{xy} y \right), -10 \left(\frac{1}{2\sqrt{|xy|}} \frac{|xy|}{xy} x \right) \right\rangle = \frac{-5xy}{|xy|^{3/2}} \langle y, x \rangle \quad \left[\text{Note that } \frac{|a|}{a} = \frac{a}{|a|} \right]$$

$$\nabla f(1, -1) = \langle -5, 5 \rangle$$

Tangent plane:

$$z = f(1, -1) + \nabla f(1, -1) \cdot \langle x - 1, y + 1 \rangle = -10 + \langle -5, 5 \rangle \cdot \langle x - 1, y + 1 \rangle = -10 + (-5x + 5 + 5y + 5)$$

$$z = -5x + 5y$$

24. Let $\mathbf{a}$ be any point of S and let $\mathbf{b}$ be any other point of S . Then for some c on the line segment between $\mathbf{a}$ and $\mathbf{b}$:

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(c) \cdot (\mathbf{b} - \mathbf{a}) = 0 \cdot (\mathbf{b} - \mathbf{a}) = 0, \text{ so}$$

$$f(\mathbf{b}) = f(\mathbf{a}) \text{ (for all } \mathbf{b} \text{ in } S).$$

$$25. \nabla f(\mathbf{p}) = \nabla g(\mathbf{p}) \Rightarrow \nabla[f(\mathbf{p}) - g(\mathbf{p})] = 0$$

$$\Rightarrow f(\mathbf{p}) - g(\mathbf{p}) \text{ is a constant.}$$

$$26. \nabla f(\mathbf{p}) = \mathbf{p} \Rightarrow \nabla f(x, y) = \langle x, y \rangle$$

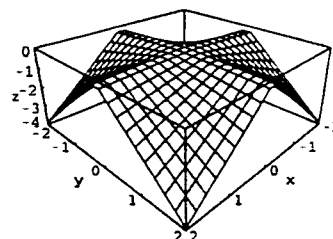
$$\Rightarrow f_x(x, y) = x, f_y(x, y) = y$$

$$\Rightarrow f(x, y) = \frac{1}{2}x^2 + \alpha(y) \text{ for any function of } y,$$

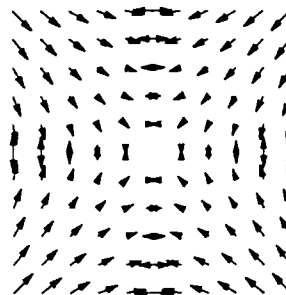
$$\text{and } f(x, y) = \frac{1}{2}y^2 + \beta(x) \text{ for any function of } x.$$

$$\Rightarrow f(x, y) = \frac{1}{2}(x^2 + y^2) + C \text{ for any } C \text{ in } \mathbb{R}.$$

27.

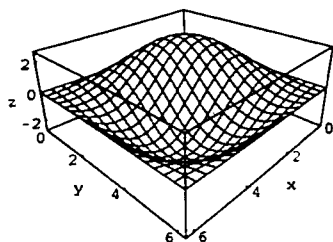


$$-|xy|$$

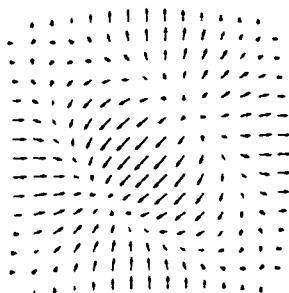


- a. The gradient points in the direction of greatest increase of the function.
- b. No. If it were, $|0 + h| - |0| = 0 + |h|\delta(h)$ where $\delta(h) \rightarrow 0$ as $h \rightarrow 0$, which is possible.

28.



$$\sin(x) + \sin(y) - \sin(x+y)$$



29. a. (i)

$$\begin{aligned}\nabla[f+g] &= \frac{\partial(f+g)}{\partial x} \mathbf{i} + \frac{\partial(f+g)}{\partial y} \mathbf{j} + \frac{\partial(f+g)}{\partial z} \mathbf{k} \\ &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} + \frac{\partial g}{\partial z} \mathbf{k} \\ &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} + \frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k} \\ &= \nabla f + \nabla g\end{aligned}$$

$$\begin{aligned}\text{(ii)} \quad \nabla[\alpha f] &= \frac{\partial[\alpha f]}{\partial x} \mathbf{i} + \frac{\partial[\alpha f]}{\partial y} \mathbf{j} + \frac{\partial[\alpha f]}{\partial z} \mathbf{k} \\ &= \alpha \frac{\partial f}{\partial x} \mathbf{i} + \alpha \frac{\partial f}{\partial y} \mathbf{j} + \alpha \frac{\partial f}{\partial z} \mathbf{k} \\ &= \alpha \nabla f\end{aligned}$$

(iii)

$$\begin{aligned}\nabla[fg] &= \frac{\partial(fg)}{\partial x} \mathbf{i} + \frac{\partial(fg)}{\partial y} \mathbf{j} + \frac{\partial(fg)}{\partial z} \mathbf{k} \\ &= \left(f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x} \right) \mathbf{i} + \left(f \frac{\partial g}{\partial y} + g \frac{\partial f}{\partial y} \right) \mathbf{j} \\ &\quad + \left(f \frac{\partial g}{\partial z} + g \frac{\partial f}{\partial z} \right) \mathbf{k} \\ &= f \left(\frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k} \right) \\ &\quad + g \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) \\ &= f \nabla g + g \nabla f\end{aligned}$$

b. (i)

$$\begin{aligned}\nabla[f+g] &= \frac{\partial(f+g)}{\partial x_1} \mathbf{i}_1 + \frac{\partial(f+g)}{\partial x_2} \mathbf{i}_2 \\ &\quad + \cdots + \frac{\partial(f+g)}{\partial x_n} \mathbf{i}_n \\ &= \frac{\partial f}{\partial x_1} \mathbf{i}_1 + \frac{\partial g}{\partial x_1} \mathbf{i}_1 + \frac{\partial f}{\partial x_2} \mathbf{i}_2 + \frac{\partial g}{\partial x_2} \mathbf{i}_2 \\ &\quad + \cdots + \frac{\partial f}{\partial x_n} \mathbf{i}_n + \frac{\partial g}{\partial x_n} \mathbf{i}_n \\ &= \frac{\partial f}{\partial x_1} \mathbf{i}_1 + \frac{\partial f}{\partial x_2} \mathbf{i}_2 + \cdots + \frac{\partial f}{\partial x_n} \mathbf{i}_n \\ &\quad + \frac{\partial g}{\partial x_1} \mathbf{i}_1 + \frac{\partial g}{\partial x_2} \mathbf{i}_2 + \cdots + \frac{\partial g}{\partial x_n} \mathbf{i}_n \\ &= \nabla f + \nabla g\end{aligned}$$

(ii)

$$\begin{aligned}\nabla[\alpha f] &= \frac{\partial[\alpha f]}{\partial x_1} \mathbf{i}_1 + \frac{\partial[\alpha f]}{\partial x_2} \mathbf{i}_2 + \cdots + \frac{\partial[\alpha f]}{\partial x_n} \mathbf{i}_n \\ &= \alpha \frac{\partial f}{\partial x_1} \mathbf{i}_1 + \alpha \frac{\partial f}{\partial x_2} \mathbf{i}_2 + \cdots + \alpha \frac{\partial f}{\partial x_n} \mathbf{i}_n \\ &= \alpha \nabla f\end{aligned}$$

(iii)

$$\begin{aligned}\nabla[fg] &= \frac{\partial(fg)}{\partial x_1} \mathbf{i}_1 + \frac{\partial(fg)}{\partial x_2} \mathbf{i}_2 + \cdots + \frac{\partial(fg)}{\partial x_n} \mathbf{i}_n \\ &= \left(f \frac{\partial g}{\partial x_1} + g \frac{\partial f}{\partial x_1} \right) \mathbf{i}_1 + \left(f \frac{\partial g}{\partial x_2} + g \frac{\partial f}{\partial x_2} \right) \mathbf{i}_2 \\ &\quad + \cdots + \left(f \frac{\partial g}{\partial x_n} + g \frac{\partial f}{\partial x_n} \right) \mathbf{i}_n \\ &= f \left(\frac{\partial g}{\partial x_1} \mathbf{i}_1 + \frac{\partial g}{\partial x_2} \mathbf{i}_2 + \cdots + \frac{\partial g}{\partial x_n} \mathbf{i}_n \right) \\ &\quad + g \left(\frac{\partial f}{\partial x_1} \mathbf{i}_1 + \frac{\partial f}{\partial x_2} \mathbf{i}_2 + \cdots + \frac{\partial f}{\partial x_n} \mathbf{i}_n \right) \\ &= f \nabla g + g \nabla f\end{aligned}$$

15.5 Concepts Review

1. $\frac{[f(\mathbf{p} + h\mathbf{u}) - f(\mathbf{p})]}{h}$
2. $u_1 f_x(x, y) + u_2 f_y(x, y)$
3. greatest increase
4. level curve

Problem Set 15.5

1. $D_{\mathbf{u}}f(x, y) = \langle 2xy, x^2 \rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$; $D_{\mathbf{u}}f(1, 2) = \frac{8}{5}$
2. $D_{\mathbf{u}}f(x, y) = \langle x^{-1}y^2, 2y \ln x \rangle \cdot \left[\left(\frac{1}{\sqrt{2}} \right) \langle 1, -1 \rangle \right]$;
 $D_{\mathbf{u}}f(1, 4) = 8\sqrt{2} \approx 11.3137$
3. $D_{\mathbf{u}}f(x, y) = f(x, y) \cdot \mathbf{u}$ (where $\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$)
 $= \langle 4x + y, x - 2y \rangle \cdot \frac{\langle 1, -1 \rangle}{\sqrt{2}}$;
 $D_{\mathbf{u}}f(3, -2) = \langle 10, 7 \rangle \cdot \frac{\langle 1, -1 \rangle}{\sqrt{2}} = \frac{3}{\sqrt{2}} \approx 2.1213$
4. $D_{\mathbf{u}}f(x, y)$
 $= \langle 2x - 3y, -3x + 4y \rangle \cdot \left[\left(\frac{1}{\sqrt{5}} \right) \langle 2, -1 \rangle \right]$;
 $D_{\mathbf{u}}f(-1, 2) = -\frac{27}{\sqrt{5}} \approx -12.0748$
5. $D_{\mathbf{u}}f(x, y) = e^x \langle \sin y, \cos y \rangle \cdot \left[\left(\frac{1}{2} \right) \langle 1, \sqrt{3} \rangle \right]$;
 $D_{\mathbf{u}}f\left(0, \frac{\pi}{4}\right) = \frac{(\sqrt{2} + \sqrt{6})}{4} \approx 0.9659$
6. $D_{\mathbf{u}}f(x, y) = \langle -ye^{-xy} - xe^{-xy}, \frac{\langle -1, \sqrt{3} \rangle}{2} \rangle$;
 $D_{\mathbf{u}}f(1, -1) = \langle e, -e \rangle \cdot \frac{\langle -1, \sqrt{3} \rangle}{2} = \frac{-e - e\sqrt{3}}{2}$
 ≈ -3.7132
7. $D_{\mathbf{u}}f(x, y, z) = \langle 3x^2y, x^3 - 2yz^2, -2y^2z \rangle \cdot \left[\left(\frac{1}{3} \right) \langle 1, -2, 2 \rangle \right]$;
 $D_{\mathbf{u}}f(-2, 1, 3) = \frac{52}{3}$
8. $D_{\mathbf{u}}f(x, y, z) = \langle 2x, 2y, 2z \rangle \cdot \left[\left(\frac{1}{2} \right) \langle \sqrt{2}, -1, -1 \rangle \right]$;
 $D_{\mathbf{u}}f(1, -1, 2) = \sqrt{2} - 1 \approx 0.4142$
9. f increases most rapidly in the direction of the gradient. $\nabla f(x, y) = \langle 3x^2, -5y^4 \rangle$;
 $\nabla f(2, -1) = \langle 12, -5 \rangle$
 $\frac{\langle 12, -5 \rangle}{13}$ is the unit vector in that direction. The rate of change of $f(x, y)$ in that direction at that point is the magnitude of the gradient.
 $|\langle 12, -5 \rangle| = 13$
10. $\nabla f(x, y) = \langle e^y \cos x, e^y \sin x \rangle$;
 $\nabla f\left(\frac{5\pi}{6}, 0\right) = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$, which is a unit vector.
The rate of change in that direction is 1.
11. $\nabla f(x, y, z) = \langle 2xyz, x^2z, x^2y \rangle$;
 $f(1, -1, 2) = \langle -4, 2, -1 \rangle$
A unit vector in that direction is $\left(\frac{1}{\sqrt{21}} \right) \langle -4, 2, -1 \rangle$. The rate of change in that direction is $\sqrt{21} \approx 4.5826$.
12. f increases most rapidly in the direction of the gradient. $\nabla f(x, y, z) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$;
 $\nabla f(2, 0, -4) = \langle 1, -8, 0 \rangle$
 $\frac{\langle 1, -8, 0 \rangle}{\sqrt{65}}$ is a unit vector in that direction.
 $|\langle 1, -8, 0 \rangle| = \sqrt{65} \approx 8.0623$ is the rate of change of $f(x, y, z)$ in that direction at that point.
13. $-\nabla f(x, y) = 2\langle x, y \rangle$; $-\nabla f(-1, 2) = 2\langle -1, 2 \rangle$ is the direction of most rapid decrease. A unit vector in that direction is $\mathbf{u} = \left(\frac{1}{\sqrt{5}} \right) \langle -1, 2 \rangle$.

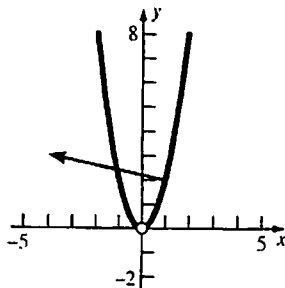
14. $-\nabla f(x, y) = \langle -3\cos(3x - y), \cos(3x - y) \rangle$;
 $-\nabla f\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \left\langle \frac{1}{\sqrt{2}} \right\rangle \langle -3, 1 \rangle$ is the direction of
 most rapid decrease. A unit vector in that
 direction is $\left\langle \frac{1}{\sqrt{10}} \right\rangle \langle -3, 1 \rangle$.

15. The level curves are $\frac{y}{x^2} = k$. For $p = (1, 2)$,
 $k = 2$, so the level curve through $(1, 2)$ is $\frac{y}{x^2} = 2$

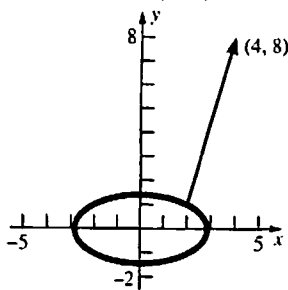
or $y = 2x^2$ ($x \neq 0$).

$$\nabla f(x, y) = \langle -2yx^{-3}, x^{-2} \rangle$$

$\nabla f(1, 2) = \langle -4, 1 \rangle$, which is perpendicular to the
 parabola at $(1, 2)$.



16. At $(2, 1)$, $x^2 + 4y^2 = 8$ is the level curve.
 $\nabla f(x, y) = \langle 2x, 8y \rangle$
 $\nabla f(2, 1) = 4\langle 1, 2 \rangle$, which is perpendicular to the
 level curve at $(2, 1)$.



17. $u = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$

$$D_u f(x, y, z) = \langle y, x, 2z \rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

$$D_u f(1, 1, 1) = \frac{2}{3}$$

18. $\left(0, \frac{\pi}{3}\right)$ is on the y -axis, so the unit vector toward
 the origin is $-\mathbf{j}$.

$$D_u(x, y) = \langle -e^{-x} \cos y, -e^{-x} \sin y \rangle \cdot \langle 0, -1 \rangle$$

$$= e^{-x} \sin y;$$

$$D_u\left(0, \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

19. a. Hottest if denominator is smallest; i.e., at the
 origin.

b. $\nabla T(x, y, z) = \frac{-200\langle 2x, 2y, 2z \rangle}{(5 + x^2 + y^2 + z^2)^2};$

$$\nabla T(1, -1, 1) = \left\langle -\frac{25}{4} \right\rangle \langle 1, -1, 1 \rangle$$

$\langle -1, 1, -1 \rangle$ is one vector in the direction of
 greatest increase.

c. Yes

20. $-\nabla V(x, y, z)$

$$= -100e^{-(x^2+y^2+z^2)} \langle -2x, -2y, -2z \rangle$$

$= 200e^{-(x^2+y^2+z^2)} \langle x, y, z \rangle$ is the direction of
 greatest decrease at (x, y, z) , and it points away
 from the origin.

21. $\nabla f(x, y, z) = \left\langle x(x^2 + y^2 + z^2)^{-\frac{1}{2}} \cos \sqrt{x^2 + y^2 + z^2}, \right.$

$$y(x^2 + y^2 + z^2)^{-\frac{1}{2}} \cos \sqrt{x^2 + y^2 + z^2},$$

$$\left. z(x^2 + y^2 + z^2)^{-\frac{1}{2}} \cos \sqrt{x^2 + y^2 + z^2} \right\rangle$$

$$= \left\langle (x^2 + y^2 + z^2)^{-\frac{1}{2}} \cos \sqrt{x^2 + y^2 + z^2} \right\rangle \langle x, y, z \rangle$$

which either points towards or away from the
 origin.

22. Let $D = \sqrt{x^2 + y^2 + z^2}$ be the distance. Then we
 have

$$\nabla T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle = \left\langle \frac{dT}{dD} \frac{\partial D}{\partial x}, \frac{dT}{dD} \frac{\partial D}{\partial y}, \frac{dT}{dD} \frac{\partial D}{\partial z} \right\rangle$$

$$= \left\langle \frac{dT}{dD} x(x^2 + y^2 + z^2)^{-\frac{1}{2}}, \frac{dT}{dD} y(x^2 + y^2 + z^2)^{-\frac{1}{2}}, \right.$$

$$\left. \frac{dT}{dD} z(x^2 + y^2 + z^2)^{-\frac{1}{2}} \right\rangle$$

$$= \left(\frac{dT}{dD} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \right) \langle x, y, z \rangle$$

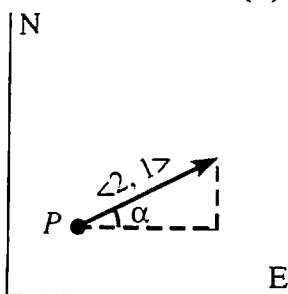
which either points towards or away from the origin.

23. He should move in the direction of

$$-\nabla f(\mathbf{p}) = -\langle f_x(\mathbf{p}), f_y(\mathbf{p}) \rangle = -\left\langle -\frac{1}{2}, -\frac{1}{4} \right\rangle$$

$$= \left\langle \frac{1}{4}, \frac{1}{2} \right\rangle. \text{ Or use } \langle 2, 1 \rangle. \text{ The angle } \alpha \text{ formed}$$

with the East is $\tan^{-1}\left(\frac{1}{2}\right) \approx 26.57^\circ$ (N63.43°E).



24. The unit vector from (2, 4) toward (5, 0) is

$$\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle. \text{ Then}$$

$$D_{\mathbf{u}}f(2, 4) = \langle -3, 8 \rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = -8.2.$$

25. The climber is moving in the direction of

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, 1 \right\rangle. \text{ Let}$$

$$f(x, y) = 3000e^{-(x^2+2y^2)/100}.$$

$$\nabla f(x, y) = 3000e^{-(x^2+2y^2)/100} \left\langle -\frac{x}{50}, -\frac{y}{25} \right\rangle;$$

$$f(10, 10) = -600e^{-3} \langle 1, 2 \rangle$$

She will move at a slope of

$$D_{\mathbf{u}}f(10, 10) = -600e^{-3} \langle 1, 2 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, 1 \right\rangle \langle -1, 1 \rangle$$

$$= (-300\sqrt{2})e^{-3} \approx -21.1229.$$

She will descend. Slope is about -21.

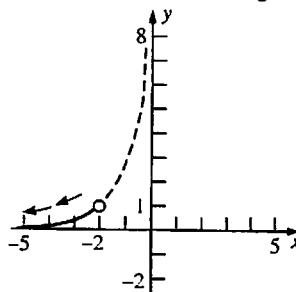
26. $\frac{dx}{2x} = \frac{dy}{-2y}; \frac{dx}{x} = \frac{dy}{-y}; \ln|x| = -\ln|y| + C$

$$\text{At } t = 0: \ln|-2| = -\ln|1| + C \Rightarrow C = \ln 2.$$

$$\ln|x| = -\ln|y| + \ln 2 = \ln \left| \frac{2}{y} \right|; |x| = \left| \frac{2}{y} \right|; |xy| = 2$$

Since the particle starts at (-2, 1) and neither x nor y can equal 0, the equation simplifies

to $xy = -2$. $\nabla T(-2, 1) = \langle -4, -2 \rangle$, so the particle moves downward along the curve.



27. $\nabla T(x, y) = \langle -4x, -2y \rangle$

$$\frac{dx}{dt} = -4x, \frac{dy}{dt} = -2y$$

$$\frac{\frac{dx}{dt}}{-4x} = \frac{\frac{dy}{dt}}{-2y} \text{ has solution } |x| = 2y^2. \text{ Since the}$$

particle starts at (-2, 1), this simplifies to $x = -2y^2$.

28. $f(1, -1) = 5 \langle -1, 1 \rangle$ (See write-up of Problem 23, Section 15.4.)

$$D_{\langle u_1, u_2 \rangle} f(1, -1) = \langle u_1, u_2 \rangle \cdot \langle -5, 5 \rangle = -5u_1 + 5u_2$$

- a. $\langle -1, 1 \rangle$ (in the direction of the gradient);

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, 1 \right\rangle.$$

- b. $\pm \langle 1, 1 \rangle$ (direction perpendicular to gradient);

$$\mathbf{u} = \left\langle \pm \frac{1}{\sqrt{2}}, 1 \right\rangle$$

- c. Want $D_{\mathbf{u}}f(1, -1) = 1$ where $|\mathbf{u}| = 1$. That is, want $-5u_1 + 5u_2 = 1$ and $u_1^2 + u_2^2 = 1$.

$$\text{Solutions are } \mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \text{ and } \left\langle -\frac{4}{5}, -\frac{3}{5} \right\rangle.$$

$$29. \text{ a. } \nabla T(x, y, z) = \left\langle -\frac{10(2x)}{(x^2 + y^2 + z^2)^2}, -\frac{10(2y)}{(x^2 + y^2 + z^2)^2}, -\frac{10(2z)}{(x^2 + y^2 + z^2)^2} \right\rangle$$

$$= -\frac{20}{(x^2 + y^2 + z^2)^2} \langle x, y, z \rangle$$

$r(t) = \langle t \cos \pi t, t \sin \pi t, t \rangle$, so $r(1) = \langle -1, 0, 1 \rangle$. Therefore, when $t = 1$, the bee is at $(-1, 0, 1)$, and $\nabla T(-1, 0, 1) = -5 \langle -1, 0, 1 \rangle$.

$r'(t) = \langle \cos \pi t - \pi t \sin \pi t, \sin \pi t + \pi t \cos \pi t, 1 \rangle$. so $r'(1) = \langle -1, -\pi, 1 \rangle$.

$U = \frac{r'(1)}{|r'(1)|} = \frac{\langle -1, -\pi, 1 \rangle}{\sqrt{2 + \pi^2}}$ is the unit tangent vector at $(-1, 0, 1)$.

$D_u T(-1, 0, 1) = u \cdot \nabla T(-1, 0, 1)$

$$= \frac{\langle -1, -\pi, 1 \rangle \cdot \langle 5, 0, -5 \rangle}{\sqrt{2 + \pi^2}} = -\frac{10}{\sqrt{2 + \pi^2}} \approx -2.9026$$

Therefore, the temperature is decreasing at about 2.9°C per meter traveled when the bee is at $(-1, 0, 1)$; i.e., when $t = 1$ s.

b. Method 1: (First express T in terms of t .)

$$T = \frac{10}{x^2 + y^2 + z^2} = \frac{10}{(t \cos \pi t)^2 + (t \sin \pi t)^2 + (t)^2} = \frac{10}{2t^2} = \frac{5}{t^2}$$

$$T(t) = 5t^{-2}; T'(t) = -10t^{-3}; t'(1) = -10$$

Method 2: (Use Chain Rule.)

$$D_t T(t) = \frac{dT}{ds} \frac{ds}{dt} = (D_u T)(|r'(t)|), \text{ so } D_t T(t) = [D_u T(-1, 0, 1)](|r'(1)|) = -\frac{10}{\sqrt{2 + \pi^2}} (\sqrt{2 + \pi^2}) = -10$$

Therefore, the temperature is decreasing at about 10°C per second when the bee is at $(-1, 0, 1)$; i.e., when $t = 1$ s.

$$30. \text{ a. } D_u f = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle \cdot \langle f_x, f_y \rangle = -6, \text{ so}$$

$$3f_x - 4f_y = -30.$$

$$D_v f = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle \cdot \langle f_x, f_y \rangle = 17, \text{ so}$$

$$4f_x + 3f_y = 85.$$

The simultaneous solution is

$$f_x = 10, f_y = 15, \text{ so } \nabla f = \langle 10, 15 \rangle.$$

In each case $\cos \phi = \sin \theta$ or $\cos \phi = -\sin \theta$,
so $\cos^2 \phi = \sin^2 \theta$.

Thus,

$$\begin{aligned} (D_u f)^2 + (D_v f)^2 &= (u \cdot \nabla f)^2 + (v \cdot \nabla f)^2 \\ &= |\nabla f|^2 \cos^2 \theta + |\nabla f|^2 \cos^2 \phi \\ &= |\nabla f|^2 (\cos^2 \theta + \cos^2 \phi) \\ &= |\nabla f|^2 (\cos^2 \theta + \sin^2 \theta) = |\nabla f|^2. \end{aligned}$$

b. Without loss of generality, let

$u = i$ and $v = j$. If θ and ϕ are the angles between u and ∇f , and between v and ∇f , then:

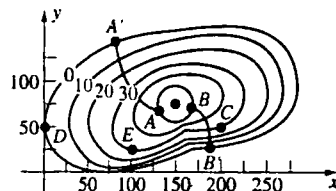
$$1. \theta + \phi = \frac{\pi}{2} \text{ (if } \nabla f \text{ is in the 1st quadrant).}$$

$$2. \theta = \frac{\pi}{2} + \phi \text{ (if } \nabla f \text{ is in the 2nd quadrant).}$$

$$3. \phi + \theta = \frac{3\pi}{2} \text{ (if } \nabla f \text{ is in the 3rd quadrant).}$$

$$4. \phi = \frac{\pi}{2} + \theta \text{ (if } \nabla f \text{ is in the 4th quadrant).}$$

31.



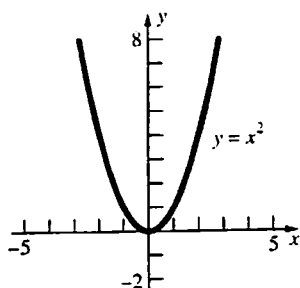
a. $A'(100, 120)$

b. $B'(190, 25)$

$$\begin{aligned} \text{c. } f_x(C) &\approx \frac{20-30}{230-200} = -\frac{1}{3}; f_y(D) = 0; \\ D_u f(E) &\approx \frac{40-30}{25} = \frac{2}{5} \end{aligned}$$

32. Graph of domain of f

$$f(x, y) = \begin{cases} 0, & \text{in shaded region} \\ 1, & \text{elsewhere} \end{cases}$$



$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist since

$(x, y) \rightarrow (0, 0)$:

along the y -axis, $f(x, y) = 0$, but along the $y = x^4$ curve, $f(x, y) = 1$.

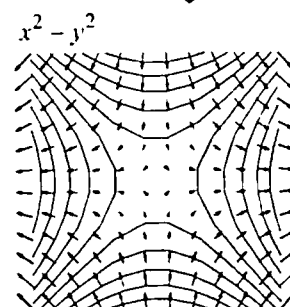
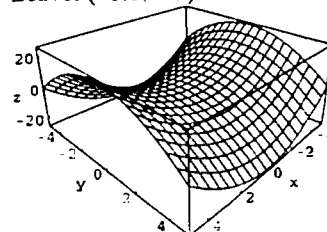
Therefore, f is not differentiable at the origin. But $D_u f(0, 0)$ exists for all u since

$$\begin{aligned} f_x(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0-0}{h} \\ &= \lim_{h \rightarrow 0} (0) = 0, \text{ and} \end{aligned}$$

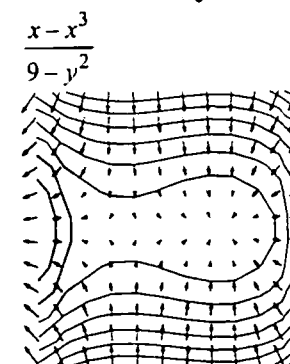
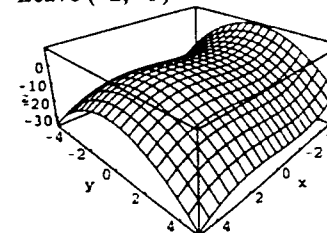
$$\begin{aligned} f_y(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0, 0+h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0-0}{h} \\ &= \lim_{h \rightarrow 0} (0) = 0, \text{ so } \nabla f(0, 0) = \langle 0, 0 \rangle = \mathbf{0}. \text{ Then} \end{aligned}$$

$$D_u f(0, 0) = \nabla f(0, 0) \cdot \mathbf{u} = \mathbf{0} \cdot \mathbf{u} = 0.$$

33. Leave: $(-0.1, -5)$



34. Leave $(-2, -5)$



35. Leave: $(3, 5)$

36. $(4.2, 4.2)$

15.6 Concepts Review

1. $\frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$

2. $y^2 \cos t + 2xy(-\sin t)$
 $= \cos^3 t - 2 \sin^2 t \cos t$

3. $\frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$

4. 12

Problem Set 15.6

- $$\frac{dw}{dt} = (2xy^3)(3t^2) + (3x^3y^2)(2t)$$

$$= (2t^9)(3t^2) + (3t^{10})(2t) = 12t^{11}$$
- $$\frac{dw}{dt} = (2xy - y^2)(-\sin t) + (x^2 - 2xy)(\cos t)$$

$$= (\sin t + \cos t)(1 - 3\sin t \cos t)$$
- $$\frac{dw}{dt} = (e^x \sin y + e^y \cos x)(3) + (e^x \cos y + e^y \sin x)(2) = 3e^{3t} \sin 2t + 3e^{2t} \cos 3t + 2e^{3t} \cos 2t + 2e^{2t} \sin 3t$$
- $$\frac{dw}{dt} = \left(\frac{1}{x}\right) \sec^2 t + \left(-\frac{1}{y}\right) (2 \sec^2 t \tan t) = \frac{\sec^2 t}{\tan t} - 2 \tan t = \frac{\sec^2 t - 2 \tan^2 t}{\tan t} = \frac{1 - \tan^2 t}{\tan t}$$
- $$\frac{dw}{dt} = [yz^2(\cos(xyz^2))](3t^2) + [xz^2 \cos(xyz^2)](2t) + [2xyz \cos(xyz^2)](1)$$

$$= (3yz^2t^2 + 2xz^2t + 2xyz) \cos(xyz^2) = (3t^6 + 2t^6 + 2t^6) \cos(t^7) = 7t^6 \cos(t^7)$$
- $$\frac{dw}{dt} = (y+z)(2t) + (x+z)(-2t) + (y+x)(-1) = 2t(2-t-t^2) - 2t(1-t+t^2) - (1) = -4t^3 + 2t - 1$$
- $$\frac{\partial w}{\partial t} = (2xy)(s) + (x^2)(-1) = 2st(s-t)s - s^2t^2 = s^2t(2s-3t)$$
- $$\frac{\partial w}{\partial t} = (2x - x^{-1}y)(-st^{-2}) + (-\ln x)(s^2) = s^2 \left[1 - 2t^{-3} - \ln\left(\frac{s}{t}\right) \right]$$
- $$\frac{\partial w}{\partial t} = e^{x^2+y^2} (2x)(s \cos t) + e^{x^2+y^2} (2y)(\sin s) = 2e^{x^2+y^2} (xs \cos t + y \sin s)$$

$$= 2(s^2 \sin t \cos t + t \sin^2 s) \exp(s^2 \sin^2 t + t^2 \sin^2 s)$$
- $$\frac{\partial w}{\partial t} = [(x+y)^{-1} - (x-y)^{-1}](e^s) + [(x+y)^{-1} + (x-y)^{-1}](se^{st}) = \frac{2e^{s(t+1)}(st-1)}{t^2 e^{2s} - e^{2st}}$$
- $$\frac{\partial w}{\partial t} = \frac{x(-s \sin st)}{(x^2 + y^2 + z^2)^{1/2}} + \frac{y(s \cos st)}{(x^2 + y^2 + z^2)^{1/2}} + \frac{z(s^2)}{(x^2 + y^2 + z^2)^{1/2}} = s^4 t (1 + s^4 t^2)^{-1/2}$$
- $$\frac{\partial w}{\partial t} = (e^{xy+z}y)(1) + (e^{xy+z}x)(-1) + (e^{xy+z})(2t) = e^{xy+z}(y-x+2t) = e^{s^2}(0) = 0$$
- $$\frac{\partial z}{\partial t} = (2xy)(2) + (x^2)(-2st) = 4(2t+s)(1-st^2) - 2st(2t+s)^2; \left(\frac{\partial z}{\partial t}\right)\bigg|_{(1,-2)} = 72$$
- $$\frac{\partial z}{\partial s} = (y+1)(1) + (x+1)(rt) = 1 + rt(1+2s+r+t); \left(\frac{\partial z}{\partial s}\right)\bigg|_{(1,-1,2)} = 5$$

$$15. \frac{dw}{dx} = (2u - \tan v)(1) + (-u \sec^2 v)(\pi) = 2x - \tan \pi x - \pi x \sec^2 \pi x$$

$$\left. \frac{dw}{dx} \right|_{x=\frac{1}{4}} = \left(\frac{1}{2} \right) - 1 - \left(\frac{\pi}{2} \right) = -\frac{1+\pi}{2} \approx -2.0708$$

$$16. \frac{\partial w}{\partial \theta} = (2xy)(-\rho \sin \theta \sin \phi) + (x^2)(\rho \cos \theta \sin \phi) + (2z)(0) = \rho^3 \cos \theta \sin^3 \phi (-2 \sin^2 \theta + \cos^2 \theta);$$

$$\left(\frac{\partial w}{\partial \theta} \right) \bigg|_{(2, \pi, \frac{\pi}{2})} = -8$$

$$17. V(r, h) = \pi r^2 h, \frac{dr}{dt} = 0.5 \text{ in./yr,}$$

$$\frac{dh}{dt} = 8 \text{ in./yr}$$

$$\frac{dV}{dt} = (2\pi r h) \left(\frac{dr}{dt} \right) + (\pi r^2) \left(\frac{dh}{dt} \right);$$

$$\left(\frac{dV}{dt} \right) \bigg|_{(20, 400)} = 11200\pi \text{ in.}^3/\text{yr}$$

$$= \frac{11200\pi \text{ in.}^3}{1 \text{ yr}} \times \frac{1 \text{ board ft}}{144 \text{ in.}^3} \approx 244.35 \text{ board ft/yr}$$

When $t = 3$, $x = 6$, $y = 12$, $s = 6\sqrt{5}$. Thus,

$$\left(\frac{ds}{dt} \right) \bigg|_{t=3} = \sqrt{20} \approx 4.47 \text{ ft/s}$$

$$20. V(r, h) = \left(\frac{1}{3} \right) \pi r^2 h, \frac{dh}{dt} = 3 \text{ in./min,}$$

$$\frac{dr}{dt} = 2 \text{ in./min}$$

$$\frac{dV}{dt} = \left(\frac{2}{3} \right) \pi r h \left(\frac{dr}{dt} \right) + \left(\frac{1}{3} \right) \pi r^2 \left(\frac{dh}{dt} \right);$$

$$\left(\frac{dV}{dt} \right) \bigg|_{(40, 100)} = \frac{20,800\pi}{3} \approx 21,782 \text{ in.}^3/\text{min}$$

$$18. \text{ Let } T = e^{-x-3y}.$$

$$\frac{dT}{dt} = e^{-x-3y}(-1) \frac{dx}{dt} + e^{-x-3y}(-3) \frac{dy}{dt}$$

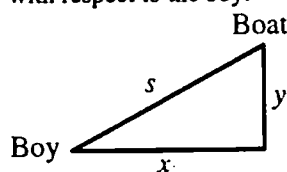
$$= e^{-x-3y}(-1)(2) + e^{-x-3y}(-3)(2)$$

$$= -8e^{-x-3y}$$

$$\left. \frac{dT}{dt} \right|_{(0, 0)} = -8, \text{ so the temperature is}$$

decreasing at $8^\circ/\text{min}$.

19. The stream carries the boat along at 2 ft/s with respect to the boy.



$$\frac{dx}{dt} = 2, \frac{dy}{dt} = 4, s^2 = x^2 + y^2$$

$$2s \left(\frac{ds}{dt} \right) = 2x \left(\frac{dx}{dt} \right) + 2y \left(\frac{dy}{dt} \right)$$

$$\frac{ds}{dt} = \frac{(2x + 4y)}{s}$$

$$21. \text{ Let } F(x, y) = x^3 + 2x^2y - y^3 = 0.$$

$$\text{Then } \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{(3x^2 + 4xy)}{2x^2 - 3y^2} = \frac{3x^2 + 4xy}{3y^2 - 2x^2}.$$

$$22. \text{ Let } F(x, y) = ye^{-x} + 5x - 17 = 0.$$

$$\frac{dy}{dx} = -\frac{(-ye^{-x} + 5)}{e^{-x}} = y - 5e^x$$

$$23. \text{ Let } F(x, y) = x \sin y + y \cos x = 0.$$

$$\frac{dy}{dx} = -\frac{(\sin y - y \sin x)}{x \cos y + \cos x} = \frac{y \sin x - \sin y}{x \cos y + \cos x}$$

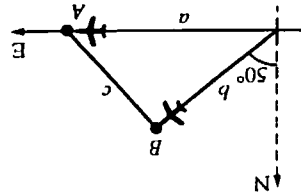
$$24. \text{ Let } F(x, y) = x^2 \cos y - y^2 \sin x = 0.$$

$$\text{Then } \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-(2x \cos y - y^2 \cos x)}{-x^2 \sin y - 2y \sin x}$$

$$= \frac{2x \cos y - y^2 \cos x}{x^2 \sin y + 2y \sin x}.$$

$$25. \text{ Let } F(x, y, z) = 3x^2z + y^3 - xyz^3 = 0.$$

$$\frac{\partial z}{\partial x} = -\frac{(6xz - yz^3)}{3x^2 - 3xyz^2} = \frac{yz^3 - 6xz}{3x^2 - 3xyz^2}$$



33. $c^2 = a^2 + b^2 - 2ab \cos 40^\circ$ (Law of Cosines) where a , b , and c are functions of t .
 $2cc' = 2aa' + 2bb' - 2(a'b + ab') \cos 40^\circ$ so $c' = \frac{aa' + bb' - (a'b + ab') \cos 40^\circ}{c}$.

$$\begin{aligned}
29. \quad y &= \left[\frac{z}{1} \right] [f(v) + f(n)] \\
\text{where } n &= x - ct, \quad v = x + ct. \\
y^x &= \left[\frac{z}{1} \right] [f(1)(n), f] + f(1)(v), f \\
y^x &= \left[\frac{z}{1} \right] [f(1)(v), f] + f(1)(n), f \\
&= \left[\frac{z}{1} \right] [f(v), f] + f(n), f \\
&= y^v [f(v), f] + f(n), f \\
&= y^v \left[\frac{z}{c} - \right] [f(v), f] + f(n), f \\
&= y^v \left[\frac{z}{1} \right] [f(v), f] + f(n), f \\
&= y^v \left[\frac{z}{c} - \right] [f(v), f] + f(n), f \\
&= y^v \left[\frac{z}{c} - \right] [f(v), f] + f(n), f \\
&= y^v \left[\frac{z}{c} - \right] [f(v), f] + f(n), f
\end{aligned}$$

28. We use the z_r notation for $\frac{\partial}{\partial z_r}$.

26. Let $f(x, y, z) = ye^{-x} + z \sin x = 0$.

$$\frac{\partial x}{\partial z} = \frac{-ye^{-x} + z \cos x}{\sin x} = \frac{-ye^{-x} - z \cos x}{\sin x}$$

27.

$$\frac{\partial T}{\partial s} = \frac{\partial x}{\partial T} \frac{\partial T}{\partial s} + \frac{\partial y}{\partial T} \frac{\partial T}{\partial s} + \frac{\partial z}{\partial T} \frac{\partial T}{\partial s} + \frac{\partial w}{\partial T} \frac{\partial T}{\partial s}$$

$$\begin{aligned}
30. \quad & \text{Let } w = f(x, y, z) \text{ where } x = r - s, y = s - t, \\
& z = t - r. \text{ Then} \\
& w_r + w_s + w_t = (w_x x_r + w_y y_s + w_z z_t) + (w_y y_s + w_z z_t) \\
& + (w_x x_r + w_z z_t) + [w_x x_t + w_y y_t + w_z z_t] + [w_x x_t + w_y y_t + w_z z_t] \\
& + [w_x x_t + w_y y_t + w_z z_t] \\
& = 0 \\
31. \quad & \text{Let } w = \int_y^x f(u) du = - \int_x^y f(u) du, \text{ where } x = g(t), y = h(t). \text{ Then} \\
& \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = -f(x)g'(t) + f(y)h'(t) \\
& = f(h(t))h'(t) - f(g(t))g'(t). \\
& \text{Thus, for the particular function given} \\
& F'(t) = \sqrt{9 + (t^2)^4} (2t) - \sqrt[4]{9 + (\sin \sqrt{2} \pi t)^4} (\sqrt{2} \pi \cos \sqrt{2} \pi t); \\
& F'(\sqrt{2}) = (5)(2\sqrt{2}) - (3)(\sqrt{2} \pi) \\
& = 10\sqrt{2} - 3\sqrt{2} \pi \approx 0.8135. \\
32. \quad & \text{If } f(x, y) = f(x, y), \text{ then} \\
& \frac{d}{dt} [f(x(t), y(t))] = \frac{d}{dt} [f(x, y)]. \\
& \text{That is,} \\
& [f_x(x, y)]x' + [f_y(x, y)]y' = f(x, y). \\
& \text{Letting } t = 1 \text{ yields the desired result.}
\end{aligned}$$

When $a = 200$ and $b = 150$, $c^2 = (200)^2 + (150)^2 - 2(200)(150)\cos 40^\circ = 62,500 - 60,000 \cos 40^\circ$.

It is given that $a' = 450$ and $b' = 400$, so at that instant,

$$c' = \frac{(200)(450) + (150)(400) - [(450)(150) + (200)(400)]\cos 40^\circ}{\sqrt{62,500 - 60,000 \cos 40^\circ}} \approx 288.$$

Thus, the distance between the airplanes is increasing at about 288 mph.

34. $r = \langle x, y, z \rangle$, so $r^2 = |r|^2 = x^2 + y^2 + z^2$.

$$F = \frac{GMm}{x^2 + y^2 + z^2}, \text{ so}$$

$$F'(t) = F_m m'(t) + F_x x'(t) + F_y y'(t) + F_z z'(t)$$

$$\begin{aligned} &= \frac{GMm'(t)}{x^2 + y^2 + z^2} - \frac{2GMmx'(t)}{(x^2 + y^2 + z^2)^2} \\ &\quad - \frac{2GMmy'(t)}{(x^2 + y^2 + z^2)^2} + \frac{2GMmz'(t)}{(x^2 + y^2 + z^2)^2} \\ &= \frac{GM[(x^2 + y^2 + z^2)m'(t) - 2m(xx'(t) + yy'(t) + zz'(t))]}{(x^2 + y^2 + z^2)^2}. \end{aligned}$$

15.7 Concepts Review

1. perpendicular

2. $\langle 3, 1, -1 \rangle$

3. $x - 4(y - 1) + 6(z - 1) = 0$

4. $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

Problem Set 15.7

1. $\nabla F(x, y, z) = 2\langle x, y, z \rangle$;

$$\nabla F(2, 3, \sqrt{3}) = 2\langle 2, 3, \sqrt{3} \rangle$$

Tangent Plane:

$$2(x - 2) + 3(y - 3) + \sqrt{3}(z - \sqrt{3}) = 0, \text{ or}$$

$$2x + 3y + \sqrt{3}z = 16$$

2. $\nabla F(x, y, z) = 2\langle 8x, y, 8z \rangle$;

$$\nabla F\left(1, 2, \frac{\sqrt{2}}{2}\right) = 4\langle 4, 1, 2\sqrt{2} \rangle$$

Tangent Plane:

$$4(x - 1) + 1(y - 2) + 2\sqrt{2}\left(z - \frac{\sqrt{2}}{2}\right) = 0, \text{ or}$$

$$4x + y + 2\sqrt{2}z = 8.$$

3. Let $F(x, y, z) = x^2 - y^2 + z^2 + 1 = 0$.

$$\nabla F(x, y, z) = \langle 2x, -2y, 2z \rangle = 2\langle x, -y, z \rangle$$

$$\nabla F(1, 3, \sqrt{7}) = 2\langle 1, -3, \sqrt{7} \rangle, \text{ so } \langle 1, -3, \sqrt{7} \rangle \text{ is}$$

normal to the surface at the point. Then the tangent plane is

$$1(x - 1) - 3(y - 3) + \sqrt{7}(z - \sqrt{7}) = 0, \text{ or more}$$

$$\text{simply, } x - 3y + \sqrt{7}z = -1.$$

4. $\nabla f(x, y, z) = 2\langle x, y, -z \rangle$;

$$\nabla f(2, 1, 1) = 2\langle 2, 1, -1 \rangle$$

Tangent plane:

$$2(x - 2) + 1(y - 1) - 1(z - 1) = 0, \text{ or } 2x + y - z = 4.$$

5. $\nabla f(x, y) = \left(\frac{1}{2}\right)\langle x, y \rangle$; $\nabla f(2, 2) = \langle 1, 1 \rangle$

Tangent plane: $z - 2 = 1(x - 2) + 1(y - 2)$, or

$$x + y - z = 2.$$

6. Let $f(x, y) = xe^{-2y}$.

$$\nabla f(x, y) = \langle e^{-2y}, -2xe^{-2y} \rangle$$

$$\nabla f(1, 0) = \langle 1, -2 \rangle$$

Then $\langle 1, -2, -1 \rangle$ is normal to the surface at

$(1, 0, 1)$, and the tangent plane is

$$1(x - 1) - 2(y - 0) - 1(z - 1) = 0, \text{ or } x - 2y - z = 0.$$

7. $\nabla f(x, y) = \langle -4e^{3y} \sin 2x, 6e^{3y} \cos 2x \rangle$;

$$\nabla f\left(\frac{\pi}{3}, 0\right) = \langle -2\sqrt{3}, -3 \rangle$$

Tangent plane: $z+1 = -2\sqrt{3}\left(x - \frac{\pi}{3}\right) - 3(y-0)$,
 or $2\sqrt{3}x + 3y + z = \frac{(2\sqrt{3}\pi - 3)}{3}$.

8. $\nabla f(x, y) = \left(\frac{1}{2}\right)\left(\frac{1}{x}, \frac{1}{y}\right)$; $\nabla f(1, 4) = \left(\frac{1}{2}, \frac{1}{8}\right)$
 Tangent plane: $z-3 = \left(\frac{1}{2}\right)(x-1) + \left(\frac{1}{8}\right)(y-4)$,
 or $\frac{1}{2}x + \frac{1}{8}y - z = -2$.

9. Let $z = f(x, y) = 2x^2y^3$;
 $dz = 4xy^3dx + 6x^2y^2dy$. For the points given,
 $dx = -0.01$, $dy = 0.02$,
 $dz = 4(-0.01) + 6(0.02) = 0.08$.
 $\Delta z = f(0.99, 1.02) - f(1, 1)$
 $= 2(0.99)^2(1.02)^3 - 2(1)^2(1)^3 \approx 0.08017992$

10. $dz = (2x - 5y)dx + (-5x + 1)dy$
 $= (-11)(0.03) + (-9)(-0.02) = -0.15$
 $\Delta z = f(2.03, 2.98) - f(2, 3) = -0.1461$

11. $dz = 2x^{-1}dx + y^{-1}dy = (-1)(0.02) + \left(\frac{1}{4}\right)(-0.04)$
 $= -0.03$
 $\Delta z = f(-1.98, 3.96) - f(-2.4)$
 $= \ln[(-1.98)^2(3.96)] - \ln 16 \approx -0.030151$

12. Let $z = f(x, y) = \tan^{-1}xy$;
 $dz = \frac{y}{1+x^2y^2}dx + \frac{x}{1+x^2y^2}dy$;
 $= \frac{(-0.5)(-0.03) + (-2)(-0.01)}{1+(4)(0.25)} = 0.0175$.
 $\Delta z = f(-2.03, -0.51) - f(-2, -0.5) \approx 0.017342$

13. Let
 $F(x, y, z) = x^2 - 2xy - y^2 - 8x + 4y - z = 0$;
 $\nabla F(x, y, z) = \langle 2x - 2y - 8, -2x - 2y + 4, -1 \rangle$
 Tangent plane is horizontal if $\nabla F = \langle 0, 0, k \rangle$ for
 any $k \neq 0$.
 $2x - 2y - 8 = 0$ and $-2x - 2y + 4 = 0$ if $x = 3$ and
 $y = -1$. Then $z = -14$. There is a horizontal
 tangent plane at $(3, -1, -14)$.

14. $\langle 8, -3, -1 \rangle$ is normal to $8x - 3y - z = 0$.
 $\nabla F(x, y, z) = \langle 4x, 6y, -1 \rangle$ is normal to

$z = 2x^2 + 3y^2$ at (x, y, z) . $4x = 8$ and
 $6y = -3$, if $x = 2$ and $y = -\frac{1}{2}$; then
 $z = 8.75$ at $\left(2, -\frac{1}{2}, 8.75\right)$.

15. For $F(x, y, z) = x^2 + 4y + z^2 = 0$,
 $\nabla F(x, y, z) = \langle 2x, 4, 2z \rangle = 2\langle x, 2, z \rangle$.
 $F(0, -1, 2) = 0$, and
 $\nabla F(0, -1, 2) = 2\langle 0, 2, 2 \rangle = 4\langle 0, 1, 1 \rangle$.
 For $G(x, y, z) = x^2 + y^2 + z^2 - 6z + 7 = 0$,
 $\nabla G(x, y, z) = \langle 2x, 2y, 2z - 6 \rangle = 2\langle x, y, z - 3 \rangle$.
 $G(0, -1, 2) = 0$, and
 $\nabla G(0, -1, 2) = 2\langle 0, -1, -1 \rangle = -2\langle 0, 1, 1 \rangle$.
 $\langle 0, 1, 1 \rangle$ is normal to both surfaces at
 $(0, -1, 2)$ so the surfaces have the same tangent
 plane; hence, they are tangent to each other at
 $(0, -1, 2)$.

16. $(1, 1, 1)$ satisfies each equation, so the surfaces
 intersect at $(1, 1, 1)$. For
 $z = f(x, y) = x^2y$; $\nabla f(x, y) = \langle 2xy, x^2 \rangle$;
 $\nabla f(1, 1) = \langle 2, 1 \rangle$, so $\langle 2, 1, -1 \rangle$ is normal at
 $(1, 1, 1)$.
 For $F(x, y, z) = x^2 - 4y + 3 = 0$;
 $\nabla f(x, y, z) = \langle 2, -4, 0 \rangle$;
 $\nabla f(1, 1, 1) = \langle 2, -4, 0 \rangle$ so $\langle 2, -4, 0 \rangle$ is normal at
 $(1, 1, 1)$.
 $\langle 2, 1, -1 \rangle \cdot \langle 2, -4, 0 \rangle = 0$, so the normals, hence
 tangent planes, and hence the surfaces, are
 perpendicular at $(1, 1, 1)$.

17. Let $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 12 = 0$;
 $\nabla F(x, y, z) = 2\langle x, 2y, 3z \rangle$ is normal to the
 plane.
 A vector in the direction of the line,
 $\langle 2, 8, -6 \rangle = 2\langle 1, 4, -3 \rangle$, is normal to the plane.
 $\langle x, 2y, 3z \rangle = k\langle 1, 4, -3 \rangle$ and (x, y, z) is on the
 surface for points $(1, 2, -1)$ [when $k=1$] and
 $(-1, -2, 1)$ [when $k=-1$].

18. Let $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
 $\nabla F(x, y, z) = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$
 $\nabla F(x_0, y_0, z_0) = 2\left\langle \frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2} \right\rangle$

The tangent plane at (x_0, y_0, z_0) is

$$\frac{x_0(x-x_0)}{a^2} + \frac{y_0(y-y_0)}{b^2} + \frac{z_0(z-z_0)}{c^2} = 0.$$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} \right) = 0$$

Therefore, $\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = 1$, since

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1.$$

19. $\nabla f(x, y, z) = 2\langle 9x, 4y, 4z \rangle;$

$$\nabla f(1, 2, 2) = 2\langle 9, 8, 8 \rangle$$

$$\nabla g(x, y, z) = 2\langle 2x, -y, 3z \rangle;$$

$$\nabla f(1, 2, 2) = 4\langle 1, -1, 3 \rangle$$

$$\langle 9, 8, 8 \rangle \times \langle 1, -1, 3 \rangle = \langle 32, -19, -17 \rangle$$

$$\text{Line: } x = 1 + 32t, y = 2 - 19t, z = 2 - 17t$$

20. Let $f(x, y, z) = x - z^2$, and $g(x, y, z) = y - z^3$.

$$\nabla f(x, y, z) = \langle 1, 0, -2z \rangle \text{ and}$$

$$\nabla g(x, y, z) = \langle 0, 1, -3z^2 \rangle$$

$$\nabla f(1, 1, 1) = \langle 1, 0, -2 \rangle \text{ and}$$

$$\nabla g(1, 1, 1) = \langle 0, 1, -3 \rangle$$

$$\langle 1, 0, -2 \rangle \times \langle 0, 1, -3 \rangle = \langle 2, 3, 1 \rangle$$

$$\text{Line: } x = 1 + 2t, y = 1 + 3t, z = 1 + t$$

21. $dS = S_A dA + S_W dW$

$$= -\frac{W}{(A-W)^2} dA + \frac{A}{(A-W)^2} dW = \frac{-WdA + AdW}{(A-W)^2}$$

$$\text{At } W = 20, A = 36:$$

$$dS = \frac{-20dA + 36dW}{256} = \frac{5dA + 9dW}{64}.$$

$$\text{Thus, } |dS| \leq \frac{5|dA| + 9|dW|}{64} \leq \frac{5(0.02) + 9(0.02)}{64}$$

$$= 0.004375$$

22. $V = lwh$, $dl = dw = \frac{1}{2}$, $dh = \frac{1}{4}$, $l = 72$, $w = 48$,

$$h = 36$$

$$dV = whdl + lhdw + lwdh = 3024 \text{ in.}^3 (1.75 \text{ ft}^3)$$

23. $V = \pi r^2 h$, $dV = 2\pi rh dr + \pi r^2 dh$

$$|dV| \leq 2\pi rh|dr| + \pi r^2|dh| \leq 2\pi rh(0.02r) + \pi r^2(0.03h)$$

$$= 0.04\pi r^2 h + 0.03\pi r^2 h = 0.07V$$

$$\text{Maximum error in } V \text{ is } 7\%.$$

24. $T = f(L, g) = 2\pi\sqrt{\frac{L}{g}}$

$$dT = f_L dL + f_g dg$$

$$= 2\pi \left(\frac{1}{2\sqrt{\frac{L}{g}}} \right) \left(\frac{1}{g} \right) dL + 2\pi \left(\frac{1}{2\sqrt{\frac{L}{g}}} \right) \left(-\frac{L}{g^2} \right) dg$$

$$= \frac{\pi(gdL - Ldg)}{g^2 \sqrt{\frac{L}{g}}}, \text{ so}$$

$$\frac{dT}{T} = \frac{\pi(gdL - Ldg)}{\left(2\pi\sqrt{\frac{L}{g}} \right) \left(g^2 \sqrt{\frac{L}{g}} \right)} = \frac{gdL - Ldg}{2gL}$$

$$= \frac{1}{2} \left(\frac{dL}{L} - \frac{dg}{g} \right).$$

Therefore,

$$\left| \frac{dT}{T} \right| \leq \frac{1}{2} \left(\left| \frac{dL}{L} \right| + \left| \frac{dg}{g} \right| \right) = \frac{1}{2} (0.5\% + 0.3\%) = 0.4\%.$$

25. Solving for R , $R = \frac{R_1 R_2}{R_1 + R_2}$, so

$$\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1 + R_2)^2} \text{ and } \frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1 + R_2)^2}.$$

$$\text{Therefore, } dR = \frac{R_2^2 dR_1 + R_1^2 dR_2}{(R_1 + R_2)^2};$$

$$|dR| \leq \frac{R_2^2 |dR_1| + R_1^2 |dR_2|}{(R_1 + R_2)^2}. \text{ Then at } R_1 = 25,$$

$$R_2 = 100, dR_1 = dR_2 = 0.5, R = \frac{(25)(100)}{25 + 100} = 20$$

$$\text{and } |dR| \leq \frac{(100)^2(0.5) + (25)^2(0.5)}{(125)^2} = 0.34.$$

26. Let $F(x, y, z) = x^2 + y^2 + 2z^2$.

$$\nabla F(x, y, z) = \langle 2x, 2y, 4z \rangle;$$

$$\nabla F(1, 2, 1) = 2\langle 1, 2, 2 \rangle; \quad \frac{\nabla F}{|\nabla F|} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

Thus, $\mathbf{u} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$ is the unit vector in the direction of flight, and

$$\langle x, y, z \rangle = \langle 1, 2, 1 \rangle + 4t \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \text{ is the location}$$

of the bee along its line of flight t seconds after takeoff. Using the parametric form of the line of flight to substitute into the equation of the plane yields $t = 3$ as the time of intersection with the

plane. Then substituting this value of t into the equation of the line yields $x = 5$, $y = 10$, $z = 9$ so the point of intersection is $(5, 10, 9)$.

27. Let $F(x, y, z) = xyz = k$; let (a, b, c) be any point on the surface of F .

$$\nabla F(x, y, z) = \langle yz, xz, xy \rangle = \left\langle \frac{k}{x}, \frac{k}{y}, \frac{k}{z} \right\rangle$$

$$= k \left\langle \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right\rangle$$

$$\nabla F(a, b, c) = k \left\langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right\rangle$$

An equation of the tangent plane at the point is

$$\left(\frac{1}{a}\right)(x-a) + \left(\frac{1}{b}\right)(x-b) + \left(\frac{1}{c}\right)(x-c) = 0, \text{ or}$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3.$$

Points of intersection of the tangent plane on the coordinate axes are $(3a, 0, 0)$, $(0, 3b, 0)$, and $(0, 0, 3c)$.

The volume of the tetrahedron is

$$\left(\frac{1}{3}\right)(\text{area of base})(\text{altitude}) = \frac{1}{3} \left(\frac{1}{2} |3a||3b| \right) (|3c|)$$

$$= \frac{9|abc|}{2} = \frac{9|k|}{2} \text{ (a constant).}$$

28. If $F(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$, then $\nabla F(x, y, z) = 0.5 \left\langle \frac{1}{\sqrt{x}}, \frac{1}{\sqrt{y}}, \frac{1}{\sqrt{z}} \right\rangle$. The equation of the tangent is

$$0.5 \left\langle \frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{z_0}} \right\rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0, \text{ or } \frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = a.$$

Intercepts are $a\sqrt{x_0}$, $a\sqrt{y_0}$, $a\sqrt{z_0}$; so the sum is $a(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}) = a^2$.

29. $f(x, y) = (x^2 + y^2)^{1/2}$; $f(3, 4) = 5$

$$f_x(x, y) = x(x^2 + y^2)^{-1/2}; f_x(3, 4) = \frac{3}{5} = 0.6$$

$$f_y(x, y) = y(x^2 + y^2)^{-1/2}; f_y(3, 4) = \frac{4}{5} = 0.8$$

$$f_{xx}(x, y) = y^2(x^2 + y^2)^{-3/2}; f_{xx}(3, 4) = \frac{16}{125} = 0.128$$

$$f_{xy}(x, y) = -xy(x^2 + y^2)^{-3/2}; f_{xy}(3, 4) = -\frac{12}{125} = -0.096$$

$$f_{yy}(x, y) = x^2(x^2 + y^2)^{-3/2}; f_{yy}(3, 4) = \frac{9}{125} = 0.072$$

Therefore, the second order Taylor approximation is

$$f(x, y) = 5 + 0.6(x-3) + 0.8(y-4) + 0.5[0.128(x-3)^2 + 2(-0.096)(x-3)(y-4) + 0.072(y-4)^2]$$

- a. First order Taylor approximation: $f(x, y) = 5 + 0.6(x-3) + 0.8(y-4)$.

$$\text{Thus, } f(3.1, 3.9) \approx 5 + 0.6(0.1) + 0.8(-0.1) = 4.98.$$

- b. $f(3.1, 3.9) \approx 5 + 0.6(-0.1) + 0.8(0.1) + 0.5[0.128(0.1)^2 + 2(-0.096)(0.1)(-0.1) + 0.072(-0.1)^2] = 4.98196$

- c. $f(3.1, 3.9) \approx 4.9819675$

15.8 Concepts Review

1. closed bounded
2. boundary; stationary; singular
3. $\nabla f(x_0, y_0) = 0$
4. $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$

Problem Set 15.8

1. $\nabla f(x, y) = \langle 2x - 4, 8y \rangle = \langle 0, 0 \rangle$ at $(2, 0)$, a stationary point.
 $D = f_{xx}f_{yy} - f_{xy}^2 = (2)(8) - (0)^2 = 16 > 0$ and $f_{xx} = 2 > 0$. Local minimum at $(2, 0)$.
2. $\nabla f(x, y) = \langle 2x - 2, 8y + 8 \rangle = \langle 0, 0 \rangle$ at $(1, -1)$, a stationary point. $D = f_{xx}f_{yy} - f_{xy}^2 = (2)(8) - (0)^2 = 16 > 0$ and $f_{xx} = 2 > 0$. Local minimum at $(1, -1)$.
3. $\nabla f(x, y) = \langle 8x^3 - 2x, 6y \rangle = \langle 2x(4x^2 - 1), 6y \rangle$
 $= (0, 0)$, at $(0, 0), (0.5, 0), (-0.5, 0)$ all stationary points.
 $f_{xx} = 24x^2 - 2$; $D = f_{xx}f_{yy} - f_{xy}^2 = (24x^2 - 2)(6) - (0)^2$
 $= 12(12x^2 - 1)$.
 At $(0, 0)$: $D = -12$, so $(0, 0)$ is a saddle point.
 At $(0.5, 0)$ and $(-0.5, 0)$: $D = 24$ and $f_{xx} = 6$, so local minima occur at these points.
4. $\nabla f(x, y) = \langle y^2 - 12x, 2xy - 6y \rangle = \langle 0, 0 \rangle$ at stationary points $(0, 0), (3, -6)$ and $(3, 6)$.
 $D = f_{xx}f_{yy} - f_{xy}^2 = (-12)(2x - 6) - (2y)^2$
 $= -4(y^2 + 6x - 18)$, $f_{xx} = -12$
 At $(0, 0)$: $D = 72$, and $f_{xx} = -12$, so local maximum at $(0, 0)$.
 At $(3, \pm 6)$: $D = -144$, so $(3, \pm 6)$ are saddle points.
5. $\nabla f(x, y) = \langle y, x \rangle = \langle 0, 0 \rangle$ at $(0, 0)$, a stationary point.
 $D = f_{xx}f_{yy} - f_{xy}^2 = (0)(0) - (1)^2 = -1$, so $(0, 0)$ is a saddle point.
6. Let $\nabla f(x, y) = \langle 3x^2 - 6y, 3y^2 - 6x \rangle = \langle 0, 0 \rangle$. Then $3x^2 - 6y = 0$ and $3y^2 - 6x = 0$. Solving simultaneously, obtain solutions $(0, 0)$ and $(2, 2)$.
 $f_{xx} = 6x$; $D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(6y) - (-6)^2 = 36(xy - 1)$ At $(0, 0)$: $D < 0$, so $(0, 0)$ is a saddle point.
 At $(2, 2)$: $D > 0$, $f_{xx} > 0$, so a local minimum occurs here.
7. $\nabla f(x, y) = \left\langle \frac{x^2y - 2}{x^2}, \frac{xy^2 - 4}{y^2} \right\rangle = \langle 0, 0 \rangle$ at $(1, 2)$.
 $D = f_{xx}f_{yy} - f_{xy}^2 = (4x - 3)(8y - 3) - (1)^2$
 $= 32x^{-3}y^{-3} - 1$, $f_{xx} = 4x^{-3}$
 At $(1, 2)$: $D > 0$, and $f_{xx} > 0$, so a local minimum at $(1, 2)$.

8. $\nabla f(x, y) = -2\exp(-x^2 - y^2 + 4y)\langle x, y - 2 \rangle = \langle 0, 0 \rangle$ at $(0, 2)$.

$$D = f_{xx}f_{yy} - f_{xy}^2 = \exp 2(-x^2 - y^2 + 4y)[(4x^2 - 2)(4y^2 - 16y + 14) - (4xy - 8x)^2],$$

$$f_{xx} = (4x^2 - 2)\exp(-x^2 - y^2 + 4y)$$

At $(0, 2)$: $D > 0$, and $f_{xx} < 0$, so local maximum at $(0, 2)$.

9. Let $\nabla f(x, y)$

$$= \langle -\sin x - \sin(x + y), -\sin y - \sin(x + y) \rangle$$

$$= \langle 0, 0 \rangle$$

Then $\begin{cases} -\sin x - \sin(x + y) = 0 \\ \sin y + \sin(x + y) = 0 \end{cases}$. Therefore,

$\sin x = \sin y$, so $x = y = \frac{\pi}{4}$. However, these

values satisfy neither equation. Therefore, the gradient is defined but never zero in its domain, and the boundary of the domain is outside the domain, so there are no critical points.

10. $\nabla f(x, y) = \langle 2x - 2a \cos y, 2ax \sin y \rangle = \langle 0, 0 \rangle$ at

$$\left(0, \pm \frac{\pi}{2}\right), (a, 0)$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = (2)(2ax \cos y) - (2a \sin y)^2,$$

$$f_{xx} = 2$$

At $\left(0, \pm \frac{\pi}{2}\right)$: $D = -4a^2 < 0$, so $\left(0, \pm \frac{\pi}{2}\right)$ are

saddle points.

At $(a, 0)$: $D = 4a^2 > 0$ and $f_{xx} > 0$, so local minimum at $(a, 0)$.

11. We do not need to use calculus for this one. $3x$ is minimum at 0 and $4y$ is minimum at -1 . $(0, -1)$ is in S , so $3x + 4y$ is minimum at $(0, -1)$; the minimum value is -4 . Similarly, $3x$ and $4y$ are each maximum at 1. $(1, 1)$ is in S , so $3x + 4y$ is maximum at $(1, 1)$; the maximum value is 7. (Use calculus techniques and compare.)

12. We do not need to use calculus for this one. Each of x^2 and y^2 is minimum at 0 and $(0, 0)$ is in S , so $x^2 + y^2$ is minimum at $(0, 0)$; the minimum value is 0. Similarly, x^2 and y^2 are maximum at $x = 3$ and $y = 4$, respectively, and $(3, 4)$ is in S , so $x^2 + y^2$ is maximum at $(3, 4)$; the maximum value is 25. (Use calculus techniques and compare.)

13. $\nabla f(x, y) = \langle 2x, -2y \rangle = \langle 0, 0 \rangle$ at $(0, 0)$.

$$D = f_{xx}f_{yy} - f_{xy}^2 = (2)(-2) - (0)^2 < 0, \text{ so } (0, 0) \text{ is}$$

a saddle point. A parametric representation of the boundary of S is $x = \cos t, y = \sin t$, t in $[0, 2\pi]$.

$$f(x, y) = f(x(t), y(t)) = \cos^2 t - \sin^2 t + 1 = \cos 2t - 1$$

$\cos 2t - 1$ is maximum if $\cos 2t = 1$, which occurs for $t = 0, \pi, 2\pi$. The points of the curve are $(\pm 1, 0)$. $f(\pm 1, 0) = 2$

$f(x, y) = \cos 2t - 1$ is minimum if $\cos 2t = -1$,

which occurs for $t = \frac{\pi}{2}, \frac{3\pi}{2}$. The points of the

curve are $(0, \pm 1)$. $f(0, \pm 1) = 0$. Global minimum of 0 at $(0, \pm 1)$; global maximum of 2 at $(\pm 1, 0)$.

14. $\nabla f(x, y) = \langle 2x - 6, 2y - 8 \rangle = 0$ at $(3, 4)$, which is outside S , so there are no stationary points. There are also no singular points.

$x = \cos t, y = \sin t$, t in $[0, 2\pi]$ is a parametric representation of the boundary of S .

$$f(x, y) = f(x(t), y(t))$$

$$= \cos^2 t - 6 \cos t + \sin^2 t - 8 \sin t + 7$$

$$= 8 - 6 \cos t - 8 \sin t = F(t)$$

$$F'(t) = 6 \sin t - 8 \cos t = 0 \text{ if } \tan t = \frac{4}{3}. t \text{ can be}$$

in the 1st or 3rd quadrants. The corresponding

points of the curve are $\left(\pm \frac{3}{5}, \pm \frac{4}{5}\right)$.

$$f\left(-\frac{3}{5}, -\frac{4}{5}\right) = 18; f\left(\frac{3}{5}, \frac{4}{5}\right) = -2$$

Global minimum of -2 at $\left(\frac{3}{5}, \frac{4}{5}\right)$; global

maximum of 18 at $\left(-\frac{3}{5}, -\frac{4}{5}\right)$.

15. Let x, y, z denote the numbers, so $x + y + z = N$. Maximize

$$P = xyz = xy(N - x - y) = Nxy - x^2y - xy^2.$$

$$\text{Let } \nabla P(x, y) = \langle Ny - 2xy - y^2, Nx - x^2 - 2xy \rangle$$

$$= \langle 0, 0 \rangle.$$

$$\text{Then } \begin{cases} Ny - 2xy - y^2 = 0 \\ Nx - x^2 - 2xy = 0 \end{cases}.$$

$$N(x, -y) = x^2 - y^2 = (x+y)(x-y). \quad x = y \text{ or } N = x + y.$$

Therefore, $x = y$ (since $N = x + y$ would mean that $P = 0$, certainly not a maximum value).

Then, substituting into $Nx - x^2 - 2xy = 0$, we obtain $Nx - x^2 - 2x^2 = 0$, from which we obtain $x(N - 3x) = 0$, so $x = \frac{N}{3}$ (since $x = 0 \Rightarrow P = 0$).

$$P_{xx} = -2y;$$

$$D = P_{xx}P_{yy} - P_{xy}^2$$

$$= (-2y)(-2x) - (N - 2x - 2y)^2 = 4xy - (N - 2x - 2y)^2$$

$$\text{At } x = y = \frac{N}{3}: D = \frac{N^2}{3} > 0, P_{xx} = -\frac{2N}{3} < 0 \text{ (so local maximum)}$$

$$\text{If } x = y = \frac{N}{3}, \text{ then } z = \frac{N}{3}.$$

Conclusion: Each number is $\frac{N}{3}$. (If the intent is to find three distinct numbers, then there is no maximum value of P that satisfies that condition.)

16. Let s be the distance from the origin to (x, y, z) on the plane. $s^2 = x^2 + y^2 + z^2$ and

$$x + 2y + 3z = 12. \text{ Minimize}$$

$$s^2 = f(y, z) = (12 - 2y - 3z)^2 + y^2 + z^2.$$

$$\nabla f(y, z) = \langle -48 + 12x + 10y, -72 + 12y + 20z \rangle$$

$$= \langle 0, 0 \rangle \text{ at } \left(\frac{12}{7}, \frac{18}{7} \right).$$

$$D = f_{yy}f_{zz} - f_{yz}^2 = 56 > 0 \text{ and } f_{yy} = 10 > 0;$$

$$\text{local maximum at } \left(\frac{12}{7}, \frac{18}{7} \right)$$

$$s^2 = \frac{504}{49}, \text{ so the shortest distance is}$$

$$s = \frac{6\sqrt{14}}{7} \approx 3.2071.$$

17. Let S denote the surface area of the box with dimensions x, y, z .

$$S = 2xy + 2xz + 2yz \text{ and } V_0 = xyz, \text{ so}$$

$$S = 2(xy + V_0y^{-1} + V_0x^{-1}).$$

Minimize $f(x, y) = xy + V_0y^{-1} + V_0x^{-1}$ subject to $x > 0, y > 0$.

$$\nabla f(x, y) = \langle y - V_0x^{-2}, x - V_0y^{-2} \rangle = \langle 0, 0 \rangle \text{ at } (V_0^{1/3}, V_0^{1/3}).$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = 4V_0^2x^{-3}y^{-3} - 1,$$

$$f_{xx} = 2V_0x^{-3}.$$

At $(V_0^{1/3}, V_0^{1/3})$: $D = 3 > 0, f_{xx} = 2 > 0$, so local minimum.

Conclusion: The box is a cube with edge $V_0^{1/3}$.

18. Let L denote the sum of edge lengths for a box of dimensions x, y, z . Minimize $L = 4x + 4y + 4z$, subject to $V_0 = xyz$.

$$L(x, y) = 4x + 4y + \frac{4V_0}{xy}, \quad x > 0, y > 0$$

Let

$$\nabla L(x, y) = 4x^{-1}y^{-1} \langle x^{-1}(x^2y - V_0), y^{-1}(xy^2 - V_0) \rangle = \langle 0, 0 \rangle.$$

Then $x^2y = V_0$ and $xy^2 = V_0$, from which it follows that $x = y$. Therefore $x = y = z = V_0^{1/3}$.

$$L_{xx} = \frac{8V_0}{x^3y};$$

$$D = L_{xx}L_{yy} - L_{xy}^2 = \left(\frac{8V_0}{x^3y} \right) \left(\frac{8V_0}{xy^3} \right) - \left(\frac{4V_0}{x^2y^2} \right)^2$$

At $(V_0^{1/3}, V_0^{1/3})$: $D > 0, L_{xx} > 0$ (so local minimum).

There are no other critical points, and as $(x, y) \rightarrow \text{boundary}, L \rightarrow \infty$. Hence, the optimal box is a cube.

19. Let S denote the area of the sides and bottom of the tank with base l by w and depth h . $S = lw + 2lh + 2wh$ and $lwh = 256$.

$$S(l, w) = lw + 2l \left(\frac{256}{lw} \right) + 2w \left(\frac{256}{lw} \right), \quad w > 0, l > 0.$$

$$S(l, w) = \langle w - 512l^{-2}, l - 512w^{-2} \rangle = \langle 0, 0 \rangle \text{ at}$$

$(8, 8)$. $h = 4$ there. At $(8, 8)$ $D > 0$ and $S_{ll} > 0$, so local minimum. Dimensions are $8' \times 8' \times 4'$.

20. Let V denote the volume of the box and (x, y, z) denote its 1st octant vertex.

$$V = (2x)(2y)(2z) = 8xyz \text{ and } 24x^2 + y^2 + z^2 = 9.$$

$$V^2 = 64 \left[\left(\frac{1}{24} \right) (9 - y^2 - z^2) \right] y^2 z^2$$

Maximize $f(y, z) = (9 - y^2 - z^2)y^2z^2, y > 0, z > 0$.

$$\nabla f(y, z) = 2\langle yz^2(9-2y^2-z^2), y^2z(9-y^2-2z^2) \rangle$$

$$= \langle 0, 0 \rangle \text{ at } (\sqrt{3}, \sqrt{3}). \quad x = \frac{\sqrt{2}}{4}$$

$$\text{At } (\sqrt{3}, \sqrt{3}), \quad D = f_{yy}f_{zz} - f_{yz}^2 > 0 \text{ and}$$

$f_{yy} < 0$, so local maximum. The greatest

$$\text{possible volume is } 8\left(\frac{\sqrt{2}}{4}\right)(\sqrt{3})(\sqrt{3}) = 6\sqrt{2}.$$

21. Let $\langle x, y, z \rangle$ denote the vector; let S be the sum of its components.

$$x^2 + y^2 + z^2 = 81, \text{ so } z = (81 - x^2 - y^2)^{1/2}.$$

$$\text{Maximize } S(x, y) = x + y + (81 - x^2 - y^2)^{1/2}, \quad 0 \leq x^2 + y^2 \leq 9.$$

$$\text{Let } \nabla S(x, y) = \langle 1 - x(81 - x^2 - y^2)^{-1/2}, 1 - y(81 - x^2 - y^2)^{-1/2} \rangle = \langle 0, 0 \rangle.$$

Therefore, $x = (81 - x^2 - y^2)^{1/2}$ and $y = (81 - x^2 - y^2)^{1/2}$. We then obtain $x = y = 3\sqrt{3}$ as the only stationary point. For these values of x and y , $z = 3\sqrt{3}$ and $S = 9\sqrt{3} \approx 15.59$.

The boundary needs to be checked. It is fairly easy to check each edge of the boundary separately. The largest value of S at a boundary point occurs at three places and turns out to be $\frac{18}{\sqrt{2}} \approx 12.73$.

Conclusion: the vector is $3\sqrt{3}\langle 1, 1, 1 \rangle$.

22. Let (x, y, z) denote a point on the cone, and s denote the distance between (x, y, z) and $(1, 2, 0)$.

$$s^2 = (x-1)^2 + (y-2)^2 + z^2 \text{ and } z^2 = x^2 + y^2. \text{ Minimize } s^2 = f(x, y) = (x-1)^2 + (y-2)^2 + (x^2 + y^2), \quad x, y \text{ in } \mathbb{R}.$$

$$\nabla f(x, y) = 2\langle 2x-1, 2y-2 \rangle = \langle 0, 0 \rangle \text{ at } \left(\frac{1}{2}, 1\right). \text{ At } \left(\frac{1}{2}, 1\right), \quad D > 0 \text{ and } f_{xx} > 0, \text{ so local minimum.}$$

$$\text{Conclusion: Minimum distance is } s = \sqrt{\frac{5}{2}} \approx 1.5811.$$

23. $A = \left(\frac{1}{2}\right)[y + (y + 2x \sin \alpha)](x \cos \alpha)$ and

$$2x + y = 12. \text{ Maximize } A(x, \alpha) = 12x \cos \alpha - 2x^2 \cos \alpha + \left(\frac{1}{2}\right)x^2 \sin 2\alpha, \quad x \text{ in } (0, 6], \alpha \text{ in } \left(0, \frac{\pi}{2}\right).$$

$$A(x, \alpha) = \langle 12 \cos \alpha - 4x \cos \alpha + 2x \sin \alpha \cos \alpha, -12x \sin \alpha + 2x^2 \sin \alpha + x^2 \cos 2\alpha \rangle = \langle 0, 0 \rangle \text{ at } \left(4, \frac{\pi}{6}\right).$$

$$\text{At } \left(4, \frac{\pi}{6}\right), \quad D > 0 \text{ and } A_{xx} < 0, \text{ so local maximum, and } A = 12\sqrt{3} \approx 20.78. \text{ At the boundary point of } x = 6, \text{ we get}$$

$$\alpha = \frac{\pi}{4}, A = 18. \text{ Thus, the maximum occurs for width of turned-up sides } = 4'', \text{ and base angle } = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}.$$

24. The lines are skew since there are no values of s and t that simultaneously satisfy $t - 1 = 3s$, $2t = s + 2$, and $t + 3 = 2s - 1$. Minimize f , the square of the distance between points on the two lines.

$$f(s, t) = (3s - t + 1)^2 + (s + 2 - 2t)^2 + (2s - 1 - t - 3)^2$$

Let

$$\nabla f(s, t) = \langle 2(3s - t + 1)(3) + 2(s + 2 - 2t)(1) + 2(2s - t - 4)(2), 2(3s - t + 1)(-1) + 2(s + 2 - 2t)(-2) + 2(2s - t - 4)(-1) \rangle \\ = \langle 28s - 14t - 6, -14s + 12t - 28 \rangle = \langle 0, 0 \rangle.$$

$$\text{Solve } 28s - 14t - 6 = 0, -14s + 12t - 28 = 0, \text{ obtaining } s = \frac{5}{7}, t = 1.$$

$$D = f_{ss}f_{tt} - f_{st}^2 = (28)(12) - (-14)^2 > 0; \quad f_{ss} = 28 > 0. \text{ (local minimum)}$$

The nature of the problem indicates the global minimum occurs here.

$$f\left(\frac{5}{7}, 1\right) = \left(\frac{15}{7}\right)^2 + \left(\frac{5}{7}\right)^2 + \left(-\frac{25}{7}\right)^2 = \frac{875}{49}$$

Conclusion: The minimum distance between the lines is $\frac{\sqrt{875}}{7} \approx 4.2258$. (For another way of doing this problem see Problem 21, Section 14.4.)

25. Let M be the maximum value of $f(x, y)$ on the polygonal region, P . Then $ax + by + (c - M) = 0$ is a line that either contains a vertex of P or divides P into two subregions. In the latter case $ax + by + (c - M)$ is positive in one of the regions and negative in the other. $ax + by + (c - M) > 0$ contradicts that M is the maximum value of $ax + by + c$ on P . (Similar argument for minimum.)

a.

x	y	$2x + 3y + 4$
-1	2	8
0	1	7
1	0	6
-3	0	-2
0	-4	-8

Maximum at $(-1, 2)$

b.

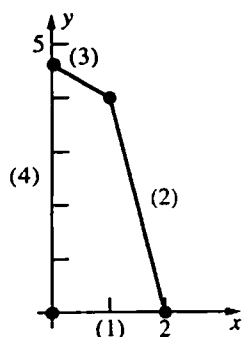
x	y	$-3x + 2y + 1$
-3	0	10
0	5	11
2	3	1
4	0	-11
1	-4	-10

Minimum at $(4, 0)$

26.

x	y	$2x + y$
0	0	0
2	0	4
1	4	6
0	14/3	14/3

Maximum of 6 occurs at $(1, 4)$



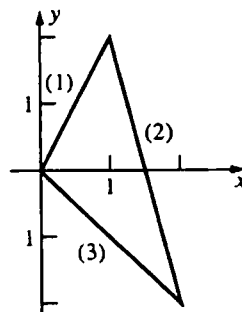
The edges of P are segments of the lines:

1. $y = 0$
2. $4x + y = 8$
3. $2x + 3y = 14$, and
4. $x = 0$

27. $z(x, y) = y^2 - x^2$

$$z(x, y) = \langle -2x, 2y \rangle = \langle 0, 0 \rangle \text{ at } (0, 0).$$

There are no stationary points and no singular points, so consider boundary points.



On side 1:

$$y = 2x, \text{ so } z = 4x^2 - x^2 = 3x^2$$

$$z'(x) = 6x = 0 \text{ if } x = 0.$$

Therefore, $(0, 0)$ is a candidate.

On side 2:

$$y = -4x + 6, \text{ so}$$

$$z = (-4x + 6)^2 - x^2 = 15x^2 - 48x + 36.$$

$$z'(x) = 30x - 48 = 0 \text{ if } x = 1.6.$$

Therefore, $(1.6, -0.4)$ is a candidate.

On side 3:

$$y = -x, \text{ so } z = (-x)^2 - x^2 = 0.$$

Also, all vertices are candidates.

x	y	z
0	0	0
1.6	-0.4	-2.4
2	-2	0
1	2	3

Minimum value of -2.4; maximum value of 3

$$\begin{aligned}
 28. \quad \text{a.} \quad \frac{\partial f}{\partial m} &= \sum_{i=1}^n \frac{\partial}{\partial m} (y_i - mx_i - b)^2 \\
 &= 2 \sum_{i=1}^n (y_i - mx_i - b)(-x_i) \\
 &= -2 \sum_{i=1}^n (x_i y_i - mx_i^2 - bx_i)
 \end{aligned}$$

Setting this result equal to zero yields

$$0 = -2 \sum_{i=1}^n (x_i y_i - mx_i^2 - bx_i)$$

$$0 = \sum_{i=1}^n (x_i y_i - mx_i^2 - bx_i)$$

or equivalently,

$$\sum_{i=1}^n x_i y_i = m \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i$$

$$\begin{aligned}
 \frac{\partial f}{\partial b} &= \sum_{i=1}^n \frac{\partial}{\partial b} (y_i - mx_i - b)^2 \\
 &= 2 \sum_{i=1}^n (y_i - mx_i - b)(-1) \\
 &= -2 \sum_{i=1}^n (y_i - mx_i - b)
 \end{aligned}$$

Setting this result equal to zero yields

$$0 = -2 \sum_{i=1}^n (y_i - mx_i - b)$$

$$0 = \sum_{i=1}^n (y_i - mx_i - b)$$

or equivalently,

$$m \sum_{i=1}^n x_i + nb = \sum_{i=1}^n y_i$$

$$\text{b.} \quad nb = \sum_{i=1}^n y_i - m \sum_{i=1}^n x_i$$

Therefore,

$$b = \frac{\sum_{i=1}^n y_i - m \sum_{i=1}^n x_i}{n}$$

$$\sum_{i=1}^n x_i y_i = m \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i$$

$$\sum_{i=1}^n x_i y_i = m \sum_{i=1}^n x_i^2 + \frac{\left(\sum_{i=1}^n y_i - m \sum_{i=1}^n x_i \right) \sum_{i=1}^n x_i}{n}$$

This simplifies into

$$m = \frac{\sum_{i=1}^n x_i y_i - \frac{1}{n} \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{\sum_{i=1}^n x_i^2 - \frac{1}{n} \left(\sum_{i=1}^n x_i \right)^2}$$

c. $\frac{\partial^2 f}{\partial m^2} = 2 \sum_{i=1}^n x_i^2$

$$\frac{\partial^2 f}{\partial b^2} = 2n$$

$$\frac{\partial^2 f}{\partial m \partial b} = 2 \sum_{i=1}^n x_i$$

Then, by Theorem C, we have

$$D = 4n \left(\sum_{i=1}^n x_i^2 - \frac{1}{n} \left(\sum_{i=1}^n x_i \right)^2 \right).$$

Assuming that all the x_i are not the same, we find that

$$D > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial m^2} > 0$$

Thus, $f(m, b)$ is minimized.

29.

x_i	y_i	x_i^2	$x_i y_i$
3	2	9	6
4	3	16	12
5	4	25	20
6	4	36	24
7	5	49	35
25	18	135	97

$m(135) + b(25) = (97)$ and $m(25) + (5)b = (18)$.
Solve simultaneously and obtain $m = 0.7$, $b = 0.1$.
The least-squares line is $y = 0.7x + 0.1$.

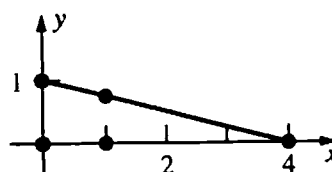
30. $z = 2x^2 + y^2 - 4x - 2y + 5$, so

$$\nabla z = \langle 4x - 4, 2y - 2 \rangle = 0.$$

$\nabla z = 0$ at $(1, 1)$ which is outside the region.

Therefore, extreme values occur on the boundary.

Three critical points are the vertices of the triangle, $(0, 0)$, $(0, 1)$, and $(4, 0)$. Others may occur on the interior of a side of the triangle.



On vertical side: $x = 0$

$z(y) = y^2 - 2y + 5$, $y \in [0, 1]$. $z'(y) = 2y - 2$, so $z'(y) = 0$ if $y = 1$. Hence, no additional critical point.

On horizontal side: $y = 0$

$z(x) = 2x^2 - 4x + 5$, $x \in [0, 4]$. $z'(x) = 4x - 4$, so $z'(x) = 0$ if $x = 1$. Hence, an additional critical point is $(1, 0)$.

On hypotenuse: $x = 4 - 4y$

$$z(y) = 2(4 - 4y)^2 + y^2 - 4(4 - 4y) - 2y + 5$$

$$= 33y^2 - 50y + 21, \quad y \in [0, 1].$$

$z'(y) = 66y - 50$, so $z'(y) = 0$ if $y = \frac{25}{33}$. Hence,

an additional critical point is $\left(\frac{32}{33}, \frac{25}{33}\right)$.

x	y	z
0	0	5
4	0	21
0	1	4
1	0	3
32/33	25/33	2.06

Maximum value of z is 21; it occurs at $(4, 0)$.

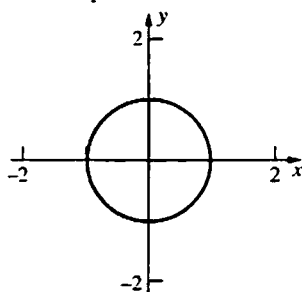
Minimum value of z is about 2.06; it occurs at

$\left(\frac{32}{33}, \frac{25}{33}\right)$.

31. $T(x, y) = 2x^2 + y^2 - y$

$\nabla T = \langle 4x, 2y - 1 \rangle = 0$

If $x = 0$ and $y = \frac{1}{2}$, so $\left(0, \frac{1}{2}\right)$ is the only interior critical point.



On the boundary $x^2 = 1 - y^2$, so T is a function of y there.

$T(y) = 2(1 - y^2) + y^2 - y = 2 - y - y^2$,

$y = [-1, 1]$

$T'(y) = -1 - 2y = 0$ if $y = -\frac{1}{2}$, so on the boundary, critical points occur where y is $-1, -\frac{1}{2}, 1$.

Thus, points to consider are $\left(0, \frac{1}{2}\right), (0, -1)$,

$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $(0, 1)$. Substituting

these into $T(x, y)$ yields that the coldest spot is

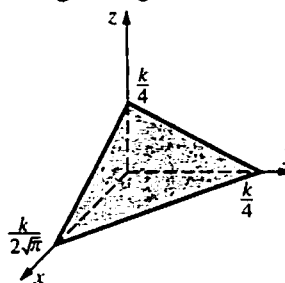
$\left(0, \frac{1}{2}\right)$ where the temperature is $-\frac{1}{4}$, and there

is a tie for the hottest spot at $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ where the temperature is $\frac{9}{4}$.

32. Let x^2, y^2, z^2 denote the areas enclosed by the circle, and the two squares, respectively. Then the radius of the circle is $\frac{x}{\sqrt{\pi}}$, and the edges of the two squares are y and z , respectively. We wish to optimize $A(x, y, z) = x^2 + y^2 + z^2$,

subject to $2\pi\left(\frac{x}{\sqrt{\pi}}\right) + 4y + 4z = k$, or

equivalently $2\sqrt{\pi}x + 4y + 4z = k$, with each of x, y , and z nonnegative. Geometrically: we seek the smallest and largest of all spheres with center at the origin and some point in common with the triangular region indicated.



Since $\frac{k}{2\sqrt{\pi}} > \frac{k}{4}$, the largest sphere will intersect

the region only at point $\left(\frac{k}{2\sqrt{\pi}}, 0, 0\right)$ and will

thus have radius $\frac{k}{2\sqrt{\pi}}$. Thus A will be maximum

if $x = \frac{k}{2\sqrt{\pi}}, y = z = 0$ (all of the wire goes into

the circle). The smallest sphere will be tangent to the triangle. The point of tangency is on the normal line through the origin,

$\langle x, y, z \rangle = t\langle \sqrt{\pi}, 2, 2 \rangle$. Substituting $x = \sqrt{\pi}$,

$y = 2, z = 2$ into the equation of the plane yields

the value $t = \frac{k}{2(\pi+8)}$, so the minimum value of

A is obtained for the values of $x = \frac{k\sqrt{\pi}}{2(\pi+8)}$,

$y = z = \frac{k}{\pi+8}$. Thus the circle will have radius

$\frac{\left[\frac{k\sqrt{\pi}}{2(\pi+8)}\right]}{\sqrt{\pi}} = \frac{k}{2(\pi+8)}$, and the squares will each

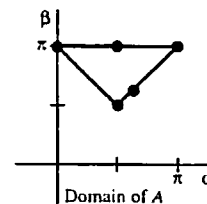
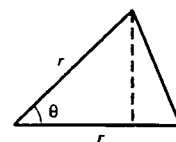
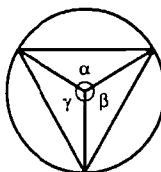
have sides $\frac{k}{(\pi+8)}$. Therefore, the circle will use

$\frac{\pi k}{(\pi+8)}$ units and the squares will each use

$\frac{4k}{(\pi+8)}$ units.

[Note: sum of the three lengths is k .]

33. Without loss of generality we will assume that $\alpha \leq \beta \leq \gamma$. We will consider it intuitively clear that for a triangle of maximum area the center of the circle will be inside or on the boundary of the triangle; i.e., α, β , and γ are in the interval $[0, \pi]$. Along with $\alpha + \beta + \gamma = 2\pi$, this implies that $\alpha + \beta \geq \pi$.



The area of an isosceles triangle with congruent sides of length r and included angle θ is $\frac{1}{2}r^2 \sin \theta$.

$$\begin{aligned} \text{Area}(\triangle ABC) &= \frac{1}{2}r^2 \sin \alpha + \frac{1}{2}r^2 \sin \beta + \frac{1}{2}r^2 \sin \gamma \\ &= \frac{1}{2}r^2 (\sin \alpha + \sin \beta + \sin [2\pi - (\alpha + \beta)]) \\ &= \frac{1}{2}r^2 [\sin \alpha + \sin \beta - \sin(\alpha + \beta)] \end{aligned}$$

$\text{Area}(\triangle ABC)$ will be maximum if (*) $A(\alpha, \beta) = \sin \alpha + \sin \beta - \sin(\alpha + \beta)$ is maximum.

Restrictions are $0 \leq \alpha \leq \beta \leq \pi$, and $\alpha + \beta \geq \pi$.

Three critical points are the vertices of the triangular domain of A : $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, $(0, \pi)$, and (π, π) . We will now search for others.

$$\begin{aligned} \Delta A(\alpha, \beta) &= \langle \cos \alpha - \cos(\alpha + \beta), \cos \beta - \cos(\alpha + \beta) \rangle = 0 \text{ if} \\ \cos \alpha &= \cos(\alpha + \beta) = \cos \beta. \end{aligned}$$

Therefore, $\cos \alpha = \cos \beta$, so $\alpha = \beta$ [due to the restrictions stated]. Then

$$\cos \alpha = \cos(\alpha + \alpha) = \cos 2\alpha = 2\cos^2 \alpha - 1, \text{ so } \cos \alpha = 2\cos^2 \alpha - 1.$$

$$\text{Solve for } \alpha: 2\cos^2 \alpha - \cos \alpha - 1 = 0; (2\cos \alpha + 1)(\cos \alpha - 1) = 0;$$

$$\cos \alpha = -\frac{1}{2} \text{ or } \cos \alpha = 1; \alpha = \frac{2\pi}{3} \text{ or } \alpha = 0.$$

(We are still in the case where $\alpha = \beta$.) $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ is a new critical point, but $(0, 0)$ is out of the domain of A .

There are no critical points in the interior of the domain of A .

On the $\beta = \pi$ edge of the domain of A :

$$A(\alpha) = \sin \alpha - \sin(\alpha - \pi) = 2\sin \alpha \text{ so } A'(\alpha) = 2\cos \alpha.$$

$$A'(\alpha) = 0 \text{ if } \alpha = \frac{\pi}{2}. \left(\frac{\pi}{2}, \pi\right) \text{ is a new critical point.}$$

On the $\beta = \pi - \alpha$ edge of the domain of A :

$$A(\alpha) = \sin \alpha + \sin(\pi - \alpha) - \sin(2\alpha - \pi) = 2 \sin \alpha + \sin 2\alpha, \text{ so}$$

$$A'(\alpha) = 2 \cos \alpha + 2 \cos 2\alpha = 2[\cos \alpha + (2 \cos^2 \alpha - 1)] = 2(2 \cos \alpha - 1)(\cos \alpha + 1).$$

$$A'(\alpha) = 0 \text{ if } \cos \alpha = \frac{1}{2} \text{ or } \cos \alpha = -1, \text{ so } \alpha = \frac{\pi}{3} \text{ or } \alpha = \pi.$$

$$\left(\frac{\pi}{3}, \frac{2\pi}{3}\right) \text{ and } (\pi, 0) \text{ are outside the domain of } A.$$

(The critical points are indicated on the graph of the domain of A .)

α	β	A
$\frac{\pi}{2}$	$\frac{\pi}{2}$	2
0	π	0
π	π	0
$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{\pi}{2}$	π	2

Maximum value of A . The triangle is equilateral.

34. If the plane through (a, b, c) is expressed as

$$Ax + By + Cz = 1, \text{ then the intercepts are } \frac{1}{A}, \frac{1}{B}, \frac{1}{C}; \text{ volume}$$

$$\text{of tetrahedron is } V = \left(\frac{1}{3}\right) \left[\left(\frac{1}{2}\right) \left(\frac{1}{A}\right) \left(\frac{1}{B}\right)\right] \left(\frac{1}{C}\right) = \frac{1}{6ABC}.$$

To maximize V subject to $Aa + Bb + Cc = 1$ is equivalent to maximizing $z = ABC$ subject to $Aa + Bb + Cc = 1$.

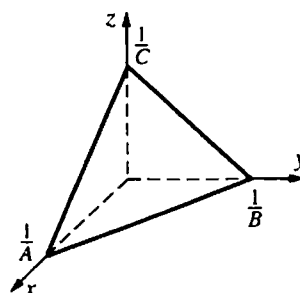
$$C = \frac{1 - aA - bB}{c}, \text{ so } z = \frac{AB(1 - aA - bB)}{c}.$$

$$\nabla z = \left(\frac{1}{c}\right) \left\langle B - 2aAB - bB^2, A - 2bAB - aA^2 \right\rangle = 0 \text{ if } A = \frac{1}{3a}, B = \frac{1}{3b} \left[C = \frac{1}{3c} \right].$$

$\left(\frac{1}{3a}, \frac{1}{3b}\right)$ is the only critical point in the first quadrant. The second partials test yields that z is maximum at this

point. The plane is $\frac{1}{3a}x + \frac{1}{3b}y + \frac{1}{3c}z = 1$, or $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.

The volume of the first quadrant tetrahedron formed by the plane is $\frac{1}{6 \left(\frac{1}{3a}\right) \left(\frac{1}{3b}\right) \left(\frac{1}{3c}\right)} = \frac{9abc}{2}.$



35. Local max: $f(1.75, 0) = 1.15$
Global max: $f(-3.8, 0) = 2.30$

36. Global max: $f(0, 1) = 0.5$
Global min: $f(0, -1) = -0.5$

37. Global min: $f(0, 1) = f(0, -1) = -0.12$

38. Global max: $f(0, 0) = 1$
Global min: $f(2, -2) = f(-2, 2) = e^{-9}$
 ≈ 0.00012341

39. Global max: $f(1.13, 0.79) = f(1.13, -0.79) = 0.53$
Global min: $f(-1.13, 0.79) = f(-1.13, -0.79)$
 $= -0.53$

40. No global maximum or global minimum

41. Global max: $f(3, 3) = f(-3, 3) \approx 74.9225$
Global min: $f(1.5708, 0) = f(-1.5708, 0) = -8$

42. Global max: $f(1, 43, 0) = 0.13$
Global min: $f(-1.82, 0) = -0.23$

43. Global max: $f(0.67, 0) = 5.06$
Global min: $f(-0.75, 0) = -3.54$

44. Global max: $f(-5.12, -4.92) = 1071$
Global min: $f(5.24, -4.96) = -658$

46. a.
$$k(\alpha, \beta) = \frac{1}{2}[80\sin\alpha + 60\sin\beta + 48\sin(2\pi - \alpha - \beta)]$$
$$= 40\sin\alpha + 30\sin\beta - 24\sin(\alpha + \beta)$$
$$L(\alpha, \beta) = (164 - 160\cos\alpha)^{1/2} + (136 - 120\cos\beta)^{1/2}$$
$$+ (100 - 96\cos(\alpha + \beta))^{1/2}$$

b. (1.95, 2.04)

c. (2.26, 2.07)

45. Global max: $f(2.1, 2.1) = 3.5$
Global min: $f(4.2, 4.2) = -3.5$

15.9 Concepts Review

1. free; constrained

2. parallel

3. $g(x, y) = 0$

4. (2, 2)

2. $-2x + y = \lambda y$

3. $x^2 + y^2 = 1$

4. $0 = \lambda x + 2\lambda y$ (From equations 1 and 2)

5. $\lambda = 0$ or $x + 2y = 0$ (4)

$\lambda = 0$: 6. $y = 2x$ (1)

7. $x = \pm \frac{1}{\sqrt{5}}$ (6, 3)

8. $y = \pm \frac{2}{\sqrt{5}}$ (7, 6)

$x + 2y = 0$: 9. $x = -2y$

10. $y = \pm \frac{1}{\sqrt{5}}$ (9, 3)

11. $x = \frac{2}{\sqrt{5}}$ (10, 9)

Critical points: $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right),$

$\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right), \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$f(x, y)$ is 0 at the first two critical points and 5 at the last two. Therefore, the maximum value of $f(x, y)$ is 5.

4. $\langle 2x + 4y, 4x + 2y \rangle = \lambda \langle 1, -1 \rangle$

$2x + 4y = \lambda, 4x + 2y = -\lambda, x - y = 6$

Critical point is (3, -3).

5. $\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 3, -2 \rangle$

$2x = \lambda, 2y = 3\lambda, 2z = -2\lambda, x + 3y - 2z = 12$

Critical point is $\left(\frac{6}{7}, \frac{18}{7}, -\frac{12}{7}\right)$.

Problem Set 15.9

1. $\langle 2x, 2y \rangle = \lambda \langle y, x \rangle$

$2x = \lambda y, 2y = \lambda x, xy = 3$

Critical points are

$(\pm\sqrt{3}, \pm\sqrt{3}), f(\pm\sqrt{3}, \pm\sqrt{3}) = 6.$

It is not clear whether 6 is the minimum or maximum, so take any other point on $xy = 3$, for example (1, 3). $f(1, 3) = 10$, so 6 is the minimum value.

2. $\langle y, x \rangle = \lambda \langle 8x, 18y \rangle$

$y = 8\lambda x, x = 18\lambda y, 4x^2 + 9y^2 = 36$

Critical points are $\left(\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right).$

Maximum value of 3 occurs at $\left(\pm\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right).$

3. Let $\nabla f(x, y) = \lambda \nabla g(x, y)$, where

$g(x, y) = x^2 + y^2 - 1 = 0.$

$\langle 8x - 4y, -4x + 2y \rangle = \lambda \langle 2x, 2y \rangle$

1. $4x - 2y = \lambda x$

$f\left(\frac{6}{7}, \frac{18}{7}, -\frac{12}{7}\right) = \frac{72}{7}$ is the minimum (e.g., $f(12, 0, 0) = 144$).

6. Let $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, where

$$g(x, y, z) = 2x^2 + y^2 - 3z = 0.$$

$$\langle 4, -2, 3 \rangle = \lambda \langle 4x, 2y, -3 \rangle$$

$$1. 4 = 4\lambda x$$

$$2. -2 = 2\lambda y$$

$$3. 3 = -3\lambda$$

$$4. 2x^2 + y^2 - 3z = 0$$

$$5. \lambda = -1 \quad (3)$$

$$6. x = -1, y = 1 \quad (5, 1, 2)$$

$$7. z = 1 \quad (6, 4)$$

Therefore, $(-1, 1, 1)$ is a critical point, and $f(-1, 1, 1) = -3$. (-3 is the minimum rather than maximum since other points satisfying $g = 0$ have larger values of f . For example, $g(1, 1, 1) = 0$, and $f(1, 1, 1) = 5$.)

7. Let l and w denote the dimensions of the base, h denote the depth. Maximize $V(l, w, h) = lwh$ subject to $g(l, w, h) = lw + 2lh + 2wh = 48$.
 $\langle wh, lh, lw \rangle = \lambda \langle w + 2h, l + 2h, 2l + 2w \rangle$
 $wh = \lambda(w + 2h), lh = \lambda(l + 2h), lw = \lambda(2l + 2w)$,
 $lw + 2lh + 2wh = 48$
 Critical point is $(4, 4, 2)$.
 $V(4, 4, 2) = 32$ is the maximum. ($V(11, 2, 1) = 22$, for example.)

8. Minimize the square of the distance to the plane,

$$f(x, y, z) = x^2 + y^2 + z^2, \text{ subject to}$$

$$x + 3y - 2z - 4 = 0.$$

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 3, -2 \rangle$$

$$2x = \lambda, 2y = 3\lambda, 2z = -2\lambda, x + 3y - 2z = 4$$

$$\text{Critical point is } \left(\frac{2}{7}, \frac{6}{7}, -\frac{4}{7}\right). \text{ The nature of the}$$

problem indicates that this will give a minimum rather than a maximum. The least distance to the

$$\text{plane is } \left[f\left(\frac{2}{7}, \frac{6}{7}, -\frac{4}{7}\right) \right]^{1/2} = \left(\frac{8}{7}\right)^{1/2} \approx 1.0690.$$

9. Let l and w denote the dimensions of the base, h the depth. Maximize $V(l, w, h) = lwh$ subject to $0.601w + 0.20(lw + 2lh + 2wh) = 12$, which simplifies to $2lw + lh + wh = 30$, or $g(l, w, h) = 2lw + lh + wh - 30$.

$$\text{Let } \nabla V(l, w, h) = \lambda \nabla g(l, w, h):$$

$$\langle wh, lh, lw \rangle = \lambda \langle 2w + h, 2l + h, l + w \rangle.$$

$$1. wh = \lambda(2w + h)$$

$$2. lh = \lambda(2l + h)$$

$$3. lw = \lambda(l + w)$$

$$4. 2lw + lh + wh = 30$$

$$5. (w - l)h = 2\lambda(w - l) \quad (1, 2)$$

$$6. w = l \text{ or } h = 2\lambda$$

$$w = 1:$$

$$7. l = 2\lambda = w \quad (3) \text{ Note: } w \neq 0, \text{ for then } V = 0.$$

$$8. h = 4\lambda \quad (7, 2)$$

$$9. \lambda = \frac{\sqrt{5}}{2} \quad (7, 8, 4)$$

$$10. l = w = \sqrt{5}, h = 2\sqrt{5} \quad (9, 7, 8)$$

$$h = 2\lambda:$$

$$11. \lambda = 0 \quad (2)$$

$$12. l = w = h = 0 \quad (11, 1 - 3)$$

(Not possible since this does not satisfy 4.)

$(\sqrt{5}, \sqrt{5}, 2\sqrt{5})$ is a critical point and

$$V(\sqrt{5}, \sqrt{5}, 2\sqrt{5}) = 10\sqrt{5} \approx 22.36 \text{ ft}^3 \text{ is the}$$

maximum volume (rather than the minimum volume since, for example, $g(1, 1, 14) = 30$ and $V(1, 1, 14) = 14$ which is less than 22.36).

10. Minimize the square of the distance,

$$f(x, y, z) = x^2 + y^2 + z^2, \text{ subject to}$$

$$g(x, y, z) = x^2 y - z^2 + 9 = 0.$$

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 2xy, x^2, -2z \rangle$$

$$2x = 2\lambda xy, 2y = \lambda x^2, 2z = -2\lambda z,$$

$$x^2 y - z^2 + 9 = 0$$

Critical points are $(0, 0, \pm 3)$ [case $x = 0$];

$(\pm\sqrt{2}, -1, \pm\sqrt{7})$ [case $x \neq 0, \lambda = -1$]; and

$(\pm 3\sqrt[6]{2/9}, -\sqrt[3]{9/2}, 0)$ [case $x \neq 0, \lambda \neq -1$].

Evaluating f at each of these eight points yields 9

(case $x = 0$), 10 (case $x \neq 0, \lambda = -1$), and

$\frac{3}{2}\sqrt[3]{2}\left(\sqrt[3]{9}\right)^2$ (case $x \neq 0, \lambda \neq -1$). The latter is

the smallest, so the least distance between the

origin and the surface is $3\sqrt[6]{\frac{3}{4}} \approx 2.8596$.

11. Maximize $f(x, y, z) = xyz$, subject to
 $g(x, y, z) = b^2 c^2 x^2 + a^2 c^2 y^2 + a^2 b^2 z^2 - a^2 b^2 c^2$
 $= 0$

$$\langle yz, xz, xy \rangle = \lambda \langle 2b^2 c^2 x, 2a^2 c^2 y, 2a^2 b^2 z \rangle$$

$$yz = 2b^2 c^2 x, xz = 2a^2 c^2 y, xy = 2a^2 b^2 z,$$

$$b^2 c^2 x^2 + a^2 c^2 y^2 + a^2 b^2 z^2 = a^2 b^2 c^2$$

$$\text{Critical point is } \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right).$$

$$V\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}, \text{ which is the}$$

maximum.

12. Maximize $V(x, y, z) = xyz$, subject to

$$g(x, y, z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0. \text{ Let}$$

$$\nabla V(x, y, z) = \lambda \nabla g(x, y, z), \text{ so}$$

$$\langle yz, xz, xy \rangle = \lambda \left\langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right\rangle. \text{ Then}$$

$$\frac{\lambda x}{a} = \frac{\lambda y}{b} = \frac{\lambda z}{c} \text{ (each equals } xyz).$$

$\lambda \neq 0$ since $\lambda = 0$ would imply $x = y = z = 0$ which would not satisfy the constraint.

$$\text{Thus, } \frac{x}{a} = \frac{y}{b} = \frac{z}{c}. \text{ These along with the}$$

$$\text{constraints yield } x = \frac{a}{3}, y = \frac{b}{3}, z = \frac{c}{3}.$$

$$\text{The maximum value of } V = \frac{abc}{27}.$$

13. A different hint, which will be used here, is to let $Ax + By + Cz = 1$ be the plane. (See write-up for Problem 34, Section 15.8.)

Maximize $f(A, B, C) = ABC$ subject to

$$g(A, B, C) = aA + bB + cC - 1 = 0.$$

$$\text{Let } \langle BC, AC, BA \rangle = \lambda \langle a, b, c \rangle.$$

$$\text{Then } BC = \lambda a, AC = \lambda b, BA = \lambda c,$$

$$aA + bB + cC = 1.$$

Therefore, $\lambda aA = \lambda bB = \lambda cC$ (since each equals ABC), so $aA = bB = cC$ (since $\lambda = 0$ implies

$A = B = C = 0$ which doesn't satisfy the

constraint equation).

15. Let $\alpha + \beta + \gamma = 1$, $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.

Maximize $P(x, y, z) = kx^\alpha y^\beta z^\gamma$, subject to $g(x, y, z) = ax + by + cz - d = 0$.

$$\text{Let } \nabla P(x, y, z) = \lambda \nabla g(x, y, z). \text{ Then } \langle k\alpha x^{\alpha-1} y^\beta z^\gamma, k\beta x^\alpha y^{\beta-1} z^\gamma, k\gamma x^\alpha y^\beta z^{\gamma-1} \rangle = \lambda \langle a, b, c \rangle.$$

$$\text{Therefore, } \frac{\lambda ax}{\alpha} = \frac{\lambda by}{\beta} = \frac{\lambda cz}{\gamma} \text{ (since each equals } kx^\alpha y^\beta z^\gamma).$$

$\lambda \neq 0$ since $\lambda = 0$ would imply $x = y = z = 0$ which would imply $P = 0$.

$$\text{Therefore, } \frac{ax}{\alpha} = \frac{by}{\beta} = \frac{cz}{\gamma} (*).$$

The constraints $ax + by + cz = d$ in the form $\alpha \left(\frac{ax}{\alpha} \right) + \beta \left(\frac{by}{\beta} \right) + \gamma \left(\frac{cz}{\gamma} \right) = d$ becomes

$$\alpha \left(\frac{ax}{\alpha} \right) + \beta \left(\frac{ax}{\alpha} \right) + \gamma \left(\frac{ax}{\alpha} \right) = d, \text{ using } (*).$$

$$\text{Then } (\alpha + \beta + \gamma) \left(\frac{ax}{\alpha} \right) = d, \text{ or } \frac{ax}{\alpha} = d \text{ (since } \alpha + \beta + \gamma = 1).$$

$$x = \frac{\alpha d}{a} (**); y = \frac{\beta d}{b} \text{ and } z = \frac{\gamma d}{c} \text{ then following using } (*) \text{ and } (**).$$

Then $3aA = 1$, so $A = \frac{1}{3a}$; similarly $B = \frac{1}{3b}$ and

$C = \frac{1}{3c}$. The rest follows as in the solution for

Problem 34, Section 15.8.

14. It is clear that the maximum will occur for a triangle which contains the center of the circle. (With this observation in mind, there are additional constraints: $0 < \alpha < \pi$, $0 < \beta < \pi$, $0 < \gamma < \pi$.)

Note that in an isosceles triangle, the side opposite the angle θ which is between the congruent sides of length r has length

$$2r \sin\left(\frac{\theta}{2}\right). \text{ Then we wish to maximize}$$

$$P(\alpha, \beta, \gamma) = 2r \left[\sin\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\beta}{2}\right) + \sin\left(\frac{\gamma}{2}\right) \right]$$

subject to $g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - 2\pi = 0$.

$$\text{Let } r \left\langle \cos\left(\frac{\alpha}{2}\right), \cos\left(\frac{\beta}{2}\right), \cos\left(\frac{\gamma}{2}\right) \right\rangle = \lambda \langle 1, 1, 1 \rangle.$$

Then $\lambda = r \cos\left(\frac{\alpha}{2}\right) = r \cos\left(\frac{\beta}{2}\right) = r \cos\left(\frac{\gamma}{2}\right)$, so

$$\alpha = \beta = \gamma \text{ (since } \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \pi).$$

$$3\alpha = 2\pi, \text{ so } \alpha = \frac{2\pi}{3}; \text{ then } \beta = \gamma = \frac{2\pi}{3}.$$

Since there is only one interior critical point, and since P is 0 on the boundary, P is maximum when

$$x = \frac{\alpha d}{a}, y = \frac{\beta d}{b}, z = \frac{\gamma d}{c}.$$

16. Let (x, y, z) denote a point of intersection. Let $f(x, y, z)$ be the square of the distance to the origin.

Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to

$$g(x, y, z) = x + y + z - 8 = 0 \text{ and}$$

$$h(x, y, z) = 2x - y + 3z - 28 = 0.$$

$$\text{Let } \nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z).$$

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 1, 1 \rangle + \mu \langle 2, -1, 3 \rangle$$

$$1. 2x = \lambda + 2\mu$$

$$2. 2y = \lambda - \mu$$

$$3. 2z = \lambda + 3\mu$$

$$4. x + y + z = 8$$

$$5. 2x - y + 3z = 28$$

$$6. 3\lambda + 4\mu = 16 \quad (1, 2, 3, 4)$$

$$7. 2\lambda + 7\mu = 28 \quad (1, 2, 3, 5)$$

$$8. \lambda = 0, \mu = 4 \quad (6, 7)$$

$$9. x = 4, y = -2, z = 6 \quad (8, 1-3)$$

$f(4, -2, 6) = 56$, and the nature of the problem indicates this is the minimum rather than the maximum.

Conclusion: The least distance is $\sqrt{56} \approx 7.4833$.

$$17. \langle -1, 2, 2 \rangle = \lambda \langle 2x, 2y, 0 \rangle + \mu \langle 0, 1, 2 \rangle$$

$$-1 = 2\lambda x, 2 = 2\lambda y + \mu, 2 = 2\mu, x^2 + y^2 = 2,$$

$$y + 2z = 1$$

Critical points are $(-1, 1, 0)$ and $(1, -1, 1)$.

$f(-1, 1, 0) = 3$, the maximum value;

$f(1, -1, 1) = -1$, the minimum value.

18. a. Maximize

$$w(x_1, x_2, \dots, x_n) = x_1 x_2 \dots x_n, (x_i > 0)$$

subject to the constraint

$$g(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n - 1 = 0. \text{ Let}$$

$$\nabla w(x_1, x_2, \dots, x_n) = \lambda \nabla g(x_1, x_2, \dots, x_n).$$

$$\langle x_2 \dots x_n, x_1 x_3 \dots x_n, x_1 \dots x_{n-1} \rangle = \lambda \langle 1, 1, \dots, 1 \rangle.$$

Therefore, $\lambda x_1 = \lambda x_2 = \dots = \lambda x_n$ (since each equals $x_1 x_2 \dots x_n$). Then $x_1 = x_2 = \dots = x_n$.

(If $\lambda = 0$, some $x_i = 0$, so $w = 0$.)

$$\text{Therefore, } nx_i = 1; x_i = \frac{1}{n}.$$

The maximum value of w is $\left(\frac{1}{n}\right)^n$, and occurs

$$\text{when each } x_i = \frac{1}{n}.$$

- b. From part a we have that $x_1 x_2 \dots x_n \leq \left(\frac{1}{n}\right)^n$.

$$\text{Therefore, } \sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{1}{n}.$$

$$\text{If } x_i = \frac{a_i}{a_1 + \dots + a_n} = \frac{a_i}{A} \text{ for each } i, \text{ then}$$

$$\sqrt[n]{\frac{a_1}{A} \frac{a_2}{A} \dots \frac{a_n}{A}} \leq \frac{1}{n}, \text{ so } \sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{A}{n}, \text{ or}$$

$$\sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n}.$$

19. Let $\langle a_1, a_2, \dots, a_n \rangle = \lambda \langle 2x_1, 2x_2, \dots, 2x_n \rangle$.

Therefore, $a_i = 2\lambda x_i$, for each $i = 1, 2, \dots, n$ (since $\lambda = 0$ implies $a_i = 0$, contrary to the hypothesis).

$$\frac{x_i}{a_i} = \frac{x_j}{a_j} \text{ for all } i, j \left(\text{since each equals } \frac{1}{2\lambda} \right).$$

The constraint equation can be expressed

$$a_1^2 \left(\frac{x_1}{a_1} \right)^2 + a_2^2 \left(\frac{x_2}{a_2} \right)^2 + \dots + a_n^2 \left(\frac{x_n}{a_n} \right)^2 = 1.$$

$$\text{Therefore, } (a_1^2 + a_2^2 + \dots + a_n^2) \left(\frac{x_1}{a_1} \right)^2 = 1.$$

$$x_1^2 = \frac{a_1^2}{a_1^2 + \dots + a_n^2}; \text{ similar for each other } x_i^2.$$

The function to be maximized is a hyperplane with positive coefficients and constant (so intercepts on all axes are positive), and the constraint is a hypersphere of radius 1, so the maximum will occur where each x_i is positive.

There is only one such critical point, the one obtained from the above by taking the principal square root to solve for x_i .

Then the maximum value of w is

$$a_1 \left(\frac{a_1}{\sqrt{A}} \right) + a_2 \left(\frac{a_2}{\sqrt{A}} \right) + \dots + a_n \left(\frac{a_n}{\sqrt{A}} \right) = \frac{A}{\sqrt{A}} = \sqrt{A}$$

$$\text{where } A = a_1^2 + a_2^2 + \dots + a_n^2.$$

$$20. \text{Max: } f(-0.71, 0.71) = f(-0.71, -0.71) = 0.71$$

$$21. \text{Min: } f(4, 0) = -4$$

$$22. \text{Max: } f(1.41, 1.41) = f(-1.41, -1.44) = 0.037$$

$$23. \text{Min: } f(0, 3) = f(0, -3) = -0.99$$

15.10 Chapter Review

Concepts Test

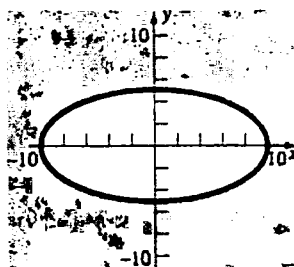
1. True: Except for the trivial case of $z = 0$, which gives a point.
2. False: Use $f(0, 0) = 0$; $f(x, y) = \frac{xy}{x^2 + y^2}$ elsewhere for counterexample.
3. True: Since $g'(0) = f_x(0, 0)$
4. True: It is the limit along the path, $y = x$.
5. True: Use "Continuity of a Product" Theorem.
6. True: Straight forward calculation of partial derivatives
7. False: See Problem 25, Section 15.4.
8. False: It is perpendicular to the level curves of f . The gradient of $F(x, y, z) = f(x, y) - z$ is perpendicular to the graph of $z = f(x, y)$.
9. True: Since $\langle 0, 0, -1 \rangle$ is normal to the tangent plane
10. False: C^{∞} : For the cylindrical surface $f(x, y) = y^3$, $f(p) = 0$ for every p on the x -axis, but $f(p)$ is not an extreme value.
11. True: It will point in the direction of greatest increase of heat, and at the origin, $\nabla T(0, 0) = \langle 1, 0 \rangle$ is that direction.
12. True: It is nonnegative for all x, y , and it has a value of 0 at $(0, 0)$.
13. True: Along the x -axis, $f(x, 0) \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$.
14. False: $|D_u f(x, y)| = |\langle 4, 4 \rangle \cdot \mathbf{u}| \leq 4\sqrt{2}$
(equality if $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)\langle 1, 1 \rangle$)
15. True: $-D_u f(x, y) = -[\nabla f(x, y) \cdot \mathbf{u}]$
 $= \nabla f(x, y) \cdot (-\mathbf{u}) = D_{-\mathbf{u}} f(x, y)$

16. True: The set (call it S , a line segment) contains all of its boundary points because for every point P not in S (i.e., not on the line segment), there is an open neighborhood of P (i.e., a circle with P as center) that contains no point of S .
17. True: By the Min-Max Existence Theorem
18. False: (x_0, y_0) could be a singular point.
19. False: $f\left(\frac{\pi}{2}, 1\right) = \sin\left(\frac{\pi}{2}\right) = 1$, the maximum value of f , and $(\pi/2, 1)$ is in the set.
20. False: The same function used in Problem 2 provides a counterexample.

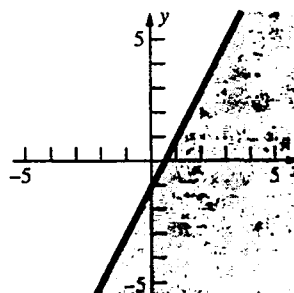
Sample Test Problems

1. a. $x^2 + 4y^2 - 100 \geq 0$

$$\frac{x^2}{100} + \frac{y^2}{25} \geq 1$$

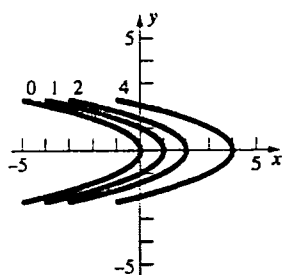


b. $2x - y - 1 \geq 0$



2. $x + y^2 = k$

$x - k = -y^2$



3. $f_x(x, y) = 12x^3y^2 + 14xy^7$

$f_{xx}(x, y) = 36x^2y^2 + 14y^7$

$f_{xy}(x, y) = 24x^3y + 98xy^6$

4. $f_x(x, y) = -2\cos x \sin x = -\sin 2x$

$f_{xx}(x, y) = -2x \cos 2x$

$f_{xy}(x, y) = 0$

5. $f_x(x, y) = e^{-y} \sec^2 x$

$f_{xx}(x, y) = 2e^{-y} \sec^2 x \tan x$

$f_{xy}(x, y) = -e^{-y} \sec^2 x$

6. $f_x(x, y) = -e^{-x} \sin y$

$f_{xx}(x, y) = e^{-x} \sin y$

$f_{xy}(x, y) = -e^{-x} \cos y$

7. $F_y(x, y) = 30x^3y^5 - 7xy^6$

$F_{yy}(x, y) = 150x^3y^4 - 42xy^5$

$F_{yyx}(x, y) = 450x^2y^4 - 42y^5$

8. $f_x(x, y, z) = y^3 - 10xyz^4$

$f_y(x, y, z) = 3xy^2 - 5x^2z^4$

$f_z(x, y, z) = -20x^2yz^3$

Therefore, $f_x(2, -1, 1) = 19$;

$f_y(2, -1, 1) = -14$; $f_z(2, -1, 1) = 80$

9. $z_y(x, y) = \frac{y}{2}$; $z_y(2, 2) = \frac{2}{2} = 1$

10. Everywhere in the plane except on the parabola $x^2 = y$.

11. No. On the path $y = x$, $\lim_{x \rightarrow 0} \frac{x-x}{x+x} = 0$. On the

path $y = 0$, $\lim_{x \rightarrow 0} \frac{x-0}{x+0} = 1$.

12. a. $\lim_{(x,y) \rightarrow (2,2)} \frac{x^2 - 2y}{x^2 + 2y} = \frac{4-4}{4+4} = 0$

b. Does not exist since $\left[\begin{matrix} \rightarrow 4 \\ \rightarrow 0 \end{matrix} \right]$.

c. $\lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + 2y^2)(x^2 - 2y^2)}{x^2 + 2y^2} = \lim_{(x,y) \rightarrow (0,0)} (x^2 - y^2) = 0$

13. a. $\nabla f(x, y, z) = \langle 2xyz^3, x^2z^3, 3x^2yz^2 \rangle$
 $f(1, 2, -1) = \langle -4, -1, 6 \rangle$

b. $\nabla f(x, y, z) = \langle y^2z \cos xz, 2y \sin xz, xy^2 \cos xz \rangle$
 $\nabla f(1, 2, -1) = -4 \langle \cos(1), \sin(1), -\cos(1) \rangle$
 $\approx \langle -2.1612, -3.3659, 2.1612 \rangle$

14. $D_u f(x, y) = \langle 3y(1+9x^2y^2)^{-1}, 3x(1+9x^2y^2)^{-1} \rangle \cdot \mathbf{u}$

$D_u f(4, 2) = \left\langle \frac{6}{577}, \frac{12}{577} \right\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$
 $= \frac{(3\sqrt{3}-6)}{577} \approx -0.001393$

15. $z = f(x, y) = x^2 + y^2$

$\langle 1, -\sqrt{3}, 0 \rangle$ is horizontal and is normal to the vertical plane that is given. By inspection, $\langle \sqrt{3}, 1, 0 \rangle$ is also a horizontal vector and is perpendicular to $\langle 1, -\sqrt{3}, 0 \rangle$ and therefore is

parallel to the vertical plane. Then $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ is the corresponding 2-dimensional unit vector.

$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$
 $= \langle 2x, 2y \rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = \sqrt{3}x + y$

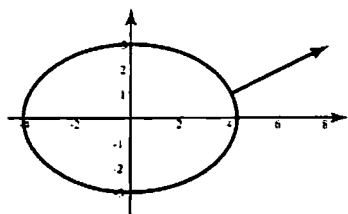
$D_u f(1, 2) = \sqrt{3} + 2 \approx 3.7321$ is the slope of the tangent to the curve.

16. In the direction of $\nabla f(1, 2) = 4\langle 9, 4 \rangle$

17. a. $f(4, 1) = 9$, so $\frac{x^2}{2} + y^2 = 9$, or $\frac{x^2}{18} + \frac{y^2}{9} = 1$.

b. $\nabla f(x, y) = \langle x, 2y \rangle$, so $f(4, 1) = \langle 4, 2 \rangle$.

c.



18. $F_x = F_u u_x + F_v v_x$

$$= \frac{v}{1+u^2 v^2} \frac{y}{2\sqrt{xy}} + \frac{u}{1+u^2 v^2} \frac{1}{2\sqrt{x}}$$

$$= \frac{v\sqrt{y} + u}{2(1+u^2 v^2)\sqrt{x}}$$

 $F_y = F_u u_y + F_v v_y$

$$= \frac{v}{1+u^2 v^2} \frac{x}{2\sqrt{xy}} + \frac{u}{1+u^2 v^2} \frac{-1}{2\sqrt{y}}$$

$$= \frac{v\sqrt{x} - u}{2(1+u^2 v^2)\sqrt{y}}$$

22. $\frac{dc}{dt} = 3, \frac{db}{dt} = -2, \frac{d\alpha}{dt} = 0.1$

Area = $A(b, c, \alpha) = \left(\frac{1}{2}\right) c(b \sin \alpha)$

$$\frac{dA}{dt} = \left[\left(\frac{b}{2}\right) (\sin \alpha) \left(\frac{dc}{dt}\right) + \left(\frac{c}{2}\right) (\sin \alpha) \left(\frac{db}{dt}\right) + \left(\frac{bc}{2}\right) (\cos \alpha) \left(\frac{d\alpha}{dt}\right) \right]$$

$$\left(\frac{dA}{dt}\right) \bigg|_{(8, 10, \frac{\pi}{6})} = \frac{(7 + 4\sqrt{3})}{2} \approx 6.9641 \text{ in.}^2/\text{s}$$

23. Let $F(x, y, z) = 9x^2 + 4y^2 + 9z^2 - 34 = 0$

$\nabla F(x, y, z) = \langle 18x, 8y, 18z \rangle$, so $\nabla f(1, 2, -1) = 2\langle 9, 8, -9 \rangle$.

Tangent plane is $9(x-1) + 8(y-2) - 9(z+1) = 0$, or $9x + 8y - 9z = 34$.

24. $V = \pi r^2 h$; $dV = V_r dr + V_h dh = 2\pi r h dr + \pi r^2 dh$

If $r = 10$, $|dr| \leq 0.02$, $h = 6$, $|dh| = 0.01$, then

$|dV| \leq 2\pi r h |dr| + \pi r^2 |dh| \leq 2\pi (10)(6)(0.02) + \pi (100)(0.01) = 3.4\pi$

19. $f_x = f_u u_x + f_v u_y = \left(\frac{1}{v}\right)(2x) + \left(-\frac{u}{v^2}\right)(yz)$
 $= x^{-2} y^{-1} z^{-1} (x^2 + 3y - 4z)$

$f_y = f_u u_y + f_v v_y = \left(\frac{1}{v}\right)(-3) + \left(-\frac{u}{v^2}\right)(xz)$
 $= -x^{-1} y^{-2} z^{-1} (x^2 + 4z)$

$f_z = f_u u_z + f_v v_z = \left(\frac{1}{v}\right)(4) + \left(-\frac{u}{v^2}\right)(xy)$
 $= x^{-1} y^{-1} z^{-2} (3y - x^2)$

20. $\frac{dF}{dt} = \frac{F}{x} \frac{dx}{dt} + \frac{F}{y} \frac{dy}{dt}$
 $= (3x^2 - y^2)(-6\sin 3t) + (-2xy - 4y^3)(2\cos t)$
 $t = 0 \Rightarrow x = 2 \text{ and } y = 0$, so $\left(\frac{dF}{dt}\right) \bigg|_{t=0} = 0$.

21. $F_t = F_x x_t + F_y y_t + F_z z_t$
 $= \left(\frac{10xy}{z^3}\right) \left(\frac{3t^{1/2}}{2}\right) + \left(\frac{5x^2}{z^3}\right) \left(\frac{1}{t}\right) + \left(-\frac{15x^2 y}{z^4}\right) (3e^{3t})$
 $= \frac{15xy\sqrt{t}}{z^3} + \frac{5x^2}{z^3 t} - \frac{45x^2 y e^{3t}}{z^4}$

$$V(10, 6) = \pi (100)(6) = 600\pi$$

Volume is $600\pi \pm 3.4\pi \approx 1884.96 \pm 10.68$

25. $df = y^2(1+z^2)^{-1}dx + 2xy(1+z^2)^{-1}dy - 2xy^2z(1+z^2)^{-2}dz$
 If $x = 1, y = 2, z = 2, dx = 0.01, dy = -0.02, dz = 0.03$, then $df = -0.0272$.
 Therefore, $f(1.01, 1.98, 2.03) \approx f(1, 2, 2) + df = 0.8 - 0.0272 = 0.7728$

26. $\nabla f(x, y) = \langle 2xy - 6x, x^2 - 12y \rangle = \langle 0, 0 \rangle$
 at $(0, 0)$ and $(\pm 6, 3)$.
 $D = f_{xx}f_{yy} - f_{xy}^2 = (2y - 6)(-12) - (2x)^2$
 $= 4(18 - 6y - x^2)$; $f_{xx} = 2(y - 3)$
 At $(0, 0)$: $D = 72 > 0$ and $f_{xx} < 0$, so local maximum at $(0, 0)$.
 At $(\pm 6, 3)$: $D < 0$, so $(\pm 6, 3)$ are saddle points.

Critical points are $\left(\frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$. Maximum of $\frac{1}{2}$ at $\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$; minimum of $-\frac{1}{2}$ at $\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

27. Let (x, y, z) denote the coordinates of the 1st octant vertex of the box. Maximize $f(x, y, z) = xyz$ subject to $g(x, y, z) = 36x^2 + 4y^2 + 9z^2 - 36 = 0$ (where $x, y, z > 0$ and the box's volume is $V(x, y, z) = f(x, y, z)$).
 Let $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.

$$\langle yz, xz, xy \rangle 8 = \lambda \langle 72x, 8y, 18z \rangle$$

$$1. 8yz = 72\lambda x$$

$$2. 8xz = 8\lambda y$$

$$3. 8xy = 18\lambda z$$

$$4. 36x^2 + 4y^2 + 9z^2 = 36$$

$$5. \frac{yz}{xz} = \frac{72\lambda x}{8\lambda y}, \text{ so } y^2 = 9x^2. \quad (1, 2)$$

$$6. \frac{yz}{xz} = \frac{72\lambda x}{18\lambda y}, \text{ so } z^2 = 4x^2. \quad (1, 3)$$

$$7. 36x^2 + 36x^2 + 36x^2 = 36, \text{ so } x = \frac{1}{\sqrt{3}}. \quad (5, 6, 4)$$

$$8. y = \frac{3}{\sqrt{3}}, z = \frac{2}{\sqrt{3}} \quad (7, 5, 6)$$

$$V\left(\frac{1}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = 8\left(\frac{1}{\sqrt{3}}\right)\left(\frac{3}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)$$

$$= \frac{16}{\sqrt{3}} \approx 9.2376$$

The nature of the problem indicates that the critical point yields a maximum value rather than a minimum value.

(For a generalization of this problem, see Problem 11 or Section 15.9.)

29. Maximize $V(r, h) = \pi r^2 h$, subject to

$$S(r, h) = 2\pi r^2 + 2\pi rh - 24\pi = 0.$$

$$\langle 2\pi rh, \pi r^2 \rangle = \lambda \langle 4\pi r + 2\pi h, 2\pi r \rangle$$

$$rh = \lambda(2r + h), r = 2\lambda, r^2 + rh = 12$$

Critical point is $(2, 4)$. The nature of the problem indicates that the critical point yields a maximum value rather than a minimum value. Conclusion: The dimensions are radius of 2 and height of 4.

28. $\langle y, x \rangle = \lambda \langle 2x, 2y \rangle$
 $y = 2\lambda x, x = 2\lambda y, x^2 + y^2 = 1$

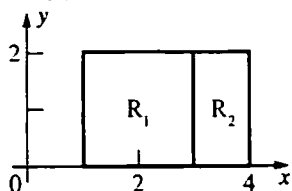
The Integral in n -Space

16.1 Concepts Review

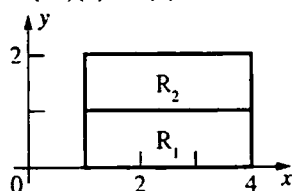
- $\sum_{k=1}^n f(\bar{x}_k, \bar{y}_k) \Delta A_k$
- the volume of the solid under $z = f(x, y)$ and above R
- continuous
- 12

Problem Set 16.1

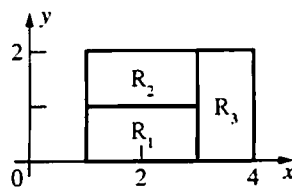
$$1. \iint_{R_1} 2 \, dA + \iint_{R_2} 3 \, dA = 2A(R_1) + 3A(R_2) \\ = 2(4) + 3(2) = 14$$



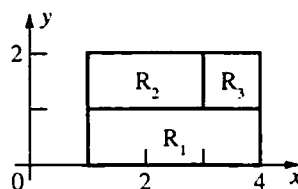
$$2. \iint_{R_1} (-1) \, dA + \iint_{R_2} 2 \, dA = (-1)A(R_1) + 2A(R_2) \\ = (-1)(3) + 2(3) = 3$$



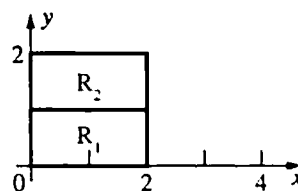
$$3. \iint_R f(x, y) \, dA = \iint_{R_1} 2 \, dA + \iint_{R_2} 1 \, dA + \iint_{R_3} 3 \, dA \\ = 2A(R_1) + 1A(R_2) + 3A(R_3) \\ = 2(2) + 1(2) + 3(2) = 12$$



$$4. \iint_{R_1} 2 \, dA + \iint_{R_2} 3 \, dA + \iint_{R_3} 1 \, dA \\ = 2A(R_1) + 3A(R_2) + 1A(R_3) \\ = 2(3) + 3(2) + 1(1) = 13$$



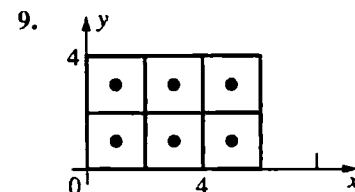
$$5. 3 \iint_R f(x, y) \, dA - \iint_R g(x, y) \, dA = 3(3) - (5) = 4$$



$$6. 2 \iint_R f(x, y) \, dA + 5 \iint_R g(x, y) \, dA \\ = 2(3) + 5(5) = 31$$

$$7. \iint_R g(x, y) \, dA - \iint_{R_1} g(x, y) \, dA = (5) - (2) = 3$$

$$8. 2 \iint_{R_1} g(x, y) \, dA + \iint_{R_1} 3 \, dA = 2(2) + 3A(R_1) \\ = 4 + 3(2) = 10$$



$$[f(1, 1) + f(3, 1) + f(5, 1) + f(1, 3) + f(3, 3) \\ + f(5, 3)](4) = [(10) + (8) + (6) + (8) + (6) \\ + (4)](4) = 168$$

$$10. 4(9 + 9 + 9 + 1 + 1 + 1) = 120$$

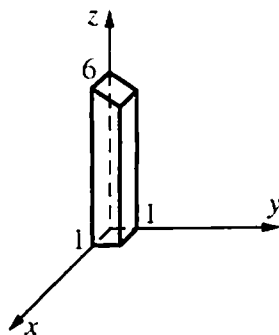
$$11. 4(3 + 11 + 27 + 19 + 27 + 43) = 520$$

$$12. \left[\left(\frac{41}{6} \right) + \left(\frac{33}{6} \right) + \left(\frac{25}{6} \right) + \left(\frac{35}{6} \right) + \left(\frac{27}{6} \right) + \left(\frac{19}{6} \right) \right] (4) = 120$$

$$13. 4(\sqrt{2} + \sqrt{4} + \sqrt{6} + \sqrt{4} + \sqrt{6} + \sqrt{8}) \approx 52.5665$$

$$14. 4(e + e^3 + e^5 + e^3 + e^9 + e^{15}) \approx 13109247$$

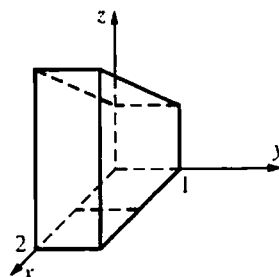
15.



$z = 6 - y$ is a plane parallel to the x -axis. Let T be the area of the front trapezoidal face; let D be the distance between the front and back faces.

$$\iint_R (6 - y) dA = \text{volume of solid} = (T)(D) \\ = \left[\left(\frac{1}{2} \right) (6 + 5) \right] (1) = 5.5$$

16.



$z = 1 + x$ is a plane parallel to the y -axis.

$\iint_R (1 + x) dA$ is the product of the area of a trapezoidal side face and the distance between the side faces.

$$= \left[\left(\frac{1}{2} \right) (1 + 3)(2) \right] (1) = 4$$

$$17. \iint_R 0 dA = 0A(R) = 0$$

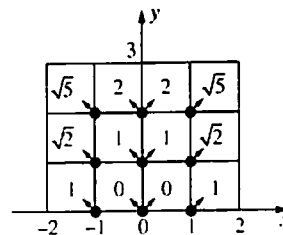
The conclusion follows.

$$18. \iint_R m dA < \iint_R f(x, y) dA < \iint_R M dA$$

(Comparison property)

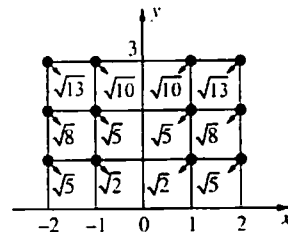
$$\text{Therefore, } m\alpha(R) < \iint_R f(x, y) dA < M\alpha(R)$$

19.



For c , take the sample point in each square to be the point of the square that is closest to the origin.

$$\text{Then } c = 2\sqrt{5} + 2\sqrt{2} + 2(2) + 4(1) \approx 15.3006$$



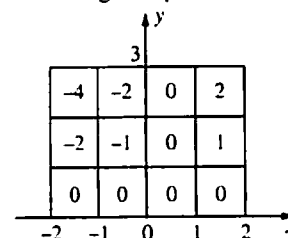
For C , take the sample point in each square to be the point of the square that is farthest from the origin. Then,

$$C = 2\sqrt{13} + 2\sqrt{10} + 2\sqrt{8} + 4\sqrt{5} + 2\sqrt{2} \approx 30.9652.$$

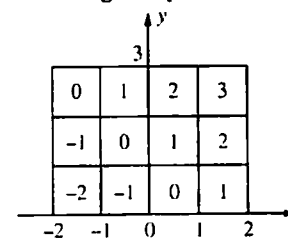
20. The integrand is symmetric with respect to the y -axis, so the value of the integral is 0.

21. The values of $\lfloor x \rfloor \lfloor y \rfloor$ and $\lfloor x \rfloor + \lfloor y \rfloor$ are indicated in the various square subregions of R . In each case the value of the integral on R is the sum of the values in the squares since the area of each square is 1.

a. The integral equals -6.



b. The integral equals 6.



22. Mass of the plate in grams

23. Total rainfall in Colorado in 1999; average rainfall in Colorado in 1999

24. For each partition of R , each subrectangle contains some points at which $f(x, y) = 0$ and some points at which $f(x, y) = 1$. Therefore, for each partition there are sample points for which the Riemann sum is 0 and others for which the Riemann sum is $(1)[\text{Area}(R)] = 12$.

25. To begin, we divide the region R (we will use the outline of the contour plot) into 16 equal squares. Then we can approximate the volume by

$$V = \iint_R f(x, y) dA \approx \sum_{k=1}^{16} f(\bar{x}_k, \bar{y}_k) \Delta A_k.$$

Each square will have $\Delta A = (1 \cdot 1) = 1$ and we will use the height at the center of each square as $f(\bar{x}_k, \bar{y}_k)$.

Therefore, we get

$$\begin{aligned} V &\approx \sum_{k=1}^{16} f(\bar{x}_k, \bar{y}_k) = 20 + 21 + 24 + 29 + 22 + 23 + 26 + 32 + 26 + 27 + 30 + 35 + 32 + 33 + 36 + 42 \\ &= 458 \text{ cubic units} \end{aligned}$$

16.2 Concepts Review

1. iterated
2. $\int_{-1}^2 \left[\int_0^2 f(x, y) dy \right] dx, \int_0^2 \left[\int_{-1}^2 f(x, y) dx \right] dy$
3. signed; plus; minus
4. is below the xy -plane

Problem Set 16.2

1. $\int_0^2 \left[\left(\frac{1}{2} \right) x^2 y^2 \right]_{y=1}^3 dx = \int_0^2 4x^2 dx = \frac{32}{3}$
2. $\int_{-1}^4 \left[xy + \left(\frac{1}{3} \right) y^3 \right]_{y=1}^2 dx = \int_{-1}^4 \left(x + \frac{7}{3} \right) dx = \frac{115}{6}$
3. $\int_1^2 \left[\frac{x^2 y}{2} + xy^2 \right]_{x=0}^3 dx = \int_1^2 \left(\frac{9y}{2} + 3y^2 \right) dy$
 $= \left[\frac{9y^2}{4} + y^3 \right]_1^2 = 17 - \frac{13}{4} = \frac{55}{4} = 13.75$
4. $\int_{-1}^1 \left[\left(\frac{1}{3} \right) x^3 + xy^2 \right]_{x=1}^2 dy = \int_{-1}^1 \left(\frac{7}{3} + y^2 \right) dy = \frac{16}{3}$
5. $\int_0^\pi \left[\left(\frac{1}{2} \right) x^2 \sin y \right]_{x=0}^1 dy = \int_0^\pi \left(\frac{1}{2} \right) \sin y dy = 1$

$$\begin{aligned} 6. \quad \int_0^{\ln 3} \int_0^{\ln 2} e^x e^y dy dx &= \int_0^{\ln 3} [e^x e^y]_{y=0}^{\ln 2} dx \\ &= \int_0^{\ln 3} [e^x (2) - e^x (1)] dx = \int_0^{\ln 3} e^x dx \\ &= [e^x]_0^{\ln 3} = 3 - 1 = 2 \end{aligned}$$

$$\begin{aligned} 7. \quad \int_0^{\pi/2} [-\cos xy]_{y=0}^1 dx &= \int_0^{\pi/2} (1 - \cos x) dx \\ &= \frac{\pi}{2} - 1 \approx 0.5708 \end{aligned}$$

$$8. \quad \int_0^1 [e^{xy}]_{y=0}^1 dx = \int_0^1 (e^x - 1) dx = e - 2 \approx 0.7183$$

$$\begin{aligned} 9. \quad \int_0^3 \left[\frac{2(x^2 + y)^{3/2}}{3} \right]_{x=0}^1 dy &= \int_0^3 \frac{2[(1+y)^{3/2} - y^{3/2}]}{3} dy \\ &= \left[\frac{4[(1+y)^{5/2} - y^{5/2}]}{15} \right]_0^3 = \frac{4(32 - 9\sqrt{3}) - 4}{15} \\ &= \frac{4(31 - 9\sqrt{3})}{15} \approx 4.1097 \end{aligned}$$

$$\begin{aligned} 10. \quad \int_0^1 [-(xy+1)^{-1}]_{x=0}^1 dy &= \int_0^1 \left(1 - \frac{1}{y+1} \right) dy \\ &= 1 - \ln 2 \approx 0.3069 \end{aligned}$$

$$\begin{aligned} 11. \quad \int_0^{\ln 3} \left[\left(\frac{1}{2} \right) \exp(xy^2) \right]_{y=0}^1 dx &= \int_0^{\ln 3} \left(\frac{1}{2} \right) (e^x - 1) dx \\ &= 1 - \left(\frac{1}{2} \right) \ln 3 \approx 0.4507 \end{aligned}$$

$$12. \int_0^1 \left[\frac{y^2}{2(1+x^2)} \right]_{y=0}^2 dx = \int_0^1 \frac{2}{1+x^2} dx$$

$$= [2 \tan^{-1} x]_0^1 = 2 \left(\frac{\pi}{4} \right) - 0 = \frac{\pi}{2}$$

$$13. \int_0^1 0 dx = 0 \text{ (since } xy^3 \text{ defines an odd function in } y).$$

$$14. \int_{-1}^1 \left[x^2 y + \left(\frac{1}{3} \right) y^3 \right]_{y=0}^2 dx = \int_{-1}^1 \left(2x^2 + \frac{8}{3} \right) dx$$

$$= \left[\left(\frac{2}{3} \right) x^3 + \left(\frac{8}{3} \right) x \right]_{-1}^1 = \frac{20}{3}$$

$$15. \int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y) dx dy$$

$$= \int_0^{\pi/2} [-\cos(x+y)]_{x=0}^{\pi/2} dy$$

$$= \int_0^{\pi/2} \left[-\cos\left(\frac{\pi}{2} + y\right) + \cos y \right] dy$$

$$= \int_0^{\pi/2} (\sin y + \cos y) dy = [-\cos y + \sin y]_0^{\pi/2}$$

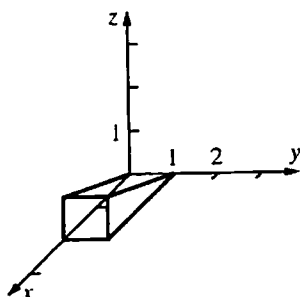
$$= (0 + 1) - (-1 + 0) = 2$$

$$16. \int_1^2 \left[\left(\frac{1}{3} \right) y(1+x^2)^{3/2} \right]_{x=0}^{\sqrt{3}} dy = \int_1^2 \left(\frac{7}{3} \right) y dy$$

$$= \left[\left(\frac{7}{6} \right) y^2 \right]_1^2 = 3.5$$

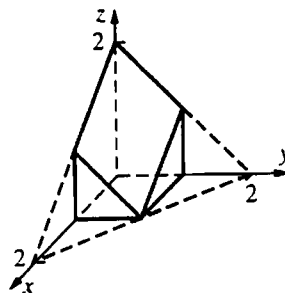
$$17. z = \frac{x}{2} \text{ is a plane.}$$

$$x - 2z = 0$$

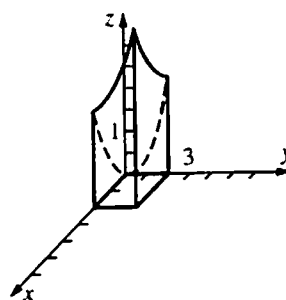


$$18. z = 2 - x - y \text{ is a plane.}$$

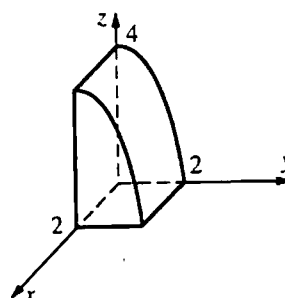
$$x + y + z = 2$$



$$19. z = x^2 + y^2 \text{ is a paraboloid opening upward with } z\text{-axis for axis.}$$



$$20. z = 4 - y^2 \text{ is a parabolic cylinder parallel to the } x\text{-axis.}$$



$$21. \int_1^3 \int_0^1 (x+y+1) dx dy = \int_1^3 \left[\left(\frac{1}{2} \right) x^2 + yx + x \right]_{x=0}^1 dy$$

$$= \int_1^3 \left(y + \frac{3}{2} \right) dy = 7$$

$$22. \int_1^2 \int_0^4 (2x+3y) dy dx = \int_1^2 \left[2xy + \left(\frac{3}{2} \right) y^2 \right]_{y=0}^4 dx$$

$$= \int_1^2 (8x + 24) dx = 36$$

$$23. x^2 + y^2 + 2 > 1$$

$$\int_{-1}^1 \int_0^1 [(x^2 + y^2 + 2) - 1] dy dx$$

$$= \int_{-1}^1 \left[x^2 y + \left(\frac{1}{3} \right) y^3 + y \right]_{y=0}^1 dx$$

$$= \int_{-1}^1 \left(x^2 + \frac{4}{3} \right) dx = \frac{10}{3}$$

$$24. \int_0^2 \int_0^2 (4 - x^2) dx dy = \int_0^2 \left[4x - \frac{x^3}{3} \right]_0^2 dy$$

$$= \int_0^2 \left(\frac{16}{3} \right) dy = \left[\frac{16y}{3} \right]_0^2 = \frac{32}{3}$$

$$25. \int_a^b \int_c^d g(x)h(y)dy dx = \int_a^b g(x) \int_c^d h(y)dy dx$$

$$= \int_c^d h(y)dy \int_a^b g(x)dx$$

(First step used linearity of integration with respect to y ; second step used linearity of integration with respect to x ; now commute.)

$$26. \int_0^{\sqrt{\ln 2}} x e^{x^2} dx \int_0^1 y(1+y^2)^{-1} dy$$

$$= \left[\left(\frac{1}{2} \right) e^{x^2} \right]_0^{\sqrt{\ln 2}} \left[\left(\frac{1}{2} \right) \ln(1+y^2) \right]_0^1$$

$$= \left[\frac{1}{2} \right] \left[\left(\frac{1}{2} \right) \ln 2 \right] = \left(\frac{1}{4} \right) \ln 2 \approx 0.1733$$

$$27. \int_0^1 \int_0^1 x y e^{x^2} e^{y^2} dy dx = \left(\int_0^1 x e^{x^2} dx \right) \left(\int_0^1 y e^{y^2} dy \right)$$

$$= \left(\int_0^1 x e^{x^2} dx \right)^2 \quad (\text{Changed the dummy variable } y)$$

$$32. \int_0^1 \int_0^{\sqrt{3}} 8x(x^2 + y^2 + 1)^{-2} dx dy = \int_0^1 [-4(x^2 + y^2 + 1)^{-1}]_{x=0}^{\sqrt{3}} dy = 4 \int_0^1 \left[\frac{-1}{4+y^2} + \frac{1}{1+y^2} \right] dy$$

$$= 4 \left[-\frac{1}{2} \arctan\left(\frac{y}{2}\right) + \arctan(y) \right]_0^1 = 4 \left[\left(-\frac{1}{2} \arctan\left(\frac{1}{2}\right) + \frac{\pi}{4} \right) - 0 \right] = \pi - 2 \arctan\left(\frac{1}{2}\right) \approx 2.2143$$

$$33. 0 \leq \int_a^b \int_a^b [f(x)g(y) - f(y)g(x)]^2 dx dy = \int_a^b \int_a^b [f^2(x)g^2(y) - 2f(x)g(x)f(y)g(y) + f^2(y)g^2(x)] dx dy$$

$$= \int_a^b f^2(x)dx \int_a^b g^2(y)dy - 2 \int_a^b f(x)g(x)dx \int_a^b f(y)g(y)dy + \int_a^b f^2(y)dy \int_a^b g^2(x)dx$$

$$= 2 \int_a^b f^2(x)dx \int_a^b g^2(x)dx - 2 \left[\int_a^b f(x)g(x)dx \right]^2$$

Therefore, $\left[\int_a^b f(x)g(x)dx \right]^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx$.

$$34. \text{ Since } f \text{ is increasing, } [y-x][f(y)-f(x)] > 0. \text{ Therefore,}$$

$$0 < \int_a^b \int_a^b [y-x][f(y)-f(x)] dx dy = \int_a^b \int_a^b y f(y) dx dy - \int_a^b \int_a^b y f(x) dx dy - \int_a^b \int_a^b x f(y) dx dy + \int_a^b \int_a^b x f(x) dx dy$$

to the dummy variable x .)

$$= \left(\left[\frac{e^{x^2}}{2} \right]_0^1 \right)^2 = \left(\frac{e-1}{2} \right)^2 \approx 0.7381$$

$$28. V = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\cos x \cos y| dx dy$$

$$= \int_{-\pi}^{\pi} |\cos x| dx \int_{-\pi}^{\pi} |\cos y| dy = \left(\int_{-\pi}^{\pi} |\cos x| dx \right)^2$$

$$= \left(4 \int_0^{\pi/2} |\cos x| dx \right)^2 = \left(4 [\sin x]_0^{\pi/2} \right)^2 = 16$$

$$29. \int_{-2}^2 x^2 dx \int_{-1}^1 |y^3| dy = 2 \int_0^2 x^2 dx \cdot 2 \int_0^1 y^3 dy$$

$$= 2 \left(\frac{8}{3} \right) 2 \left(\frac{1}{4} \right) = \frac{8}{3}$$

$$30. \int_{-2}^2 [x^2] dx \int_{-1}^1 y^3 dy = 0 \quad (\text{since the second integral equals 0}).$$

$$31. \int_{-2}^2 [x^2] dx \int_{-1}^1 |y^3| dy = 2 \int_0^2 [x^2] dx \cdot 2 \int_0^1 y^3 dy$$

$$= 2 \left[\int_0^1 0 dx + \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 dx + \int_{\sqrt{3}}^2 3 dx \right] \left[2 \left(\frac{1}{4} \right) \right]$$

$$= 2 \left[0 + (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) \right] \left[\frac{1}{2} \right]$$

$$= 5 - \sqrt{3} - \sqrt{2} \approx 1.8537$$

$$\begin{aligned}
 &= (b-a) \int_a^b y f(y) dy - \frac{b^2-a^2}{2} \int_a^b f(x) dx - \frac{b^2-a^2}{2} \int_a^b f(y) dy + (b-a) \int_a^b x f(x) dx \\
 &= 2(b-a) \int_a^b x f(x) dx - (b^2-a^2) \int_a^b f(x) dx = (b-a) \left[2 \int_a^b x f(x) dx - (b+a) \int_a^b f(x) dx \right]
 \end{aligned}$$

Therefore, $(b+a) \int_a^b f(x) dx < 2 \int_a^b x f(x) dx$. Now divide each side by the positive number $2 \int_a^b f(x) dx$ to obtain the desired result.

Interpretation:

If f is increasing on $[a, b]$ and $f(x) \geq 0$, then the x -coordinate of the centroid (of the region between the graph of f and the x -axis for x in $[a, b]$) is to the right of the midpoint between a and b .

Another interpretation:

If $f(x)$ is the density at x of a wire and the density is increasing as x increases for x in $[a, b]$, then the center of mass of the wire is to the right of the midpoint of $[a, b]$.

16.3 Concepts Review

1. A rectangle containing S ; 0

2. $\phi_1(x) \leq y \leq \phi_2(x)$

3. $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy dx$

4. $\int_0^1 \int_0^{1-x} 2x dy dx; \frac{1}{3}$

Problem Set 16.3

1. $\int_0^1 [x^2 y]_{y=0}^{3x} dx = \int_0^1 3x^3 dx = \frac{3}{4}$

2. $\int_1^2 \left[\left(\frac{1}{2} \right) y^2 \right]_{y=0}^{x-1} dx = \int_1^2 \left(\frac{1}{2} \right) (x-1)^2 dx = \frac{1}{6}$

3. $\int_{-1}^3 \left[\frac{x^3}{3} + y^2 x \right]_{x=0}^{3y} dy = \int_{-1}^3 (9y^3 + 3y^3) dy$
 $= [3y^4]_{-1}^3 = 243 - 3 = 240$

4. $\int_{-3}^1 \left[x^2 y - \left(\frac{1}{4} \right) y^4 \right]_{y=0}^x dx = \int_{-3}^1 \left[x^3 - \left(\frac{1}{4} \right) x^4 \right] dx$
 $= -32.2$

5. $\int_1^3 \left[\left(\frac{1}{2} \right) x^2 \exp(y^3) \right]_{x=-y}^{2y} dy = \int_1^3 \left(\frac{3}{2} \right) y^2 \exp(y^3) dy$
 $= \left(\frac{1}{2} \right) (e^{27} - e) \approx 2.660 \times 10^{11}$

6. $\int_1^5 \left[\frac{3}{x} \tan^{-1} \left(\frac{y}{x} \right) \right]_{y=0}^x dx = \int_1^5 \frac{3}{x} \frac{\pi}{4} dx$
 $= \left[\frac{3\pi \ln x}{4} \right]_1^5 = \frac{3\pi \ln 5}{4} \approx 3.7921$

7. $\int_{1/2}^1 [y \cos(\pi x^2)]_{y=0}^{2x} dx = \int_{1/2}^1 2x \cos(\pi x^2) dx$
 $= -\frac{\sqrt{2}}{2\pi} \approx -0.2251$

8. $\int_0^{\pi/4} \left[\left(\frac{1}{2} \right) r^2 \right]_{r=\sqrt{2}}^{\sqrt{2} \cos \theta} d\theta = \int_0^{\pi/4} (\cos^2 \theta - 1) d\theta$
 $= \frac{(2-\pi)}{8} \approx -0.1427$

9. $\int_0^{\pi/9} [\tan \theta]_{\theta=\pi/4}^{3r} dr = \int_0^{\pi/9} (\tan 3r - 1) dr$
 $= \left[-\frac{\ln |\cos 3r|}{3} - r \right]_0^{\pi/9}$
 $= \left(-\frac{\ln(\frac{1}{2})}{3} - \frac{\pi}{9} \right) - \left(-\frac{\ln(1)}{3} - 0 \right)$
 $= \frac{3 \ln 2 - \pi}{9} \approx -0.1180$

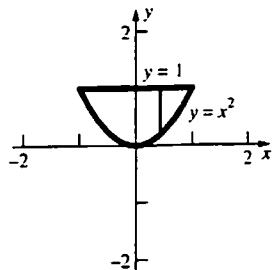
10. $\int_0^{\pi/2} [e^x \cos y]_{x=0}^{\sin y} dy = \int_0^{\pi/2} (e^{\sin y} \cos y - \cos y) dy$
 $= e - 2 \approx 0.7183$

11. $\int_0^2 \left[xy + \left(\frac{1}{2} \right) y^2 \right]_{y=0}^{\sqrt{4-x^2}} dx$
 $= \int_0^2 \left[x(4-x^2)^{1/2} + 2 - \left(\frac{1}{2} \right) x^2 \right] dx = \frac{16}{3}$

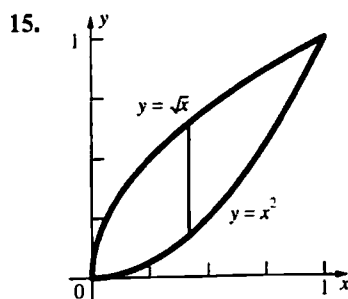
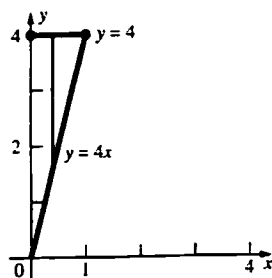
$$12. \int_{\pi/6}^{\pi/2} [3r^2 \cos \theta]_{r=0}^{\sin \theta} d\theta = \int_{\pi/6}^{\pi/2} 3 \sin^2 \theta \cos \theta d\theta$$

$$= [\sin^3 \theta]_{\pi/6}^{\pi/2} = \frac{7}{8} = 0.875$$

$$13. \int_{-1}^1 \int_{x^2}^1 xy dy dx = 0$$



$$14. \int_0^1 \int_{4x}^4 (x+y) dy dx = 6$$



$$\int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + 2y) dy dx = \int_0^1 [x^2 y + y^2]_{y=x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 [(x^{5/2} + x) - (x^4 + x^4)] dx$$

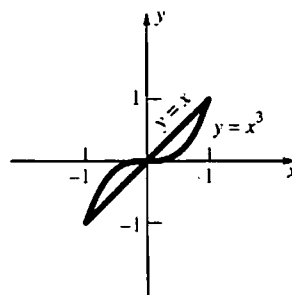
$$= \left[\frac{2x^{7/2}}{7} + \frac{x^2}{2} - \frac{2x^5}{5} \right]_0^1 = \frac{2}{7} + \frac{1}{2} - \frac{2}{5}$$

$$= \frac{27}{70} \approx 0.3857$$

$$16. \int_0^2 \int_x^{3x-x^2} (x^2 - xy) dy dx = \frac{52}{15}$$

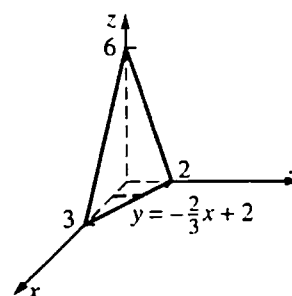
$$17. \int_0^2 \int_x^2 2(1+x^2)^{-1} dy dx = 4 \tan^{-1} 2 - \ln 5 \approx 2.8192$$

18.



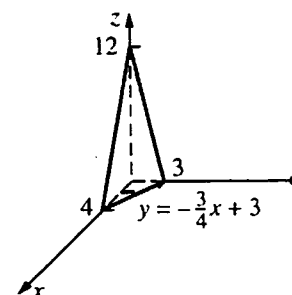
Since S is symmetric with respect to the origin and the integrand is an odd function in x , the value of the integral is 0.

19.



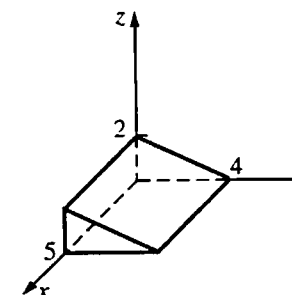
$$\int_0^3 \int_0^{(-2/3)x+2} (6-2x-3y) dy dx = 6$$

20.



$$\int_0^4 \int_0^{(-3/4)x+3} (12-3x-4y) dy dx = 24$$

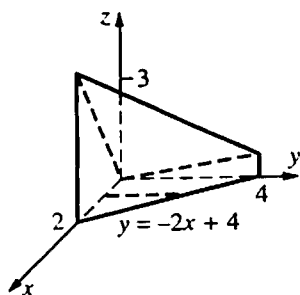
21.



$$\int_0^5 \int_0^4 \frac{4-y}{2} dy dx = \left(\int_0^5 1 dx \right) \left(\int_0^4 \frac{4-y}{2} dy \right)$$

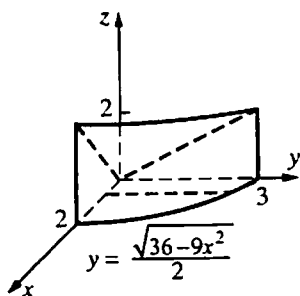
$$= 5 \left[2y - \frac{y^2}{4} \right]_0^4 = 5(8-4) = 20$$

22.



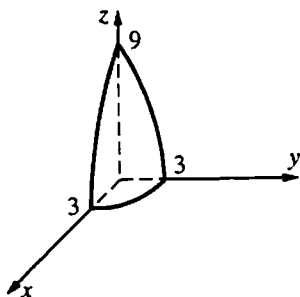
$$\int_0^2 \int_0^{-2x+4} \left[2x + \left(\frac{1}{4} \right) y \right] dy dx = \frac{20}{3}$$

23.



$$\int_0^2 \int_0^{(1/2)\sqrt{36-9x^2}} \left(\frac{1}{6} \right) (9x + 4y) dy dx = 10$$

24.



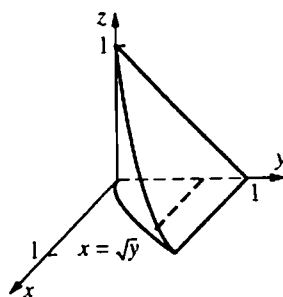
$$\begin{aligned} & \int_0^3 \int_0^{\sqrt{9-x^2}} (9 - x^2 - y^2) dy dx \\ &= \int_0^3 \left[(9 - x^2)y - \frac{y^3}{3} \right]_{y=0}^{\sqrt{9-x^2}} dx \\ &= \int_0^3 \frac{2(9-x^2)^{3/2}}{3} dx = \int_0^{\pi/2} 18 \cos^3 t \cdot 3 \cos t dt \\ &= \int_0^{\pi/2} 54 \cos^4 t dt \\ &= \int_0^{\pi/2} \left(\frac{81}{4} + 27 \cos 2t + \frac{27 \cos 4t}{4} \right) dt \\ &= \left[\frac{81t}{4} + \frac{27 \sin 2t}{2} + \frac{27 \sin 4t}{16} \right]_0^{\pi/2} \\ &= \frac{81\pi}{8} \approx 31.8086 \end{aligned}$$

(At the third step, the substitution $x = 3 \sin t$ was

used. At the 5th step the identity

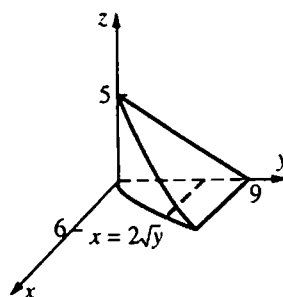
$$\cos^2 A = \left(\frac{1}{2} \right) (1 + \cos 2A) \text{ was used a few times.})$$

25.

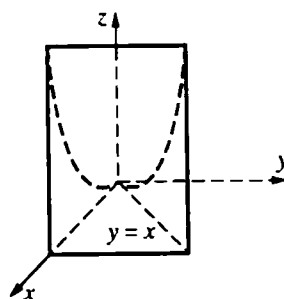


$$\int_0^1 \int_0^{\sqrt{y}} (1 - y) dx dy = \frac{4}{15}$$

$$26. \int_0^9 \int_0^{2\sqrt{y}} \left[5 - \left(\frac{5}{9} \right) y \right] dx dy = 72$$

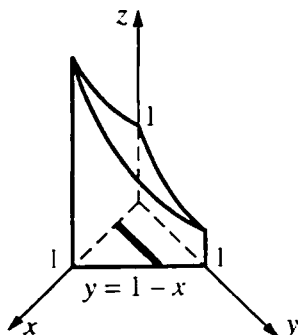


27.

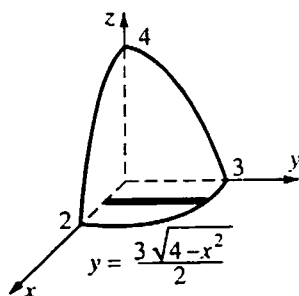


$$\begin{aligned} & \int_0^1 \int_0^x \tan x^2 dy dx = \int_0^1 [y \tan x^2]_{y=0}^x dx \\ &= \int_0^1 x \tan x^2 dx = \left[-\frac{\ln |\cos x^2|}{2} \right]_0^1 = \left(-\frac{1}{2} \right) \ln(\cos 1) \\ &\approx 0.3078 \end{aligned}$$

28. $\int_0^1 \int_0^{1-x} e^{x-y} dy dx = \left(\frac{1}{2}\right)(e + e^{-1} - 2) \approx 0.5431$



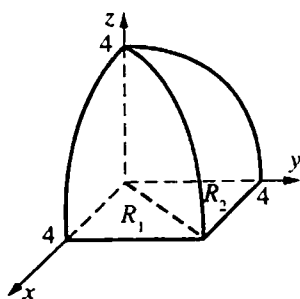
29.



$$\int_0^2 \int_0^{(3/2)\sqrt{4-x^2}} \left[4 - x^2 - \left(\frac{4}{9}\right)y^2 \right] dy dx = 3\pi$$

$$\approx 9.4248$$

30.



Making use of symmetry, the volume is

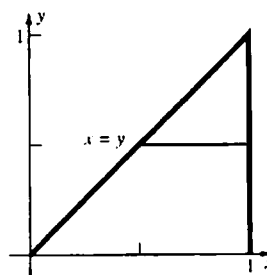
$$2 \iint_{R_1} (16 - x^2)^{1/2} dA = 2 \int_0^4 \int_0^x (16 - x^2)^{1/2} dy dx$$

$$= 2 \int_0^4 [(16 - x^2)^{1/2} y]_{y=0}^x dx$$

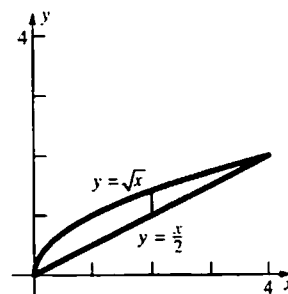
$$= 2 \int_0^4 (16 - x^2)^{1/2} x dx = \left[\frac{-2(16 - x^2)^{3/2}}{3} \right]_0^4$$

$$= 0 + \frac{2(64)}{3} = \frac{128}{3} \approx 42.6667$$

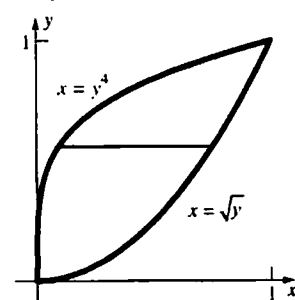
31. $\int_0^1 \int_y^1 f(x, y) dx dy$



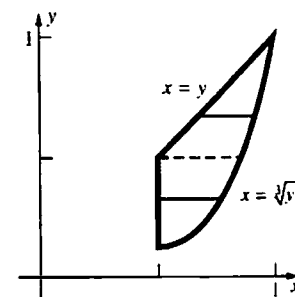
32. $\int_0^4 \int_{x/2}^{\sqrt{x}} f(x, y) dy dx$



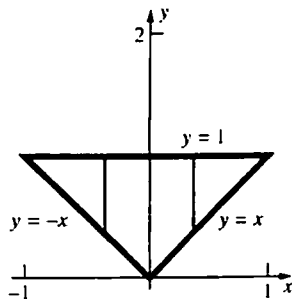
33. $\int_0^1 \int_{y^4}^{\sqrt{y}} f(x, y) dx dy$



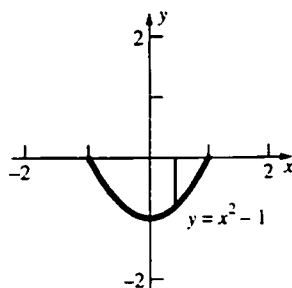
34. $\int_{1/8}^{1/2} \int_{1/2}^{y^{1/3}} f(x, y) dx dy + \int_{1/2}^1 \int_y^{y^{1/3}} f(x, y) dx dy$



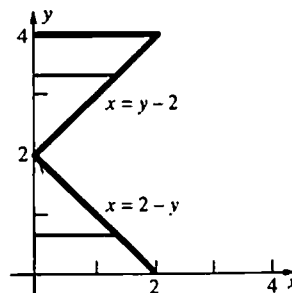
$$35. \int_{-1}^0 \int_{-x}^1 f(x, y) dy dx + \int_0^1 \int_x^1 f(x, y) dy dx$$



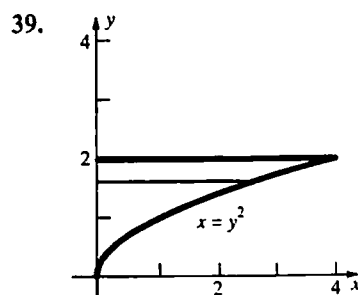
$$36. \int_{-1}^1 \int_{x^2-1}^0 f(x, y) dy dx$$



$$37. \int_0^2 \int_0^{2-y} xy^2 dx dy + \int_2^4 \int_0^{y-2} xy^2 dx dy = \frac{256}{15} \approx 17.0667$$

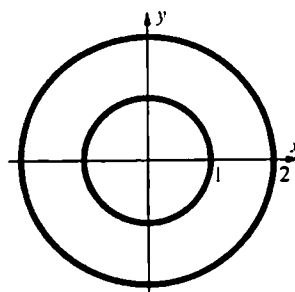


$$38. \int_{-2}^1 \int_{x^2}^{-x+2} xy dy dx - \int_{1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2}^{1/2} xy dy dx = -\frac{45}{8} = -5.625$$



$$\begin{aligned} \int_0^2 \int_0^{y^2} \sin(y^3) dx dy &= \int_0^2 [x \sin(y^3)]_{x=0}^{y^2} dy \\ &= \int_0^2 y^2 \sin(y^3) dy = \left[-\frac{\cos(y^3)}{3} \right]_0^2 \\ &= \frac{1 - \cos 8}{3} \approx 0.3818 \end{aligned}$$

40.



$z = f(x, y) = \sin(xy^2)$ is symmetric with respect to the x -axis, as is the annulus. Therefore, the integral equals 0.

41. The integral over S of $x^4 y$ is 0 (see Problem 40).

Therefore,

$$\begin{aligned} \iint_S (x^2 + x^4 y) dA &= \iint_S x^2 dA \\ &= 4 \left(\iint_{S_1} x^2 dA + \iint_{S_2} x^2 dA \right) \\ &= 4 \left(\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x^2 dy dx + \int_1^2 \int_0^{\sqrt{4-x^2}} x^2 dy dx \right) \\ &= 4 \left(\int_0^1 x^2 \sqrt{4-x^2} dx - \int_0^1 x^2 \sqrt{1-x^2} dx \right) \\ &= 4 \left(16 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta - \int_0^{\pi/2} \sin^2 \phi \cos^2 \phi d\phi \right) \end{aligned}$$

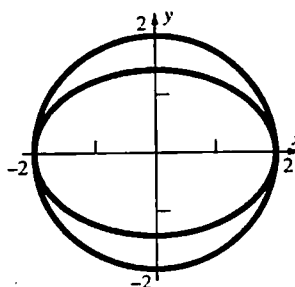
(using $x = 2 \sin \theta$ in 1st integral; $x = \sin \phi$ in 2nd)

$$= 60 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta = \frac{15\pi}{4}$$

(See work in Problem 42.)

$$\approx 11.7810$$

42.



$$\iint_S x^2 dA = \int_0^2 \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^2 dy dx$$

$$\begin{aligned}
 &= \int_0^2 x^2 \left(\sqrt{4-x^2} - \frac{\sqrt{4-x^2}}{\sqrt{2}} \right) dx \\
 &= \int_0^2 x^2 \sqrt{4-x^2} \left(1 - \frac{1}{\sqrt{2}} \right) dx \\
 &= \frac{(2-\sqrt{2})}{2} \int_0^2 x^2 \sqrt{4-x^2} dx
 \end{aligned}$$

$$\text{Let } x = 2 \sin \theta, \theta \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

$$\text{Then } dx = 2 \cos \theta d\theta$$

$$x = 2 \Rightarrow \theta = \frac{\pi}{2}$$

$$x = 0 \Rightarrow \theta = 0$$

$$\begin{aligned}
 &= \frac{(2-\sqrt{2})}{2} \int_0^{\pi/2} (2 \sin \theta)^2 (2 \cos \theta) 2 \cos \theta d\theta \\
 &= 8(2-\sqrt{2}) \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta \\
 &= 8(2-\sqrt{2}) \left(\frac{\pi}{16} \right) = \frac{\pi(2-\sqrt{2})}{2}
 \end{aligned}$$

Therefore,

$$\iint_S x^2 dA = 4 \left[\frac{\pi(2-\sqrt{2})}{2} \right] = 2\pi(2-\sqrt{2}) \approx 3.6806$$

$$\begin{aligned}
 * &= \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta \\
 &= \int_0^{\pi/2} \left[\frac{1}{2}(1 - \cos 2\theta) \right] \left[\frac{1}{2}(1 + \cos 2\theta) \right] d\theta
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{4} \int_0^{\pi/2} (1 - \cos^2 2\theta) d\theta \\
 &= \frac{1}{4} \frac{\pi}{2} - \frac{1}{4} \int_0^{\pi/2} \frac{1}{2} (1 + \cos 4\theta) d\theta \\
 &= \frac{\pi}{8} - \frac{1}{8} \frac{\pi}{2} + \frac{1}{8} \left[\frac{\sin 4\theta}{4} \right]_0^{\pi/2} \\
 &= \frac{\pi}{8} - \frac{\pi}{16} + 0 = \frac{\pi}{16}
 \end{aligned}$$

43. We first slice the river into eleven 100' sections parallel to the bridge. We will assume that the cross-section of the river is roughly the shape of an isosceles triangle and that the cross-sectional area is uniform across a slice. We can then approximate the volume of the water by

$$\begin{aligned}
 V &\approx \sum_{k=1}^{11} A_k(y_k) \Delta y = \sum_{k=1}^{11} \frac{1}{2} (w_k)(d_k) 100 \\
 &= 50 \sum_{k=1}^{11} (w_k)(d_k)
 \end{aligned}$$

where w_k is the width across the river at the left side of the k th slice, and d_k is the center depth of the river at the left side of the k th slice. This gives

$$\begin{aligned}
 V &\approx 50[300 \cdot 40 + 300 \cdot 39 + 300 \cdot 35 + 300 \cdot 31 \\
 &\quad + 290 \cdot 28 + 275 \cdot 26 + 250 \cdot 25 + 225 \cdot 24 \\
 &\quad + 205 \cdot 23 + 200 \cdot 21 + 175 \cdot 19] \\
 &= 4,133,000 \text{ ft}^3
 \end{aligned}$$

16.4 Concepts Review

1. $a \leq r \leq b; \alpha \leq \theta \leq \beta$

2. $r dr d\theta$

3. $\int_0^\pi \int_0^2 r^3 dr d\theta$

4. 4π

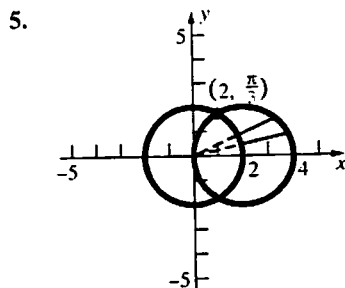
Problem Set 16.4

$$\begin{aligned}
 1. & \int_0^{\pi/2} \left[\left(\frac{1}{3} \right) r^3 \sin \theta \right]_{r=0}^{\cos \theta} d\theta \\
 &= \int_0^{\pi/2} \left(\frac{1}{3} \right) \cos^3 \theta \sin \theta d\theta \\
 &= \frac{1}{12} \approx 0.0833
 \end{aligned}$$

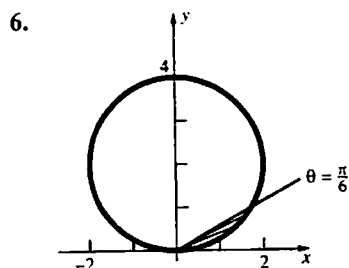
$$\begin{aligned}
 2. & \int_0^{\pi/2} \left[\left(\frac{1}{2} \right) r^2 \right]_{r=0}^{\sin \theta} d\theta = \int_0^{\pi/2} \left(\frac{1}{2} \right) \sin^2 \theta d\theta \\
 &= \frac{\pi}{8} \approx 0.3927
 \end{aligned}$$

$$\begin{aligned}
 3. & \int_0^\pi \left[\frac{r^3}{3} \right]_{r=0}^{\sin \theta} d\theta = \int_0^\pi \frac{\sin^3 \theta}{3} d\theta \\
 &= \int_0^\pi \frac{(1 - \cos^2 \theta) \sin \theta}{3} d\theta \\
 &= \left[\frac{-\cos \theta}{3} + \frac{\cos^3 \theta}{9} \right]_0^\pi \\
 &= \left(\frac{1}{3} - \frac{1}{9} \right) - \left(-\frac{1}{3} + \frac{1}{9} \right) = \frac{4}{9}
 \end{aligned}$$

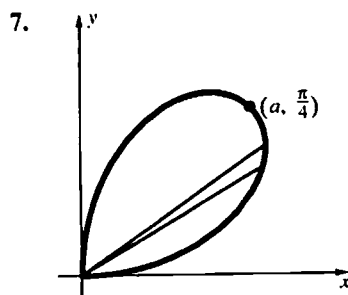
$$\begin{aligned}
 4. \quad & \int_0^{\pi} \left[\left(\frac{1}{2} \right) r^2 \sin \theta \right]_{r=0}^{1-\cos \theta} d\theta \\
 &= \int_0^{\pi} \left(\frac{1}{2} \right) (1-\cos \theta)^2 \sin \theta d\theta \\
 &= \frac{4}{3}
 \end{aligned}$$



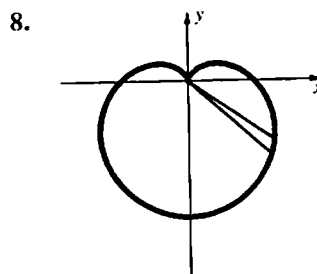
$$2 \int_0^{\pi/3} \int_2^{4\cos \theta} r dr d\theta = 2 \left[\frac{2\pi}{3} + \sqrt{3} \right] \approx 7.6529$$



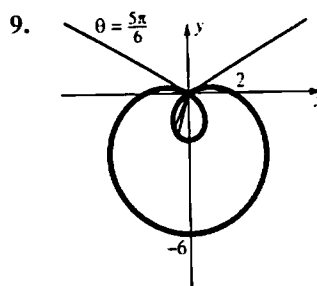
$$\begin{aligned}
 \int_0^{\pi/6} \int_0^{4\sin \theta} r dr d\theta &= \int_0^{\pi/6} \left[\frac{r^2}{2} \right]_0^{4\sin \theta} d\theta \\
 &= \int_0^{\pi/6} 8\sin^2 \theta d\theta = \int_0^{\pi/6} 4(1-\cos 2\theta) d\theta \\
 &= [4\theta - 2\sin 2\theta]_0^{\pi/6} = \frac{2\pi}{3} - \sqrt{3} \approx 0.3623
 \end{aligned}$$



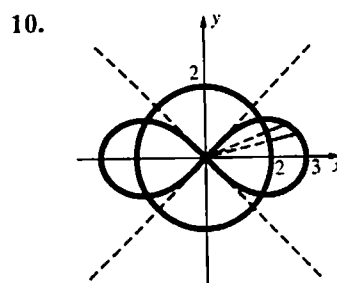
$$\int_0^{\pi/2} \int_0^{a\sin 2\theta} r dr d\theta = \frac{a^2 \pi}{8}$$



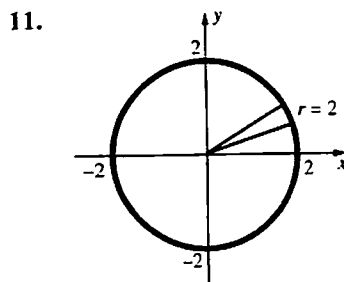
$$\int_0^{2\pi} \int_0^{6-6\sin \theta} r dr d\theta = 54\pi \approx 169.6460$$



$$\begin{aligned}
 2 \int_{5\pi/6}^{3\pi/2} \int_0^{2-4\sin \theta} r dr d\theta &= 2 \int_{5\pi/6}^{3\pi/2} \left[\frac{r^2}{2} \right]_0^{2-4\sin \theta} d\theta \\
 &= 2 \int_{5\pi/6}^{3\pi/2} (6-8\sin \theta-4\cos 2\theta) d\theta \\
 &= 2[6\theta+8\cos \theta-2\sin 2\theta]_{5\pi/6}^{3\pi/2} \\
 &= 2(4\pi+3\sqrt{3}) \approx 35.525
 \end{aligned}$$

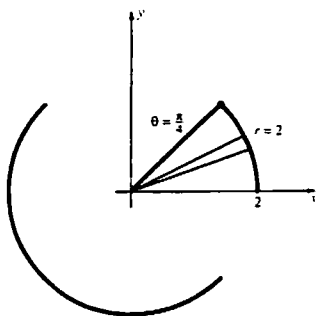


$$\begin{aligned}
 4 \int_0^{(1/2)\cos^{-1}(4/9)} \int_2^{\sqrt{3\cos 2\theta}} r dr d\theta &= \sqrt{65} - 4\cos^{-1}\left(\frac{4}{9}\right) \\
 &\approx 3.6213
 \end{aligned}$$



$$2 \int_0^{\pi} \int_0^2 e^{r^2} dr d\theta = \pi(e^4 - 1) \approx 168.3836$$

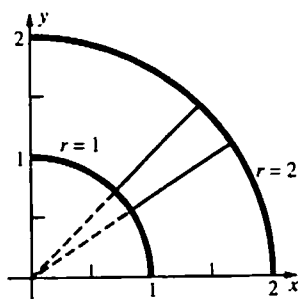
12.



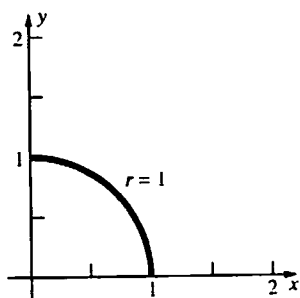
$$\begin{aligned} & \int_0^{\pi/4} \int_0^2 (4-r^2)^{1/2} r \, dr \, d\theta \\ &= \int_0^{\pi/4} \left[\frac{(4-r^2)^{3/2}}{-3} \right]_0^2 d\theta \\ &= \int_0^{\pi/4} \left(\frac{8}{3} \right) d\theta = \left[\frac{8\theta}{3} \right]_0^{\pi/4} = \frac{2\pi}{3} \approx 2.0944 \end{aligned}$$

$$13. \int_0^{\pi/4} \int_0^2 (4+r^2)^{-1} r \, dr \, d\theta = \left(\frac{\pi}{8} \right) \ln 2 \approx 0.2722$$

$$14. \int_0^{\pi/2} \int_1^2 r \sin \theta \, dr \, d\theta = \frac{7}{3}$$

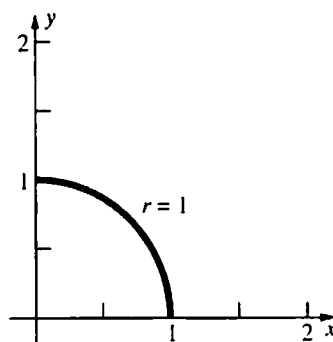


15.

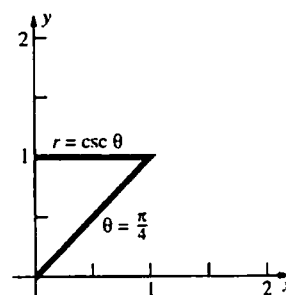


$$\begin{aligned} & \int_0^{\pi/2} \int_0^1 (4-r^2)^{-1/2} r \, dr \, d\theta \\ &= \int_0^{\pi/2} \left[-(4-r^2)^{1/2} \right]_0^1 d\theta \\ &= \int_0^{\pi/2} (-\sqrt{3} + 2) d\theta = (-\sqrt{3} + 2) \left(\frac{\pi}{2} \right) \approx 0.4209 \end{aligned}$$

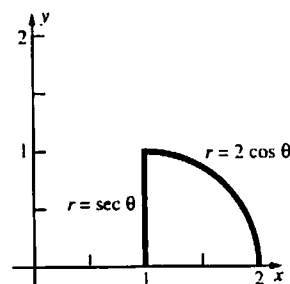
$$16. \int_0^{\pi/2} \int_0^1 [\sin(r^2)] r \, dr \, d\theta = \left(\frac{\pi}{4} \right) (1 - \cos 1) \approx 0.3610$$



$$17. \int_{\pi/4}^{\pi/2} \int_0^{\csc \theta} r^2 \cos^2 \theta \, r \, dr \, d\theta = \frac{1}{12} \approx 0.0833$$

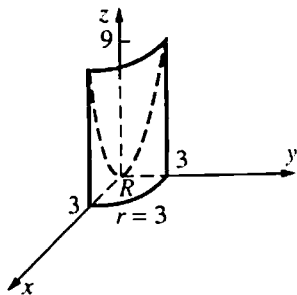


18.



$$\begin{aligned} & \int_0^{\pi/4} \int_{\sec \theta}^{2 \cos \theta} r^{-1} r \, dr \, d\theta = \int_0^{\pi/4} [r]_{\sec \theta}^{2 \cos \theta} d\theta \\ &= \int_0^{\pi/4} (2 \cos \theta - \sec \theta) d\theta \\ &= \left[2 \sin \theta - \ln |\sec \theta + \tan \theta| \right]_0^{\pi/4} \\ &= \left[\sqrt{2} - \ln(\sqrt{2} + 1) \right] - [0 - \ln(1 + 0)] \\ &= \sqrt{2} - \ln(\sqrt{2} + 1) \approx 0.5328 \end{aligned}$$

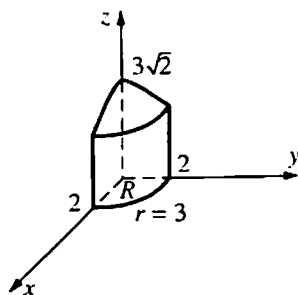
19.



$$\iint_R (x^2 + y^2) dA = \int_0^{\pi/2} \int_0^3 r^2 r dr d\theta$$

$$= \frac{81\pi}{8} \approx 31.8086$$

20.

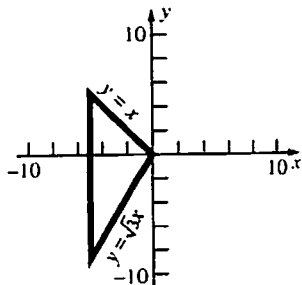


$$4 \iint_R (18 - 2x^2 - 2y^2)^{1/2} dA$$

$$= 4 \int_0^{\pi/2} \int_0^2 (18 - 2r^2)^{1/2} r dr d\theta$$

$$= \left(\frac{\pi}{3}\right) (18^{3/2} - 10^{3/2}) \approx 46.8566$$

21.



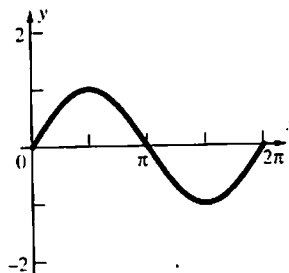
$$\int_{-5}^0 \int_{\sqrt{3}x}^x (y^2) dy dx = \int_{-5}^0 \left[\frac{y^3}{3} \right]_{\sqrt{3}x}^x dx$$

$$= \int_{-5}^0 \frac{-1-3\sqrt{3}}{3} x^3 dx = \left[\frac{(-1-3\sqrt{3})x^4}{12} \right]_{-5}^0$$

$$= \frac{(1+3\sqrt{3})625}{12} \approx 322.7163$$

22. a. The solid bounded by the xy -plane and $z = \sin \sqrt{x^2 + y^2}$ for $x^2 + y^2 \leq 4\pi^2$ is the solid of revolution obtained by revolving

about the z -axis the region in the xz -plane that is bounded by the x -axis and the graph of $z = \sin x$ for $0 \leq x \leq 2\pi$.



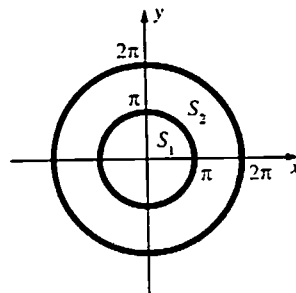
Regions A and B are congruent but region B is farther from the origin, so it generates a larger solid than region A generates. Therefore, the integral is negative.

b. $V = \int_0^{2\pi} \int_0^{2\pi} (\sin r) r dr d\theta = 2\pi \int_0^{2\pi} (\sin r) r dr$

Now use integration by parts.

$$= 2\pi(-2\pi) = -4\pi^2 \approx -39.4784$$

c.



$$W = \iint_{S_1} \sin \sqrt{x^2 + y^2} dA + \iint_{S_2} -\sin \sqrt{x^2 + y^2} dA$$

$$= \int_0^{2\pi} \left[\int_0^{\pi} (\sin r) r dr - \int_0^{2\pi} (\sin r) r dr \right] d\theta$$

$$= 2\pi[(\pi) - (-3\pi)] = 8\pi^2 \approx 78.9568$$

23. This can be done by the methods of this section, but an easier way to do it is to realize that the intersection is the union of two congruent segments (of one base) of the spheres, so (see Problem 20, Section 6.2, with $d = h$ and $a = r$) the volume is $2 \left[\left(\frac{1}{3} \right) \pi d^2 (3a - d) \right] = 2\pi d^2 \frac{(3a - d)}{3}$.

24. $100 = \int_0^{2\pi} \int_0^{10} k e^{-r/10} r dr d\theta = 2\pi \int_0^{10} k e^{-r/10} r dr$

Let $u = r$ and $dv = e^{-r/10} dr$.

Then $du = dr$ and $v = -10e^{-r/10}$.

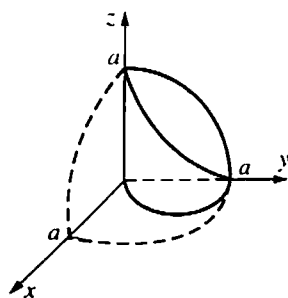
$$= 2\pi k \left(\left[-10re^{-r/10} \right]_0^{10} + \int_0^{10} 10e^{-r/10} dr \right)$$

$$= 2\pi k \left(-100e^{-1} - \left[100e^{-r/10} \right]_0^{10} \right)$$

$$= 2\pi k(-100e^{-1} - 100e^{-1} + 100)$$

$$= 200\pi k(1 - 2e^{-1}), \text{ so } k = \frac{e}{2\pi(e-2)} \approx 0.6023.$$

25.



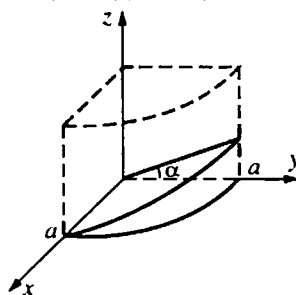
$$\text{Volume} = 4 \int_0^{\pi/2} \int_0^{a \sin \theta} \sqrt{a^2 - r^2} r dr d\theta$$

$$= \int_0^{\pi/2} \left[-\frac{1}{3} (a^3 \cos^3 \theta - a^3) \right] d\theta$$

$$= \left(-\frac{4}{3} \right) a^3 \left[\frac{2}{3} - \frac{\pi}{2} \right] = \left(\frac{2}{9} \right) a^3 (3\pi - 4)$$

26. Normal vector to plane is $\langle 0, -\sin \alpha, \cos \alpha \rangle$.

Therefore, an equation of the plane is $(-\sin \alpha)y + (\cos \alpha)z = 0$, or $z = (\tan \alpha)y$, or $z = (\tan \alpha)(r \sin \theta)$.



$$\text{Volume} = 2 \int_0^{\pi/2} \int_0^a (\tan \alpha) r \sin \theta dr d\theta$$

$$= 2(\tan \alpha) \int_0^{\pi/2} \sin \theta d\theta \int_0^a r^2 dr$$

$$= 2(\tan \alpha) [1] \left[\frac{a^3}{3} \right] = \left(\frac{2}{3} \right) a^3 \tan \alpha$$

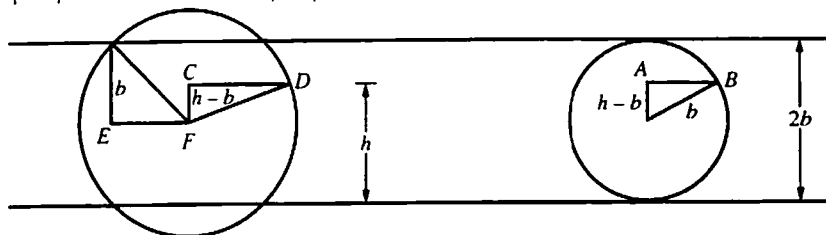
27. Choose a coordinate system so the center of the sphere is the origin and the axis of the part removed is the z-axis.

Volume (Ring) = Volume (Sphere of radius a) - Volume (Part removed)

$$= \frac{4}{3} \pi a^3 - 2 \int_0^{2\pi} \int_0^{\sqrt{a^2 - b^2}} \sqrt{a^2 - r^2} r dr d\theta = \frac{4}{3} \pi a^3 - 2(2\pi) \int_0^{\sqrt{a^2 - b^2}} (a^2 - r^2)^{1/2} r dr$$

$$= \frac{4}{3} \pi a^3 - 4\pi \left[\frac{1}{3} (a^2 - r^2)^{3/2} \right]_0^{\sqrt{a^2 - b^2}} = \frac{4}{3} \pi a^3 + 4\pi \frac{1}{3} (b^3 - a^3) = \frac{4}{3} \pi b^3$$

$$28. |EF|^2 = a^2 - b^2 \quad |CD| = a^2 - (h-b)^2 \quad |AB|^2 = b^2 - (h-b)^2$$



$$\text{Area of left cross-sectional region} = \pi[a^2 - (h-b)^2] - \pi[a^2 - b^2]$$

$$= \pi[b^2 - (h-b)^2] = \text{area of right cross-sectional region}$$

$$\text{Volume} = \left(\frac{4}{3} \right) \pi b^3 - \left(\frac{1}{3} \right) \pi (2b-h)^2 [3b - (2b-h)] \quad (\text{using problem 23})$$

$$= \left(\frac{1}{3} \right) \pi h^2 (3b-h)$$

$$29. \int_0^{\pi/2} \left[\lim_{b \rightarrow \infty} \int_0^b (1+r^2)^{-2} r dr \right] d\theta = \int_0^{\pi/2} \left(\lim_{b \rightarrow \infty} \left[\left(-\frac{1}{2} \right) (1+b^2)^{-1} - \left(-\frac{1}{2} \right) \right] \right) d\theta = \int_0^{\pi/2} \left(\frac{1}{2} \right) d\theta = \frac{\pi}{4} \approx 0.7854$$

$$\begin{aligned}
 30. \quad A &= \frac{1}{2}r_2^2(\theta_2 - \theta_1) - \frac{1}{2}r_1^2(\theta_2 - \theta_1) \\
 &= \frac{1}{2}(\theta_2 - \theta_1)(r_2^2 - r_1^2) \\
 &= \frac{1}{2}(\theta_2 - \theta_1)(r_2 - r_1)(r_2 + r_1) \\
 &= \frac{r_1 + r_2}{2}(r_2 - r_1)(\theta_2 - \theta_1)
 \end{aligned}$$

$$\begin{aligned}
 31. \quad &\text{From symmetry we have} \\
 &\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx \\
 &= 2 \int_0^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx.
 \end{aligned}$$

Using the substitution $u = \frac{x-\mu}{\sigma\sqrt{2}}$ we get

$du = \frac{dx}{\sigma\sqrt{2}}$. Our integral then becomes

$$\begin{aligned}
 &\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx \\
 &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du
 \end{aligned}$$

Using the result from Example 4, we see that

$$\begin{aligned}
 &\int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}. \text{ Thus we have} \\
 &\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx \\
 &= \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1.
 \end{aligned}$$

16.5 Concepts Review

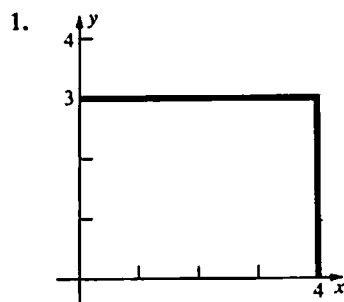
$$1. \iint_S x^2 y^4 dA$$

$$2. \iint_S \frac{x^2 y^5 dA}{m}$$

$$3. \iint_S x^4 y^4 dA$$

4. greater

Problem Set 16.5

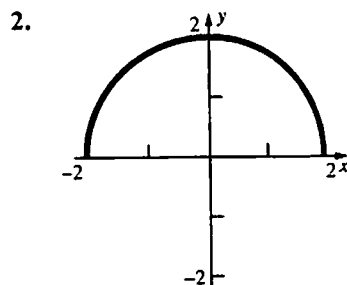


$$m = \int_0^3 \int_0^4 (y+1) dx dy = 30$$

$$M_y = \int_0^3 \int_0^4 x(y+1) dx dy = 60$$

$$M_x = \int_0^3 \int_0^4 y(y+1) dx dy = 54$$

$$(\bar{x}, \bar{y}) = (2, 1.8)$$

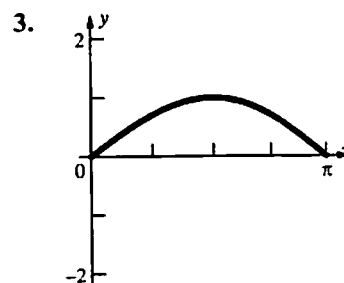


$$m = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} y dy dx = \frac{16}{3}$$

$$M_y = 0 \text{ (symmetry)}$$

$$M_x = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} yy dx dy = 2\pi$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{3\pi}{8}\right)$$



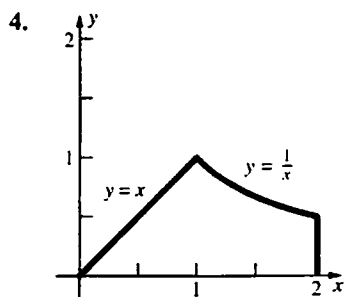
$$m = \int_0^{\pi} \int_0^{\sin x} y dy dx = \int_0^{\pi} \left[\frac{y^2}{2} \right]_0^{\sin x} dx$$

$$= \int_0^{\pi} \frac{\sin^2 x}{2} dx = \int_0^{\pi} \frac{1 - \cos 2x}{4} dx$$

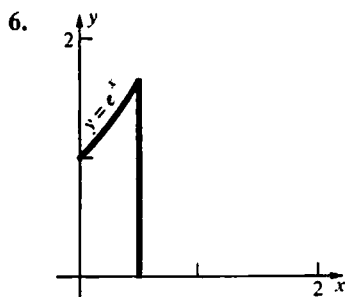
$$= \left[\frac{x}{4} - \frac{\sin 2x}{8} \right]_0^{\pi} = \frac{\pi}{4}$$

$$\begin{aligned}
 M_x &= \int_0^{\pi} \int_0^{\sin x} yy \, dy \, dx = \int_0^{\pi} \left[\frac{y^3}{3} \right]_0^{\sin x} dx \\
 &= \int_0^{\pi} \frac{\sin^3 x}{3} dx = \frac{1}{3} \int_0^{\pi} (1 - \cos^2 x) \sin x \, dx \\
 &= \frac{1}{3} \left[-\cos x + \frac{\cos^3 x}{3} \right]_0^{\pi} = \frac{4}{9} \\
 \bar{y} &= \frac{M_x}{m} = \frac{\frac{4}{9}}{\frac{\pi}{4}} = \frac{16}{9\pi} \approx 0.5659; \\
 \bar{x} &= \frac{\pi}{2} \text{ (by symmetry)}
 \end{aligned}$$

$$\text{Thus, } M_y = \bar{x} \cdot m = \frac{\pi}{4} \cdot \frac{\pi}{2} = \frac{\pi^2}{8}$$



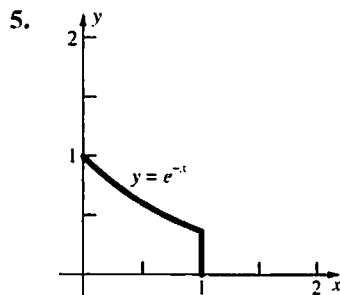
$$\begin{aligned}
 m &= \int_0^1 \int_0^x x \, dy \, dx + \int_1^2 \int_0^{1/x} x \, dy \, dx = \frac{4}{3} \\
 M_y &= \int_0^1 \int_0^x x^2 \, dy \, dx + \int_1^2 \int_0^{1/x} x^2 \, dy \, dx = \frac{7}{4} \\
 M_x &= \int_0^1 \int_0^x xy \, dy \, dx + \int_1^2 \int_0^{1/x} xy \, dy \, dx
 \end{aligned}$$



$$\begin{aligned}
 m &= \int_0^1 \int_0^{e^x} (2-x+y) \, dy \, dx = \int_0^1 \left[(2-x)y + \frac{y^2}{2} \right]_{y=0}^{e^x} dx = \int_0^1 \left(2e^x - xe^x + \frac{e^{2x}}{2} \right) dx \\
 &= \left[2e^x - (xe^x - e^x) + \frac{e^{2x}}{4} \right]_0^1 = \frac{e^2 + 8e - 13}{4} \\
 M_x &= \int_0^1 \int_0^{e^x} (2-x+y)y \, dy \, dx = \int_0^1 \left[y^2 - \frac{xy^2}{2} + \frac{y^3}{3} \right]_{y=0}^{e^x} dx = \int_0^1 \left[e^{2x} - \frac{xe^{2x}}{2} + \frac{e^{3x}}{3} \right] dx
 \end{aligned}$$

$$= \left(\frac{1}{8} \right) (1 + 4 \ln 2)$$

$$(\bar{x}, \bar{y}) = \left(\frac{21}{16}, \left(\frac{3}{32} \right) (1 + 4 \ln 2) \right) \approx (1.3125, 0.3537)$$



$$m = \int_0^1 \int_0^{e^{-x}} y^2 \, dy \, dx = \left(\frac{1}{9} \right) (1 - e^{-3})$$

$$M_x = \int_0^1 \int_0^{e^{-x}} y^3 \, dy \, dx = \left(\frac{1}{16} \right) (1 - e^{-4})$$

$$M_y = \int_0^1 \int_0^{e^{-x}} xy^2 \, dy \, dx = \left(\frac{1}{27} \right) (1 - 4e^{-3}) \approx 0.1056$$

$$(\bar{x}, \bar{y}) =$$

$$\begin{aligned}
 &\left(\left(\frac{1}{3} \right) (e^3 - 4)(e^3 - 1)^{-1}, \left(\frac{9}{16} \right) e^{-1} (e^4 - 1)(e^3 - 1)^{-1} \right) \\
 &\approx (0.2809, 0.5811)
 \end{aligned}$$

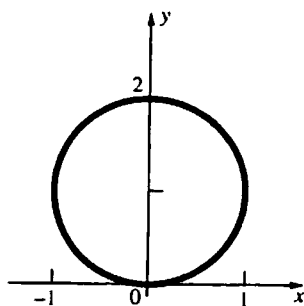
$$= \left[\frac{e^{2x}}{2} - \left(\frac{xe^{2x}}{4} - \frac{e^{2x}}{8} \right) + \frac{e^{3x}}{9} \right]_0^1 = \left(\frac{e^2}{2} - \frac{e^2}{4} + \frac{e^2}{8} + \frac{e^3}{9} \right) - \left(\frac{1}{2} - 0 + \frac{1}{8} + \frac{1}{9} \right) = \frac{8e^3 + 27e^2 - 53}{72}$$

$$M_y = \int_0^1 \int_0^{e^x} (2-x+y)x \, dy \, dx = \int_0^1 \left[2xy - x^2y + \frac{xy^2}{2} \right]_{y=0}^{e^x} dx = \int_0^1 \left(2xe^x - x^2e^x + \frac{xe^{2x}}{2} \right) dx$$

$$= \left[(2xe^x - 2e^x) - (x^2e^x - 2xe^x + 2e^x) + \left(\frac{xe^{2x}}{4} - \frac{e^{2x}}{8} \right) \right]_0^1 = \frac{e^2 - 8e + 33}{8}$$

$$\bar{x} = \frac{M_y}{m} = \frac{e^2 - 8e + 33}{2(e^2 + 8e - 13)} \approx 0.5777; \bar{y} = \frac{M_x}{m} = \frac{8e^3 + 27e^2 - 53}{18(e^2 + 8e - 13)} \approx 1.0577$$

7.



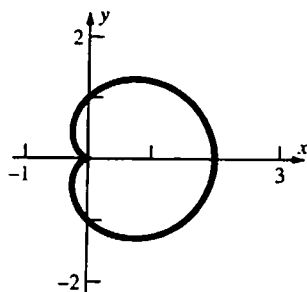
$$m = \int_0^\pi \int_0^{2\sin\theta} r \, r \, dr \, d\theta = \frac{32}{9}$$

$$M_x = \int_0^\pi \int_0^{2\sin\theta} (r \sin\theta) r \, r \, dr \, d\theta = \frac{64}{15}$$

$$M_y = 0 \text{ (symmetry)}$$

$$(\bar{x}, \bar{y}) = (0, 1.2)$$

8.



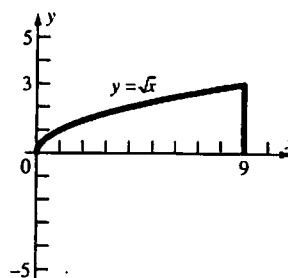
$$m = 2 \int_0^\pi \int_0^{1+\cos\theta} r \, r \, dr \, d\theta = \frac{5\pi}{3}$$

$$M_y = 2 \int_0^\pi \int_0^{1+\cos\theta} (r \cos\theta) r \, r \, dr \, d\theta = \frac{7\pi}{4}$$

$$M_x = 0 \text{ (symmetry)}$$

$$(\bar{x}, \bar{y}) = (1.05, 0)$$

9.



$$I_x = \int_0^9 \int_0^{\sqrt{x}} y^2 (x+y) \, dy \, dx$$

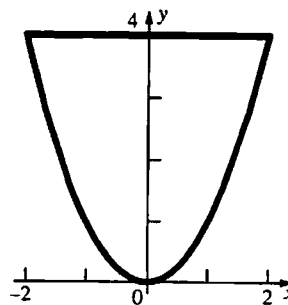
$$= \int_0^9 \left(\frac{81y^2}{2} + 9y^3 - \frac{y^6}{2} - y^5 \right) dy = \frac{7533}{28} \approx 269$$

$$I_y = \int_0^9 \int_0^{\sqrt{x}} x^2 (x+y) \, dy \, dx = \int_0^9 \left(x^{7/2} + \frac{x^3}{2} \right) dx$$

$$= \frac{41553}{8} \approx 5194$$

$$I_z = I_x + I_y = \frac{305937}{56} \approx 5463$$

10.

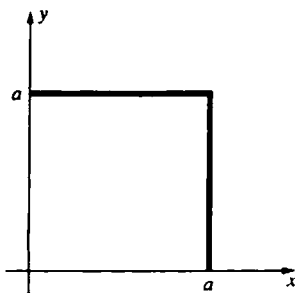


$$I_x = \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} y^3 \, dx \, dy = \frac{2048}{9} \approx 227.56$$

$$I_y = \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} x^2 y \, dx \, dy = \frac{512}{21} \approx 24.38$$

$$I_z = I_x + I_y = \frac{15872}{63} \approx 251.94$$

11.

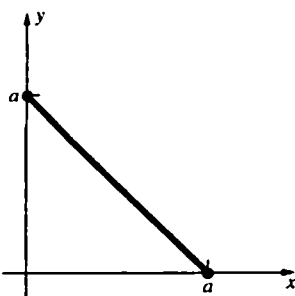


$$I_x = \int_0^a \int_0^a (x+y)y^2 dx dy = \left(\frac{5}{12}\right)a^5$$

$$I_y = \left(\frac{5}{12}\right)a^5$$

$$I_z = \left(\frac{5}{6}\right)a^5$$

12.



$$I_x = \int_0^a \int_0^{a-y} (x^2 + y^2)y^2 dx dy$$

$$= \frac{1}{3} \int_0^a (a^3 y^2 - 3a^2 y + 6ay^2 - 4y^5) dy = \frac{7a^6}{180}$$

$$I_y = \frac{7a^6}{180}; I_z = \frac{7a^6}{90} \text{ (Same result for } a < 0 \text{)}$$

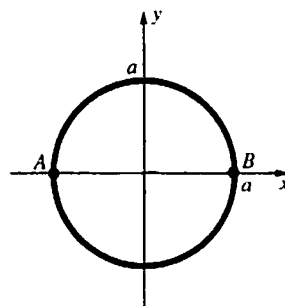
$$13. \quad m = \int_0^a \int_0^a (x+y) dx dy = a^3$$

$$\bar{r} = \left(\frac{I_x}{m}\right)^{1/2} = \left(\frac{5}{12}\right)^{1/2} a \approx 0.6455a$$

$$14. \quad m = \int_0^a \int_0^{a-y} (x^2 + y^2) dx dy = \left(\frac{1}{6}\right)a^4$$

$$\bar{r} = \left(\frac{I_y}{m}\right)^{1/2} = \left(\frac{7}{30}\right)^{1/2} a \approx 0.4830a$$

15.



$$m = \delta \pi a^2$$

The moment of inertia about diameter AB is

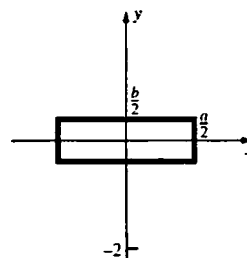
$$I = I_x = \int_0^{2\pi} \int_0^a \delta r^2 \sin^2 \theta r dr d\theta$$

$$= \int_0^{2\pi} \frac{\delta a^4 \sin^2 \theta}{4} d\theta = \frac{\delta a^4}{8} \int_0^{2\pi} (1 - \cos 2\theta) d\theta$$

$$= \frac{\delta a^4}{8} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} = \frac{\delta a^4 \pi}{4}$$

$$\bar{r} = \left(\frac{I}{m}\right)^{1/2} = \left(\frac{\frac{\delta a^4 \pi}{4}}{\delta \pi a^2}\right)^{1/2} = \frac{a}{2}$$

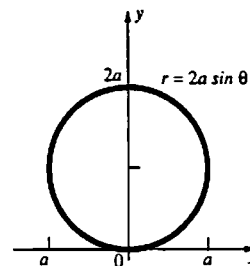
16.



$$I = I_z = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (x^2 + y^2) k dx dy$$

$$= \left(\frac{k}{12}\right)(a^3 b + ab^3)$$

17.



$$I_x = \iint_S \delta y^2 dA$$

$$= 2\delta \int_0^{\pi/2} \int_0^{2a \sin \theta} (r \sin \theta)^2 r dr d\theta$$

$$= 2\delta \int_0^{\pi/2} 4a^4 \sin^6 \theta d\theta$$

$$= 8a^4 \delta \frac{(1)(3)(5)}{(2)(4)(6)} \frac{\pi}{2} = \frac{5a^4 \delta \pi}{4}$$

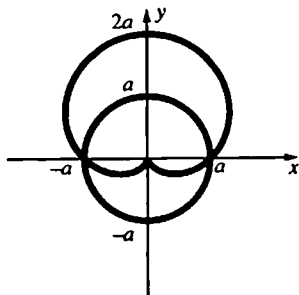
18. $\bar{x} = 0$ (by symmetry)

$$\begin{aligned}
 M_x &= \iint_S 1y \, dA = 2k \int_{-\pi/2}^{\pi/2} \int_0^{a(1+\sin\theta)} (r \sin\theta) r \, dr \, d\theta = 2k \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3} \sin\theta \right]_{r=0}^{a(1+\sin\theta)} d\theta \\
 &= \frac{2ka^3}{3} \int_{-\pi/2}^{\pi/2} (1+\sin\theta)^3 \sin\theta \, d\theta \\
 &= \frac{2ka^3}{3} \int_{-\pi/2}^{\pi/2} (\sin\theta + 3\sin^2\theta + 3\sin^3\theta + \sin^4\theta) d\theta \\
 &= \frac{4ka^3}{3} \int_0^{\pi/2} (3\sin^2\theta + \sin^4\theta) d\theta \quad (\text{using the symmetry property for odd and even functions.}) \\
 &= \frac{4ka^3}{3} \left(3 \frac{1}{2} \frac{\pi}{2} + \frac{1 \cdot 3}{2 \cdot 4} \frac{\pi}{2} \right) = \frac{5\pi ka^3}{4} \quad (\text{using Formula 113})
 \end{aligned}$$

$$\text{Therefore, } \bar{y} = \frac{M_x}{m} = \frac{5a}{6}.$$

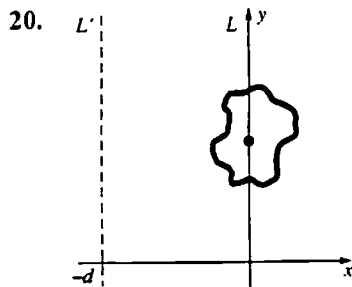
$$\begin{aligned}
 I_x &= \iint_S ky^2 \, dA = 2k \int_{-\pi/2}^{\pi/2} \int_0^{a(1+\sin\theta)} (r \sin\theta)^2 r \, dr \, d\theta = 2k \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \sin^2\theta \right]_{r=0}^{a(1+\sin\theta)} d\theta \\
 &= \frac{ka^4}{2} \int_{-\pi/2}^{\pi/2} (1+\sin\theta)^4 \sin^2\theta \, d\theta = \frac{ka^4}{2} \int_{-\pi/2}^{\pi/2} (\sin^2\theta + 4\sin^3\theta + 6\sin^4\theta + 4\sin^5\theta + \sin^6\theta) d\theta \\
 &= ka \int_0^{\pi/2} (\sin^2\theta + 6\sin^4\theta + \sin^6\theta) d\theta \quad (\text{symmetry property for odd and even functions}) \\
 &= ka^4 \left[\frac{1}{2} \frac{\pi}{2} + 6 \frac{1 \cdot 3}{2 \cdot 4} \frac{\pi}{2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{\pi}{2} \right] = \frac{49\pi ka^4}{32} \quad (\text{using Formula 113})
 \end{aligned}$$

19.



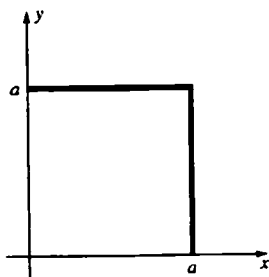
$x = 0$ (by symmetry)

$$\begin{aligned}
 M_x &= \iint_S ky \, dA = 2k \int_{-\pi/2}^{\pi/2} \int_0^{a(1+\sin\theta)} (r \sin\theta) r \, dr \, d\theta = 2k \int_{-\pi/2}^{\pi/2} \left(\frac{a^3}{3} \right) (3\sin^2\theta + 3\sin^3\theta + \sin^4\theta) d\theta \\
 &= \left(\frac{2}{3} \right) ka^3 \left[\frac{(15\pi + 32)}{16} \right] = \left(\frac{1}{24} \right) ka^3 (15\pi + 32) \\
 m &= \iint_S k \, dA = 2k \int_{-\pi/2}^{\pi/2} \int_0^{a(1+\sin\theta)} r \, dr \, d\theta = 2k \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} \right) a^2 (2\sin\theta + \sin^2\theta) d\theta = ka^2 \left[\frac{(8 + \pi)}{4} \right] = \left(\frac{1}{4} \right) ka^2 (\pi + 8) \\
 \text{Therefore, } \bar{y} &= \frac{M_x}{m} = \frac{\left(\frac{1}{24} \right) ka^3 (15\pi + 32)}{\left(\frac{1}{4} \right) ka^2 (\pi + 8)} = \frac{a(15\pi + 32)}{6(\pi + 8)} \approx 1.1836a
 \end{aligned}$$



$$\begin{aligned}
 I' &= \iint_S (x+d)^2 \delta(x, y) dA = \iint_S (x^2 + 2xd + d^2) \delta(x, y) dA \\
 &= \iint_S x^2 \delta(x, y) dA + \iint_S 2xd \delta(x, y) dA + \iint_S d^2 \delta(x, y) dA \\
 &= I + M_y + d^2 m = I + 0 + d^2 m = I + d^2 m
 \end{aligned}$$

21. a.



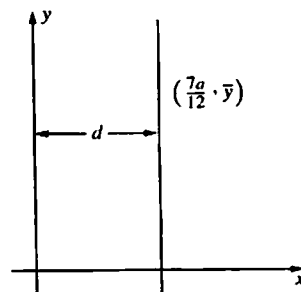
$$\begin{aligned}
 m &= \iint_S (x+y) dA = \int_0^a \int_0^a (x+y) dx dy \\
 &= \int_0^a \left[\frac{x^2}{2} + xy \right]_{x=0}^a dy = \int_0^a \left(\frac{a^2}{2} + ay \right) dy \\
 &= \left[\frac{a^2 y}{2} + \frac{ay^2}{2} \right]_0^a = a^3
 \end{aligned}$$

b. $M_y = \iint_S x(x+y) dA = \int_0^a \int_0^a (x^2 + xy) dy dx$

$$\begin{aligned}
 &= \int_0^a \left[x^2 y + \frac{xy^2}{2} \right]_{y=0}^a dx = \int_0^a \left(ax^2 + \frac{a^2 x}{2} \right) dx \\
 &= \left[\frac{ax^3}{3} + \frac{a^2 x^2}{4} \right]_0^a = \frac{7a^4}{12}
 \end{aligned}$$

Therefore, $\bar{x} = \frac{M_y}{m} = \frac{7a}{12}$.

c.

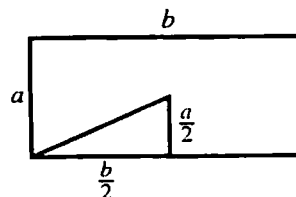


$$\begin{aligned}
 I_y &= I_L + d^2 m, \text{ so } \frac{5a^5}{12} = I_L + \left(\frac{7a}{12} \right)^2 (a^3); \\
 I_L &= \frac{11a^5}{144}
 \end{aligned}$$

22. $I_{17} = I_{15} + md^2 = 0.25\delta a^4 \pi + (\delta \pi a^2) a^2$
 $= 1.25\delta a^4 \pi$

23. $I_x = 2[I_{15}] = \frac{ka^4 \pi}{2}$
 $I_y = 2[I_{15} + md^2]$
 $= 2[0.25a^4 \pi + (k\pi a^2)(2a)^2] = 8.5ka^4 \pi$
 $I_z = I_x + I_y = 9ka^4 \pi$

24.



The square of the distance of the corner from the center of mass is $d^2 = \frac{a^2 + b^2}{4}$.

$$I = I(\text{Prob. 16}) + md^2$$

$$= \frac{k(a^3b + ab^3)}{12} + (kab) \frac{a^2 + b^2}{4} = \frac{k(a^3b + ab^3)}{3}$$

$$25. \quad M_y = \iint_{S_1 \cup S_2} x \delta(x, y) dA$$

$$= \iint_{S_1} x \delta(x, y) dA + \iint_{S_2} x \delta(x, y) dA$$

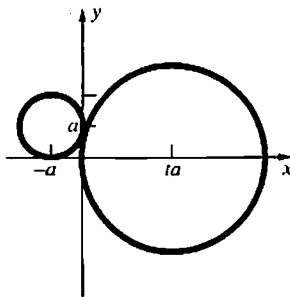
$$= \frac{m_1 \iint_{S_1} x \delta(x, y) dA}{m_1} + \frac{m_2 \iint_{S_2} x \delta(x, y) dA}{m_2}$$

$$= m_1 \bar{x}_1 + m_2 \bar{x}_2$$

$$\text{Thus, } \bar{x} = \frac{M_y}{m} = \frac{m_1 \bar{x}_1 + m_2 \bar{x}_2}{m_1 + m_2} \text{ which is equal to}$$

what we are to obtain and which is what we would obtain using the center of mass formula for two point masses. (Similar result can be obtained for $\bar{y}$.)

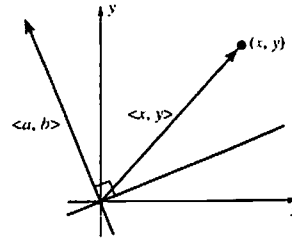
26.



$$\bar{x} = \frac{(-a)(\delta\pi a^2) + (ta)[\delta\pi(ta)^2]}{\delta\pi a^2 + \delta\pi(ta)^2} = \frac{a(t^3 - 1)}{t^2 + 1}$$

$$\bar{y} = \frac{(a)(\delta\pi a^2) + (0)}{\delta\pi a^2 + \delta\pi(ta)^2} = \frac{a}{t^2 + 1}$$

27.



$\langle a, b \rangle$ is perpendicular to the line $ax + by = 0$.

Therefore, the (signed) distance of (x, y) to L is the scalar projection of $\langle x, y \rangle$ onto $\langle a, b \rangle$, which

$$\text{is } d(x, y) = \frac{\langle x, y \rangle \cdot \langle a, b \rangle}{\|\langle a, b \rangle\|} = \frac{ax + by}{\|\langle a, b \rangle\|}.$$

$$M_L = \iint_S d(x, y) \delta(x, y) dA$$

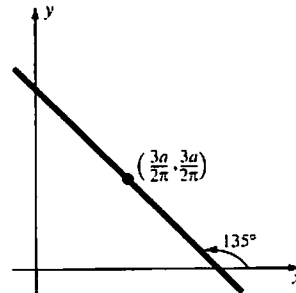
$$= \iint_S \frac{ax + by}{\|\langle a, b \rangle\|} \delta(x, y) dA$$

$$= \frac{a}{\|\langle a, b \rangle\|} \iint_S x \delta(x, y) dA + \frac{b}{\|\langle a, b \rangle\|} \iint_S y \delta(x, y) dA$$

$$= \frac{a}{\|\langle a, b \rangle\|} (0) + \frac{b}{\|\langle a, b \rangle\|} (0) = 0$$

[since $(\bar{x}, \bar{y}) = (0, 0)$]

28.



The equation has the form $x + y = b$.

$$\frac{3a}{2\pi} + \frac{3a}{2\pi} = b \text{ so } b = \frac{3a}{\pi}.$$

Therefore, the equation is $x + y = \frac{3a}{\pi}$, or

$$\pi x + \pi y = 3a.$$

16.6 Concepts Review

1. $|\mathbf{u} \times \mathbf{v}|$

2. $\iint_S \sqrt{f_x^2 + f_y^2 + 1} dA$

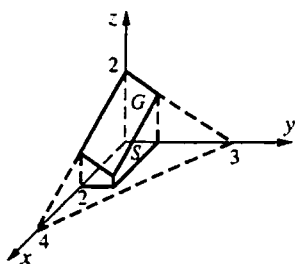
3. $\int_{-a}^a \int_{\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\frac{a}{\sqrt{a^2-x^2-y^2}} \right) dy dx$

$$= \int_0^{2\pi} \int_0^a \left(\frac{ar}{\sqrt{a^2-r^2}} \right) dr d\theta; \quad 2\pi a^2$$

4. $2\pi ah$

Problem Set 16.6

1.



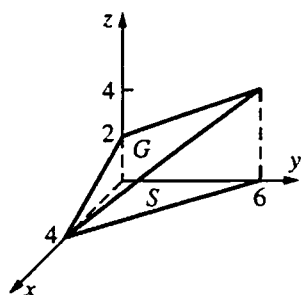
$$z = 2 - \frac{1}{2}x - \frac{2}{3}y$$

$$f_x(x, y) = -\frac{1}{2}; f_y(x, y) = -\frac{2}{3}$$

$$A(G) = \int_0^2 \int_0^{\frac{3}{2}(2-y)} \sqrt{\frac{1}{4} + \frac{4}{9}} + 1 \, dy \, dx$$

$$= \frac{\sqrt{61}}{3} \approx 2.6034$$

2.



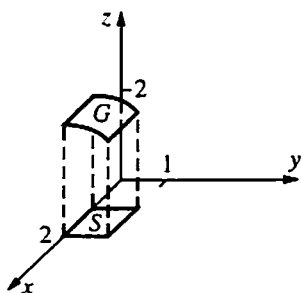
$$z = 2 - \frac{1}{2}x + \frac{1}{3}y$$

$$f_x(x, y) = -\frac{1}{2}; f_y(x, y) = \frac{1}{3}$$

$$A(G) = \int_0^4 \int_0^{\frac{3}{2}x+6} \sqrt{\frac{1}{4} + \frac{1}{9}} + 1 \, dy \, dx$$

$$= \frac{7}{6} \int_0^4 \left(-\frac{3}{2}x + 6 \right) dx = \left(\frac{7}{6} \right) (12) = 14$$

3.



$$z = f(x, y) = (4 - y^2)^{1/2}; f_x(x, y) = 0;$$

$$f_y(x, y) = -y(4 - y^2)^{-1/2}$$

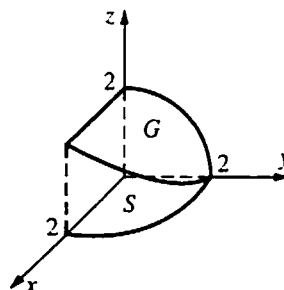
$$A(G) = \int_0^1 \int_0^2 \sqrt{y^2(4 - y^2)^{-1} + 1} \, dx \, dy$$

$$= \int_0^1 \int_0^2 \frac{2}{\sqrt{4 - y^2}} \, dx \, dy$$

$$= \int_0^1 \frac{2}{\sqrt{4 - y^2}} \, dy = \left[2 \sin^{-1} \left(\frac{y}{2} \right) \right]_0^1 = 2 \left(\frac{\pi}{6} \right) - 2(0)$$

$$= \frac{\pi}{3} \approx 1.0472$$

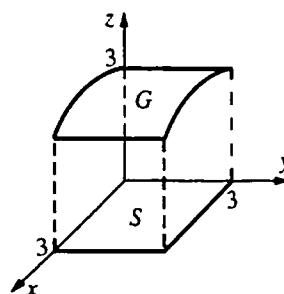
4.



$$A(G) = \int_0^2 \int_0^{\sqrt{4-y^2}} 2(4 - y^2)^{-1/2} \, dx \, dy = 4$$

(See problem 3 for the integrand.)

5.



$$\text{Let } z = f(x, y) = (9 - x^2)^{1/2}.$$

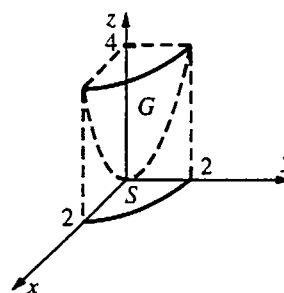
$$f_x(x, y) = -x(9 - x^2)^{-1/2}; f_y(x, y) = 0$$

$$A(G) = \int_0^2 \int_0^3 [x^2(9 - x^2)^{-1} + 1] \, dy \, dx$$

$$= \int_0^2 \int_0^3 3(9 - x^2)^{-1/2} \, dx \, dy = 9 \sin^{-1} \left(\frac{2}{3} \right)$$

$$\approx 6.5675$$

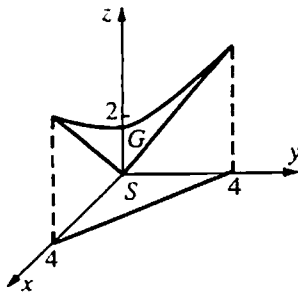
6.



Let $z = f(x, y) = x^2 + y^2$; $f_x(x, y) = 2x$;
 $f_y(x, y) = 2y$.

$$\begin{aligned} A(G) &= 4 \int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{4x^2 + 4y^2 + 1} \, dy \, dx \\ &= 4 \int_0^{\pi/2} \int_0^2 (4r^2 + 1)^{1/2} r \, dr \, d\theta \\ &= 4 \int_0^{\pi/2} \left[\frac{(4r^2 + 1)^{3/2}}{12} \right]_0^2 d\theta = \frac{(17^{3/2} - 1)\pi}{3} \frac{1}{2} \\ &\approx 36.1769 \end{aligned}$$

7.

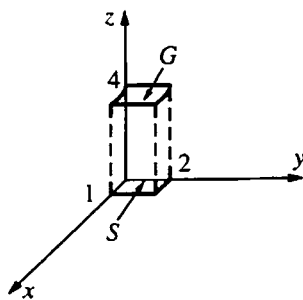


$$z = f(x, y) = (x^2 + y^2)^{1/2}$$

$$f_x(x, y) = x(x^2 + y^2)^{-1/2}, \quad f_y(x, y) = y(x^2 + y^2)^{-1/2}$$

$$A(G) = \int_0^4 \int_0^{4-x} [x^2(x^2 + y^2)^{-1} + y^2(x^2 + y^2)^{-1} + 1]^{1/2} dy \, dx = \int_0^4 \int_0^{4-x} \sqrt{2} dy \, dx = 8\sqrt{2}$$

8.

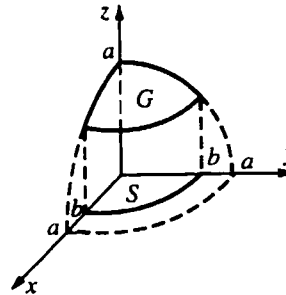


$$z = f(x, y) = \left(\frac{1}{4}\right)x^2 + 4$$

$$f_x(x, y) = \frac{x}{2}; \quad f_y(x, y) = 0$$

$$\begin{aligned} A(G) &= \int_0^1 \int_0^2 \left[\left(\frac{1}{4}\right)x^2 + 1 \right]^{1/2} dy \, dx \\ &= \frac{\sqrt{5}}{2} + 2 \ln \left[\frac{(\sqrt{5} + 1)}{2} \right] \approx 2.0805 \end{aligned}$$

9.

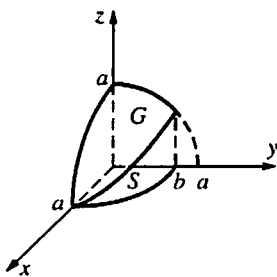


$$f_x(x, y) = \frac{x^2}{a^2 - x^2 - y^2}; \quad f_y(x, y) = \frac{y^2}{a^2 - x^2 - y^2}$$

(See Example 3.)

$$\begin{aligned} A(G) &= 8 \iint_S \frac{a}{\sqrt{a^2 - x^2 - y^2}} dA \\ &= 8 \int_0^{\pi/2} \int_0^b \frac{a}{\sqrt{a^2 - r^2}} r \, dr \, d\theta \\ &= 8a \left(\frac{\pi}{2} \right) \int_0^b (a^2 - r^2)^{-1/2} r \, dr \\ &= -4a\pi \left[(a^2 - r^2)^{1/2} \right]_0^b = 4\pi a \left(a - \sqrt{a^2 - b^2} \right) \end{aligned}$$

10.



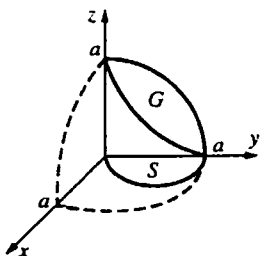
$$f_x(x, y) = \frac{x^2}{a^2 - x^2 - y^2}; f_y(x, y) = \frac{y^2}{a^2 - x^2 - y^2}$$

(See Example 3.)

$$A(G) = 8 \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}} dy dx$$

$$= 8a \int_0^a \sin^{-1}\left(\frac{b}{a}\right) dx = 8a^2 \sin^{-1}\left(\frac{b}{a}\right)$$

11.



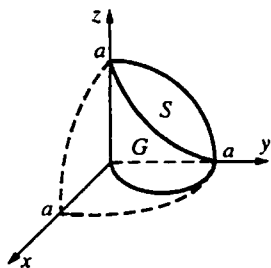
$$f_x(x, y) = \frac{x^2}{a^2 - x^2 - y^2}; f_y(x, y) = \frac{y^2}{a^2 - x^2 - y^2}$$

(See Example 3.)

$$A(G) = 4 \int_0^{\pi/2} \int_0^{a \sin \theta} \frac{a}{\sqrt{a^2 - r^2}} r dr d\theta$$

$$= 4a^2 \int_0^{\pi/2} (1 - \cos \theta) d\theta = 2a^2(\pi - 2)$$

12.



$$\text{Let } F(x, y, z) = x^2 + y^2 - ay.$$

$$\sec \gamma = \frac{\sqrt{(2x)^2 + (2y - a)^2}}{|2x|} \quad (\text{Interchange roles of } x \text{ and } z.)$$

$$= \frac{\sqrt{4(x^2 + y^2) - 4ay + a^2}}{2|x|} = \frac{\sqrt{4(ay) - 4ay + a^2}}{2\sqrt{y(a - y)}}$$

$$= \frac{a}{2\sqrt{y(a - y)}}$$

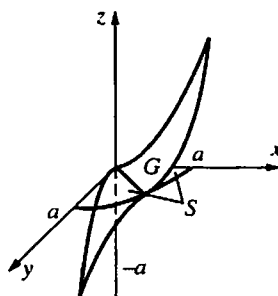
$$A(G) = 4 \iint_S \frac{a}{2\sqrt{y(a - y)}} dA$$

$$= 2a \int_0^a \int_0^{\sqrt{a(a - y)}} \frac{1}{\sqrt{y(a - y)}} dz dy$$

$$= 2a \int_0^a \frac{\sqrt{a(a - y)}}{\sqrt{y(a - y)}} dy = 2a\sqrt{a} \int_0^a y^{-1/2} dy$$

$$= 2a\sqrt{a} \left[\lim_{k \rightarrow 0^+} \int_k^a y^{-1/2} dy \right] = 2a\sqrt{a} (2\sqrt{a}) = 4a^2$$

13.



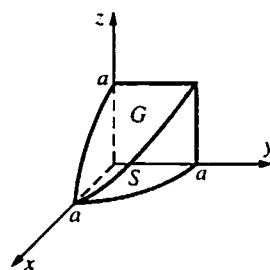
$$\text{Let } F(x, y, z) = x^2 - y^2 - az.$$

$$f_x(x, y) = \frac{2x}{a}; f_y(x, y) = \frac{-2y}{a}$$

$$A(G) = \int_0^{2\pi} \int_0^a \frac{\sqrt{4r^2 + a^2}}{a} r dr d\theta$$

$$= \frac{2\pi}{a} \int_0^a (4r^2 + a^2)^{1/2} r dr = \frac{\pi a^2 (5\sqrt{5} - 1)}{6}$$

14.



$$f_x(x, y) = \frac{-x}{\sqrt{a^2 - x^2}}; f_y(x, y) = 0$$

$$\sqrt{(f_x(x, y))^2 + (f_y(x, y))^2} + 1 = \frac{a}{\sqrt{a^2 - x^2}}$$

$$= \frac{a}{\sqrt{a^2 - r^2 \cos^2 \theta}}$$

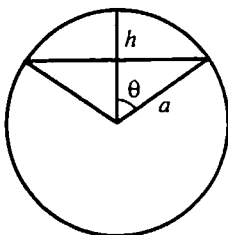
$$A(\text{all sides}) = 16 \int_0^{\pi/2} \int_0^a \frac{a}{\sqrt{a^2 - r^2 \cos^2 \theta}} d\theta dr$$

$$= 16 \int_0^{\pi/2} \frac{a^2}{1 + \sin \theta} d\theta = 16a^2(1) = 16a^2$$

15. $\bar{x} = \bar{y} = 0$ (by symmetry)

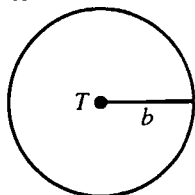
Let $h = \frac{h_1 + h_2}{2}$. Planes $z = h_1$ and $z = h_2$ cut out the same surface area as planes $z = h$ and $z = h$. Therefore, $\bar{z} = h$, the arithmetic average of h_1 and h_2 .

16.

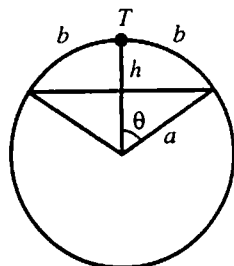


$$\begin{aligned}\text{Area} &= 2\pi ah \\ &= 2\pi a(a - a \cos \phi) = 2\pi a^2(1 - \cos \phi)\end{aligned}$$

17. a. $A = \pi b^2$

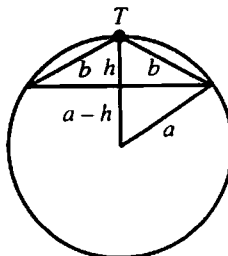


b.



$$\begin{aligned}B &= 2\pi a^2(1 - \cos \phi) \quad (\text{Problem 16}) \\ &= 2\pi a^2 \left[1 - \cos \left(\frac{b}{a} \right) \right] \\ &= 2\pi a^2 \left[\frac{b^2}{2!a^2} - \frac{b^4}{4!a^4} + \frac{b^6}{6!a^6} - \dots \right] \\ &= \pi b^2 \left[1 - \frac{b^2}{12a^2} + \frac{b^4}{360a^4} - \dots \right] \leq \pi b^2\end{aligned}$$

c.

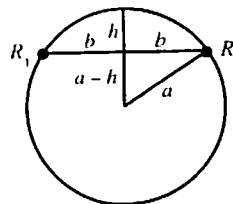


$$a^2 - (a - h)^2 = b^2 - h^2, \text{ so } h = \frac{b^2}{2a}.$$

$$\text{Thus, } C = 2\pi ah$$

$$= 2\pi a \left(\frac{b^2}{2a} \right) = \pi b^2.$$

d.



$$\begin{aligned}D &= 2\pi ah \\ &= 2\pi a \left(a - \sqrt{a^2 - b^2} \right) = \frac{2\pi a[a^2 - (a^2 - b^2)]}{a + \sqrt{a^2 - b^2}} \\ &= \frac{2\pi ab^2}{a + \sqrt{a^2 - b^2}} > \pi b^2\end{aligned}$$

Therefore, $B < A = C < D$.

$$\begin{aligned}18. & [A(S_{yz})]^2 + [A(S_{xz})]^2 + [A(S_{xy})]^2 \\ &= [A(S) \cos \alpha]^2 + [A(S) \cos \beta]^2 + [A(S) \cos \gamma]^2 \\ & \quad (\text{where } \alpha, \beta, \text{ and } \gamma \text{ are direction angles for a normal to } S.) \\ &= [A(S)]^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = [A(S)]^2\end{aligned}$$

19. In the following, each double integral is over S_{xy}

$$\begin{aligned}A(S_{xy})f(\bar{x}, \bar{y}) &= A(S_{xy})(a\bar{x} + b\bar{y} + c) \\ &= \iint dA \left[a \frac{\iint x dA}{\iint dA} + b \frac{\iint y dA}{\iint dA} + c \right] \\ &= a \iint x dA + b \iint y dA + c \iint dA \\ &= \iint (ax + by + c) dA \\ &= \text{Volume of solid cylinder under } S_{xy}\end{aligned}$$

20. Because the slopes of both roofs are the same, the area of T_m will be the same for both roofs. (Essentially we will be integrating over a constant). Therefore, the area of the roofs will be the same.

21. Let G denote the surface of that part of the plane $z = Ax + By + C$ over the region S . First, suppose that S is the rectangle $a \leq x \leq b$, $c \leq y \leq d$. Then the vectors u and v that form the edge of the parallelogram G are $u = (b - a)i + 0j + A(b - a)k$ and $v = 0i + (d - c)j + B(d - c)k$. The surface area of

G is thus

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}| &= \\ | -A(b-a)(d-c)\mathbf{i} - B(b-a)(d-c)\mathbf{j} + (b-a)(d-c)\mathbf{k} | \\ &= (b-a)(d-c)\sqrt{A^2 + B^2 + 1} \end{aligned}$$

A normal vector to the plane is $\mathbf{n} = -A\mathbf{i} - B\mathbf{j} + \mathbf{k}$.

Thus,

$$\cos \gamma = \frac{\mathbf{n} \cdot \mathbf{k}}{|\mathbf{n}| |\mathbf{k}|} = \frac{\langle -A, -B, 1 \rangle \cdot \langle 0, 0, 1 \rangle}{\sqrt{A^2 + B^2 + 1} \cdot 1} = \frac{1}{\sqrt{A^2 + B^2 + 1}}$$

$$\sec \gamma = \frac{1}{\cos \gamma} = \sqrt{A^2 + B^2 + 1} = |\mathbf{u} \times \mathbf{v}| = A(G).$$

If S is not a rectangle, then make a partition of S with rectangles $R_1, R_2, \dots, R_n$. The Riemann sum will be

$$\sum_{m=1}^n A(G_m) \Delta x_m \Delta y_m = \sum_{m=1}^n \sec \gamma \Delta x_m \Delta y_m.$$

$$= \sec \gamma \sum_{m=1}^n A(R_m). \text{ As we take the limit as}$$

$|P| \rightarrow 0$ the sum converges to the area of S .

Thus the surface area will be

$$A(G) = \lim_{|P| \rightarrow 0} \sec \gamma \sum_{m=1}^n A(R_m) = \sec \gamma A(S).$$

22. Let $\gamma = \gamma(x, y, f(x, y))$ be the acute angle between a unit vector $\mathbf{n}$ that is normal to the surface and makes an acute angle with the z -axis. Let $F(x, y, z) = z - f(x, y)$. Then the normal vector to the surface $F(x, y, z) = 0 = z - f(x, y)$ is parallel to the gradient

$\nabla F(x, y, z) = -f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}$. The unit normal vector is thus $\mathbf{n} = (-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}) / \sqrt{f_x^2 + f_y^2 + 1}$

The cosine of the angle γ is thus

$$\cos \gamma = \frac{\mathbf{n} \cdot \mathbf{k}}{|\mathbf{n}| |\mathbf{k}|} = \frac{-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k} \cdot \mathbf{k}}{\sqrt{f_x^2 + f_y^2 + 1} \cdot 1} = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$\text{Hence, } \sec \gamma = \sqrt{f_x^2 + f_y^2 + 1}$$

16.7 Concepts Review

1. volume
2. $\iiint_S |xyz| dV$
3. $y; \sqrt{y}$
4. 0

Problem Set 16.7

1. $\int_3^7 \int_0^{2x} (x-1-y) dy dx = \int_3^7 -2x dx = -40$
2. $\int_0^2 \int_1^4 (3y+x) dy dx = \int_0^2 \left(\frac{45}{2} + 5x \right) dx = 55$
3. $\int_1^4 \int_{z-1}^{2z} \int_0^{y+2z} dx dy dz = \int_1^4 \int_{z-1}^{2z} (y+2z) dy dz$
 $= \int_1^4 \left[\frac{y^2}{2} + 2yz \right]_{y=z-1}^{2z} dz$
 $= \int_1^4 \left(\frac{7z^2}{2} + 3z - \frac{1}{2} \right) dz$
 $= \left[\frac{7z^3}{6} + \frac{3z^2}{2} - \frac{z}{2} \right]_1^4 = \frac{189}{2} = 94.5$

$$4. \int_0^5 z^3 dz \int_2^4 y^2 dy \int_1^2 x dx = \left(\frac{625}{4} \right) (72)(3) = 33,750$$

$$5. \int_0^2 \int_1^z x^2 dx dz = \int_0^2 \left(\frac{1}{3} \right) (z^3 - 1) dz = \frac{2}{3}$$

$$6. \int_0^{\pi/2} \int_0^z \int_0^y \sin(x+y+z) dx dy dz$$

$$= \int_0^{\pi/2} \int_0^z [-\cos(2y+z) + \cos(y+z)] dy dz$$

$$= \int_0^{\pi/2} \left(-\frac{\sin 3z}{2} + \sin 2z - \frac{\sin z}{2} \right) dz = \frac{1}{3}$$

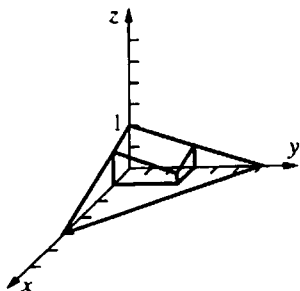
$$7. \int_{-2}^4 \int_{x-1}^{x+1} 3y^2 dy dx = \int_{-2}^4 (6x^2 + 2) dx = 156$$

$$8. \int_0^{\pi/2} \int_{\sin 2z}^0 y(1 - \cos 2z) dy dz$$

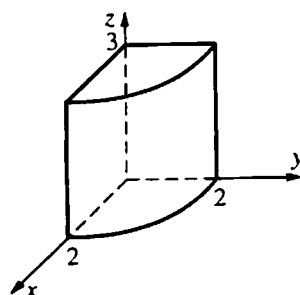
$$= \int_0^{\pi/2} \left(-\frac{1}{2} \right) (\sin^2 2z)(1 - \cos 2z) dz$$

$$= -\frac{\pi}{8} \approx 0.3927$$

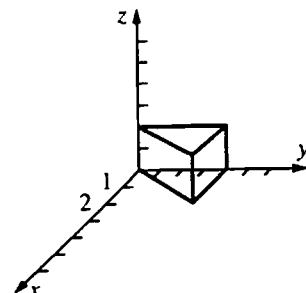
$$9. \int_0^1 \int_0^3 \int_0^{(12-3x-2y)/6} f(x, y, z) dz dy dx$$



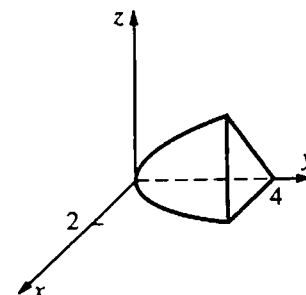
$$10. \int_0^3 \int_0^2 \int_0^{\sqrt{4-y^2}} f(x, y, z) dx dy dz$$



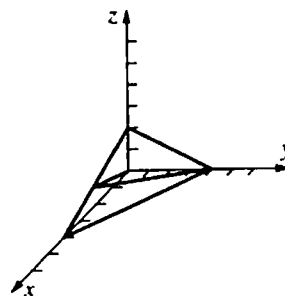
$$11. \int_0^2 \int_0^4 \int_0^{y/2} f(x, y, z) dx dy dz$$



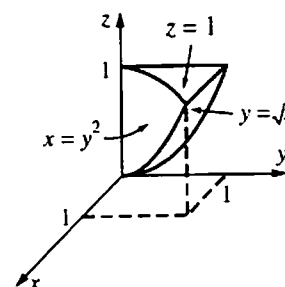
$$12. \int_0^4 \int_0^{\sqrt{y}} \int_0^{3x/2} f(x, y, z) dz dx dy$$



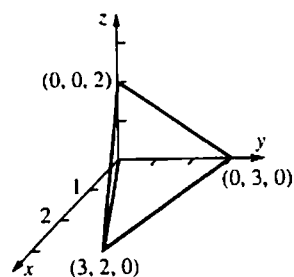
$$13. \int_0^{2.4} \int_{x/3}^{(4-x)/2} \int_0^{4-x-2z} f(x, y, z) dy dz dx$$



$$14. \int_0^1 \int_0^{\sqrt{z}} \int_0^{y^2} f(x, y, z) dx dy dz$$



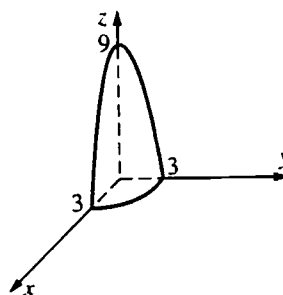
15.



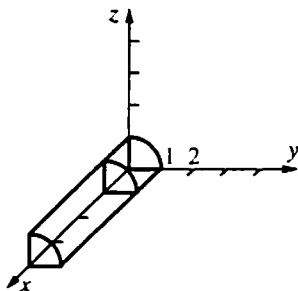
Using the cross product of vectors along edges, it is easy to show that $\langle 2, 6, 9 \rangle$ is normal to the upward face. Then obtain that its equation is $2x + 6y + 9z = 18$.

$$\int_0^3 \int_{2x/3}^{(9-x)/3} \int_0^{(18-2x-6y)/9} f(x, y, z) dz dy dx$$

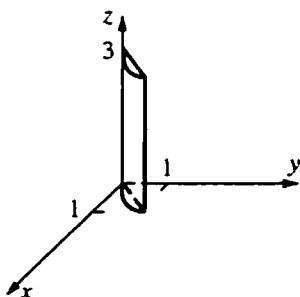
$$16. \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} f(x, y, z) dz dy dx$$



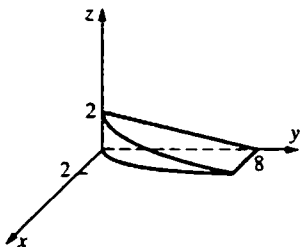
$$17. \int_1^4 \int_0^1 \int_0^{\sqrt{1-y^2}} f(x, y, z) dz dy dx$$



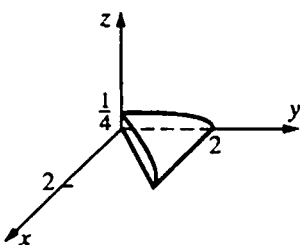
$$18. \int_0^3 \int_0^1 \int_y^{\sqrt{2y-y^2}} f(x, y, z) dx dy dz$$



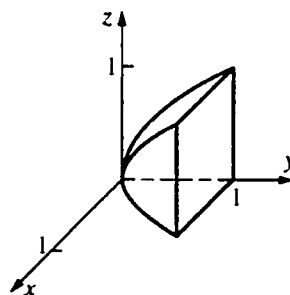
$$19. \int_0^2 \int_{2x^2}^8 \int_0^{2-y/4} 1 dz dy dx = \frac{128}{15}$$



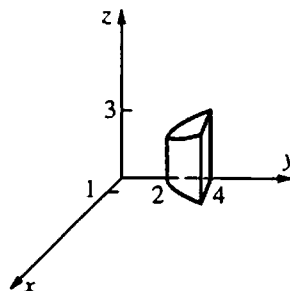
$$20. \int_0^2 \int_0^y \int_0^{\sqrt{4-y^2}/8} 1 dz dx dy = \frac{1}{3}$$



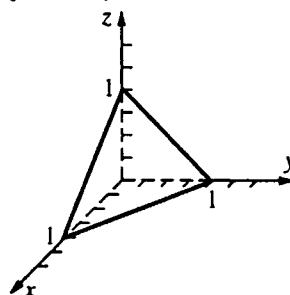
$$21. V = 4 \int_0^1 \int_0^{\sqrt{y}} \int_0^{\sqrt{y}} 1 dz dx dy = 4 \int_0^1 \int_0^{\sqrt{y}} \sqrt{y} dx dy \\ = 4 \int_0^1 \sqrt{y} \sqrt{y} dy = [2y^2]_0^1 = 2$$



$$22. 2 \int_0^{\sqrt{2}} \int_{x^2+2}^4 \int_0^{3y/4} 1 dz dy dx = 32 \frac{\sqrt{2}}{5} \approx 9.0510$$



23. Let $\delta(x, y, z) = x + y + z$. (See note with next problem.)



$$m = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z) dz dy dx = \frac{1}{8}$$

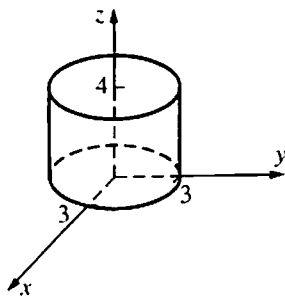
$$M_{yz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x(x + y + z) dz dy dx = \frac{1}{30}$$

$$\bar{x} = \frac{4}{15}$$

$$\text{Then } \bar{y} = \bar{z} = \frac{4}{15} \text{ (symmetry).}$$

$$24. (x, y, z) = k(x^2 + y^2 + z^2)$$

In evaluating the coordinates of the center of mass, k is a factor of the numerator and denominator and so may be canceled. Hence, for sake of convenience we may just let $k = 1$ when determining the center of mass. Note that this is not valid if we are concerned with values of moments or mass.



$$\begin{aligned}
 m &= 4 \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^4 (x^2 + y^2 + z^2) dz dy dx \\
 &= 4 \int_0^3 \int_0^{\sqrt{9-x^2}} \left[4(x^2 + y^2) + \frac{64}{3} \right] dy dx \\
 &= 4 \int_0^{\pi/2} \int_0^3 \left(4r^2 + \frac{64}{3} \right) r dr d\theta \quad (\text{changing to polar})
 \end{aligned}$$

$$\begin{aligned}
 &= 4 \int_0^{\pi/2} \left[r^4 + \frac{32r^2}{3} \right]_0^3 d\theta = 4 \int_0^{\pi/2} 177 d\theta \\
 &= 354\pi
 \end{aligned}$$

$$\begin{aligned}
 M_{xy} &= 4 \int_0^{\pi/2} \int_0^3 \int_0^4 z(r^2 + z^2) r dz dr d\theta \\
 (\text{polar coordinates})
 \end{aligned}$$

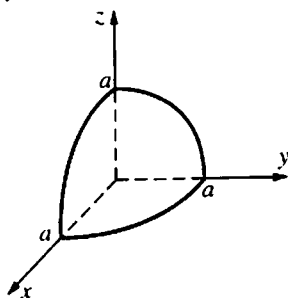
$$= 4 \int_0^{\pi/2} \int_0^3 (8r^3 + 64r) dr d\theta$$

$$= 4 \int_0^{\pi/2} 450 d\theta = 900\pi$$

$$\bar{z} = \frac{900\pi}{354\pi} = \frac{150}{59} \approx 2.5425;$$

$$\bar{x} = \bar{y} = 0 \quad (\text{by symmetry})$$

25. Let $\delta(x, y, z) = 1$. (See note with previous problem.)



$$m = \left(\frac{1}{8} \right) (\text{volume of sphere}) = \left(\frac{\pi}{6} \right) a^3$$

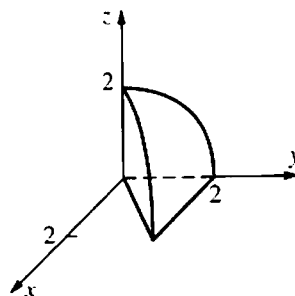
$$M_{xy} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} z dz dy dx$$

$$= \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} zr dz dr d\theta = \left(\frac{\pi}{16} \right) a^4$$

$$\bar{x} = \left(\frac{3}{8} \right) a$$

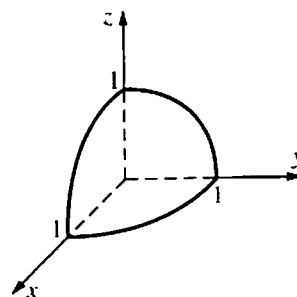
$$\bar{y} = \bar{z} = \left(\frac{3}{8} \right) a \quad (\text{by symmetry})$$

26. $y^2 + z^2$ is the distance of (x, y, z) from the x -axis.



$$\begin{aligned}
 I_x &= \iiint_S (y^2 + z^2)(x, y, z) dV \\
 &= \int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^y (y^2 + z^2)z dx dy dz = \frac{16}{3}
 \end{aligned}$$

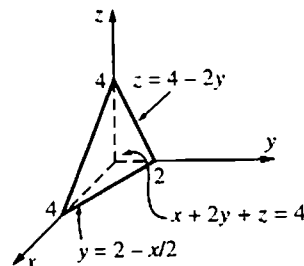
- 27.



The limits of integration are those for the first octant part of a sphere of radius 1.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} f(x, y, z) dz dy dx$$

- 28.

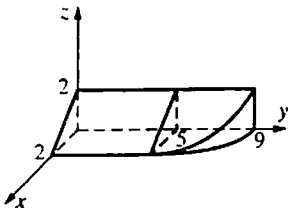


$$\int_0^2 \int_0^{2-x/2} \int_0^{4-x-2y} f(x, y, z) dz dy dx$$

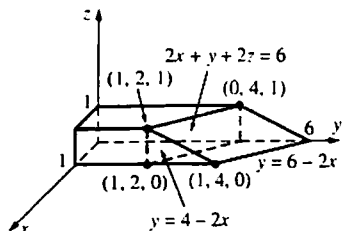
29. $\int_0^2 \int_0^{2-z} \int_0^{9-x^2} f(x, y, z) dy dx dz$

Figure is same as for Problem 30 except that the solid doesn't need to be divided into two parts.

$$30. \int_0^5 \int_0^2 \int_0^{2-x} f(x, y, z) dz dx dy + \int_5^9 \int_0^{\sqrt{9-y}} \int_0^{2-x} f(x, y, z) dz dx dy$$



31.



$$a. \int_0^1 \int_0^{4-2x} \int_0^{6-2x-2z} dz dy dx + \int_0^1 \int_{4-2x}^{6-2x} \int_0^{3-x-y/2} dz dy dx = 3 + 1 = 4$$

$$b. \int_0^1 \int_0^1 \int_0^{6-2x-2z} 1 dy dx dz = 4$$

$$c. A(S_{xz})f(\bar{x}, \bar{z}) \quad (S_{xz} \text{ is the unit square in the corner of } xz\text{-plane; and } (\bar{x}, \bar{z}) = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ is the centroid of } S_{xz}.)$$

$$= (1) \left[6 - 2\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) \right] = 4$$

32. The moment of inertia with respect to the y -axis is the integral (over the solid) of the function which gives the square of the distance of each point in the solid from the y -axis.

$$\int_0^1 \int_0^1 \int_0^{6-2x-2z} k(x^2 + z^2) dy dx dz = \frac{7}{3} k$$

$$33. \int_0^1 \int_0^1 \int_0^{6-2x-2z} (30-z) dy dx dz = \int_0^1 \int_0^1 (30-z)(6-2x-2z) dx dz = \int_0^1 [(30-z)(6x-x^2-2xz)]_{x=0}^1 dz$$

$$= \int_0^1 (30-z)(5-2z) dz = \int_0^1 (150-65z+2z^2) dz = \left[150z - \frac{65z^2}{2} + \frac{2z^3}{3} \right]_0^1 = \frac{709}{6}$$

The volume of the solid is 4 (from Problem 31).

Hence, the average temperature of the solid is $\frac{\frac{709}{6}}{4} = \frac{709}{24} \approx 29.54^\circ$.

$$\int_0^1 \int_0^1 \int_0^{6-2x-2z} z dy dx dz = \int_0^1 \int_0^1 z(6-2x-2z) dx dz = \int_0^1 [z(6x-x^2-2xz)]_{x=0}^1 dz$$

$$= \int_0^1 (5z-2z^2) dz = \left[\frac{5z^2}{2} - \frac{2z^3}{3} \right]_0^1 = \frac{11}{6}$$

Hence, $\bar{z} = \frac{\frac{11}{6}}{4} = \frac{11}{24} \approx 0.4583$.

34.

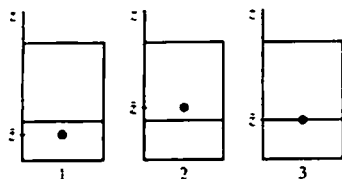


Figure 1: When the center of mass is in this position, it will go lower when a little more soda leaks out since mass above the center of mass is being removed.

Figure 2: When the center of mass is in this position, it was lower moments before since mass that was below the center of mass was removed,

causing the center of mass to rise.

Therefore, the center of mass is lowest when it is at the height of the soda, as in Figure 3. The same argument would hold for a soda bottle.

35. The result obtained from Mathematica is:

$$\int_0^c \int_0^b \int_0^{\sqrt{1-z^2/c^2}} \int_0^{\sqrt{1-y^2/b^2-z^2/c^2}} 8(xy+xz+yz) dx dy dz = \frac{8}{15} a^2 b^2 c + \frac{8}{15} a^2 b c^2 + \frac{8}{15} a b^2 c^2$$

$$= \frac{8}{15} acb(ca+cb+ab)$$

16.8 Concepts Review

1. $r \, dz \, dr \, d\theta, \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$

2. $\int_0^{\pi/2} \int_0^1 \int_0^3 r^3 \cos \theta \sin \theta \, dz \, dr \, d\theta$

3. $\int_0^\pi \int_0^{2\pi} \int_0^1 \rho^4 \cos^2 \phi \sin \phi \, d\rho \, d\theta \, d\phi$

4. $\frac{4\pi}{15}$

Problem Set 16.8

1. $\int_0^{2\pi} \int_0^2 \int_2^4 r \, dz \, dr \, d\theta = 8\pi \approx 25.1327$

2. $\int_0^{2\pi} \int_0^2 \int_0^{\sqrt{9-r^2}} r \, dz \, dr \, d\theta = \left(\frac{2}{3}\right)\pi(27-5^{3/2})$
 ≈ 33.1326

3. $\int_0^{2\pi} \int_0^2 \int_{r^2/4}^{\sqrt{5-r^2}} r \, dz \, dr \, d\theta$
 $= \int_0^{2\pi} \int_0^2 \left[4r(5-r^2)^{1/2} - \frac{r^3}{4} \right] dr \, d\theta$
 $= \int_0^{2\pi} \frac{5^{3/2}-4}{3} d\theta = \frac{2\pi(5^{3/2}-4)}{3} \approx 15.0385$

4. $\int_0^{2\pi} \int_p^{2\cos\theta} \int_0^{r^2\sin\theta\cos\theta} r \, dz \, dr \, d\theta = \frac{2}{3}$

5. Let $\delta(x, y, z) = 1$.
 (See write-up of Problem 24, Section 16.7.)

$$m = \int_0^{2\pi} \int_0^2 \int_2^{12-2r^2} r \, dz \, dr \, d\theta = 24\pi$$

$$M_{xy} = \int_0^{2\pi} \int_0^2 \int_2^{12-2r^2} zr \, dz \, dr \, d\theta = 128\pi$$

$$\bar{z} = \frac{16}{3}$$

$$\bar{x} = \bar{y} = 0 \text{ (by symmetry)}$$

6. Let $\delta(x, y, z) = 1$.
 (See comment at beginning of write-up of Problem 24 of the previous section.)

$$m = \int_0^{2\pi} \int_1^2 \int_0^{12-r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r(12-r^2) dr \, d\theta$$

$$= \int_0^{2\pi} \left[6r^2 - \frac{r^4}{4} \right]_0^2 d\theta = \int_0^{2\pi} \left(\frac{57}{4} \right) d\theta = \frac{57\pi}{2}$$

$$M_{xy} = \int_0^{2\pi} \int_1^2 \int_0^{12-r^2} zr \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_1^2 \frac{(12-r^2)^2(-2r)}{-4} dr \, d\theta$$

$$= \int_0^{2\pi} \frac{11^3-8^3}{12} d\theta = \frac{273\pi}{2}$$

$$\text{Therefore, } \bar{z} = \frac{\frac{273\pi}{2}}{\frac{57\pi}{2}} = \frac{91}{19} \approx 4.7895.$$

$$\bar{x} = \bar{y} = 0 \text{ (by symmetry)}$$

7. Let $\delta(x, y, z) = k\rho$

$$m = \int_0^\pi \int_0^{2\pi} \int_a^b k\rho\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = k\pi(b^4 - a^4)$$

8. $8 \int_{\pi/6}^{\pi/2} \int_0^{\pi/2} \int_{a\csc\phi}^{2a} k\rho^2 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \left(\frac{56}{5}\right)k\pi a^5$

9. Let $\delta(x, y, z) = \rho$.
 (Letting $k = 1$ - see comment at the beginning of the write-up of Problem 24 of the previous section.)

$$m = \int_0^{\pi/2} \int_0^{2\pi} \int_0^a \rho^3 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \frac{a^4 \sin \phi}{4} d\theta \, d\phi$$

$$= \int_0^{\pi/2} \frac{\pi a^4 \sin \phi}{2} d\phi = \frac{\pi a^4}{2}$$

$$M_{xy} = \int_0^{\pi/2} \int_0^{2\pi} \int_0^a \rho^4 \sin \phi \cos \phi d\rho d\theta d\phi$$

$$(z = \rho \cos \phi)$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \frac{a^5 \sin 2\phi}{10} d\theta d\phi$$

$$= \int_0^{\pi/2} \frac{\pi a^5 \sin 2\phi}{5} d\phi = \left(\frac{\pi}{5}\right) a^5$$

$$\bar{z} = \frac{\frac{\pi a^5}{5}}{\frac{\pi a^4}{2}} = \frac{2}{5} a; \bar{x} = \bar{y} = 0 \text{ (by symmetry)}$$

10. $\alpha(x, y, z) = \rho \sin \phi$ (letting $k = 1$)

$$m = \int_0^{\pi/2} \int_0^{2\pi} \int_0^a \rho^3 \sin^2 \phi d\rho d\theta d\phi = \left(\frac{1}{8}\right) \pi^2 a^4$$

$$M_{xy} = \int_0^{\pi/2} \int_0^{2\pi} \int_0^a \rho^4 \sin^2 \phi \cos \phi d\rho d\theta d\phi$$

$$= \left(\frac{2}{15}\right) \pi a^5$$

$$\bar{z} = \frac{16a}{15\pi} \approx 0.3395a$$

$$\bar{x} = \bar{y} = 0 \text{ (by symmetry)}$$

$$\begin{aligned} 11. I_z &= \iiint_S (x^2 + y^2) k (x^2 + y^2)^{1/2} dV \\ &= \int_0^{\pi/2} \int_0^{2\pi} \int_0^a k \rho^5 \sin^4 \phi d\rho d\theta d\phi = \left(\frac{k}{16}\right) \pi^2 a^6 \end{aligned}$$

$$\begin{aligned} 12. \text{Volume} &= \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^4 \rho^2 \sin \phi d\rho d\theta d\phi \\ &= \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \frac{64 \sin \phi}{3} d\theta d\phi = \int_{\pi/4}^{\pi/2} \frac{128\pi \sin \phi}{3} d\phi \\ &= \frac{64\sqrt{2}\pi}{3} \approx 94.7815 \end{aligned}$$

$$13. \int_0^{\pi} \int_0^{\pi/6} \int_0^1 \rho^2 \sin \phi d\rho d\theta d\phi = \frac{\pi}{9} \approx 0.3491$$

$$14. \int_0^{\pi} \int_0^{2\pi} \int_0^3 \rho^3 \rho^2 \sin \phi d\rho d\phi d\theta = 486\pi \approx 1526.81$$

$$\begin{aligned} 15. \text{Volume} &= \int_0^{\pi} \int_0^{\sin \theta} \int_{r^2}^{\sin \theta} r dz dr d\theta \\ &= \int_0^{\pi} \int_0^{\sin \theta} r(r \sin \theta - r^2) dr d\theta = \int_0^{\pi} \frac{\sin^4 \theta}{12} d\theta \\ &= \frac{1}{48} \int_0^{\pi} \left[1 - 2 \cos 2\theta + \frac{1 + \cos 4\theta}{2} \right] d\theta \\ &= \frac{\pi}{32} \approx 0.0982 \end{aligned}$$

$$16. \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^{2\sqrt{2} \cos \theta} \rho^2 \sin \phi d\rho d\theta d\phi = \left(\frac{\pi}{3}\right) \frac{2\sqrt{2}}{3} \pi \approx 2.9619$$

17. a. Position the ball with its center at the origin. The distance of (x, y, z) from the origin is $(x^2 + y^2 + z^2)^{1/2} = \rho$.

$$\iiint_S (x^2 + y^2 + z^2)^{1/2} dV = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho (\sin \theta \rho^2) d\rho d\theta d\phi = \pi a^4$$

$$\text{Then the average distance from the center is } \frac{\pi a^4}{\left[\left(\frac{4}{3}\right) \pi a^3\right]} = \frac{3a}{4}.$$

- b. Position the ball with its center at the origin and consider the diameter along the z -axis. The distance of (x, y, z) from the z -axis is $(x^2 + y^2)^{1/2} = \rho \sin \phi$.

$$\iiint_S (x^2 + y^2)^{1/2} dV = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a (\rho \sin \phi) (\rho^2 \sin \theta) d\rho d\theta d\phi = \frac{a^4 \pi^2}{4}$$

$$\text{Then the average distance from a diameter is } \frac{\left[\frac{a^4 \pi^2}{4}\right]}{\left[\left(\frac{4}{3}\right) \pi a^3\right]} = \frac{3\pi a}{16}.$$

- c. Position the sphere above and tangent to the xy -plane at the origin and consider the point on the boundary to be the origin. The equation of the sphere is $\rho = 2a \cos \phi$, and the distance of (x, y, z) from the origin is ρ .

$$\iiint_S (x^2 + y^2 + z^2)^{1/2} dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2a \cos \phi} \rho (\rho^2 \sin \theta) d\rho d\theta d\phi = \frac{8\pi a^4}{5}$$

$$\text{Then the average distance from the origin is } \frac{\left[\frac{8\pi a^4}{5} \right]}{\left[\left(\frac{4}{3} \right) \pi a^3 \right]} = \frac{6a}{5}.$$

18. Average value of $ax + by + cz$ on S is

$$\frac{\iiint_S (ax + by + cz + d) dV}{\iiint_S dV} = \frac{a \iiint_S kx dV + b \iiint_S ky dV + c \iiint_S kz dV + d \iiint_S k dV}{\iiint_S k dV}$$

$$= \frac{aM_{yz} + bM_{xz} + cM_{xy} + dm}{m} = a\bar{x} + b\bar{y} + c\bar{z} + d = f(\bar{x}, \bar{y}, \bar{z}).$$

19. a. $M_{yz} = \iiint_S kx dV = 4k \int_0^{\pi/2} \int_0^\alpha \int_0^\alpha (\rho \sin \phi \cos \theta) (\rho^2 \sin \phi) d\rho d\theta d\phi = ka^4 \pi \frac{(\sin \alpha)}{4}$

$$m = \iiint_S k dV = 4k \int_0^{\pi/2} \int_0^\alpha \int_0^\alpha \rho^2 \sin \phi d\rho d\theta d\phi = \frac{4a^3 k \alpha}{3}$$

$$\text{Therefore, } \bar{x} = \frac{\left[\frac{ka^4 \pi (\sin \alpha)}{4} \right]}{\left[\frac{4a^3 k \alpha}{3} \right]} = \frac{3a\pi (\sin \alpha)}{16\alpha}.$$

b. $\frac{3\pi a}{16}$ (See Problem 17b.)

20. a. (See Problem 17b.) $I_z = \iiint_S k[(x^2 + y^2)^{1/2}]^2 dV$

$$= 8k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^\alpha (\rho \sin \phi)^2 (\rho^2 \sin \phi) d\rho d\theta d\phi = \frac{8a^5 \pi k}{15} = \frac{2a^2 m}{5} \quad (\text{since } m = \left(\frac{4}{3} \right) \pi a^3 k)$$

b. $I' = I = d^2 m = \frac{2a^2 m}{5} + a^2 m = \frac{7a^2 m}{5}$

c. $I = 2 \left[\frac{2a^2 m}{5} + (a+b)^2 m \right] = \frac{2m(7a^2 + 10ab + 5b^2)}{5}$

21. Let m_1 and m_2 be the masses of the left and right balls, respectively. Then $m_1 = \frac{4}{3} \pi a^3 k$ and

$$m_2 = \frac{4}{3} \pi a^3 (ck), \text{ so } m_2 = cm_1.$$

$$\bar{y} = \frac{m_1(-a-b) + m_2(a+b)}{m_1 + m_2}$$

$$= \frac{m_1(-a-b) + cm_1(a+b)}{m_1 + cm_1} = \frac{-a-b+c(a+b)}{1+c}$$

$$= \frac{(a+b)(-1+c)}{1+c} = \frac{c-1}{c+1}(a+b)$$

(Analogue)

$$\bar{y} = \frac{m_1 \bar{y}_1 + m_2 \bar{y}_2}{m_1 + m_2} = \bar{y}_1 \frac{m_1}{m_1 + m_2} + \bar{y}_2 \frac{m_2}{m_1 + m_2}$$

22. $x = r \cos \theta, y = r \sin \theta, z = z$

$$J(r, \theta, z) = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix}$$

$$= \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= |[(\cos \theta)(-r \cos \theta) - (\sin \theta)(-r \sin \theta)]|$$

$$= r(\cos^2 \theta + \sin^2 \theta) = r$$

23. $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$

$$J(\rho, \phi, \theta) = \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix}$$

$$= (\cos \phi)(\rho^2 \cos \phi \sin \phi)(\cos^2 \theta + \sin^2 \theta) + (\rho \sin \phi)(\rho \sin^2 \phi)(\cos^2 \theta + \sin^2 \theta)$$

$$= \rho^2 \sin \phi (\cos^2 \phi + \sin^2 \phi) = \rho^2 \sin \phi$$

(Expansion was along the third row of the determinant.)

24. Volume = $\iiint_S 1 dV$

Let $x = au, y = bv, z = cw$

Then $J(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

$$= \iiint_T abc dV$$

$$S: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \text{ (ellipsoid)}$$

$$= abc \frac{4}{3} \pi (1)^3 = \frac{4}{3} \pi abc$$

$$\Rightarrow T: u^2 + v^2 + w^2 \leq 1 \text{ (ball)}$$

$$I_z = \iiint_S k(x^2 + y^2) dV = k \iiint_T (a^2 u^2 + b^2 v^2) abc dV$$

$$= kabc \left[a^2 \iiint_T u^2 dV + b^2 \iiint_T v^2 dV \right] \quad (\text{By symmetry, these integrals are equals.})$$

$$= kabc(a^2 + b^2) \iiint_T w^2 dV = kabc(a^2 + b^2) \iiint_T w^2 dV = 8kabc(a^2 + b^2) \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 (\rho \cos \phi)^2 \rho^2 \sin \phi d\rho d\theta d\phi$$

$$= 8kabc(a^2 + b^2) \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{\rho^5}{5} \cos^2 \phi \sin \phi \right]_{\rho=0}^1 d\theta d\phi = 8kabc(a^2 + b^2) \int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{1}{5} \cos^2 \phi \sin \phi \right) d\theta d\phi$$

$$= 8kabc(a^2 + b^2) \int_0^{\pi/2} \frac{1}{5} \frac{\pi}{2} \cos^2 \phi \sin \phi d\phi = 8kabc(a^2 + b^2) \left[\frac{1}{5} \frac{\pi}{2} \frac{(-\cos^3 \phi)}{3} \right]_0^{\pi/2}$$

$$= 8kabc(a^2 + b^2) \frac{1}{5} \frac{\pi}{2} \frac{1}{3} = \frac{4kabc(a^2 + b^2)\pi}{15}$$

16.9 Chapter Review

Concepts Test

1. True: Use result of Problem 25, Section 16.2, and then change dummy variable y to dummy variable x .
2. False: Let $f(x, y) = x$. 1st integral is $\frac{1}{3}$; 2nd is $\frac{1}{6}$.
3. True: Inside integral is 0 since $\sin(x^3 y^2)$ is an odd function in x .

4. True: Use Problem 25, Section 16.2. Each integrand, e^{x^2} and e^{2y^2} , determines and even function.
5. True: It is less than or equal to $\int_1^2 \int_0^2 1 dx dy$ which equals 2.
6. True: $f(x, y) \geq \frac{f(x_0, y_0)}{2}$ in some neighborhood N of (x_0, y_0) due to the continuity. Then
$$\iint_R f(x, y) dA \geq \iint_N \left(\frac{1}{2} \right) f(x_0, y_0) dA = \left(\frac{1}{2} \right) f(x_0, y_0) (\text{Area } N) > 0.$$

7. False: Let $f(x, y) = x$, $g(x, y) = x^2$,
 $R = \{(x, y): x \text{ in } [0, 2], y \text{ in } [0, 1]\}$.
 The inequality holds for the integrals
 but $f(0.5, 0) > g(0.5, 0)$.
8. False: Let $f(0, 0) = 1$, $f(x, y) = 0$ elsewhere
 for $x^2 + y^2 \leq 1$.
9. True: See the write-up of Problem 24,
 Section 16.7.
10. True: For each x , the density increases as y
 increases, so the top half of R is more
 dense than the bottom half. For each
 y , the density decreases as the x
 increases, so the right half of R is
 more dense than the left half.
11. True: The integral is the volume between
 concentric spheres of radii 4 and 1.
 That volume is 84π .
12. True: See Section 16.6.
 $A(T) = (\text{Area of base})(\sec 30^\circ)$
 $= \pi(1)^2 \left(\frac{2}{\sqrt{3}} \right) = \frac{2\sqrt{3}\pi}{3}$
13. False: There are 6.
14. False: The integrand should be r .
15. True: $|\nabla f|$ is the magnitude of the greatest
 increase in f .
 $|D_u f| = |\nabla f \cdot \mathbf{u}| = |\langle f_x, f_y \rangle \cdot \mathbf{u}|$
 $= \sqrt{f_x^2 + f_y^2} (1) \cos \theta \leq \sqrt{4 + 4} = \sqrt{8}$
 Therefore,
 $\text{Area}(G) \leq \text{Area}(R) \max \{\sec \gamma\}$
 $\leq (1) \sec(\tan^{-1} \sqrt{8}) = 3$.

Sample Test Problems

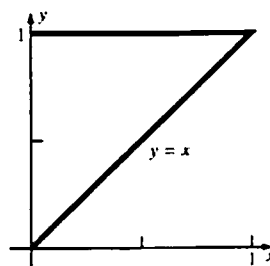
1. $\int_0^1 \left(\frac{1}{2} \right) (x^2 - x^3) dx = \frac{1}{24} \approx 0.0417$
2. $\int_{-2}^2 0 dy = 0$ (Integrand determines an odd
 function in x .)

$$3. \int_0^{\pi/2} \left[\frac{r^2 \cos \theta}{2} \right]_{r=0}^{2 \sin \theta} d\theta = \int_0^{\pi/2} 2 \sin^2 \theta \cos \theta d\theta$$

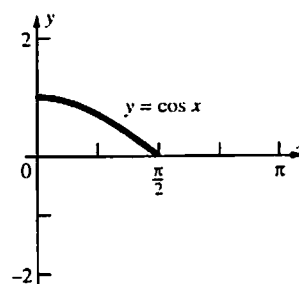
$$= \left[\frac{2 \sin^3 \theta}{3} \right]_0^{\pi/2} = \frac{2}{3}$$

$$4. \int_1^2 \int_3^x \left(\frac{\pi}{3} \right) dy dx = \int_1^2 \left(\frac{\pi}{3} \right) (x - 3) dx = -\frac{\pi}{2}$$

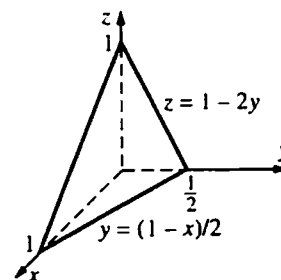
$$5. \int_0^1 \int_0^y f(x, y) dx dy$$



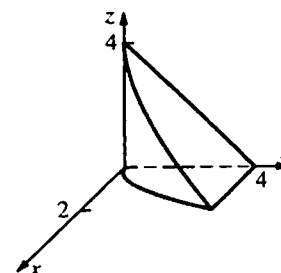
$$6. \int_0^{\pi/2} \int_0^{\cos x} f(x, y) dx dy$$



$$7. \int_0^{1/2} \int_0^{1-2y} \int_0^{1-2y-z} f(x, y, z) dx dz dy$$



$$8. \int_0^4 \int_0^{4-z} \int_0^{\sqrt{y}} f(x, y, z) dx dy dz$$

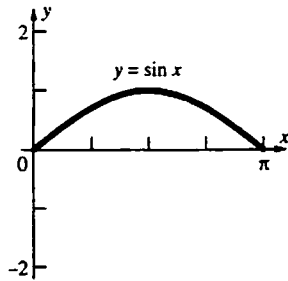


$$9. \text{ a. } 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz dy dx$$

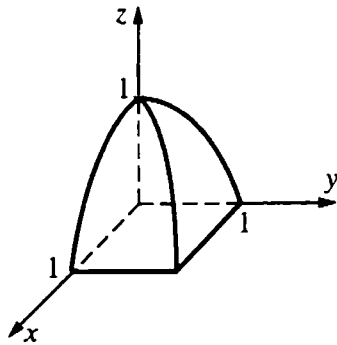
$$\text{b. } 8 \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} r dz dr d\theta$$

$$\text{c. } 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^2 \sin \phi d\rho d\theta d\phi$$

$$10. \int_0^\pi \int_0^{\sin x} (x+y) dy dx = \frac{5\pi}{4} \approx 3.9270$$



$$11. 8 \int_0^1 \int_0^x \int_0^{1-y^2} z^2 dz dy dx = \frac{31}{35} \approx 0.8857$$



$$12. \int_0^{2\pi} \int_2^3 (r^{-2}) r dr d\theta = \int_0^{2\pi} [\ln r]_2^3 d\theta$$

$$= \int_0^{2\pi} \ln\left(\frac{3}{2}\right) d\theta = 2\pi \ln\left(\frac{3}{2}\right) \approx 2.5476$$

$$13. m = \int_0^2 \int_1^3 xy^2 dx dy = \frac{32}{3}$$

$$M_x = \int_0^2 \int_1^3 xy^3 dx dy = 16$$

$$M_y = \int_0^2 \int_1^3 x^2 y^2 dx dy = \frac{208}{9}$$

$$(\bar{x}, \bar{y}) = \left(\frac{13}{6}, \frac{3}{2}\right)$$

$$14. I_x = \int_0^2 \int_1^3 xy^4 dx dy = \frac{128}{5} = 25.6$$

$$15. z = f(x, y) = (9 - y^2)^{1/2}; f_x(x, y) = 0;$$

$$f_y(x, y) = -y(9 - y^2)^{-1/2}$$

$$\text{Area} = \int_0^3 \int_{y/3}^y \sqrt{y^2(9 - y^2)^{-1} + 1} dx dy$$

$$= \int_0^3 \int_{y/3}^y 3(9 - y^2)^{-1/2} dx dy$$

$$= \int_0^3 (9 - y^2)^{-1/2} (2y) dy = [-2(9 - y^2)^{1/2}]_0^3 = 6$$

$$16. \text{ a. } \int_0^{\pi/2} \int_0^2 \int_0^3 r r dr dz d\theta = 9\pi \approx 28.2743$$

b.

$$\int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{4-r^2}} z(4 - r^2)^{1/2} r dz dr d\theta = \left(\frac{8}{5}\right)\pi$$

$$\approx 5.0265$$

$$17. \delta(x, y, z) = k\rho$$

$$m = \int_0^\pi \int_0^{2\pi} \int_1^3 k\rho \rho^2 \sin \phi d\rho d\theta d\phi = 80\pi k$$

$$18. m = \iint_R 1 dA = \int_0^{2\pi} \int_0^{4(1+\sin \theta)} r dr d\theta$$

$$= 16 \int_0^{2\pi} \left(1 + 2\sin \theta + \frac{1 - \cos 2\theta}{2}\right) d\theta = 24\pi$$

$$M_x = \iint_R y dA = \int_0^{2\pi} \int_0^{4(1+\sin \theta)} (r \sin \theta) r dr d\theta$$

$$= 80\pi$$

$$\bar{y} = \frac{80\pi}{24\pi} = \frac{10}{3}; \bar{x} = 0 \text{ (by symmetry)}$$

$$19. m = \int_0^a \int_0^{(b/a)(a-x)} \int_0^{(c/ab)(ab-bx-ay)} kx dz dy dx$$

$$= \left(\frac{k}{24}\right) a^2 bc$$

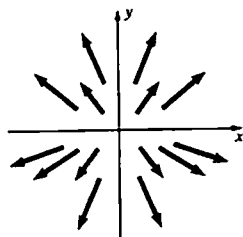
$$20. \int_0^\pi \int_0^{2\sin \theta} \int_0^2 r dz dr d\theta = \frac{3\pi}{2} \approx 4.7124$$

17.1 Concepts Review

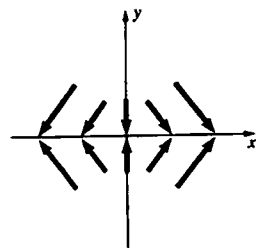
1. vector-valued function of three real variables or a vector field
2. gradient field
3. gravitational fields; electric fields
4. $\nabla \cdot \mathbf{F}, \nabla \times \mathbf{F}$

Problem Set 17.1

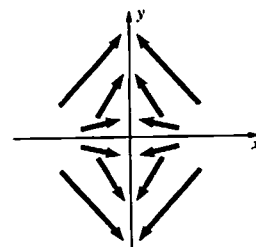
1.



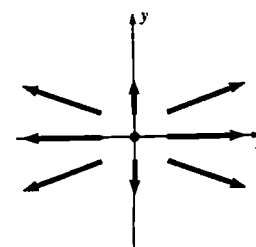
2.



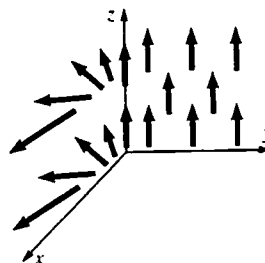
3.



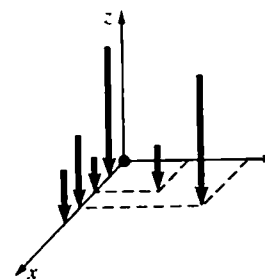
4.



5.



6.



7. $\langle 2x - 3y, -3x, 2 \rangle$

8. $(\cos xyz) \langle yz, xz, xy \rangle$

9. $f(x, y, z) = \ln|x| + \ln|y| + \ln|z|$
 $\nabla f(x, y, z) = \langle x^{-1}, y^{-1}, z^{-1} \rangle$

10. $\langle x, y, z \rangle$

11. $e^y \langle \cos z, x \cos z, -x \sin z \rangle$

12. $\nabla f(x, y, z) = \langle 0, 2ye^{-2z}, -2y^2e^{-2z} \rangle$
 $= 2ye^{-2z} \langle 0, 1, -y \rangle$

13. $\text{div } \mathbf{F} = 2x - 2x + 2yz = 2yz$
 $\text{curl } \mathbf{F} = \langle z^2, 0, -2y \rangle$

14. $\text{div } \mathbf{F} = 2x + 2y + 2z$
 $\text{curl } \mathbf{F} = \langle 0, 0, 0 \rangle = \mathbf{0}$

15. $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = 0 + 0 + 0 = 0$
 $\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \langle x - x, y - y, z - z \rangle = \mathbf{0}$

16. $\text{div } \mathbf{F} = -\sin x + \cos y + 0$
 $\text{curl } \mathbf{F} = \langle 0, 0, 0 \rangle = \mathbf{0}$
17. $\text{div } \mathbf{F} = e^x \cos y + e^x \cos y + 1 = 2e^x \cos y + 1$
 $\text{curl } \mathbf{F} = \langle 0, 0, 2e^x \sin y \rangle$
18. $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = 0 + 0 + 0 = 0$
 $\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \langle 1 - 1, 1 - 1, 1 - 1 \rangle = \mathbf{0}$
19. a. meaningless
 b. vector field
 c. vector field
 d. scalar field
 e. vector field
 f. vector field
 g. vector field
 h. meaningless
 i. meaningless
 j. scalar field
 k. meaningless

20. a. $\text{div}(\text{curl } \mathbf{F}) = \text{div} \cdot \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle = (P_{yx} - N_{zx}) + (M_{zy} - P_{xy}) + (N_{xz} - M_{yz}) = 0$

b. $\text{curl}(\text{grad } f) = \text{curl} \langle f_x, f_y, f_z \rangle = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle = \mathbf{0}$

c. $\text{div}(f\mathbf{F}) = \text{div} \langle fM, fN, fP \rangle = (fM_x + f_x M) + (fN_y + f_y N) + (fP_z + f_z P)$
 $= (f)(M_x + N_y + P_z) + (f_x M + f_y N + f_z P) = (f)(\text{div } \mathbf{F}) + (\text{grad } f) \cdot \mathbf{F}$

d. $\text{curl}(f\mathbf{F}) = \text{curl} \langle fM, fN, fP \rangle$
 $= \langle (fP_y + f_y P) - (fN_z + f_z N), (fM_z + f_z M) - (fP_x + f_x P), (fN_x + f_x N) - (fM_y + f_y M) \rangle$
 $= (f) \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle + \langle f_x, f_y, f_z \rangle \times \langle M, N, P \rangle = (f)(\text{curl } \mathbf{F}) + (\text{grad } f) \times \mathbf{F}$

21. Let $f(x, y, z) = -c|r|^{-3}$, so
 $\text{grad}(f) = 3c|r|^{-4} \frac{\mathbf{r}}{|r|} = 3c|r|^{-5} \mathbf{r}$.
 Then $\text{curl } \mathbf{F} = \text{curl} \left[(-c|r|^{-3}) \mathbf{r} \right]$
 $= (-c|r|^{-3})(\text{curl } \mathbf{r}) + (3c|r|^{-5} \mathbf{r}) \times \mathbf{r}$ (by 20d)
 $= (-c|r|^{-3})(\mathbf{0}) + (3c|r|^{-5})(\mathbf{r} \times \mathbf{r}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$
 $\text{div } \mathbf{F} = \text{div} \left[(-c|r|^{-3}) \mathbf{r} \right]$
 $= (-c|r|^{-3})(\text{div } \mathbf{r}) + (3c|r|^{-5} \mathbf{r}) \cdot \mathbf{r}$ (by 20c)
 $= (-c|r|^{-3})(1 + 1 + 1) + (3c|r|^{-5})|r|^2$
 $= (-3c|r|^{-3}) + 3c|r|^{-3} = 0$

22. $\text{curl} \left[-c|r|^{-m} \mathbf{r} \right] = (-c|r|^{-m})(\mathbf{0}) + mc|r|^{-m-2}(\mathbf{0})$
 $= \mathbf{0}$
 $\text{div} \left[-c|r|^{-m} \mathbf{r} \right] = (-c|r|^{-m})(3) + mc|r|^{-m-2}|r|^2$
 $= (m-3)c|r|^{-m}$

23. $\text{grad } f = \langle f'(r)xr^{-1/2}, f'(r)yr^{-1/2}, f'(r)zr^{-1/2} \rangle$
 (if $r \neq 0$)
 $= f'(r)r^{-1/2} \langle x, y, z \rangle = f'(r)r^{-1/2} \mathbf{r}$
 $\text{curl } \mathbf{F} = [f(r)][\text{curl } \mathbf{r}] + [f'(r)r^{-1/2} \mathbf{r}] \times \mathbf{r}$
 $= [f(r)][\text{curl } \mathbf{r}] + [f'(r)r^{-1/2} \mathbf{r}] \times \mathbf{r}$
 $= \mathbf{0} + \mathbf{0} = \mathbf{0}$

24. $\text{div } \mathbf{F} = \text{div}[f(r)\mathbf{r}] = [f(r)](\text{div } \mathbf{r}) + \text{grad}[f(r)] \cdot \mathbf{r}$
 $= [f(r)](\text{div } \mathbf{r}) + [f'(r)r^{-1} \mathbf{r}] \cdot \mathbf{r}$
 $= [f(r)](3) + [f'(r)r^{-1}](\mathbf{r} \cdot \mathbf{r})$
 $= 3f(r) + [f'(r)r^{-1}](r^2) = 3f(r) + rf'(r)$

Now if $\text{div } \mathbf{F} = 0$, and we let $y = f(r)$, we have the differential equation $3y + r \frac{dy}{dr} = 0$, which can be solved as follows:

$$\frac{dy}{y} = -3 \frac{dr}{r}; \ln|y| = -3 \ln|r| + \ln|C| = \ln|Cr^{-3}|,$$

for each $C \neq 0$. Then $y = Cr^{-3}$, or

$f(r) = Cr^{-3}$, is a solution (even for $C = 0$).

25. a. Let $P = (x_0, y_0)$.

$\text{div } \mathbf{F} = \text{div } \mathbf{H} = 0$ since there is no tendency toward P except along the line $x = x_0$, and along that line the tendencies toward and away from P are balanced; $\text{div } \mathbf{G} < 0$ since there is no tendency toward P except along the line $x = x_0$, and along that line there is more tendency toward than away from P ; $\text{div } \mathbf{L} > 0$ since the tendency away from P is greater than the tendency toward P .

- b. No rotation for $\mathbf{F}$, $\mathbf{G}$, $\mathbf{L}$; clockwise rotation for $\mathbf{H}$ since the magnitudes of the forces to the right of P are less than those to the left.

- c. $\text{div } \mathbf{F} = 0$; $\text{curl } \mathbf{F} = 0$

$\text{div } \mathbf{G} = -2ye^{-y^2} < 0$ since $y > 0$ at P ; $\text{curl } \mathbf{G} = 0$

29. a. $\mathbf{F} \times \mathbf{G} = (f_y g_z - f_z g_y)\mathbf{i} - (f_x g_z - f_z g_x)\mathbf{j} + (f_x g_y - f_y g_x)\mathbf{k}$

Therefore,

$$\text{div}(\mathbf{F} \times \mathbf{G}) = \frac{\partial}{\partial x}(f_y g_z - f_z g_y) - \frac{\partial}{\partial y}(f_x g_z - f_z g_x) + \frac{\partial}{\partial z}(f_x g_y - f_y g_x).$$

Using the product rule for partials and some algebra gives

$$\begin{aligned} \text{div}(\mathbf{F} \times \mathbf{G}) &= g_x \left[\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right] + g_y \left[\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right] + g_z \left[\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right] \\ &\quad + f_x \left[\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z} \right] + f_y \left[\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x} \right] + f_z \left[\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} \right] \\ &= \mathbf{G} \cdot \text{curl}(\mathbf{F}) - \mathbf{F} \cdot \text{curl}(\mathbf{G}) \end{aligned}$$

b. $\nabla f \times \nabla g = \left(\frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial y} \right) \mathbf{i} - \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) \mathbf{k}$

Therefore,

$$\text{div}(\nabla f \times \nabla g) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right).$$

Using the product rule for partials and some algebra will yield the result

$$\text{div}(\nabla f \times \nabla g) = 0$$

$$\text{div } \mathbf{L} = (x^2 + y^2)^{-1/2}; \text{curl } \mathbf{L} = \mathbf{0}$$

$$\text{div } \mathbf{H} = 0; \text{curl } \mathbf{H} = \langle 0, 0, -2xe^{-x^2} \rangle \text{ which}$$

points downward at P , so the rotation is clockwise in a right-hand system.

26. $\text{div } \mathbf{v} = 0 + 0 + 0 = 0$;

$$\text{curl } \mathbf{v} = \langle 0, 0, w + w \rangle = 2\omega \mathbf{k}$$

27. $\nabla f(x, y, z) = \frac{1}{2} m \omega^2 \langle 2x, 2y, 2z \rangle = m \omega^2 \langle x, y, z \rangle$
 $= \mathbf{F}(x, y, z)$

28. $\nabla^2 f = \text{div}(\text{grad } f) = \text{div} \langle f_x, f_y, f_z \rangle$
 $= f_{xx} + f_{yy} + f_{zz}$

a. $\nabla^2 f = 4 - 2 - 2 = 0$

b. $\nabla^2 f = 0 + 0 + 0 = 0$

c. $\nabla^2 f = 6x - 6x + 0 = 0$

d. $\nabla^2 f = \text{div}(\text{grad } f) = \text{div}(\text{grad } |\mathbf{r}|^{-1})$
 $= \text{div}(-|\mathbf{r}|^{-3} \mathbf{r}) = 0$ (by problem 21)

Hence, each is harmonic.

30. $\lim_{(x, y, z) \rightarrow (a, b, c)} F(x, y, z) = L$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < \|(x, y, z) - (a, b, c)\| < \delta$ implies that $|F(x, y, z) - L| < \varepsilon$.
 F is continuous at (a, b, c) if and only if $\lim_{(x, y, z) \rightarrow (a, b, c)} F(x, y, z) = F(a, b, c)$.

17.2 Concepts Review

1. Increasing values of t

2. $\sum_{i=1}^n f(\bar{x}_i, \bar{y}_i) \Delta s_i$

3. $f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2}$

4. $F \cdot \frac{dr}{dt}$

3. Let $x = t, y = 2t, t$ in $[0, \pi]$.

Then

$$\int_C (\sin x + \cos y) ds = \int_0^\pi (\sin t + \cos 2t) \sqrt{1+4} dt$$

$$= 2\sqrt{5} \approx 4.4721$$

4. Vector equation of the segment is

$$\langle x, y \rangle = \langle -1, 2 \rangle + t \langle 2, -1 \rangle, t \text{ in } [0, 1].$$

$$\int_0^1 (-1+2t)e^{2-t}(4+1)^{1/2} dt = \sqrt{5}e^2(1-3e^{-1})$$

$$\approx -1.7124$$

5. $\int_0^1 (2t+9t^3)(1+4t^2+9t^4)^{1/2} dt = \left(\frac{1}{6}\right)(14^{3/2}-1)$
 ≈ 8.5639

Problem Set 17.2

1. $\int_0^1 (27t^3 + t^3)(9+9t^4)^{1/2} dt = 14(2\sqrt{2}-1)$
 ≈ 25.5980

2. $\int_0^1 \left(\frac{t}{2}\right) \left(t\right) \left(\frac{1}{4} + \frac{25t^3}{4}\right)^{1/2} dt = \left(\frac{1}{450}\right)(26^{3/2}-1)$
 ≈ 0.2924

6. $\int_0^{2\pi} (16\cos^2 t + 16\sin^2 t + 9t^2)(16\sin^2 t + 16\cos^2 t + 9)^{1/2} dt = \int_0^{2\pi} (16+9t^2)(5) dt$
 $= \left[80t + 15t^3\right]_0^{2\pi} = 160\pi + 120\pi^3 \approx 4878.11$

7. $\int_0^2 [(t^2-1)(2) + (4t^2)(2t)] dt = \frac{100}{3}$

8. $\int_0^4 (-1) dx + \int_{-1}^3 (4)^2 dy = 60$

9. $\int_C y^3 dx + x^3 dy = \int_{C_1} y^3 dx + x^3 dy$
 $+ \int_{C_2} y^3 dx + x^3 dy$
 $= \int_1^{-2} (-4)^3 dy + \int_{-4}^2 (-2)^3 dx = 192 + (-48) = 144$

10. $\int_{-2}^1 [(t^2-3)^3(2) + (2t)^3(2t)] dt = \frac{828}{35} \approx 23.6571$

11. $y = -x + 2$
 $\int_1^3 ([x+2(-x+2)](1) + [x-2(-x+2)](-1)) dx = 0$

12. $\int_0^1 [x^2 + (x)2x] dx = \int_0^1 3x^2 dx = 1$
 (letting x be the parameter; i.e., $x = x, y = x^2$)

$$13. \langle x, y, z \rangle = \langle 1, 2, 1 \rangle + t \langle 1, -1, 0 \rangle$$

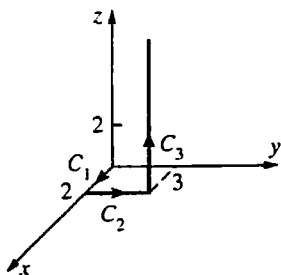
$$\int_0^1 [(4-t)(1) + (1+t)(-1) - (2-3t+t^2)(-1)] dt = \frac{17}{6} \\ \approx 2.8333$$

$$14. \int_0^1 [(e^{3t})(e') + (e^{-t} + e^{2t})(-3^{-t}) + (e')(2e^{2t})] dt \\ = \left(\frac{1}{4}\right)e^4 + \left(\frac{2}{3}\right)e^3 - e + \left(\frac{1}{2}\right)e^{-2} - \frac{5}{12} \\ \approx 23.9726$$

$$15. \text{ On } C_1: y = z = dy = dz = 0$$

$$\text{ On } C_2: x = 2, z = dx = dz = 0$$

$$\text{ On } C_3: x = 2, y = 3, dx = dy = 0$$



$$\int_0^2 x dx + \int_0^3 (2-2y) dy + \int_0^4 (4+3-z) dz \\ = \left[\frac{x^2}{2}\right]_0^2 + [2y - y^2]_0^3 + \left[7z - \frac{z^2}{2}\right]_0^4 \\ = 2 + (-3) + 20 = 19$$

$$16. \langle x, y, z \rangle = t \langle 2, 3, 4 \rangle, t \text{ in } [0, 1].$$

$$\int_0^1 [(9t)(2) + (8t)(3) + (3t)(4)] dt = 27$$

$$17. m = \int_C k|x| ds = \int_{-2}^2 k|x|(1+4x^2)^{1/2} dx \\ = \left(\frac{k}{6}\right)(17^{3/2} - 1) \approx 11.6821k$$

$$18. \text{ Let } \delta(x, y, z) = k \text{ (a constant).}$$

$$m = k \int_C 1 ds = k \int_0^{3\pi} 1(a^2 \sin^2 t + a^2 \cos^2 t + b^2)^{1/2} dt \\ = 3\pi k(a^2 + b^2)^{1/2}$$

$$M_{xy} = k \int_C z ds = k(a^2 + b^2)^{1/2} \int_0^{3\pi} bt dt \\ = \frac{9\pi^2 bk(a^2 + b^2)^{1/2}}{2}$$

$$M_{xz} = k \int_C y ds = k(a^2 + b^2)^{1/2} \int_0^{3\pi} a \sin t dt \\ = ak(a^2 + b^2)^{1/2}(2) = 2ak(a^2 + b^2)^{1/2}$$

$$M_{yz} = k \int_C x ds = k(a^2 + b^2)^{1/2} \int_0^{3\pi} a \cos t dt \\ = ak(a^2 + b^2)^{1/2}(0) = 0$$

$$\text{ Therefore, } \bar{x} = \frac{M_{yz}}{m} = 0; \bar{y} = \frac{M_{xz}}{m} = \frac{2a}{3\pi};$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{3\pi b}{2}.$$

$$19. \int_C (x^3 - y^3) dx + xy^2 dy \\ = \int_{-1}^0 [(t^6 - t^9)(2t) + (t^2)(t^6)(3t^2)] dt \\ = -\frac{7}{44} \approx -0.1591$$

$$20. \int_C e^x dx - e^{-y} dy = \int_1^5 \left[(t^3) \left(\frac{3}{t} \right) - \left(\frac{1}{2t} \right) \left(\frac{1}{t} \right) \right] dt \\ = 123.6$$

$$21. W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x+y) dx + (x-y) dy = \int_0^{\pi/2} [(a \cos t + b \sin t)(-a \sin t) + (a \cos t - b \sin t)(b \cos t)] dt \\ = \int_0^{\pi/2} [-(a^2 + b^2) \sin t \cos t + ab(\cos^2 t - \sin^2 t)] dt = \int_0^{\pi/2} \frac{-(a^2 + b^2) \sin 2t}{2} + ab \cos 2t dt \\ = \left[\frac{(a^2 + b^2) \cos 2t}{4} + \frac{ab \sin 2t}{2} \right]_0^{\pi/2} = \frac{a^2 + b^2}{-2}$$

$$22. \langle x, y, z \rangle = t \langle 1, 1, 1 \rangle, t \text{ in } [0, 1].$$

$$\int_C (2x - y) dx + 2z dy + (y - z) dz = \int_0^1 (t + 2t + 0) dt = 1.5$$

$$23. \int_0^{\pi} \left[\left(\frac{\pi}{2} \right) \sin \left(\frac{\pi t}{2} \right) \cos \left(\frac{\pi t}{2} \right) + \pi t \cos \left(\frac{\pi t}{2} \right) + \sin \left(\frac{\pi t}{2} \right) - t \right] dt = 2 - \frac{2}{\pi} \approx 1.3634$$

$$24. \quad W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y \, dx + z \, dy + x \, dz = \int_0^2 [(t^2)(1) + (t^3)(2t) + (t)(3t^2)] dt$$

$$= \int_0^2 (2t^4 + 3t^3 + t^2) dt = \frac{64}{5} + 12 + \frac{8}{3} = \frac{412}{15} \approx 27.4667$$

$$25. \quad \int_C \left(1 + \frac{y}{3}\right) ds = \int_0^2 (1 + 10 \sin^3 t) [(-90 \cos^2 t \sin t)^2 + (90 \sin^2 t \cos t)^2]^{1/2} dt = 225$$

Christy needs $\frac{450}{200} = 2.25$ gal of paint.

$$26. \quad \int_C \langle 0, 0, 1.2 \rangle \cdot \langle dx, dy, dz \rangle = \int_C 1.2 \, dz$$

$$= \int_0^{8\pi} 1.2(4) dt = 38.4\pi \approx 120.64 \text{ ft-lb}$$

Trivial way: The squirrel ends up 32π ft immediately above where it started.
 $(32\pi \text{ ft})(1.2 \text{ lb}) \approx 120.64 \text{ ft-lb}$

$$27. \quad C: x + y = a$$

Let $x = t, y = a - t, t$ in $[0, a]$.

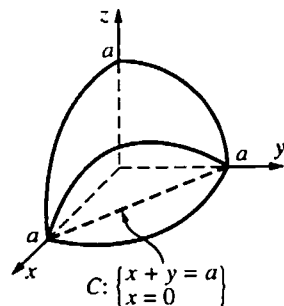
Cylinder: $x + y = a; (x + y)^2 = a^2;$

$$x^2 + 2xy + y^2 = a^2$$

Sphere: $x^2 + y^2 + z^2 = a^2$

The curve of intersection satisfies:

$$z^2 = 2xy; z = \sqrt{2xy}.$$



$$\text{Area} = 8 \int_C \sqrt{2xy} \, ds = 8 \int_0^a \sqrt{2t(a-t)} \sqrt{(1)^2 + (-1)^2} \, dt$$

$$= 16 \int_0^a \sqrt{at - t^2} \, dt$$

$$= 16 \left[\frac{t - \frac{a}{2}}{2} \sqrt{at - t^2} + \frac{\left(\frac{a}{2}\right)^2}{2} \sin^{-1} \left(\frac{t - \frac{a}{2}}{\frac{a}{2}} \right) \right]_0^a$$

$$= 16 \left[\left[0 + \left(\frac{a^2}{8} \right) \left(\frac{\pi}{2} \right) \right] - \left[0 + \left(\frac{a^2}{8} \right) \left(\frac{-\pi}{2} \right) \right] \right] = 2a^2\pi$$

Trivial way: Each side of the cylinder is part of a plane that intersects the sphere in a circle. The radius of each circle is the value of z in

$$z = \sqrt{2xy} \text{ when } x = y = \frac{a}{2}. \text{ That is, the radius is}$$

$$\sqrt{2 \left(\frac{a}{2} \right) \left(\frac{a}{2} \right)} = \frac{a\sqrt{2}}{2}. \text{ Therefore, the total area of}$$

$$\text{the part cut out is } r \left[\pi \left(\frac{a\sqrt{2}}{2} \right)^2 \right] = 2a^2\pi.$$

$$28. \quad I_y = \int_C kx^2 \, ds = 4k \int_0^a t^2 \sqrt{2} \, dt = 4\sqrt{2} \frac{ka^3}{3}$$

(using same parametric equations as in Problem 27)

$$I_x = I_y \text{ (symmetry)}$$

$$I_z = I_x + I_y = 8\sqrt{2} \frac{ka^3}{3}$$

$$29. \quad \text{Note that } r = a \cos \theta \text{ along } C.$$

$$\text{Then } (a^2 - x^2 - y^2)^{1/2} = (a^2 - r^2)^{1/2} = a \cos \theta.$$

$$\text{Let } \begin{cases} x = r \cos \theta = (a \sin \theta) \cos \theta \\ y = r \sin \theta = (a \sin \theta) \sin \theta \end{cases}, \theta \text{ in } \left[0, \frac{\pi}{2} \right].$$

$$\text{Therefore, } x'(\theta) = a \cos 2\theta; y'(\theta) = a \sin 2\theta.$$

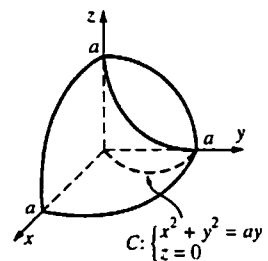
$$\text{Then Area} = 4 \int_C (a^2 - x^2 - y^2)^{1/2} \, ds$$

$$= 4 \int_0^{\pi/2} (a \cos \theta) [(a \sin 2\theta)^2 + (a \cos 2\theta)^2]^{1/2} \, d\theta$$

$$= 4a^2.$$

$$30. \quad C: x^2 + y^2 = a^2$$

$$\text{Let } x = a \cos \theta, y = a \sin \theta, \theta \text{ in } \left[0, \frac{\pi}{2} \right].$$



$$\text{Area} = 8 \int_C \sqrt{a^2 - x^2} \, ds$$

$$= 8 \int_0^{\pi/2} (a \sin \theta) \sqrt{(-a \sin \theta)^2 + (a \cos \theta)^2} \, d\theta$$

$$= 8 \int_0^{\pi/2} (a \sin \theta) \sqrt{a^2} d\theta = 8a^2 [-\cos \theta]_0^{\pi/2} \\ = 8a^2$$

(Note: The result of Problem 26, Section 12.8, could be used to arrive at this result more quickly.)

$$31. \text{ a. } \int_C x^2 y \, ds = \int_0^{\pi/2} (3 \sin t)^2 (3 \cos t) [(3 \cos t)^2 + (-3 \sin t)^2]^{1/2} dt = 81 \int_0^{\pi/2} \sin^2 t \cos t \, dt = 81 \left[\left(\frac{1}{3} \right) \sin^3 t \right]_0^{\pi/2} = 27$$

$$\text{ b. } \int_{C_4} xy^2 \, dx + xy^2 \, dy = \int_0^3 (3-t)(5-t)^2 (-1) dt + \int_0^3 (3-t)(5-t)^2 (-1) dt = 2 \int_0^3 (t^3 - 13t^2 + 55t - 75) dt = -148.5$$

17.3 Concepts Review

1. $f(b) - f(a)$
2. gradient; $\nabla f(\mathbf{r})$
3. 0; 0
4. $\mathbf{F}$ is conservative.

Problem Set 17.3

1. $M_y = -7 = N_x$, so $\mathbf{F}$ is conservative.
 $f(x, y) = 5x^2 - 7xy + y^2 + C$
2. $M_y = 6y + 5 = N_x$, so $\mathbf{F}$ is conservative.
 $f(x, y) = 4x^3 + 3xy^2 + 5xy - y^3 + C$
3. $M_y = 90x^4 y - 36y^5 \neq N_x$ since
 $N_x = 90x^4 y - 12y^5$, so $\mathbf{F}$ is not conservative.
4. $M_y = -12x^2 y^3 + 9y^8 = N_x$, so $\mathbf{F}$ is conservative.
 $f(x, y) = 7x^5 - x^3 y^4 + xy^9 + C$
5. $M_y = \left(-\frac{12}{5} \right) x^2 y^{-3} = N_x$, so $\mathbf{F}$ is conservative.
 $f(x, y) = \left(\frac{2}{5} \right) x^3 y^{-2} + C$
6. $M_y = (4y^2)(-2xy \sin xy^2) + (8y)(\cos xy^2) \neq N_x$
since $N_x = (8x)(-y^2 \sin xy^2) + (8)(\cos xy^2)$, so $\mathbf{F}$ is not conservative.

7. $M_y = 2e^y - e^x = N_x$ so $\mathbf{F}$ is conservative.
 $f(x, y) = 2xe^y - ye^x + C$
8. $M_y = -e^{-x} y^{-1} = N_x$, so $\mathbf{F}$ is conservative.
 $f(x, y) = e^{-x} \ln y + C$
9. $M_y = 0 = N_x$, $M_z = 0 = P_x$, and $N_z = 0 = P_y$,
so $\mathbf{F}$ is conservative. f satisfies
 $f_x(x, y, z) = 3x^2$, $f_y(x, y, z) = 6y^2$, and
 $f_z(x, y, z) = 9z^2$.
Therefore, f satisfies
1. $f(x, y, z) = x^3 + C_1(y, z)$,
2. $f(x, y, z) = 2y^3 + C_2(x, z)$, and
3. $f(x, y, z) = 3z^3 + C_3(x, y)$.
A function with an arbitrary constant that
satisfies 1, 2, and 3 is
 $f(x, y, z) = x^3 + 2y^3 + 3z^3 + C$.
10. $M_y = 2x = N_y$, $M_z = 2z = P_x$, and
 $N_z = 0 = P_y$, so $\mathbf{F}$ is conservative.
 $f(x, y, z) = x^2 y + xz^2 + \sin \pi z + C$
11. $M_y = 2y + 2x = N_x$, so integral is path
independent. $f(x, y) = xy^2 + x^2 y$
 $\int_{(-1, 2)}^{(3, 1)} (y^2 + 2xy) dx + (x^2 + 2xy) dy = [xy^2 + x^2 y]_{(-1, 2)}^{(3, 1)}$
 $= 14$ (Or use paths.)
12. $M_y = e^x \cos y = N_x$, so the line integral is
independent of the path.
Let $f_x(x, y) = e^x \sin y$ and $f_y(x, y) = e^x \cos y$.
Then $f(x, y) = e^x \sin y + C_1(y)$ and

$$f(x, y) = e^x \sin y + C_2(x).$$

Choose $f(x, y) = e^x \sin y$.

By Theorem A,

$$\int_{(0,0)}^{(1,\pi/2)} e^x \sin y \, dx + e^x \cos y \, dy = [e^x \sin y]_{(0,0)}^{(1,\pi/2)} = e.$$

(Or use line segments (0, 0) to (1, 0), then (1, 0)

$$\text{to } \left(1, \frac{\pi}{2}\right).$$

13. $M_y = 18xy^2 = N_x, M_z = 4x = P_x, N_z = 0 = P_y.$

By paths (0, 0, 0) to (1, 0, 0); (1, 0, 0) to (1, 1, 0); (1, 1, 0) to (1, 1, 1)

$$\int_0^1 0 \, dx + \int_0^1 9y^2 \, dy + \int_0^1 (4z + 1) \, dz = 6$$

(Or use $f(x, y) = 3x^2y^3 + 2xz^2 + z$.)

14. $M_y = z = N_x, M_z = y = P_x, N_z = x = P_y.$

$$f(x, y) = xyz + x + y + z$$

Thus, the integral equals

$$[xyz + x + y + z]_{(0,0,0)}^{(1,1,1)} = 3. \text{ (Or use paths.)}$$

15. $M_y = 1 = N_x, M_z = 1 = P_x, N_z = 1 = P_y$ (so path

independent). From inspection observe that

$$f(x, y, z) = xy + xz + yz \text{ satisfies}$$

$f = \langle y + z, x + z, x + y \rangle$, so the integral equals

$$[xy + xz + yz]_{(0,0,0)}^{(1,1,1)} = -\pi. \text{ (Or use line}$$

segments (0, 1, 0) to (1, 1, 0), then (1, 1, 0) to

(1, 1, 1).)

16. $M_y = 2z = N_x, M_z = 2y = P_x, N_z = 2x = P_y$ by

paths (0, 0, 0) to $(\pi, 0, 0)$, $(\pi, 0, 0)$ to $(\pi, \pi, 0)$.

$$\int_0^\pi \cos x \, dx + \int_0^\pi \sin y \, dy = 2$$

Or use $f(x, y, z) = \sin x + 2xyz - \cos y + \frac{z^2}{2}$.

17. $f_x = M, f_y = N, f_z = P$

$$f_{xy} = M_y, \text{ and } f_{yx} = N_x, \text{ so } M_y = N_x.$$

$$f_{xz} = M_z \text{ and } f_{zx} = P_x, \text{ so } M_z = P_x.$$

$$f_{yz} = N_z \text{ and } f_{zy} = P_y, \text{ so } N_z = P_y.$$

18. $f_x(x, y, z) = \frac{-kx}{x^2 + y^2 + z^2}$, so

$$f(x, y, z) = \frac{-k}{2} \ln(x^2 + y^2 + z^2) + C_1(y, z).$$

Similarly,

$$f(x, y, z) = \frac{-k}{2} \ln(x^2 + y^2 + z^2) + C_2(y, z),$$

using f_y ; and

$$f(x, y, z) = \frac{-k}{2} \ln(x^2 + y^2 + z^2) + C_3(y, z),$$

using f_z .

Thus, one potential function for F is

$$f(x, y, z) = \frac{-k}{2} \ln(x^2 + y^2 + z^2).$$

19. $F(x, y, z) = k \frac{\mathbf{r}}{|\mathbf{r}|} = k\mathbf{r} = k \langle x, y, z \rangle$

$$f(x, y, z) = \left(\frac{k}{2}\right)(x^2 + y^2 + z^2) \text{ works.}$$

20. Let $f = \left(\frac{1}{2}\right)h(u)$ where $u = x^2 + y^2 + z^2$.

$$\text{Then } f_x = \left(\frac{1}{2}\right)h'(u)u_x = \left(\frac{1}{2}\right)g(u)(2x) = xg(u).$$

Similarly, $f_y = yg(u)$ and $f_z = zg(u)$.

Therefore, $f(x, y, z) = g(u) \langle x, y, z \rangle$

$$= g(x^2 + y^2 + z^2) \langle x, y, z \rangle = F(x, y, z).$$

21. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b (m\mathbf{r}' \cdot \mathbf{r}') \, dt$

$$= m \int_a^b (x'^2 + y'^2 + z'^2) \, dt$$

$$= m \left[\frac{(x')^2}{2} + \frac{(y')^2}{2} + \frac{(z')^2}{2} \right]_a^b$$

$$= \frac{m}{2} \left[|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2 \right]$$

22. The force exerted by Matt is not the only force acting on the object. There is also an equal but opposite force due to friction. The work done by the sum of the (equal but opposite) forces is zero since the sum of the forces is zero.

23. $f(x, y, z) = -gmz$ satisfies

$$\nabla f(x, y, z) = \langle 0, 0, -gm \rangle = \mathbf{F}. \text{ Then, assuming}$$

the path is piecewise smooth,

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = [-gmz]_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)}$$

$$= -gm(z_2 - z_1) = gm(z_1 - z_2).$$

24. a. Place the earth at the origin.

$$GMm = 7.92(10^{44})$$

$$f(\mathbf{r}) = \frac{-GMm}{|\mathbf{r}|} \text{ is a potential function of}$$

$\mathbf{F}(\mathbf{r})$. (See Example 1.)

$$\text{Work} = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \left[\frac{-GMm}{|\mathbf{r}|} \right]_{|\mathbf{r}|=152.1(10^9)}^{147.1(10^9)}$$

$$\approx -1.77(10^{32}) \text{ joules}$$

b. Zero

25. a. $M = \frac{y}{(x^2 + y^2)}; M_y = \frac{(x^2 - y^2)}{(x^2 + y^2)^2}$

$$N = -\frac{x}{(x^2 + y^2)}; N_x = \frac{(x^2 - y^2)}{(x^2 + y^2)^2}$$

b. $M = \frac{y}{(x^2 + y^2)} = \frac{(\sin t)}{(\cos^2 t + \sin^2 t)} = \sin t$

$$N = -\frac{x}{(x^2 + y^2)} = \frac{(-\cos t)}{(\cos^2 t + \sin^2 t)} = -\cos t$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Mdx + Ndy$$

$$= \int_0^{2\pi} [(\sin t)(-\sin t) + (-\cos t)(\cos t)] dt$$

$$= -\int_0^{2\pi} 1 dt = -2\pi \neq 0$$

26. f is not continuously differentiable on C since f is undefined at two points of C (where x is 0).

17.4 Concepts Review

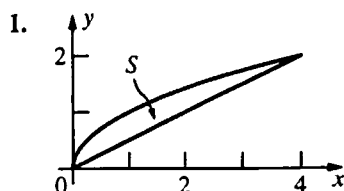
1. $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

2. -2; -2

3. source; sink

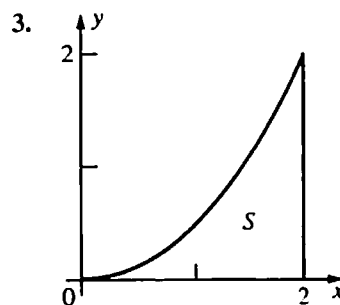
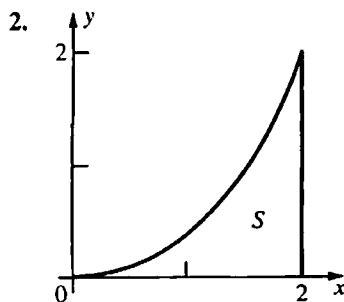
4. rotate; irrotational

Problem Set 17.4



$$\oint_C 2xy dx + y^2 dy = \iint_S (0 - 2x) dA$$

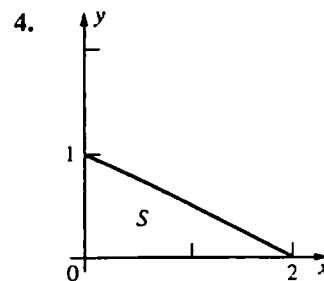
$$= \int_0^2 \int_{y^2}^{2-y} -2x dx dy = -\frac{64}{15} \approx -4.2667$$



$$\oint_C (2x + y^2) dx + (x^2 + 2y) dy = \iint_S (2x - 2y) dA$$

$$= \int_0^2 \int_0^{x^{3/4}} (2x - 2y) dy dx = \int_0^2 \left[\frac{x^4}{2} - \frac{x^6}{16} \right] dx$$

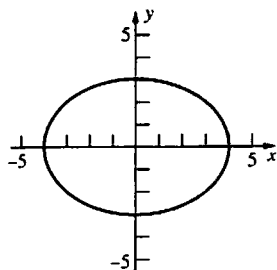
$$= \frac{16}{5} - \frac{8}{7} = \frac{72}{35} \approx 2.0571$$



$$\oint_C xy dx + (x + y) dy = \iint_S (1 - x) dA$$

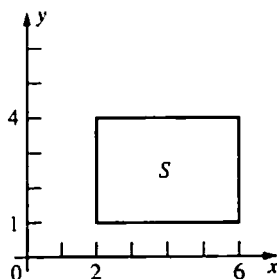
$$= \int_0^1 \int_0^{1-y} (-2y + 2) dx dy = \frac{1}{3}$$

5.



$$\oint_C (x^2 + 4xy)dx + (2x^2 + 3y)dy = \iint_S (4x - 4x)dA = 0$$

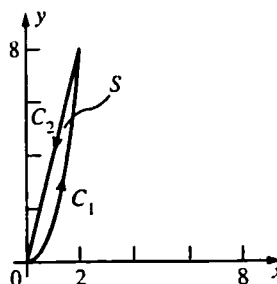
6.



$$\oint_C (e^{3x} + 2y)dx + (x^2 + \sin y)dy = \iint_S (2x - 2)dA$$

$$\int_1^4 \int_2^6 (2x - 2)dx dy = \int_1^4 24dy = 24(3) = 72$$

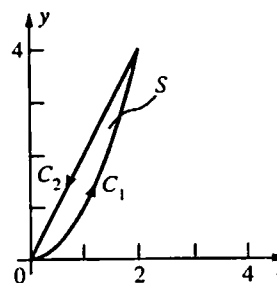
7.



$$A(S) = \left(\frac{1}{2}\right) \oint_C x dy - y dx$$

$$= \left(\frac{1}{2}\right) \int_0^2 [4x^2 - 2x^2]dx + \left(\frac{1}{2}\right) \int_2^0 [4x - 4x]dx = \frac{8}{3}$$

8.

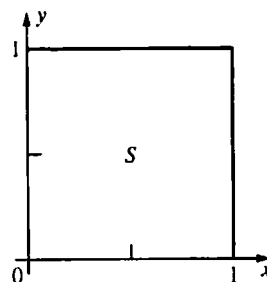


$$A(S) = \left(\frac{1}{2}\right) \oint_C x dy - y dx$$

$$= \left(\frac{1}{2}\right) \int_0^2 \left[\left(\frac{3}{2}\right)x^3 - \left(\frac{1}{2}\right)x^3 \right] dx - \left(\frac{1}{2}\right) \int_2^0 [2x^2 - x^2] dx$$

$$= \frac{2}{3}$$

9.



$$\text{a. } \iint_S \text{div } \mathbf{F} dA = \iint_S (M_x + N_y) dA$$

$$= \iint_S (0 + 0) dA = 0$$

$$\text{b. } \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA = \iint_S (N_x - M_y) dA$$

$$= \iint_S (2x - 2y) dA = \int_0^1 \int_0^1 (2x - 2y) dx dy$$

$$= \int_0^1 (1 - 2y) dy = 0$$

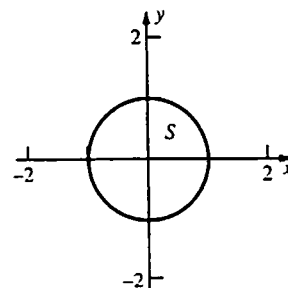
$$10. \text{ a. } \iint_S (0 + 0) dA = 0$$

$$\text{b. } \iint_S (b - a) dA = \int_0^1 \int_0^1 (b - a) dx dy = b - a$$

$$11. \text{ a. } \iint_S (0 + 0) dA = 0$$

$$\text{b. } \iint_S (3x^2 - 3y^2) dA = 0, \text{ since for the integrand, } f(y, x) = -f(x, y).$$

12.



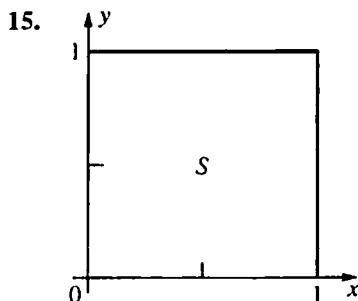
$$\text{a. } \iint_S \text{div } \mathbf{F} dA = \iint_S (M_x + N_y) dA$$

$$= \iint_S (1 + 1) dA = 2[A(S)] = 2\pi$$

$$\begin{aligned} \text{b. } \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA &= \iint_S (N_x - M_y) \, dA \\ &= \iint_S (0 - 0) \, dA = 0 \end{aligned}$$

$$\begin{aligned} 13. \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA &= \int_{C_1} \mathbf{F} \cdot \mathbf{T} \, ds - \int_{C_2} \mathbf{F} \cdot \mathbf{T} \, ds \\ &= 30 - (-20) = 50 \end{aligned}$$

$$14. \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S (2x + 2x) \, dA = \int_0^1 \int_0^1 4x \, dx \, dy = 2$$



$$\begin{aligned} W &= \oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S (N_x - M_y) \, dA \\ &= \iint_S (-2y - 2y) \, dA = \int_0^1 \int_0^1 -4y \, dx \, dy \\ &= \int_0^1 -4y \, dy = -2 \end{aligned}$$

19. a. Each equals $(x^2 - y^2)(x^2 + y^2)^{-2}$.

$$\begin{aligned} \text{b. } \oint_C y(x^2 + y^2)^{-1} \, dx - x(x^2 + y^2)^{-1} \, dy &= \int_0^{2\pi} (-\sin^2 t - \cos^2 t) \, dt \\ &= \int_0^{2\pi} -1 \, dt \end{aligned}$$

c. M and N are discontinuous at $(0, 0)$.

20. a. Parameterization of the ellipse: $x = 3 \cos t$, $y = 2 \sin t$, t in $[0, 2\pi]$.

$$\int_0^{2\pi} \left[\frac{2 \sin t}{9 \cos^2 t + 4 \sin^2 t} (-3 \sin t) - \frac{3 \cos t}{9 \cos^2 t + 4 \sin^2 t} (2 \cos t) \right] dt = -2\pi$$

$$\text{b. } \int_{-1}^1 -(1 + y^2)^{-1} \, dy + \int_1^{-1} (x^2 + 1)^{-1} \, dx + \int_1^{-1} (1 + y^2)^{-1} \, dy + \int_{-1}^1 -(x^2 + 1)^{-1} \, dx = -2\pi$$

c. Green's Theorem applies here. The integral is 0 since $N_x - M_y$.

21. Use Green's Theorem with $M(x, y) = -y$ and $N(x, y) = 0$.

$$\oint_C (-y) \, dx = \iint_S [0 - (-1)] \, dA = A(S)$$

Now use Green's Theorem with $M(x, y) = 0$ and $N(x, y) = x$.

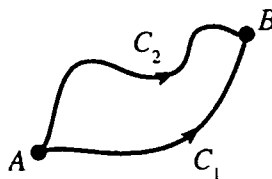
$$\oint_C x \, dy = \iint_S (1 - 0) \, dA = A(S)$$

$$16. \oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S (2y - 2y) \, dA = 0$$

17. $\mathbf{F}$ is a constant, so $N_x = M_y = 0$.

$$\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S (N_x - M_y) \, dA = 0$$

$$18. \oint M \, dx + N \, dy = \iint_S (N_x - M_y) \, dA = 0$$



Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path since

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

(Where C is the loop C_1 followed by $-C_2$.)

Therefore, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, so $\mathbf{F}$ is conservative.

$$22. \oint_C \left(-\frac{1}{2}\right) y^2 dx = \iint_S (0+y) dA = M_x \cdot \oint_C \left(\frac{1}{2}\right) x^2 dy$$

$$\iint_S (x-0) dA = M_y$$

$$23. A(S) = \left(\frac{1}{2}\right) \oint_C x dy - y dx = \left(\frac{1}{2}\right) \int_0^{2\pi} [(a \cos^3 t)(3a \sin^2 t)(\cos t) - (a \sin^3 t)(3a \cos^2 t)(-\sin t)] dt = \left(\frac{3}{8}\right) a^2 \pi$$

$$24. W = \oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA = \iint_S (N_x - M_y) dA = \iint_S (-3-2) dA$$

$$= -5[A(S)] = -5\left(\frac{3a^2\pi}{8}\right) = -\frac{15a^2\pi}{8}, \text{ using the result of Problem 23.}$$

$$25. a. \mathbf{F} \cdot \mathbf{n} = \frac{x^2 + y^2}{(x^2 + y^2)^{3/2}} = \frac{1}{(x^2 + y^2)^{1/2}} = \frac{1}{a}$$

$$\text{Therefore, } \int_C \mathbf{F} \cdot \mathbf{n} ds = \frac{1}{a} \int_C 1 ds = \frac{1}{a} (2\pi a) = 2\pi.$$

$$b. \text{div } \mathbf{F} = \frac{(x^2 + y^2)(1) - (x)(2x)}{(x^2 + y^2)^2} + \frac{(x^2 + y^2)(1) - (y)(2y)}{(x^2 + y^2)^2} = 0$$

$$c. M = \frac{x}{(x^2 + y^2)} \text{ is not defined at } (0, 0) \text{ which is inside } C.$$

$$d. \text{ If origin is outside } C, \text{ then } \oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_S \text{div } \mathbf{F} dA = \iint_S 0 dA = 0.$$

If origin is inside C , let C' be a circle (centered at the origin) inside C and oriented clockwise. Let S be the region between C and C' .

$$\text{Then } 0 = \iint_S \text{div } \mathbf{F} dA \text{ (by "origin outside } C' \text{" case)}$$

$$= \int_C \mathbf{F} \cdot \mathbf{n} ds - \int_{C'} \mathbf{F} \cdot \mathbf{n} ds \text{ (by Green's Theorem)}$$

$$= \int_C \mathbf{F} \cdot \mathbf{n} ds - 2\pi \text{ (by part a), so } \int_C \mathbf{F} \cdot \mathbf{n} ds = 2\pi.$$

$$26. a. \text{Equation of } C:$$

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + t \langle x_1 - x_0, y_1 - y_0 \rangle,$$

$$t \text{ in } [0, 1].$$

Thus;

$$\int_C x dy = \int_0^1 [x_0 + t(x_1 - x_0)](y_1 - y_0) dt,$$

which equals the desired result.

$$b. \text{Area}(P) = \int_C x dy \text{ where}$$

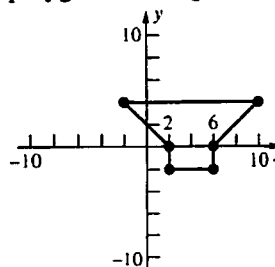
$C = C_1 \cup C_2 \cup \dots \cup C_n$ and C_i is the i th edge. (by Problem 21)

$$= \int_{C_1} x dy + \int_{C_2} x dy + \dots + \int_{C_n} x dy$$

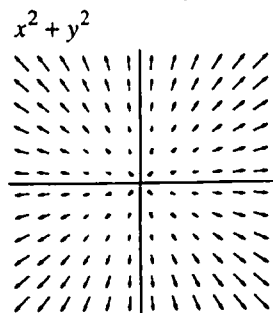
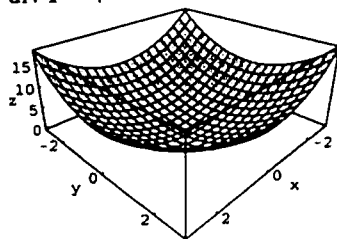
$$= \sum_{i=1}^n \frac{(x_i - x_{i-1})(y_i - y_{i-1})}{2} \text{ (by part a)}$$

c. Immediate result of part b if each x_i and each y_i is an integer.

d. Formula gives 40 which is correct for the polygon in the figure below.

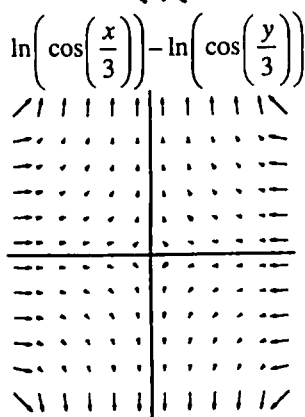
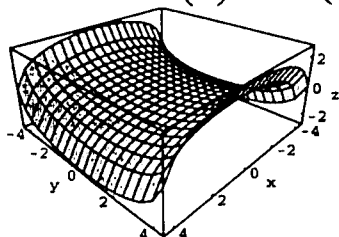


27. a. $\text{div } \mathbf{F} = 4$



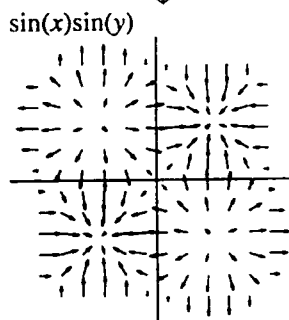
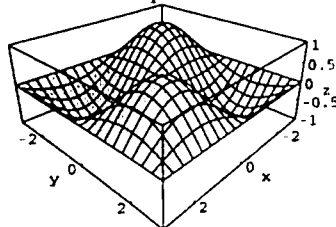
b. $4(36) = 144$

28. a. $\text{div } \mathbf{F} = -\frac{1}{9}\sec^2\left(\frac{x}{3}\right) + \frac{1}{9}\sec^2\left(\frac{y}{3}\right)$



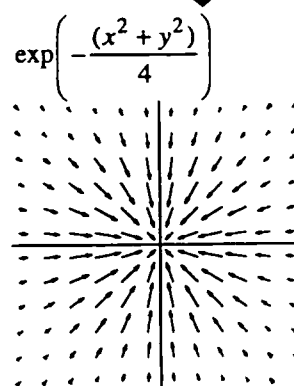
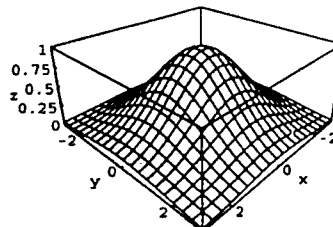
b. $\frac{1}{9} \int_{-3}^3 \int_{-3}^3 \left[-\sec^2\left(\frac{x}{3}\right) + \sec^2\left(\frac{y}{3}\right) \right] dy dx = 0$

29. a. $\text{div } \mathbf{F} = -2 \sin x \sin y$
 $\text{div } \mathbf{F} < 0$ in quadrants I and III
 $\text{div } \mathbf{F} > 0$ in quadrants II and IV



b. Flux across boundary of S is 0.
 Flux across boundary T is $-2(1 - \cos 3)^2$.

30. $\text{div } \mathbf{F} = \frac{1}{4}e^{-(x^2+y^2)/4}(x^2 + y^2 - 4)$
 so $\text{div } \mathbf{F} < 0$ when $x^2 + y^2 < 4$ and $\text{div } \mathbf{F} > 0$
 when $x^2 + y^2 > 4$.

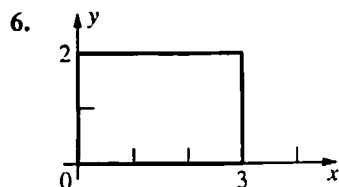


17.5 Concepts Review

1. surface integral
2. $\sum_{i=1}^n g(\bar{x}_i, \bar{y}_i, \bar{z}_i) \Delta S_i$
3. $\sqrt{f_x^2 + f_y^2 + 1}$
4. 2; 18π

Problem Set 17.5

1.
$$\iint_R [x^2 + y^2 + (x + y + 1)](1 + 1 + 1)^{1/2} dA$$
$$= \int_0^1 \int_0^1 \sqrt{3}(x^2 + y^2 + x + y + 1) dx dy = \frac{8\sqrt{3}}{3}$$
2.
$$\iint_R x \left(\frac{1}{4} + \frac{1}{4} + 1 \right)^{1/2} dA = \int_0^1 \int_0^1 \left(\frac{\sqrt{6}}{2} \right) x dx dy$$
$$= \frac{\sqrt{6}}{4}$$
3.
$$\iint_R (x + y) \sqrt{[-x(4 - x^2)^{-1/2}]^2 + 0 + 1} dA$$
$$= \int_0^{\sqrt{3}} \int_0^1 \frac{2(x + y)}{(4 - x^2)^{1/2}} dy dx$$
$$= \int_0^{\sqrt{3}} \frac{2x + 1}{(4 - x^2)^{1/2}} dx$$
$$= \left[-2(4 - x^2)^{1/2} + \sin^{-1} \left(\frac{x}{2} \right) \right]_0^{\sqrt{3}}$$
$$= \frac{\pi + 6}{3} = 2 + \frac{\pi}{3} \approx 3.0472$$
4.
$$\int_0^{2\pi} \int_0^1 r^2 (4r^2 + 1)^{1/2} r dr d\theta = \left(\frac{\pi}{60} \right) (25\sqrt{5} + 1)$$
$$\approx 2.9794$$
5.
$$\int_0^{\pi} \int_0^{\sin \theta} (4r^2 + 1) r dr d\theta = \left(\frac{5}{8} \right) \pi \approx 1.9635$$



$$\iint_R y(4y^2 + 1)^{1/2} dA = \int_0^3 \int_0^2 (4y^2 + 1)^{1/2} y dy dx$$
$$= \int_0^3 \frac{(17^{3/2} - 1)}{12} dx = \frac{17^{3/2} - 1}{4} \approx 17.2732$$

7. $\iint_R (x + y)(0 + 0 + 1)^{1/2} dA$

Bottom ($z = 0$): $\int_0^1 \int_0^1 (x + y) dx dy = 1$

Top ($z = 1$): Same integral

Left side ($y = 0$): $\int_0^1 \int_0^1 (x + 0) dx dz = \frac{1}{2}$

Right side ($y = 1$): $\int_0^1 \int_0^1 (x + 1) dx dz = \frac{3}{2}$

Back ($x = 0$): $\int_0^1 \int_0^1 (0 + y) dy dz = \frac{1}{2}$

Front ($x = 1$): $\int_0^1 \int_0^1 (1 + y) dy dz = \frac{3}{2}$

Therefore, the integral equals

$$1 + 1 + \frac{1}{2} + \frac{3}{2} + \frac{1}{2} + \frac{3}{2} = 6.$$

8. Bottom ($z = 0$): The integrand is 0 so the integral is 0.

Left face ($y = 0$): $\int_0^4 \int_0^{8-2x} z \sqrt{1} dz dx = \frac{128}{3}$

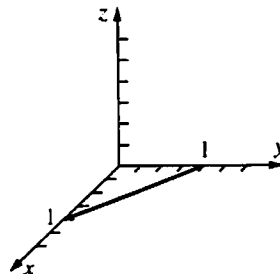
Right face ($z = 8 - 2x - 4y$):

$$\int_0^2 \int_0^{4-2y} (8 - 2x - 4y)(4 + 16 + 1)^{1/2} dx dy$$
$$= \left(\frac{32}{3} \right) \sqrt{21}$$

Back face ($x = 0$): $\int_0^2 \int_0^{8-4y} z \sqrt{1} dz dy = \frac{64}{3}$

Therefore, integral = $64 + \left(\frac{32}{3} \right) \sqrt{21} \approx 112.88$.

9.



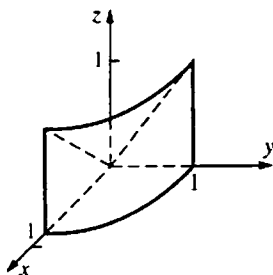
$$\iint_G \mathbf{F} \cdot \mathbf{n} ds = \iint_R (-Mf_x - Nf_y + P) dA$$
$$= \int_0^1 \int_0^{1-y} (8y + 4x + 0) dx dy$$
$$= \int_0^1 [8(1 - y)y + 2(1 - y)^2] dy$$
$$= \int_0^1 (-6y^2 + 4y + 2) dy = 2$$

$$10. \int_0^3 \int_0^{(6-2x)^{1/3}} (x^2 - 9) \left(-\frac{1}{2}\right) dy dx = 11.25$$

$$11. \int_0^5 \int_{-1}^1 [-xy(1-y^2)^{-1/2} + 2] dy dx = 20$$

(In the inside integral, note that the first term is odd in y .)

12.



$$\begin{aligned} & \iint_R [-Mf_x - Nf_y + P] dA \\ &= \iint_R [-2x(x^2 + y^2)^{-1/2} - 5y(x^2 + y^2)^{-1/2} + 3] dA \\ &= \int_0^{2\pi} \int_0^1 [(-2r \cos \theta - 5r \sin \theta)r^{-1} + 3] r dr d\theta \\ &= \int_0^{2\pi} (-2 \cos \theta - 5 \sin \theta + 3) d\theta \int_0^1 r dr \\ &= (6\pi) \left(\frac{1}{2}\right) = 3\pi \approx 9.4248 \end{aligned}$$

$$13. m = \iint_G kx^2 ds = \iint_R kx^2 \sqrt{3} dA$$

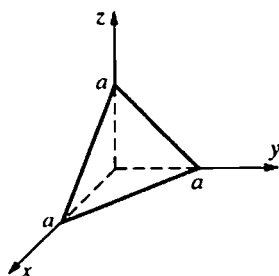
$$= \sqrt{3}k \int_0^a \int_0^{a-x} x^2 dy dx = \left(\frac{\sqrt{3}k}{12}\right) a^4$$

$$14. m = \iint_G kxy ds = \iint_R kxy(x^2 + y^2 + 1)^{1/2} dA$$

$$= \int_0^1 \int_0^1 kxy(x^2 + y^2 + 1)^{1/2} dx dy$$

$$= \left(\frac{k}{15}\right) (9\sqrt{3} - 8\sqrt{2} + 1) \approx 0.3516k$$

15.



Let $\delta = 1$.

$$\begin{aligned} m &= \iint_S 1 ds = \iint_R (1+1+1)^{1/2} dA \\ &= \sqrt{3} \int_0^a \int_0^{a-y} dx dy = \sqrt{3} \int_0^a (a-y) dy = \frac{a^2 \sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} M_{xy} &= \iint_S z ds = \iint_R (a-x-y)\sqrt{3} dA \\ &= \sqrt{3} \int_0^a \int_0^{a-y} (a-x-y) dx dy \\ &= \sqrt{3} \int_0^a \left[a(a-y) - \frac{(a-y)^2}{2} - y(a-y) \right] dy \\ &= \sqrt{3} \int_0^a \left(\frac{a^2}{2} - ay + \frac{y^2}{2} \right) dy = \frac{a^3 \sqrt{3}}{6} \\ \bar{z} &= \frac{M_{xy}}{m} = \frac{a}{3}; \text{ then } \bar{x} = \bar{y} = \frac{a}{3} \text{ (by symmetry).} \end{aligned}$$

$$16. m = \iint_G z ds = \iint_R 3 dA$$

$$(= 3A(R) = 3\pi(3)^2 = 27\pi, \text{ ignoring the subtlety})$$

$$= \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \int_0^{3-\epsilon} 3r dr d\theta = \lim_{\epsilon \rightarrow 0} 3(3-\epsilon)^2 \pi = 27\pi$$

17. a. 0 (By symmetry, since $g(x, y, -z) = -g(x, y, z)$.)

b. 0 (By symmetry, since $g(x, y, -z) = -g(x, y, z)$.)

c. $\iint_G (x^2 + y^2 + z^2) dS = \iint_G a^2 dS$
 $= a^2 \text{Area}(G) = a^2 (4\pi a^2) = 4\pi a^4$

d. Note:

$$\begin{aligned} \iint_G (x^2 + y^2 + z^2) dS &= \iint_G x^2 dS + \iint_G y^2 dS \\ &= \iint_G z^2 dS = 3 \iint_G x^2 dS \end{aligned}$$

(due to symmetry of the sphere with respect to the origin.)

Therefore,

$$\begin{aligned} \iint_G x^2 dS &= \left(\frac{1}{3}\right) \iint_G (x^2 + y^2 + z^2) dS \\ &= \left(\frac{1}{3}\right) 4\pi a^4 = \frac{4\pi a^4}{3}. \end{aligned}$$

e. $\iint_G (x^2 + y^2) dS = \left(\frac{2}{3}\right) 4\pi a^4 = \frac{8\pi a^4}{3}$

18. a. Let the diameter be along the z -axis.

$$I_z = \iint_G k(x^2 + y^2) dS$$

1. $\iint_G x^2 dS = \iint_G y^2 dS = \iint_G z^2 dS$ (by symmetry of the sphere)

2. $\iint_G (x^2 + y^2 + z^2) dS = \iint_G a^2 dS$
 $= a^2 (\text{Area of sphere}) = a^2 (4\pi a^2) = 4\pi a^4$
 Thus,

$$I_z = \iint_G k(x^2 + y^2) dS = \frac{2}{3} k(4\pi a^4) = \frac{8\pi a^4 k}{3}.$$

(using 1 and 2)

- b. Let the tangent line be parallel to the z -axis.

$$\begin{aligned} \text{Then } I &= I_z + ma^2 = \frac{8\pi a^4 k}{3} + [k(4\pi a^2)]a^2 \\ &= \frac{20\pi a^4 k}{3}. \end{aligned}$$

19. a. Place center of sphere at the origin.

$$\begin{aligned} F &= \iint_G k(a - z) dS = ka \iint_G 1 dS - k \iint_G z dS \\ &= ka(4\pi a^2) - 0 = 4\pi a^3 k \end{aligned}$$

- b. Place hemisphere above xy -plane with center at origin and circular base in xy -plane.

$$\begin{aligned} F &= \text{Force on hemisphere} + \text{Force on circular base} \\ &= \iint_G k(a - z) dS + ka(\pi a^2) \\ &= ka \iint_G 1 dS - k \iint_G z dS + \pi a^3 k \\ &= ka(2\pi a^2) - k \iint_R z \sqrt{\frac{a^2}{a^2 - x^2 - y^2}} dA + \pi a^3 k \\ &= 3\pi a^3 k - k \iint_R z \frac{a}{z} dA \\ &= 3\pi a^3 k - ka(\pi a^2) = 2\pi a^3 k \end{aligned}$$

- c. Place the cylinder above xy -plane with circular base in xy -plane with the center at the origin.

$$\begin{aligned} F &= \text{Force on top} + \text{Force on cylindrical side} + \text{Force on base} \\ &= 0 + \iint_G k(h - z) dS + kh(\pi a^2) \\ &= kh \iint_G 1 dS - k \iint_G z dS + \pi a^2 hk \end{aligned}$$

$$= kh(2\pi ah) - 4k \iint_R z \sqrt{\frac{a^2}{a^2 - y^2} + 0 + 1} dA + \pi a^2 hk$$

(where R is a region in the yz -plane:

$$0 \leq y = a, 0 \leq z \leq h)$$

$$\begin{aligned} &= 2\pi ah^2 k + \pi a^2 hk - 4k \int_0^a \int_0^h \frac{az}{\sqrt{a^2 - y^2}} dz dy \\ &= 2\pi ah^2 k + \pi a^2 hk - \pi kah^2 \\ &= \pi ah^2 k + \pi a^2 hk = \pi ahk(h + a) \end{aligned}$$

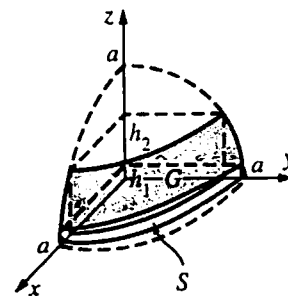
20. $\bar{x} = \bar{y} = 0$

Now let G' be the 1st octant part of G .

$$M_{xy} = \iint_G k dS = 4 \iint_{G'} kz dS = 4k \iint_{R'} z \left(\frac{a}{z} \right) dA$$

(See Problem 19b.)

$$\begin{aligned} &= 4ak [\text{Area}(R')] \\ &= 4ak\pi \left[\frac{(a^2 - h_1^2)}{4} - \frac{(a^2 - h_2^2)}{4} \right] \\ &= ak\pi(h_2^2 - h_1^2) \end{aligned}$$



$$m(G) = \iint_G k dS = k[\text{Area}(G)]$$

$$= k[2\pi a(h_2 - h_1)] = 2\pi ak(h_2 - h_1)$$

$$\text{Therefore, } \bar{z} = \frac{\pi ak(h_2^2 - h_1^2)}{2\pi ak(h_2 - h_1)} = \frac{h_1 + h_2}{2}.$$

17.6 Concepts Review

1. boundary; ∂S

2. $F \cdot n$

3. $\text{div } F$

4. flux; the shape

Problem Set 17.6

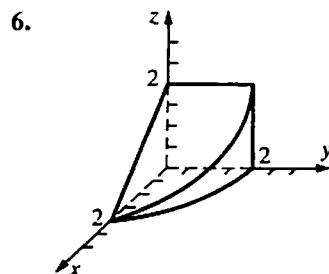
1. $\iiint_S (0 + 0 + 0) dV = 0$

2. $\iiint_S (1 + 2 + 3) dV = 6V(S) = 6$

$$\begin{aligned}
 3. \quad \iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dS &= \iiint_S (M_x + N_y + P_z) \, dV \\
 &= \int_0^c \int_0^b \int_0^a (2xyz + 2xyz + 2xyz) \, dx \, dy \, dz \\
 &= \int_0^c \int_0^b 3a^2 yz \, dy \, dz = \int_0^c \frac{3a^2 b^2 z}{2} \, dz \\
 &= \frac{3a^2 b^2 c^2}{4}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \iiint_S (3 - 2 + 4) \, dV &= 5V(S) = 5 \left[\left(\frac{4}{3} \right) \pi (3)^3 \right] \\
 &= 180\pi = 565.49
 \end{aligned}$$

$$5. \quad 2 \iiint_S (x + y + z) \, dV = 2 \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (r \cos \theta + r \sin \theta + z) r \, dz \, dr \, d\theta = \frac{64\pi}{3} \approx 67.02$$



$$\begin{aligned}
 \iiint_S (M_x + N_y + P_z) \, dV &= \iiint_S (2x + 1 + 2z) \, dV = \int_0^{2\pi} \int_0^2 \int_0^{2-r \cos \theta} (2r \cos \theta + 1 + 2z) r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 [(2r^2 \cos \theta + r)(2 - r \cos \theta) + r(2 - r \cos \theta)^2] \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (6r - r^3 \cos^2 \theta - r^2 \cos \theta) \, dr \, d\theta \\
 &= \int_0^{2\pi} \left(12 - 4 \cos^2 \theta - \frac{8 \cos \theta}{3} \right) d\theta = 20\pi
 \end{aligned}$$

$$7. \quad \iiint_S (1 + 1 + 0) \, dV = 2(\text{volume of cylinder}) = 2\pi(1)^2(2) = 4\pi \approx 12.5664$$

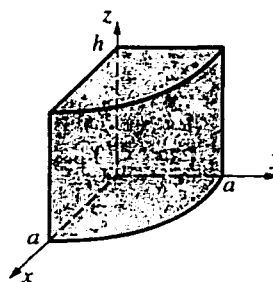
$$8. \quad \iiint_S (2x + 2y + 2z) \, dV = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} (2x + 2y + 2z) \, dz \, dy \, dx = 64$$

$$\begin{aligned}
 9. \quad \iiint_S (M_x + N_y + P_z) \, dV &= \iiint_S (2 + 3 + 4) \, dV \\
 &= 9(\text{Volume of spherical shell}) \\
 &= 9 \left(\frac{4\pi}{3} \right) (5^3 - 3^3) = 1176\pi \approx 3694.51
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \iiint_S (0 + 0 + 2z) \, dV &= \int_0^{2\pi} \int_1^2 \int_0^2 2zr \, dz \, dr \, d\theta \\
 &= 12\pi \approx 37.6991
 \end{aligned}$$

$$11. \quad \left(\frac{1}{3} \right) \iiint_S (1 + 1 + 1) \, dV = V(S)$$

12.



$$\begin{aligned}
 V(S) &= \frac{1}{3} \iint_{\partial S} \mathbf{F} \cdot \mathbf{n} dS \text{ for } \mathbf{F} = \langle x, y, z \rangle \\
 &= \frac{1}{3} \iiint_S 3 dV = \int_0^{2\pi} \int_0^a \int_0^h r dz dr d\theta \\
 &= \int_0^{2\pi} \int_0^a rh dr d\theta = \int_0^{2\pi} \frac{a^2 h}{2} d\theta = 2\pi \frac{a^2 h}{2} \\
 &= \pi a^2 h
 \end{aligned}$$

13. Note:

$$1. \iint_R (ax + by + cz) dS = \iint_R d dS = dD \quad (R \text{ is the slanted face.})$$

$$2. \mathbf{n} = \frac{\langle a, b, c \rangle}{(a^2 + b^2 + c^2)^{1/2}} \quad (\text{for slanted face})$$

$$3. \mathbf{F} \cdot \mathbf{n} = 0 \text{ on each coordinate-plane face.}$$

$$\begin{aligned}
 \text{Volume} &= \left(\frac{1}{3} \right) \iint_S \mathbf{F} \cdot \mathbf{n} dS \quad (\text{where } \mathbf{F} = \langle x, y, z \rangle). \\
 &= \left(\frac{1}{3} \right) \iint_R \mathbf{F} \cdot \mathbf{n} dS \quad (\text{by Note 3}) \\
 &= \left(\frac{1}{3} \right) \iint_R \frac{(ax + by + cz)}{\sqrt{a^2 + b^2 + c^2}} dS \\
 &= \frac{dD}{3\sqrt{a^2 + b^2 + c^2}}
 \end{aligned}$$

$$14. \iiint_S \operatorname{div} \mathbf{F} dV = \iiint_S 0 dV = 0 \quad (\text{"Nice" if there is an outer normal vector at each point of } \partial S.)$$

$$15. \text{ a. } \operatorname{div} \mathbf{F} = 2 + 3 + 2z = 5 + 2z$$

$$\begin{aligned}
 \iint_{\partial S} \mathbf{F} \cdot \mathbf{n} dS &= \iiint_S (5 + 2z) dV = \iiint_S 5 dV + 2 \iiint_S z dV = 5 (\text{Volume of } S) + 2M_{xy} \\
 &= 5 \left(\frac{4\pi}{3} \right) + 2z (\text{Volume of } S) = \frac{20\pi}{3} + 2(0)(\text{Volume}) = \frac{20\pi}{3}
 \end{aligned}$$

$$\text{b. } \mathbf{F} \cdot \mathbf{n} = (x^2 + y^2 + z^2)^{3/2} \langle x, y, z \rangle \cdot \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 + z^2)^2 = 1 \text{ on } \partial S.$$

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} dS = \iiint_S 1 dV = 4\pi(1)^2 = 4\pi$$

$$\text{c. } \operatorname{div} \mathbf{F} = 2x + 2y + 2z$$

$$\begin{aligned}
 \iint_{\partial S} \mathbf{F} \cdot \mathbf{n} dS &= \iiint_S 2(x + y + z) dV \\
 &= 2 \iiint_S x dV \quad (\text{Since } \bar{x} = \bar{y} = \bar{z} = 0 \text{ as in a.}) \\
 &= 2M_{yz} = 2(\bar{x})(\text{Volume of } S) = 2(2) \left(\frac{4\pi}{3} \right) = \frac{16\pi}{3}
 \end{aligned}$$

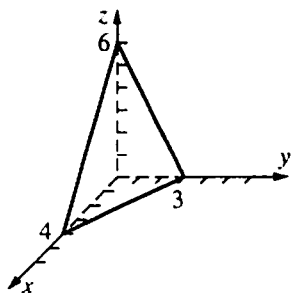
$$\text{d. } \mathbf{F} \cdot \mathbf{n} = 0 \text{ on each face except the face } R \text{ in the plane } x = 1.$$

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} dS = \iint_R \mathbf{F} \cdot \mathbf{n} dS = \iint_R \langle 1, 0, 0 \rangle \cdot \langle 1, 0, 0 \rangle dS = \iint_R 1 dS = (1)^2 = 1$$

$$\text{e. } \operatorname{div} \mathbf{F} = 1 + 1 + 1 = 3$$

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} dS = \iiint_S 3 dV = 3(\text{Volume of } S) = 3 \left(\frac{1}{3} \left[\frac{1}{2} (4)(3) \right] (6) \right) = 36$$

f.



$$\operatorname{div} \mathbf{F} = 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2) = 3 \text{ on } \partial S.$$

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dS = 3 \iiint_S (x^2 + y^2 + z^2) \, dV = 3 \left(\frac{3}{2} \frac{8\pi}{15} \right) = \frac{12\pi}{5}$$

(That answer can be obtained by making use of symmetry and a change to spherical coordinates. Or you could go to the solution for Problem 24, Section 16.8, and realize that the value of the integral in this problem is $\frac{3}{2}$ (there are three terms instead of two) times the answer obtained there for I_z , letting $kabc = 1$.)

g. $\mathbf{F} \cdot \mathbf{n} = [\ln(x^2 + y^2)] \langle x, y, 0 \rangle \cdot \langle 0, 0, 1 \rangle = 0$ on top and bottom.

$$\mathbf{F} \cdot \mathbf{n} = (\ln 4) \langle x, y, 0 \rangle \cdot \frac{\langle x, y, 0 \rangle}{\sqrt{x^2 + y^2}} = (\ln 4) \sqrt{x^2 + y^2} = (\ln 4) \sqrt{4} = 2 \ln 4 = 4 \ln 2 \text{ on the side.}$$

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R 4 \ln 2 \, dS = (4 \ln 2) [2\pi(2)(2)] = 32\pi \ln 2$$

16. a. $\operatorname{div} \mathbf{F} = 0$ (See Problem 21, Section 17.1.)

$$\text{Therefore, } \iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_S \operatorname{div} \mathbf{F} \, dV = 0.$$

b. $\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dS = 4\pi$ (by Gauss's law with $-cM = 1$ as in Example 5).

c. $\mathbf{F} \cdot \mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{|\mathbf{r}|} = \frac{1}{a}$ on ∂S .

$$\text{Thus, } \iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dS = \left(\frac{1}{a} \right) \iint_{\partial S} 1 \, dS$$

$$= \left(\frac{1}{a} \right) (\text{Surface area of sphere})$$

$$= \left(\frac{1}{a} \right) (4\pi a^2) = 4\pi a.$$

d. $\mathbf{F} \cdot \mathbf{n} = f(|r|) \mathbf{r} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = |r| f(|r|) = af(a)$ on ∂S .

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dS = af(a) \iint_{\partial S} 1 \, dS = [af(a)](4\pi a^2) = 4\pi a^3 f(a)$$

e. The ball is above the xy -plane, is tangent to the xy -plane at the origin, and has radius $\frac{a}{2}$.

$$\operatorname{div} \mathbf{F} = |\mathbf{r}|^n \operatorname{div} \mathbf{r} + (\operatorname{grad} |\mathbf{r}|^n) \cdot \mathbf{r} \quad (\text{See Problem 20c, Section 17.1.})$$

$$= |\mathbf{r}|^n (1+1+1) + n |\mathbf{r}|^{n-1} \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{r} = 3|\mathbf{r}|^n + n|\mathbf{r}|^n = (3+n)|\mathbf{r}|^n$$

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dS = (3+n) \iiint_S |\mathbf{r}|^n \, dV = (3+n) \int_0^{2\pi} \int_0^{\pi/2} \int_0^{a \cos \phi} \rho^n (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta = \frac{2\pi a^{n+3}}{n+4}$$

$$17. \iint_{\partial S} D_n f \, dS = \iint_{\partial S} \nabla f \cdot \mathbf{n} \, dS = \iiint_S \operatorname{div}(\nabla f) \, dV = \iiint_S \nabla^2 f \, dV \quad (\text{See next problem.})$$

$$\begin{aligned} 18. \iint_{\partial S} f(\nabla f \cdot \mathbf{n}) \, dS &= \iint_{\partial S} (f \nabla f) \cdot \mathbf{n} \, dS = \iiint_S \operatorname{div}(f \nabla f) \, dV \\ &= \iiint_S \operatorname{div}(\nabla f) + (\nabla f) \cdot (\nabla f) \, dV \quad (\text{See Problem 20c, Section 17.1.}) \\ &= \iiint_S [(f_{xx} + f_{yy} + f_{zz}) + |\nabla f|^2] \, dV \\ &= \iiint_S [(\nabla^2 f) + |\nabla f|^2] \, dV = \iiint_S |\nabla f|^2 \, dV \quad (\text{Since it is given that } \nabla^2 f = 0 \text{ on } S.) \end{aligned}$$

$$\begin{aligned} 19. \iint_{\partial S} f D_n g \, dS &= \iint_{\partial S} f(\nabla g \cdot \mathbf{n}) \, dS = \iint_{\partial S} (f \nabla g) \cdot \mathbf{n} \, dS = \iiint_S \operatorname{div}(f \nabla g) \, dV \quad (\text{Gauss}) \\ &= \iiint_S [f(\operatorname{div} \nabla g) + (\nabla f) \cdot (\nabla g)] \, dV = \iiint_S (f \nabla^2 g + \nabla f \cdot \nabla g) \, dV \quad (\text{See Problem 20c, Section 17.1.}) \end{aligned}$$

$$\begin{aligned} 20. \iint_{\partial S} (f D_n g - g D_n f) \, dS &= \iint_{\partial S} f D_n g \, dS - \iint_{\partial S} g D_n f \, dS \\ &= \iiint_S (f \nabla^2 g + \nabla f \cdot \nabla g) \, dV - \iiint_S (g \nabla^2 f + \nabla g \cdot \nabla f) \, dV \quad (\text{by Green's 1st identity}) \\ &= \iiint_S (f \nabla^2 g - g \nabla^2 f) \, dV \end{aligned}$$

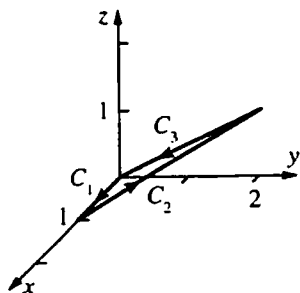
17.7 Concepts Review

1. $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}$
2. Möbius band
3. $\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, dS$
4. $\operatorname{curl} \mathbf{F}$

Problem Set 17.7

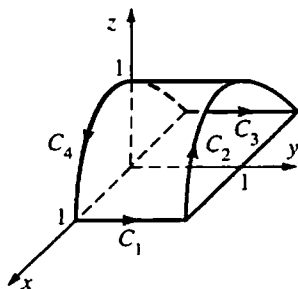
$$1. \oint_{\partial S} \mathbf{F} \cdot \mathbf{T} \, dS = \iint_R (N_x - M_y) \, dA = \iint_R 0 \, dA = 0$$

2.



$$\oint_{\partial S} \mathbf{F} \cdot \mathbf{T} \, ds = \int_{C_1} 0 \, dx + \int_{C_2} xy \, dx + yz \, dy + xz \, dz + \int_{C_3} yz \, dy = \int_0^1 (t^2 + 7t - 4) \, dt = -\frac{1}{6}$$

3.



$$\begin{aligned}\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS &= \oint_{\partial S} \mathbf{F} \cdot \mathbf{T} \, ds = \oint_{\partial S} (y+z)dx + (x^2+z^2)dy + y \, dz \\ &= \int_0^1 1 \, dt + \int_0^\pi [(1+\sin t)(-\sin t) + \cos t] \, dt + \int_0^1 -1 \, dt + \int_0^\pi \sin^2 t \, dt \quad (*) \\ &= \int_0^\pi (-\sin t + \cos t) \, dt = -2\end{aligned}$$

The result at (*) was obtained by integrating along S by doing so along C_1, C_2, C_3, C_4 in that order.

Along C_1 : $x = 1, y = t, z = 0, dx = dz = 0, dy = dt, t$ in $[0, 1]$

Along C_2 : $x = \cos t, y = 1, z = \sin t, dx = -\sin t \, dt, dy = 0, dz = \cos t \, dt, t$ in $[0, \pi]$

Along C_3 : $x = -1, y = 1 - t, z = 0, dx = dz = 0, dy = -dt, t$ in $[0, 1]$

Along C_4 : $x = -\cos t, y = 0, z = \sin t, dx = \sin t \, dt, dy = 0, dz = \cos t \, dt, t$ in $[0, \pi]$

4. ∂S is the circle $x^2 + y^2 = 1, z = 0$ (in the xy -plane).

$$\begin{aligned}\oint_{\partial S} \mathbf{F} \cdot \mathbf{T} \, ds &= \oint_{\partial S} xy^2 \, dx + x^3 \, dy = (\cos xz) \, dz \\ &= \oint_S x^3 \, dy = \int_0^{2\pi} (\cos^3 t)(-\cos t) \, dt = \left(-\frac{3}{4}\right)\pi \\ &\approx -2.3562\end{aligned}$$

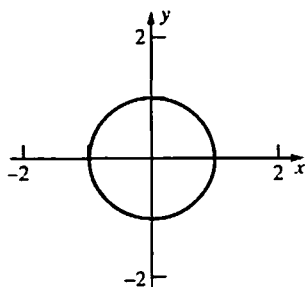
5. ∂S is the circle $x^2 + y^2 = 12, z = 2$.

Parameterization of circle:

$$x = \sqrt{12} \cos t, y = \sqrt{12} \sin t, z = 2, t \text{ in } [0, 2\pi]$$

$$\begin{aligned}\oint_{\partial S} \mathbf{F} \cdot \mathbf{T} \, ds &= \oint_{\partial S} yz \, dx + 3xz \, dy + z^2 \, dz \\ &= \int_0^{2\pi} (24 \sin^2 t - 72 \cos^2 t) \, dt = -48\pi \approx -150.80\end{aligned}$$

6. ∂S is the circle $x^2 + y^2 = 1, z = 0$



$$\begin{aligned}\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS &= \oint_{\partial S} \mathbf{F} \cdot \mathbf{T} \, ds \\ &= \oint_S (z-y)dx + (z+x)dy + (-x-y)dz \\ &= \int_0^{2\pi} [(-\sin t)(-\sin t) + (\cos t)(\cos t)] \, dt = 2\pi \approx 6.2832\end{aligned}$$

$$\begin{aligned}x &= \cos t \\ y &= \sin t \\ z &= 0 \\ t &\text{ in } [0, 2\pi]\end{aligned}$$

$$7. (\text{curl } \mathbf{F}) \cdot \mathbf{n} = \langle 3, 2, 1 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, 1, 0, -1 \right\rangle = \sqrt{2}$$

$$\begin{aligned}\oint_{\partial S} \mathbf{F} \cdot \mathbf{T} \, ds &= \iint_S \sqrt{2} \, dS = \sqrt{2} A(S) \\ &= \sqrt{2} [\sec(45^\circ)] (\text{Area of a circle}) = 8\pi \approx 25.1327\end{aligned}$$

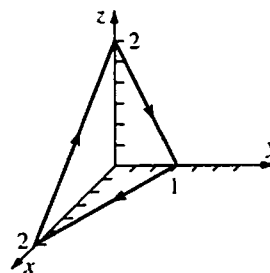
$$8. (\text{curl } \mathbf{F}) \cdot \mathbf{n} = \langle -1, -1, -1 \rangle \cdot \left[\left(\frac{1}{\sqrt{2}} \right) \langle 0, 1, -1 \rangle \right] = 0,$$

so the integral is 0.

$$9. (\text{curl } \mathbf{F}) = \langle -1+1, 0-1, 1-1 \rangle = \langle 0, -1, 0 \rangle$$

The unit normal vector that is needed to apply Stokes' Theorem points downward. It is

$$\mathbf{n} = \frac{\langle -1, -2, -1 \rangle}{\sqrt{6}}.$$



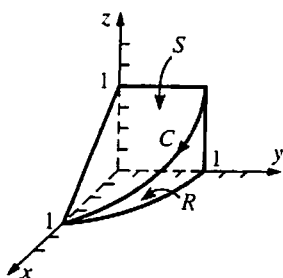
$$\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$$

$$\begin{aligned}
 &= \iint_S \left(\frac{2}{\sqrt{6}} \right) dS = \iint_R \left(\frac{2}{\sqrt{6}} \right) (1+4+1)^{1/2} dA \\
 &= \iint_R 2 dA = 2(\text{Area of triangle in } xy\text{-plane}) \\
 &= 2(1) = 2
 \end{aligned}$$

$$\begin{aligned}
 10. \quad (\text{curl } \mathbf{F}) \cdot \mathbf{n} &= \langle 0, 0, -4x^2 - 4y^2 \rangle \cdot \left[\left(\frac{1}{\sqrt{2}} \right) \langle -1, 0, 1 \rangle \right] \\
 &= -2\sqrt{2}(x^2 + y^2) \\
 \iint_S -\frac{2}{\sqrt{2}}(x^2 + y^2) dS \\
 &= -4 \int_0^1 \int_0^1 (x^2 + y^2) dx dy = -\frac{8}{3}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad (\text{curl } \mathbf{F}) \cdot \mathbf{n} &= \langle 0, 0, 1 \rangle \cdot \langle x, y, z \rangle = z \\
 \iint_S z dS &= \iint_R 1 dA = \text{Area of } R \\
 \pi \left(\frac{1}{2} \right)^2 &= \frac{\pi}{4} \approx 0.7854
 \end{aligned}$$

$$\begin{aligned}
 12. \quad (\text{curl } \mathbf{F}) &= \langle -1-1, -1-1, -1-1 \rangle = -2\langle 1, 1, 1 \rangle, \\
 \mathbf{n} &= \frac{\langle 1, 0, 1 \rangle}{\sqrt{2}}, \text{ so } (\text{curl } \mathbf{F}) \cdot \mathbf{n} = -2\sqrt{2}.
 \end{aligned}$$



$$\begin{aligned}
 \oint_C \mathbf{F} \cdot \mathbf{T} ds &= \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS \\
 &= -2\sqrt{2} \iint_S 1 dS = -2\sqrt{2} \iint_R (\sec 45^\circ) dA \\
 &= -\frac{2}{\sqrt{2}\sqrt{2}} [A(R)] = -4\pi
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \text{Let } H(x, y, z) &= z - g(x, y) = 0. \\
 \text{Then } \mathbf{n} &= \frac{\nabla H}{|\nabla H|} = \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{1+g_x^2+g_y^2}} \text{ points upward.}
 \end{aligned}$$

$$\begin{aligned}
 \text{Thus,} \\
 \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS &= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \mathbf{n} \sec \gamma dA \\
 &= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{g_x^2+g_y^2+1}} \sqrt{g_x^2+g_y^2+1} dA \\
 (\text{Theorem A, Section 17.5}) \\
 &= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \langle -g_x, -g_y, 1 \rangle dA
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \text{curl } \mathbf{F} &= \langle z^2, 0, -2y \rangle \\
 \oint_C \mathbf{F} \cdot \mathbf{T} ds &= \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS \quad (\text{Stoke's Theorem}) \\
 &= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \langle -g_x, -g_y, 1 \rangle dA \quad (\text{Problem 13}) \\
 &= \iint_{S_{xy}} \langle z^2, 0, -2y \rangle \cdot \langle -y, -x, 1 \rangle dA \\
 (\text{where } z = xy) \\
 &= \int_0^1 \int_0^1 (-x^2 y^3 - 2y) dx dy = -\frac{13}{12}
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \text{curl } \mathbf{F} &= \langle 0-x, 0-0, z-0 \rangle = \langle -x, 0, z \rangle \\
 \oint_C \mathbf{F} \cdot \mathbf{T} ds &= \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS \\
 &= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \langle -g_x, -g_y, 1 \rangle dA \text{ where} \\
 z &= g(x, y) = xy^2. \quad (\text{Problem 13}) \\
 &= \iint_{S_{xy}} \langle -x, 0, z \rangle \cdot \langle -y^2, -2xy, 1 \rangle dA \\
 &= \int_0^1 \int_0^1 (xy^2 + 0 + xy^2) dx dy \\
 &= \int_0^1 ([x^2 y^2]_{x=0}^1) dy = \int_0^1 y^2 dy = \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \oint_C \mathbf{F} \cdot \mathbf{T} ds &= \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS \\
 &= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \langle -g_x, -g_y, 1 \rangle dA \\
 &= \iint_{S_{xy}} \langle -x, 0, z \rangle \cdot \langle -2xy^2, -2x^2 y, 1 \rangle dA \\
 (\text{where } z &= x^2 y^2) \\
 &= \iint_{S_{xy}} 3x^2 y^2 dA \\
 &= 12 \int_0^{\pi/2} \int_0^a (r \cos \theta)^2 (r \sin \theta)^2 r dr d\theta = \frac{\pi a^6}{8}
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \oint_{\partial S} \mathbf{F} \cdot \mathbf{T} ds &= \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS \\
 &= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \langle -g_x, -g_y, 1 \rangle dA \\
 &= \iint_{S_{xy}} \langle 2, 2, 0 \rangle \cdot \left[\frac{\langle x, y, (a^2 - x^2 - y^2)^{-1/2} \rangle}{(a^2 - x^2 - y^2)^{-1/2}} \right] dA \\
 &= 2 \iint_{S_{xy}} (x+y)(a^2 - x^2 - y^2)^{-1/2} dA \\
 &= 2 \iint_{S_{xy}} y(a^2 - x^2 - y^2)^{-1/2} dA \\
 &= 4 \int_0^{\pi/2} \int_0^{a \sin \theta} (r \sin \theta)(a^2 - r^2)^{-1/2} dr d\theta \\
 &= \frac{4a^2}{3} \text{ joules}
 \end{aligned}$$

18. $\text{curl } \mathbf{F} = 0$ by Problem 23, Section 17.1. The result then follows from Stokes' Theorem since the left-hand side of the equation in the theorem is the work and the integrand of the right-hand side equals 0.

19. a. Let C be any piecewise smooth simple closed oriented curve C that separates the "nice" surface into two "nice" surfaces, S_1 and S_2 .

$$\begin{aligned} & \iint_{\partial S} (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS \\ &= \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS + \iint_{S_2} (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS \end{aligned}$$

$$= \oint_C \mathbf{F} \cdot \mathbf{T} \, ds + \oint_{-C} \mathbf{F} \cdot \mathbf{T} \, ds = 0 \quad (-C \text{ is } C \text{ with opposite orientation.})$$

- b. $\text{div}(\text{curl } \mathbf{F}) = 0$ (See Problem 20, Section 17.1.) Result follows.

$$\begin{aligned} 20. \quad & \oint_S (f \nabla g) \cdot \mathbf{T} \, ds = \iint_S \text{curl}(f \nabla g) \cdot \mathbf{n} \, dS \\ &= \iint_S [f(\text{curl } \nabla g) + (\nabla f \times \nabla g)] \cdot \mathbf{n} \, dS \\ &= \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n} \, dS, \text{ since } \text{curl } \nabla g = 0. \\ & \text{(See 20b, Section 17.1.)} \end{aligned}$$

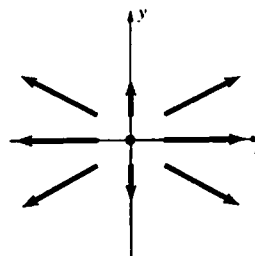
17.8 Chapter Review

Concepts Test

- True: See Example 4, Section 17.1
- False: It is a scalar field.
- False: $\text{grad}(\text{curl } \mathbf{F})$ is not defined since $\text{curl } \mathbf{F}$ is not a scalar field.
- True: See Problem 20b, Section 17.1.
- True: See the three equivalent conditions in Section 17.3.
- True: See the three equivalent conditions in Section 17.3.
- False: $N_z = 0 \neq z^2 = P_y$
- True: See discussion on text page 774.
- True: It is the case in which the surface is in a plane.
- False: See the Mobius band in Figure 7, Section 17.5.
- True: See discussion on text page 777.
- True: $\text{div } \mathbf{F} = 0$, so by Gauss's Divergence Theorem, the integral given equals $\iiint_D 0 \, dV$ where D is the solid sphere for which $S = \partial D$.

Sample Test Problems

1.



2. $\text{div } \mathbf{F} = 2yz - 6y + 2y^2$

$$\text{curl } \mathbf{F} = \langle 4yz, 2xy, -2xz \rangle$$

$$\text{grad}(\text{div } \mathbf{F}) = \langle 0, 2z - 6 + 4y, 2y \rangle$$

$$\text{div}(\text{curl } \mathbf{F}) = 0$$

(See 20a, Section 17.1.)

3. $\text{curl}(f \nabla f) = (f)(\text{curl } \nabla f) + (\nabla f \times \nabla f)$
 $= (f)(0) + 0 = 0$

4. a. $f(x, y) = x^2 y + xy + \sin y + C$

b. $f(x, y, z) = xyz + e^{-x} + e^y + C$

5. a. Parameterization is $x = \sin t, y = -\cos t, t$ in $\left[0, \frac{\pi}{2}\right]$.

$$\begin{aligned} & \int_0^{\pi/2} (1 - \cos^2 t)(\sin^2 t + \cos^2 t)^{1/2} dt = \frac{\pi}{4} \\ & \approx 0.7854 \end{aligned}$$

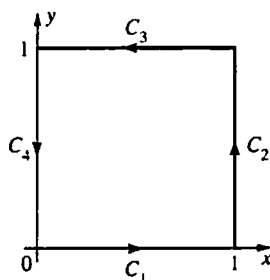
b. $\int_0^{\pi/2} [t \cos t - \sin^2 t \cos t + \sin t \cos t] dt$
 $= \frac{(3\pi - 5)}{6} \approx 0.7375$

6. $M_x = 2y = N_y$ so the integral is independent of the path. Find any function $f(x, y)$ such that $f_x(x, y) = y^2$ and $f_y(x, y) = 2xy$.
 $f(x, y) = xy^2 + C_1(y)$ and
 $f(x, y) = xy^2 + C_2(x)$, so let $f(x, y) = xy^2$.
 Then the given integral equals $[xy^2]_{(0,0)}^{(1,2)} = 4$.

7. $[xy^2]_{(1,1)}^{(3,4)} = 47$

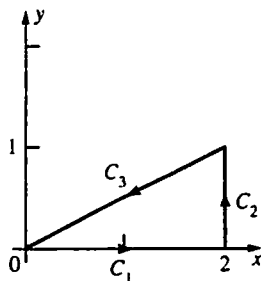
8. $[xyz + e^{-x} + e^y]_{(0,0,0)}^{(1,1,4)} = 2 + e^{-1} + e \approx 5.0862$

9. a.



$$\int_0^1 0 dx + \int_0^1 (1 + y^2) dy + \int_1^0 x dx + \int_1^0 y^2 dy = 0 + \frac{4}{3} - \frac{1}{2} - \frac{1}{3} = \frac{1}{2}$$

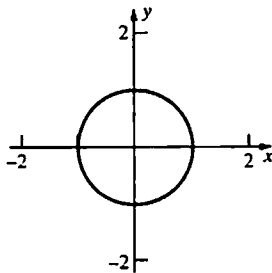
b.



A vector equation of C_3 is $\langle x, y \rangle = \langle 2, 1 \rangle + t\langle -2, -1 \rangle$ for t in $[0, 1]$, so let $x = 2 - 2t$, $y = 1 - t$ for t in $[0, 1]$ be parametric equations of C_3 .

$$\int_0^2 0 dx + \int_0^1 (4 + y^2) dy + \int_0^1 [2(1-t)^2(-2) + 5(1-t)^2(-1)] dt = 0 + \frac{13}{3} - 3 = \frac{4}{3}$$

c.



$$\begin{aligned} x &= \cos t \\ y &= \sin t \\ t &\text{ in } [2, \pi] \end{aligned}$$

$$\int_0^{2\pi} [(\cos t)(\sin t)(-\sin t) + (\cos^2 t + \sin^2 t)(\cos t)] dt = \int_0^{2\pi} (1 - \sin^2 t) \cos t dt = \left[\sin t - \frac{\sin^3 t}{3} \right]_0^{2\pi} = 0$$

$$10. \iint_S \operatorname{div} \mathbf{F} \, dA = \iint_S 2 \, dA = 2A(S) = 8$$

11. Let $f(x, y) = (1 - x^2 - y^2)$ and $g(x, y) = -(1 - x^2 - y^2)$, the upper and lower hemispheres.

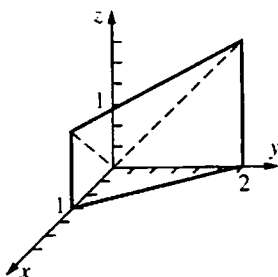
$$\text{Then Flux} = \iint_G \mathbf{F} \cdot \mathbf{n} \, dS$$

$$= \iint_R [-Mf_x - Nf_y + P] \, dA + \iint_R [-Mg_x - Ng_y + P] \, dA = \iint_R 2P \, dA \quad (\text{since } f_x = -g_x \text{ and } f_y = -g_y)$$

$$= \iint_R 6 \, dA = 6 \quad (\text{Area of } R, \text{ the circle } x^2 + y^2 = 1, z = 0)$$

$$= 6\pi \approx 18.8496$$

12.



$$\iint_G xyz \, dS = \iint_R xy(x+y)(\sec) \, dA = \sqrt{3} \int_0^1 \int_0^{-2x+2} (x^2y + xy^2) \, dy \, dx = \sqrt{3} \int_0^1 \frac{4x^2(1-x)^2}{2} + \frac{8x(1-x)^3}{3} \, dx$$

$$= -\frac{2\sqrt{3}}{3} \int_0^1 (x^4 - 6x^3 + 9x^2 - 4x) \, dx = -\frac{2\sqrt{3}}{3} \left[\frac{1}{5} - \frac{3}{2} + 3 - 2 \right] = \frac{3}{5} \approx 0.3464$$

$$\cos \nu = \frac{\langle -1, -1, 1 \rangle \cdot \langle 0, 0, 1 \rangle}{\sqrt{3}}$$

$$\text{Therefore, } \sec \nu = \sqrt{3}.$$

13. ∂S is the circle $x^2 + y^2 = 1, z = 1$.

A parameterization of the circle is $x = \cos t$,
 $y = \sin t, z = 1, t$ in $[0, 2\pi]$.

$$\oint_{\partial S} \mathbf{F} \cdot \mathbf{T} \, ds = \oint_{\partial S} \left[x^3 y \, dx + e^y \, dy + z \tan \left(\frac{xyz}{4} \right) \, dz \right] = \oint_{\partial S} (x^3 y + e^y \, dy)$$

$$= \int_0^{2\pi} [(\cos t)^3 (\sin t)(-\sin t) + (e^{\sin t})(\cos t)] \, dt = 0$$

$$14. \iiint_S \operatorname{div} \mathbf{F} \, dv = \iiint_S [(\cos x) + (1 - \cos x) + (4)] \, dV = \iiint_S 5 \, dV = 5V(S) = 5 \left(\frac{1}{2} \right) \left[\left(\frac{4}{3} \right) \pi (3)^3 \right]$$

$$= 90\pi \approx 282.7433$$

15. $\operatorname{curl} \mathbf{F} = \langle 3 - 0, 0 - 0, -1 - 1 \rangle = \langle 3, 0, -2 \rangle$

$$\mathbf{n} = \frac{\langle a, b, 1 \rangle}{\sqrt{a^2 + b^2 + 1}}$$

$$\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_S \frac{3a - 2}{\sqrt{a^2 + b^2 + 1}} \, dS = \frac{3a - 2}{\sqrt{a^2 + b^2 + 1}} [A(S)]$$

$$= \frac{3a - 2}{\sqrt{a^2 + b^2 + 1}} (9\pi) \quad (S \text{ is a circle of radius } 3.)$$

$$= \frac{9\pi(3a - 2)}{\sqrt{a^2 + b^2 + 1}}$$

18.1 Concepts Review

1. $r^2 + a_1r + a_2 = 0$; complex conjugate roots
2. $C_1e^{-x} + C_2e^x$
3. $(C_1 + C_2x)e^x$
4. $C_1 \cos x + C_2 \sin x$

Problem Set 18.1

1. Roots are 2 and 3. General solution is $y = C_1e^{2x} + C_2e^{3x}$.
2. Roots are -6 and 1. General solution is $y = C_1e^{-6x} + C_2e^x$.
3. Auxiliary equation: $r^2 + 6r - 7 = 0$, $(r + 7)(r - 1) = 0$ has roots -7, 1.
General solution: $y = C_1e^{-7x} + C_2e^x$
 $y' = -7C_1e^{-7x} + C_2e^x$
If $x = 0$, $y = 0$, $y' = 4$, then $0 = C_1 + C_2$ and $4 = -7C_1 + C_2$, so $C_1 = -\frac{1}{2}$ and $C_2 = \frac{1}{2}$.
Therefore, $y = \frac{e^x - e^{-7x}}{2}$.
4. Roots are -2 and 5. General solution is $y = C_1e^{-2x} + C_2e^{5x}$. Particular solution is $y = \left(\frac{12}{7}\right)e^{5x} - \left(\frac{5}{7}\right)e^{-2x}$.
5. Repeated root 2. General solution is $y = (C_1 + C_2x)e^{2x}$.
6. Auxiliary equation: $r^2 + 10r + 25 = 0$, $(r + 5)^2 = 0$ has one repeated root -5.

General solution: $y = C_1e^{-5x} + C_2xe^{-5x}$ or
 $y = (C_1 + C_2x)e^{-5x}$

7. Roots are $2 \pm \sqrt{3}$. General solution is $y = e^{2x}(C_1e^{\sqrt{3}x} + C_2e^{-\sqrt{3}x})$.
8. Roots are $-3 \pm \sqrt{11}$. General solution is $y = e^{-3x}(C_1e^{\sqrt{11}x} + C_2e^{-\sqrt{11}x})$.
9. Auxiliary equation: $r^2 + 4 = 0$ has roots $\pm 2i$.
General solution: $y = C_1 \cos 2x + C_2 \sin 2x$
If $x = 0$ and $y = 2$, then $2 = C_1$; if $x = \frac{\pi}{4}$ and $y = 3$, then $3 = C_2$.
Therefore, $y = 2 \cos 2x + 3 \sin 2x$.
10. Roots are $\pm 3i$. General solution is $y = (C_1 \cos 3x + C_2 \sin 3x)$. Particular solution is $y = -\sin 3x - 3 \cos 3x$.
11. Roots are $-1 \pm i$. General solution is $y = e^{-x}(C_1 \cos x + C_2 \sin x)$.
12. Auxiliary equation: $r^2 + r + 1 = 0$ has roots $\frac{-1 \pm \sqrt{3}}{2}i$.
General solution:
 $y = C_1e^{(-1/2)x} \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2e^{(-1/2)x} \sin\left(\frac{\sqrt{3}}{2}x\right)$
 $y = e^{-x/2} \left[C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}x\right) \right]$
13. Roots are 0, 0, -4, 1.
General solution is $y = C_1 + C_2x + C_3e^{-4x} + C_4e^x$.
14. Roots are -1, 1, $\pm i$. General solution is $y = C_1e^{-x} + C_2e^x + C_3 \cos x + C_4 \sin x$.

15. Auxiliary equation: $r^4 + 3r^2 - 4 = 0$,

$$(r+1)(r-1)(r^2+4) = 0 \text{ has roots } -1, 1, \pm 2i.$$

General solution:

$$y = C_1 e^{-x} + C_2 e^x + C_3 \cos 2x + C_4 \sin 2x$$

16. Roots are $-2, 3, \pm i$. General solution is

$$y = C_1 e^{-2x} + C_2 e^{3x} + C_3 \cos x + C_4 \sin x.$$

18. $e^u = \cosh u + \sinh u$ and $e^{-u} = \cosh u - \sinh u$. (See Problems 1 and 2, Section 7.8.)

$$\text{Auxiliary equation: } r^2 - 2br - c^2 = 0$$

$$\text{Roots of auxiliary equation: } \frac{2b \pm \sqrt{4b^2 + 4c^2}}{2} = b \pm \sqrt{b^2 + c^2}$$

$$\text{General solution: } y = C_1 e^{(b+\sqrt{b^2+c^2})x} + C_2 e^{(b-\sqrt{b^2+c^2})x}$$

$$= e^{bx} \left[C_1 \left(\cosh(\sqrt{b^2+c^2}x) + \sinh(\sqrt{b^2+c^2}x) \right) + C_2 \left(\cosh(\sqrt{b^2+c^2}x) - \sinh(\sqrt{b^2+c^2}x) \right) \right]$$

$$= e^{bx} \left[(C_1 + C_2) \cosh(\sqrt{b^2+c^2}x) + (C_1 - C_2) \sinh(\sqrt{b^2+c^2}x) \right]$$

$$= e^{bx} \left[D_1 \cosh(\sqrt{b^2+c^2}x) + D_2 \sinh(\sqrt{b^2+c^2}x) \right]$$

19. Repeated roots $\left(-\frac{1}{2}\right) \pm \left(\frac{\sqrt{3}}{2}\right)i$.

$$\text{General solution is } y = e^{-x/2} \left[(C_1 + C_2 x) \cos\left(\frac{\sqrt{3}}{2}x\right) + (C_3 + C_4 x) \sin\left(\frac{\sqrt{3}}{2}x\right) \right].$$

20. Roots $1 \pm i$. General solution is

$$y = e^x (C_1 \cos x + C_2 \sin x)$$

$$= e^x (c \sin \gamma \cos x + c \cos \gamma \sin x) = ce^x \sin(x + \gamma).$$

21. (*) $x^2 y'' + 5xy' + 4y = 0$

Let $x = e^z$. Then $z = \ln x$;

$$y' = \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{dy}{dz} \frac{1}{x};$$

$$y'' = \frac{dy'}{dx} = \frac{d}{dx} \left(\frac{dy}{dz} \frac{1}{x} \right) = \frac{dy}{dz} \frac{-1}{x^2} + \frac{1}{x} \frac{d^2 y}{dz^2} \frac{dz}{dx}$$

$$= \frac{dy}{dz} \frac{-1}{x^2} + \frac{1}{x} \frac{d^2 y}{dz^2} \frac{1}{x}$$

$$\left(-\frac{dy}{dz} + \frac{d^2 y}{dz^2} \right) + \left(5 \frac{dy}{dz} \right) + 4y = 0$$

(Substituting y' and y'' into (*))

$$\frac{d^2 y}{dz^2} + 4 \frac{dy}{dz} + 4y = 0$$

17. Roots are $-2, 2$. General solution is

$$y = C_1 e^{-2x} + C_2 e^{2x}.$$

$$y = C_1 (\cosh 2x - \sinh 2x) + C_2 (\sinh 2x + \cosh 2x)$$

$$= (-C_1 + C_2) \sinh 2x + (C_1 + C_2) \cosh 2x$$

$$= D_1 \sinh 2x + D_2 \cosh 2x$$

Auxiliary equation: $r^2 + 4r + 4 = 0$, $(r+2)^2 = 0$
has roots $-2, -2$.

General solution: $y = (C_1 + C_2 z) e^{-2z}$,

$$y = (C_1 + C_2 \ln x) e^{-2 \ln x}$$

$$y = (C_1 + C_2 \ln x) x^{-2}$$

22. As done in Problem 21,

$$\left[-a \left(\frac{dy}{dz} \right) + a \left(\frac{d^2 y}{dz^2} \right) \right] + b \left(\frac{dy}{dz} \right) + cy = 0.$$

$$\text{Therefore, } a \left(\frac{d^2 y}{dz^2} \right) + (b-a) \left(\frac{dy}{dz} \right) + cy = 0.$$

23. We need to show that $y'' + a_1 y' + a_2 y = 0$ if r_1 and r_2 are distinct real roots of the auxiliary equation.

We have,

$$y' = C_1 r_1 e^{r_1 x} + C_2 r_2 e^{r_2 x}$$

$$y'' = C_1 r_1^2 e^{r_1 x} + C_2 r_2^2 e^{r_2 x}.$$

When put into the differential equation, we

obtain

$$y'' + a_1 y' + a_2 y = C_1 r_1^2 e^{r_1 x} + C_2 r_2^2 e^{r_2 x} + a_1 (C_1 r_1 e^{r_1 x} + C_2 r_2 e^{r_2 x}) + a_2 (C_1 e^{r_1 x} + C_2 e^{r_2 x}) \quad (*)$$

The solutions to the auxiliary equation are given by

$$r_1 = \frac{1}{2} \left(-a_1 - \sqrt{a_1^2 - 4a_2} \right) \text{ and}$$

$$r_2 = \frac{1}{2} \left(-a_1 + \sqrt{a_1^2 - 4a_2} \right).$$

Putting these values into (*) and simplifying yields the desired result: $y'' + a_1 y' + a_2 y = 0$.

24. We need to show that $y'' + a_1 y' + a_2 y = 0$ if $\alpha \pm \beta i$ are complex conjugate roots of the auxiliary equation. We have,

$$y' = e^{\alpha x} ((\alpha C_1 + \beta C_2) \cos(\beta x) + (\alpha C_2 - \beta C_1) \sin(\beta x))$$

$$y'' = e^{\alpha x} ((\alpha^2 C_1 - \beta^2 C_1 + 2\alpha\beta C_2) \cos(\beta x) + (\alpha^2 C_2 - \beta^2 C_2 - 2\alpha\beta C_1) \sin(\beta x))$$

When put into the differential equation, we obtain

$$y'' + a_1 y' + a_2 y = e^{\alpha x} ((\alpha^2 C_1 - \beta^2 C_1 + 2\alpha\beta C_2) \cos(\beta x) + (\alpha^2 C_2 - \beta^2 C_2 - 2\alpha\beta C_1) \sin(\beta x)) + a_1 e^{\alpha x} ((\alpha C_1 + \beta C_2) \cos(\beta x) + (\alpha C_2 - \beta C_1) \sin(\beta x)) + a_2 (C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)) \quad (*)$$

From the solutions to the auxiliary equation, we find that

$$\alpha = \frac{-a_1}{2} \text{ and } \beta = -\frac{1}{2} i \sqrt{a_1^2 - 4a_2}.$$

Putting these values into (*) and simplifying yields the desired result: $y'' + a_1 y' + a_2 y = 0$.

25. a.
$$e^{bi} = 1 + (bi) + \frac{(bi)^2}{2!} + \frac{(bi)^3}{3!} + \frac{(bi)^4}{4!} + \frac{(bi)^5}{5!} + \dots = \left(1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \frac{b^6}{6!} + \dots \right) + i \left(b - \frac{b^3}{3!} + \frac{b^5}{5!} - \frac{b^7}{7!} + \dots \right)$$

$$= \cos(b) + i \sin(b)$$

b.
$$e^{a+bi} = e^a e^{bi} = e^a [\cos(b) + i \sin(b)]$$

c.
$$D_x [e^{(\alpha+\beta i)x}] = D_x [e^{\alpha x} (\cos \beta x + i \sin \beta x)] = \alpha e^{\alpha x} (\cos \beta x + i \sin \beta x) + e^{\alpha x} (-i \beta \sin \beta x + i \beta \cos \beta x)$$

$$= e^{\alpha x} [(\alpha + \beta i) \cos \beta x + (\alpha i - \beta) \sin \beta x]$$

$$(\alpha + \beta i) e^{(\alpha+\beta i)x} = (\alpha + \beta i) [e^{\alpha x} (\cos \beta x + i \sin \beta x)] = e^{\alpha x} [(\alpha + \beta i) \cos \beta x + (\alpha i - \beta) \sin \beta x]$$

 Therefore,
$$D_x [e^{(\alpha+\beta i)x}] = (\alpha + i \beta) e^{(\alpha+\beta i)x}$$

26. $c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha+\beta i)x}$ [c_1 and c_2 are complex constants.]

$$= c_1 e^{\alpha x} [\cos \beta x + i \sin \beta x] + c_2 e^{\alpha x} [\cos(-\beta x) + i \sin(-\beta x)] = e^{\alpha x} [(c_1 + c_2) \cos \beta x + (c_1 - c_2) i \sin \beta x]$$

$$= e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x], \text{ where } C_1 = c_1 + c_2, \text{ and } C_2 = c_1 - c_2.$$

Note: If c_1 and c_2 are complex conjugates, then C_1 and C_2 are real.

27. $y = 0.5e^{5.16228x} + 0.5e^{-1.162278x}$

28. $y = 3.5xe^{-2.5x} + 2e^{-2.5x}$

29. $y = 1.29099e^{-0.25x} \sin(0.968246x)$

30. $y = e^{0.333333x} [2.5 \cos(0.471405x) - 4.94975 \sin(0.471405x)]$

18.2 Concepts Review

1. particular solution to the nonhomogeneous equation; homogeneous equation
2. $-6 + C_1 e^{-2x} + C_2 e^{3x}$
3. $y = Ax^2 + Bx + C$
4. $y = Bxe^{\frac{1}{3}x}$

Problem Set 18.2

1. $y_h = C_1 e^{-3x} + C_2 e^{3x}$
 $y_p = \left(-\frac{1}{9}\right)x + 0$
 $y = \left(-\frac{1}{9}\right)x + C_1 e^{-3x} + C_2 e^{3x}$
2. $y_h = C_1 e^{-3x} + C_2 e^{2x}$
 $y_p = \left(-\frac{1}{3}\right)x^2 + \left(-\frac{1}{9}\right)x + \left(-\frac{7}{54}\right)$
 $y = \left(-\frac{1}{3}\right)x^2 - \left(\frac{1}{9}\right)x - \left(\frac{7}{54}\right) + C_1 e^{-3x} + C_2 e^{2x}$
3. Auxiliary equation: $r^2 - 2r + 1 = 0$ has roots 1, 1.
 $y_h = (C_1 + C_2 x)e^x$
Let $y_p = Ax^2 + Bx + C$; $y'_p = 2Ax + B$;
 $y''_p = 2A$.
Then $(2A) - 2(2Ax + B) + (Ax^2 + Bx + C) = x^2 + x$.
 $Ax^2 + (-4A + B)x + (2A - 2B + C) = x^2 + x$
Thus, $A = 1$, $-4A + B = 1$, $2A - 2B + C = 0$, so
 $A = 1$, $B = 5$, $C = 8$.
General solution: $y = x^2 + 5x + 8 + (C_1 + C_2 x)e^x$
4. $y_h = C_1 e^{-x} + C_2 \cdot y_p = 2x^2 + (-4)x$
 $y = 2x^2 - 4x + C_1 e^{-x} + C_2$
5. $y_h = C_1 e^{2x} + C_2 e^{3x} \cdot y_p = \left(\frac{1}{2}\right)e^x \cdot y$
 $= \left(\frac{1}{2}\right)e^x + C_1 e^{2x} + C_2 e^{3x}$
6. Auxiliary equation: $r^2 + 6r + 9 = 0$, $(r + 3)^2 = 0$ has roots -3, -3.
 $y_h = (C_1 + C_2 x)e^{-3x}$
Let $y_p = Be^{-x}$; $y'_p = -Be^{-x}$; $y''_p = Be^{-x}$.

Then $(Be^{-x}) + 6(-Be^{-x}) + 9(Be^{-x}) = 2e^{-x}$; $4Be^{-x} = 2e^{-x}$; $B = \frac{1}{2}$

General solution: $y = \left(\frac{1}{2}\right)e^{-x} + (C_1 + C_2x)e^{-3x}$

7. $y_h = C_1e^{-3x} + C_2e^{-x}$

$$y_p = \left(-\frac{1}{2}\right)xe^{-3x}$$

$$y = \left(-\frac{1}{2}\right)xe^{-3x} + C_1e^{-3x} + C_2e^{-x}$$

8. $y_h = e^{-x}(C_1 \cos x + C_2 \sin x)$

$$y_p = \left(\frac{3}{2}\right)e^{-2x}$$

$$y = \left(\frac{3}{2}\right)e^{-2x} + e^{-x}(C_1 \cos x + C_2 \sin x)$$

9. Auxiliary equation: $r^2 - r - 2 = 0$,

$(r+1)(r-2) = 0$ has roots $-1, 2$.

$$y_h = C_1e^{-x} + C_2e^{2x}$$

Let $y_p = B \cos x + C \sin x$; $y'_p = -B \sin x + C \cos x$; $y''_p = -B \cos x - C \sin x$.

Then $(-B \cos x - C \sin x) - (-B \sin x + C \cos x)$

$$-2(B \cos x + C \sin x) = 2 \sin x.$$

$(-3B - C) \cos x + (B - 3C) \sin x = 2 \sin x$, so $-3B - C = 0$ so $-3B - C = 0$ and $B - 3C = 2$; $B = \frac{1}{5}$; $C = \frac{-3}{5}$.

General solution: $\left(\frac{1}{5}\right)\cos x - \left(\frac{3}{5}\right)\sin x + C_1e^{2x} + C_2e^{-x}$

10. $y_h = C_1e^{-4x} + C_2$

$$y_p = \left(-\frac{1}{17}\right)\cos x + \left(\frac{4}{17}\right)\sin x$$

$$y = \left(-\frac{1}{17}\right)\cos x + \left(\frac{4}{17}\right)\sin x + C_1e^{-4x} + C_2$$

11. $y_h = C_1 \cos 2x + C_2 \sin 2x$

$$y_p = (0)x \cos 2x + \left(\frac{1}{2}\right)x \sin 2x$$

$$y = \left(\frac{1}{2}\right)x \sin 2x + C_1 \cos 2x + C_2 \sin 2x$$

12. Auxiliary equation: $r^2 + 9 = 0$ has roots $\pm 3i$, so $y_h = C_1 \cos 3x + C_2 \sin 3x$.

Let $y_p = Bx \cos 3x + Cx \sin 3x$; $y'_p = (-3Bx + C) \sin 3x + (B + 3Cx) \cos 3x$;

$$y''_p = (-9Bx + 6C) \cos 3x + (-9Cx - 6B) \sin 3x.$$

Then substituting into the original equation and simplifying, obtain $6C \cos 3x - 6B \sin 3x = \sin 3x$, so $C = 0$ and

$$B = -\frac{1}{6}.$$

General solution: $y = \left(-\frac{1}{6}\right)x \cos 3x + C_1 \cos 3x + C_2 \sin 3x$

13. $y_h = C_1 \cos 3x + C_2 \sin 3x$

$$y_p = (0) \cos x + \left(\frac{1}{8}\right) \sin x + \left(\frac{1}{13}\right) e^{2x}$$

$$y = \left(\frac{1}{8}\right) \sin x + \left(\frac{1}{13}\right) e^{2x} + C_1 \cos 3x + C_2 \sin 3x$$

14. $y_h = C_1 e^{-x} + C_2$

$$y_p = \left(\frac{1}{2}\right) e^x + \left(\frac{3}{2}\right) x^2 + (-3)x$$

$$y = \left(\frac{1}{2}\right) e^x + \left(\frac{3}{2}\right) x^2 - 3x + C_1 e^{-x} + C_2$$

15. Auxiliary equation: $r^2 - 5r + 6 = 0$ has roots 2 and 3, so $y_h = C_1 e^{2x} + C_2 e^{3x}$.

Let $y_p = Be^x$; $y'_p = Be^x$; $y''_p = Be^x$.

Then $(Be^x) - 5(Be^x) + 6(Be^x) = 2e^x$; $2Be^x = 2e^x$; $B = 1$.

General solution: $y = e^x + C_1 e^{2x} + C_2 e^{3x}$

$$y' = e^x + 2C_1 e^{2x} + 3C_2 e^{3x}$$

If $x = 0, y = 1, y' = 0$, then $1 = 1 + C_1 + C_2$ and $0 = 1 + 2C_1 + 3C_2$; $C_1 = 1, C_2 = -1$.

Therefore, $y = e^x + e^{2x} - e^{3x}$.

16. $y_h = C_1 e^{-2x} + C_2 e^{2x}$

$$y_p = (0) \cos x + \left(-\frac{4}{5}\right) \sin x$$

$$y = \left(-\frac{4}{5}\right) \sin x + C_1 e^{-2x} + C_2 e^{2x}$$

$$y = \left(-\frac{4}{5}\right) \sin x + \left(\frac{9}{5}\right) e^{-2x} + \left(\frac{11}{5}\right) e^{2x} \text{ satisfies the conditions.}$$

17. $y_h = C_1 e^x + C_2 e^{2x}$

$$y_p = \left(\frac{1}{4}\right) (10x + 19)$$

$$y = \left(\frac{1}{4}\right) (10x + 19) + C_1 e^x + C_2 e^{2x}$$

18. Auxiliary equation: $r^2 - 4 = 0$ has roots 2, -2, so $y_h = C_1 e^{2x} + C_2 e^{-2x}$.

Let $y_p = v_1 e^{2x} + v_2 e^{-2x}$, subject to $v'_1 e^{2x} + v'_2 e^{-2x} = 0$, and $v'_1 (2e^{2x}) + v'_2 (-2e^{-2x}) = e^{2x}$.

Then $v'_1 (4e^{2x}) = e^{2x}$ and $v'_2 (-4e^{-2x}) = e^{2x}$; $v'_1 = \frac{1}{4}$ and $v'_2 = -e^{4x/4}$; $v_1 = \frac{x}{4}$ and $v_2 = -\frac{e^{4x}}{16}$.

General solution: $y = \frac{x e^{2x}}{4} - \frac{e^{2x}}{16} + C_1 e^{2x} + C_2 e^{-2x}$

19. $y_h = C_1 \cos x + C_2 \sin x$

$$y_p = -\cos \ln |\sin x| - \cos x - x \sin x$$

$$y = -\cos x \ln |\sin x| - x \sin x + C_3 \cos x + C_2 \sin x \text{ (combined cos x terms)}$$

20. $y_h = C_1 \cos x + C_2 \sin x$

$$y_p = -\sin x \ln |\csc x + \cot x|$$

$$y = -\sin x \ln |\csc x + \cot x| + C_1 \cos x + C_2 \sin x$$

21. Auxiliary equation: $r^2 - 3r + 2 = 0$ has roots 1, 2, so $y_h = C_1 e^x + C_2 e^{2x}$.

Let $y_p = v_1 e^x + v_2 e^{2x}$ subject to $v_1' e^x + v_2' e^{2x} = 0$, and $v_1'(e^x) + v_2'(2e^{2x}) = e^x (ex + 1)^{-1}$.

Then $v_1' = \frac{-e^x}{e^x(e^x + 1)}$ so $v_1 = \int \frac{-e^x}{e^x(e^x + 1)} dx = \int \frac{-1}{u(u+1)} du$

$$= \int \left(\frac{-1}{u} + \frac{1}{u+1} \right) du = -\ln u + \ln(u+1) = \ln \left(\frac{u+1}{u} \right) = \ln \frac{e^x + 1}{e^x} = \ln(1 + e^{-x})$$

$$v_2' = \frac{e^x}{e^{2x}(e^x + 1)} \text{ so } v_2 = -e^{-x} + \ln(1 + e^{-x})$$

(similar to finding v_1)

General solution: $y = e^x \ln(1 + e^{-x}) - e^x + e^{2x} \ln(1 + e^{-x}) + C_1 e^x + C_2 e^{2x}$

$$y = (e^x + e^{2x}) \ln(1 + e^{-x}) + D_1 e^x + D_2 e^{2x}$$

22. $y_h = C_1 e^{2x} + C_2 e^{3x}$; $y_p = e^x$

$$y = e^x + C_1 e^{2x} + C_2 e^{3x}$$

23. $L(y_p) = (v_1 u_1 + v_2 u_2)'' + b(v_1 u_1 + v_2 u_2)' + c(v_1 u_1 + v_2 u_2)$

$$= (v_1' u_1 + v_1 u_1' + v_2' u_2 + v_2 u_2') + b(v_1' u_1 + v_1 u_1' + v_2' u_2 + v_2 u_2') + c(v_1 u_1 + v_2 u_2)$$

$$= (v_1'' u_1 + v_1' u_1' + v_1 u_1'' + v_2'' u_2 + v_2' u_2' + v_2 u_2'') + b(v_1' u_1 + v_1 u_1' + v_2' u_2 + v_2 u_2') + c(v_1 u_1 + v_2 u_2)$$

$$= v_1(u_1'' + b u_1' + c u_1) + v_2(u_2'' + b u_2' + c u_2) + b(v_1' u_1 + v_2' u_2) + (v_1' u_1 + v_1 u_1' + v_2' u_2 + v_2 u_2')$$

$$= v_1(u_1'' + b u_1' + c u_1) + v_2(u_2'' + b u_2' + c u_2) + b(v_1' u_1 + v_2' u_2) + (v_1' u_1 + v_1 u_1' + v_2' u_2 + v_2 u_2')$$

$$= v_1(0) + v_2(0) + b(0) + (0) + k(x) = k(x)$$

24. Auxiliary equation: $r^2 + 4 = 0$ has roots $\pm 2i$.

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

Now write $\sin^3 x$ in a form involving $\sin$ βx 's or $\cos$ βx 's.

$$\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

(C.R.C. Standard Mathematical Tables, or derive it using half-angle and product identities.)

Let $y_p = A \sin x + B \cos x + C \sin 3x + D \cos 3x$;

$$y_p' = A \cos x - B \sin x + 3C \cos 3x - 3D \sin 3x$$

$$y_p'' = -A \sin x - B \cos x - 9C \sin 3x - 9D \cos 3x$$

Then

$$y_p'' + 4y_p = 3A \sin x + 3B \cos x - 5C \sin 3x - 5D \cos 3x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x, \text{ so}$$

$$A = \frac{1}{4}, B = 0, C = \frac{1}{20}, D = 0.$$

General solution: $y = \frac{1}{4} \sin x + \frac{1}{20} \sin 3x + C_1 \cos 2x + C_2 \sin 2x$

18.3 Concepts Review

1. $3; \pi$
2. π ; decreases
3. 0
4. electric circuit

Problem Set 18.3

1. $k = 250, m = 10, B^2 = k/m = 250/10 = 25, B = 5$

(the problem gives the mass as $m = 10 \text{ kg}$)

Thus, $y'' = -25y$. The general solution is $y = C_1 \cos 5t + C_2 \sin 5t$. Apply the initial condition to get $y = 0.1 \cos 5t$.

The period is $\frac{2\pi}{5}$ seconds.

2. $k = 100 \text{ lb/ft}, w = 1 \text{ lb}, g = 32 \text{ ft/s}^2, y_0 = \frac{1}{12} \text{ ft},$

$$B = 40\sqrt{2}. \text{ Then } y = \left(\frac{1}{12}\right) \cos(40\sqrt{2}t).$$

Amplitude is $\frac{1}{12} \text{ ft} = 1 \text{ in.}$

Period is $\frac{2\pi}{40\sqrt{2}} \approx 0.1111 \text{ s.}$

3. $y = 0.1 \cos 5t = 0$ whenever $5t = \frac{\pi}{2} + \pi k$ or $t = \frac{\pi}{10} + \frac{\pi}{5}k$.

$$\begin{aligned} \left| y' \left(\frac{\pi}{10} + \frac{\pi}{5}k \right) \right| &= 0.5 \left| \sin 5 \left(\frac{\pi}{10} + \frac{\pi}{5}k \right) \right| \\ &= 0.5 \left| \sin \left(\frac{\pi}{2} + \pi k \right) \right| \\ &= 0.5 \text{ meters per second} \end{aligned}$$

4. $|10| = k \left(\frac{1}{3} \right)$, so $k = 30 \text{ lb/ft}, w = 20 \text{ lb},$

$$g = 32 \text{ ft/s}^2, y_0 = -1 \text{ ft}, v_0 = 2 \text{ ft/s}, B = 4\sqrt{3}$$

Then $y = C_1 \cos(4\sqrt{3}t) + C_2 \sin(4\sqrt{3}t)$.

$y = \cos(4\sqrt{3}t) + \left(\frac{\sqrt{3}t}{6} \right) \sin(4\sqrt{3}t)$ satisfies the initial conditions.

5. $k = 20 \text{ lb/ft}; w = 10 \text{ lb}; y_0 = 1 \text{ ft}; q = \frac{1}{10} \text{ s-lb/ft}, B = 8, E = 0.32$

$E^2 - 4B^2 < 0$, so there is damped motion. Roots of auxiliary equation are approximately $-0.16 \pm 8i$.

General solution is $y \approx e^{-0.16t} (C_1 \cos 8t + C_2 \sin 8t)$. $y \approx e^{-0.16t} (\cos 8t + 0.02 \sin 8t)$ satisfies the initial conditions.

6. $k = 20 \text{ lb/ft}; w = 10 \text{ lb}; y_0 = 1 \text{ ft}; q = 4 \text{ s-lb/ft}$

$$B = \sqrt{\frac{(20)(32)}{10}} = 8; E = \frac{(4)(32)}{10} = 12.8; E^2 - 4B^2 < 0, \text{ so damped motion.}$$

Roots of auxiliary equation are $\frac{-E \pm \sqrt{E^2 - 4B^2}}{2} = -6.4 \pm 4.8i$.

General solution is $y = e^{-6.4t}(C_1 \cos 4.8t + C_2 \sin 4.8t)$.

$$y' = e^{-6.4t}(-4.8C_1 \sin 4.8t + 4.8C_2 \cos 4.8t) - 6.4e^{-6.4t}(C_1 \cos 4.8t + C_2 \sin 4.8t)$$

If $t = 0, y = 1, y' = 0$, then $1 = C_1$ and $0 = 4.8C_2 - 6.4C_1$, so $C_1 = 1$ and $C_2 = \frac{4}{3}$.

$$\text{Therefore, } y = e^{-6.4t} \left[\cos 4.8t + \left(\frac{4}{3} \right) \sin 4.8t \right].$$

7. Original amplitude is 1 ft. Considering the contribution of the sine term to be negligible due to the 0.02 coefficient, the amplitude is approximately $e^{-0.16t}$.

$e^{-0.16t} \approx 0.1$ if $t \approx 14.39$, so amplitude will be about one-tenth of original in about 14.4 s.

8. $C_1 = 1$ and $C_2 = -0.105$, so $y = e^{-0.16t}(\cos 8t + 0.105 \sin 8t)$.

9. $LQ'' + RQ' + \frac{Q}{C} = E(t)$; $10^6 Q' + 10^6 Q = 1$; $Q' + Q = 10^{-6}$

Integrating factor: e^t

$$D[Qe^t] = 10^{-6}e^t; Qe^t = 10^{-6}e^t + C;$$

$$Q = 10^{-6} + Ce^{-t}$$

If $t = 0, Q = 0$, then $C = -10^{-6}$.

$$\text{Therefore, } Q(t) = 10^{-6} - 10^{-6}e^{-t} = 10^{-6}(1 - e^{-t}).$$

10. Same as Problem 9, except $C = 4 - 10^{-6}$, so $Q(t) = 10^{-6} + (4 - 10^{-6})e^{-t}$.

$$\text{Then } I(t) = Q'(t) = -(4 - 10^{-6})e^{-t}.$$

11. $\frac{Q}{[2(10^{-6})]} = 120 \sin 377t$

a. $Q(t) = 0.00024 \sin 377t$

b. $I(t) = Q'(t) = 0.09048 \cos 377t$

12. $LQ'' + RQ' + \frac{Q}{C} = E$; $10^{-2}Q'' + \frac{Q}{10^{-7}} = 20$; $Q'' + 10^9 Q = 2000$

The auxiliary equation, $r^2 + 10^9 = 0$, has roots $\pm 10^{9/2}i$.

$$Q_h = C_1 \cos 10^{9/2}t + C_2 \sin 10^{9/2}t$$

$$Q_p = 2000(10^{-9}) = 2(10^{-6}) \text{ is a particular solution (by inspection).}$$

$$\text{General solution: } Q(t) = 2(10^{-6}) + C_1 \cos 10^{9/2}t + C_2 \sin 10^{9/2}t$$

$$\text{Then } I(t) = Q'(t) = -10^{9/2}C_1 \sin 10^{9/2}t + 10^{9/2}C_2 \cos 10^{9/2}t.$$

If $t = 0, Q = 0, I = 0$, then $0 = 2(10^{-6}) + C_1$ and $0 = C_2$.

$$\text{Therefore, } I(t) = -10^{9/2}(-2[10^{-6}]) \sin 10^{9/2}t = 2(10^{-3/2}) \sin 10^{9/2}t.$$

13. $3.5Q'' + 1000Q' + \frac{Q}{[2(10^{-6})]} = 120 \sin 377t$

(Values are approximated to 6 significant figures for the remainder of the problem.)

$$Q'' + 285.714Q' + 142857Q = 34.2857 \sin 377t$$

Roots of the auxiliary equation are

$$-142.857 \pm 349.927i.$$

$$Q_h = e^{-142.857t} (C_1 \cos 349.927t + C_2 \sin 349.927t)$$

$$Q_p = -3.18288(10^{-4}) \cos 377t + 2.15119(10^{-6}) \sin 377t$$

$$\text{Then, } Q = -3.18288(10^{-4}) \cos 377t + 2.15119(10^{-6}) \sin 377t + Q_h.$$

$$I = Q' = 0.119995 \sin 377t + 0.000810998 \cos 377t + Q'_h$$

$0.000888 \cos 377t$ is small and $Q'_h \rightarrow 0$ as $t \rightarrow \infty$, so the steady-state current is $I \approx 0.12 \sin 377t$.

14. a. Roots of the auxiliary equation are $\pm Bi$.

$$y_h = C_1 \cos Bt + C_2 \sin Bt.$$

$$y_p = \left[\frac{c}{(B^2 - A^2)} \right] \sin At$$

The desired result follows.

b. $y_p = \left(-\frac{c}{2B} \right) t \cos Bt$, so

$$y = C_1 \cos Bt + C_2 \sin Bt - \left(\frac{c}{2B} \right) t \cos Bt.$$

- c. Due to the t factor in the last term, it increases without bound.

15. $A \sin(\beta t + \gamma) = A(\sin \beta t \cos \gamma + \cos \beta t \sin \gamma)$
 $= (A \cos \gamma) \sin \beta t + (A \sin \gamma) \cos \beta t$
 $= C_1 \sin \beta t + C_2 \cos \beta t$, where $C_1 = A \cos \gamma$ and $C_2 = A \sin \gamma$.

[Note that

$$C_1^2 + C_2^2 = A^2 \cos^2 \gamma + A^2 \sin^2 \gamma = A^2.)$$

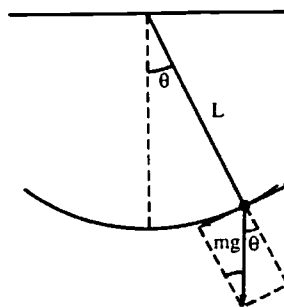
16. The first two terms have period $\frac{2\pi}{B}$ and the last

has period $\frac{2\pi}{A}$. Then the sum of the three terms

is periodic if $m\left(\frac{2\pi}{B}\right) = n\left(\frac{2\pi}{A}\right)$ for some integers

m, n ; equivalently, if $\frac{B}{A} = \frac{m}{n}$, a rational number.

17. The magnitudes of the tangential components of the forces acting on the pendulum bob must be equal.



Therefore, $-m \frac{d^2 s}{dt^2} = mg \sin \theta$.

$s = L\theta$, so $\frac{d^2 s}{dt^2} = L \frac{d^2 \theta}{dt^2}$.

Therefore, $-mL \frac{d^2 \theta}{dt^2} = mg \sin \theta$.

Hence, $\frac{d^2 \theta}{dt^2} = -\frac{g}{L} \sin \theta$.

18. a. Since the roots of the auxiliary equation are $\pm \sqrt{\frac{g}{L}}i$, the solution of $\theta''(t) + \left(\frac{g}{L}\right)\theta = 0$ is

$$\theta = C_1 \cos \sqrt{\frac{g}{L}}t + C_2 \sin \sqrt{\frac{g}{L}}t, \text{ which can be}$$

$$\text{written as } \theta = C \left(\sqrt{\frac{g}{L}}t + \gamma \right)$$

(by Problem 15).

The period of this function is

$$\frac{2\pi}{\sqrt{\frac{g}{L}}} = 2\pi \frac{L}{\sqrt{g}} = 2\pi \sqrt{\frac{LR^2}{GM}} = 2\pi R \sqrt{\frac{L}{GM}}.$$

Therefore, $\frac{p_1}{p_2} = \frac{2\pi R_1 \sqrt{\frac{L_1}{GM}}}{2\pi R_2 \sqrt{\frac{L_2}{GM}}} = \frac{R_1 \sqrt{L_1}}{R_2 \sqrt{L_2}}.$

- b. To keep perfect time at both places, require

$$p_1 = p_2. \text{ Then } 1 = \frac{R_2 \sqrt{80.85}}{3960 \sqrt{81}}, \text{ so}$$

$$R_2 \approx 3963.67.$$

The height of the mountain is about

$$3963.67 - 3960 = 3.67 \text{ mi (about 19,387 ft).}$$

18.4 Chapter Review

Concepts Test

- False: y^2 is not linear in y .
- True: y and y'' are linear in y and y'' , respectively.
- True: $y' = \sec^2 x + \sec x \tan x$
 $2y' - y^2 = (2\sec^2 x + 2\sec x \tan x)$
 $-(\tan^2 x + 2\sec x \tan x + \sec^2 x)$
 $= \sec^2 x - \tan^2 x = 1$
- False: It should involve 6.
- True: D^2 adheres to the conditions for linear operators.
 $D^2(kf) = kD^2(f)$
 $D^2(f+g) = D^2 f + D^2 g$
- False: Replacing y by $C_1 u_1(x) + C_2 u_2(x)$ would yield, on the left side,
 $C_1 f(x) + C_2 f(x) = (C_1 + C_2)f(x)$
which is $f(x)$ only if $C_1 + C_2 = 1$ or $f(x) = 0$.
- True: -1 is a repeated root, with multiplicity 3, of the auxiliary equation.
- True: $L(u_1 - u_2) = L(u_1) - L(u_2)$
 $= f(x) - f(x) = 0$
- False: That is the form of y_h . y_p should have the form $Bx \cos 3x + Cx \sin 3x$.
- True: See Problem 15, Section 18.3.

Sample Test Problems

- $u' + 3u = e^x$. Integrating factor is e^{3x} .
 $D[ue^{3x}] = e^{4x}$

$$y = \left(\frac{1}{4}\right)e^x + C_1 e^{-3x}$$

$$y' = \left(\frac{1}{4}\right)e^x + C_1 e^{-3x}$$

$$y = \left(\frac{1}{4}\right)e^x + C_3 e^{-3x} + C_2$$

- Roots are $-1, 1$. $y = C_1 e^{-x} + C_2 e^x$
- (Second order homogeneous)
The auxiliary equation, $r^2 - 3r + 2 = 0$, has roots $1, 2$. The general solution is $y = C_1 e^x + C_2 e^{2x}$.
 $y' = C_1 e^x + 2C_2 e^{2x}$
If $x = 0, y = 0, y' = 3$, then $0 = C_1 + C_2$ and $3 = C_1 + 2C_2$, so $C_1 = -3, C_2 = 3$.
Therefore, $y = -3e^x + 3e^{2x}$.
- Repeated root $-\frac{3}{2}$. $y = (C_1 + C_2 x)e^{(-3/2)x}$
- $y_h = C_1 e^{-x} + C_2 e^x$ (Problem 2)
 $y_p = -1 + C_1 e^{-x} + C_2 e^x$
- (Second-order nonhomogeneous)
The auxiliary equation, $r^2 + 4r + 4 = 0$, has roots $-2, -2$.
 $y_h = C_1 e^{-2x} + C_2 x e^{-2x} = (C_1 + C_2 x)e^{-2x}$
Let $y_p = Be^x; y'_p = Be^x; y''_p = Be^x$.
 $(Be^x) + 4(Be^x) + 4(Be^x) = 3e^x$, so $B = \frac{1}{3}$.
General solution: $y = \frac{e^x}{3} + (C_1 + C_2 x)e^{-2x}$

- $y_h = (C_1 + C_2 x)e^{-2x}$ (Problem 12)

$$y_p = \left(\frac{1}{2}\right)x^2 e^{-2x}$$

$$y = \left[\left(\frac{1}{2}\right)x^2 + C_1 + C_2 x\right]e^{-2x}$$

8. Roots are $\pm 2i$.

$$y = C_1 \cos 2x + C_2 \sin 2x$$

$y = \sin 2x$ satisfies the conditions.

9. (Second-order homogeneous)

The auxiliary equation, $r^2 + 6r + 25 = 0$, has roots $-3 \pm 4i$. General solution:

$$y = e^{-3x}(C_1 \cos 4x + C_2 \sin 4x)$$

10. Roots are $\pm i$. $y_h = C_1 \cos x + C_2 \sin x$

$$y_p = x \cos x - \sin x + \sin x \ln |\cos x|$$

$$y = x \cos x - \sin x \ln |\cos x| + C_1 \cos x + C_2 \sin x$$

(combining the sine terms)

11. Roots are $-4, 0, 2$. $y = C_1 e^{-4x} + C_2 + C_3 e^{2x}$

12. (Fourth-order homogeneous)

The auxiliary equation, $r^4 - 3r^2 - 10 = 0$ or $(r^2 - 5)(r^2 + 2) = 0$, has roots $-\sqrt{5}, \sqrt{5}, \pm \sqrt{2}i$.

General solution:

$$y = C_1 e^{\sqrt{5}x} + C_2 e^{-\sqrt{5}x} + C_3 \cos \sqrt{2}x + C_4 \sin \sqrt{2}x$$

13. Repeated roots $\pm \sqrt{2}$

$$y = (C_1 + C_2 x)e^{-\sqrt{2}x} + (C_3 + C_4 x)e^{\sqrt{2}x}$$

14. a. $Q'(t) = 3 - 0.02Q$

- b. $Q'(t) + 0.02Q = 3$

Integrating factor is $e^{0.02t}$

$$D[Qe^{0.02t}] = 3e^{0.02t}$$

$$Q(t) = 150 + Ce^{-0.02t}$$

$$Q(t) = 150 - 30e^{-0.02t} \text{ goes through } (0, 120).$$

- c. $Q \rightarrow 150$ g, as $t \rightarrow \infty$.

15. (Simple harmonic motion)

$$k = 5; w = 10; y_0 = -1$$

$$B = \sqrt{\frac{(5)(32)}{10}} = 4$$

Then the equation of motion is $y = -\cos 4t$.

The amplitude is $|-1| = 1$; the period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

16. It is at equilibrium when $y = 0$ or $-\cos 4t = 0$, or

$$t = \frac{\pi}{8}, \frac{3\pi}{8}, \dots$$

$$y'(t) = 4 \sin 4t, \text{ so at equilibrium } |y'| = |\pm 4| = 4.$$

17. $Q'' + 2Q' + 2Q = 1$

Roots are $-1 \pm i$.

$$Q_h = e^{-t}(C_1 \cos t + C_2 \sin t) \text{ and } Q_p = \frac{1}{2};$$

$$Q = e^{-t}(C_1 \cos t + C_2 \sin t) + \frac{1}{2}$$

$$I(t) = Q'(t) = -e^{-t}[(C_1 - C_2)\cos t + (C_1 + C_2)\sin t]$$

$$I(t) = e^{-t} \sin t \text{ satisfies the initial conditions.}$$

Maple and Mathematica Code for some Technology Projects

Technology Project 12.1 Rotations in the Plane

Maple:

```
with(plots):  
f := (x,y) -> x^2 + 24*x*y + 8*y^2 - 136;  
implicitplot(f(x,y) = 0, x=-16..16, y=-16..16);
```

Mathematica:

```
<<Graphics`ImplicitPlot`  
f[x_,y_] = x^2 + 24 x y + 8 y^2 -136;  
ImplicitPlot[ f[x,y]==0 , {x,-16,16} , {y,-16,16} ]
```

Technology Project 12.1 Roses

Maple:

```
rect := (theta,r) ->[ r*cos(theta) , r*sin(theta) ] :  
n := 5:  
d := 49:  
degr := 2*Pi/360:  
rose := k -> evalf( rect( k*degr , sin(n*k*degr) ) ):  
path := [ seq ( rose(d*k) , k=1..360 ) ]:  
PLOT( CURVES(path) );
```

Mathematica:

```
rect[theta_,r_] := N[{r Cos[theta],r Sin[theta]}]  
Attributes[rose]={Listable}  
n = 5; d = 13;  
rose[k_] = rect[k Degree,Sin[n k Degree] ];  
path = Line[ rose[Range[0,360 d,d]] ];  
Show[Graphics[{Thickness[.0005],path}],AspectRatio->1]
```

Technology Project 13.2 Measuring Home Run Distance

Maple:

```
degr := 2*Pi/360:  
g := 3:  
d := 430:  
h := 80:
```

```

phi := 30*degr:
theta := arctan( tan(phi) + 2*(h-g)/d );
v0 := evalf( sqrt( 32*d / ((cos(theta))^2*(tan(phi)+tan(theta))) ) );
y := x ->-16*x^2/(v0^2 * (cos(theta))^2) + (tan(theta))*x + g;
plot(y(x), x=0..d, scaling=constrained);
HRdist := evalf(
( - tan(theta) - sqrt( (tan(theta))^2 + 64*g/(v0^2 * (cos(theta))^2) ) ) /
(-32/(v0^2 * (cos(theta))^2)) );

```

Technology Project 15.1 Newton's Method for Two Equations in Two Unknowns

Maple:

```

f := (x,y) ->x^2 + y^2 - 48:
g := (x,y) ->x + y - 6:
dfx := unapply(diff(f(x,y),x),(x,y)):
dfy := unapply(diff(f(x,y),y),(x,y)):
dgy := unapply(diff(g(x,y),x),(x,y)):
dgy := unapply(diff(g(x,y),y),(x,y)):
x1 := 2: y1 := 0.5:
for n from 1 to 5 do
  x2 := x1 - ( f(x1,y1)*dgy(x1,y1) - g(x1,y1)*dfy(x1,y1) ) /
  ( dfx(x1,y1)*dgy(x1,y1) - dgx(x1,y1)*dfy(x1,y1) ):1
  y2 := y1 - ( g(x1,y1)*dfx(x1,y1) - f(x1,y1)*dgx(x1,y1) ) /
  ( dfx(x1,y1)*dgy(x1,y1) - dgx(x1,y1)*dfy(x1,y1) ):
  x1 := x2:
  y1 := y2:
od;

```

Mathematica:

```

f[x_,y_] = x^2 + y^2 - 48;
g[x_,y_] = x + y - 6;
dfx[x_,y_] = D[f[x,y],x];
dfy[x_,y_] = D[f[x,y],y];
dgx[x_,y_] = D[g[x,y],x];
dgy[x_,y_] = D[g[x,y],y];
x1 = 2; y1 = 0.5;
Do[ x2 = x1 - ( f[x1,y1] dgy[x1,y1] - g[x1,y1] dfy[x1,y1] ) /
  ( dfx[x1,y1] dgy[x1,y1] - dgx[x1,y1] dfy[x1,y1] );
  y2 = y1 - ( g[x1,y1] dfx[x1,y1] - f[x1,y1] dgx[x1,y1] ) /
  ( dfx[x1,y1] dgy[x1,y1] - dgx[x1,y1] dfy[x1,y1] );
  x1 = x2;
  y1 = y2;

```

```
Print[x2, ' ' , y2],
{n, 1, 7} ]
```

Technology Project 15.2 Visualizing the Directional Derivative

Maple

```
with(plots):
f := (x,y) -> 3*x + 4*y + 4:
theta := -70: phi := 53:
x0 := 1: y0 := -1:
e := 0.2:
a := t -> (f(x0+e*cos(t),y0+e*sin(t))-f(x0,y0)) / e:
h := t -> f(x0+e*cos(t),y0+e*sin(t)):
p1 := plot3d( f(x,y) , x=-2..2 , y=-2..2 , axes=frame , color=grey ,
orientation=[theta,phi] ):
p2 := spacecurve( {[x0+e*cos(t),y0+e*sin(t) , h(t)]} , t=0..2*Pi ,
axes=FRAME , thickness=2 , color=black , orientation=[theta,phi]):
p3 := contourplot( f(x,y) , x=-2..2 , y=-2..2 , grid=[21,21] ,
color=black , scaling=constrained , contours=20 ):
p4 := plot( [ (x0+e*cos(t),y0+e*sin(t)) , t=0..2*Pi] ,
scaling=constrained , color=black , thickness=2 ):
display([p1,p2]);
display([p3,p4]);
plot( h(t) , t=0..2*Pi );
plot( a(t) , t=0..2*Pi );
evalf( int(a(t),t=0..2*Pi) );
```

Mathematica

```
f[x_,y_] = 3 x + 4 y + 4;
x0 = 1; y0 = -1;
e = 0.2;
a[t_] = ( f[x0+e*Cos[t],y0+e*Sin[t]] - f[x0,y0] ) / e;
h[t_] = f[x0+e*Cos[t],y0+e*Sin[t]];
p1 = Plot3D[ f[x,y] , {x,-2,2} , {y,-2,2} , AxesLabel->{'x','y','z'} ,
Boxed->True , ViewPoint ->{5, -15, 8} , DisplayFunction->Identity ];
p2 = ParametricPlot3D[ {x0+e*Cos[t] , y0+e*Sin[t] , h[t]} , {t,0,2Pi} ,
DisplayFunction->Identity ];
p3 = ContourPlot[ f[x,y] , {x,-2,2} , {y,-2,2} , PlotPoints->21 ,
ContourShading->False , DisplayFunction->Identity ];
p4 = ParametricPlot[ {x0+e*Cos[t] , y0+e*Sin[t]} , {t,0,2Pi} ,
DisplayFunction->Identity ];
Show[ {p1,p2} , DisplayFunction->$DisplayFunction ];
Show[ {p3,p4} , DisplayFunction->$DisplayFunction];
```

```

Plot[ h[t] , {t,0,2Pi} ];
Plot[ a[t] , {t,0,2Pi} ];
NIntegrate[ a[t] , {t,0,2Pi} ]

```

Technology Project 16.2 Monte Carlo Integration

Mathematica:

```

f[x_] := x (1-x) (Sin[100 x (1-x)])^2
xmin = 0; xmax = 1;
ymin = 0; ymax = 0.25;
numtrials = 1000;
p1 := Plot[f[x],{x,0,1},PlotStyle->Thickness[0.012],
  DisplayFunction->Identity]
x = Table[ Random[Real,{xmin,xmax}],{numtrials} ];
y = Table[ Random[Real,{ymin,ymax}],{numtrials} ];
data = Transpose[{x,y}];
p2 := ListPlot[data,PlotStyle->PointSize[0.012],
  DisplayFunction->Identity];
Show[p1,p2,DisplayFunction->$DisplayFunction]
numsucc = 0;
Do[If[ y[[k]] < f[x[[k]]],numsucc=numsucc+1],
  {k,1,numtrials}]
areaestimate = (xmax-xmin)(ymax-ymin)numsucc/numtrials;
Print[ areaestimate ]

```

Maple

```

with(plots):
with(stats):
with(linalg):
f := x -> x*(1-x)*sin(100*x*(1-x))^2:
xmin := 0: xmax := 1:
ymin := 0: ymax := 0.25:
numtrials := 1000:
p1 := plot( f(x) , x=xmin..xmax , y=ymin..ymax ):
randomize(Seed);
xx := [ stats[random, uniform](numtrials) ]:
yy := [ stats[random, uniform](numtrials) ]:
xx := [seq(xmin,i=1..numtrials)] + (xmax-xmin)*xx:
yy := [seq(ymin,i=1..numtrials)] + (ymax-ymin)*yy:
p2 := pointplot( transpose( [xx,yy] ) ):
display( [ p1,p2 ] );
numsucc := 0:
for i from 1 to numtrials do

```

```

if yy[i] < f(xx[i]) then numsucc := numsucc+1 fi:
od;
areaestimate := (xmax-xmin)*(ymax-ymin)*numsucc/numtrials;

```

Technology Project 17.2 Parametrized Surfaces

Maple

```

plot3d( [sin(s)*cos(t) , sin(s)*sin(t) , cos(s)] , s=0..Pi , t=0..2*Pi );

```

Mathematica

```

ParametricPlot3D[ { Sin[s]Cos[t] , Sin[s]Sin[t] , Cos[s] } ,
{ s,0,Pi} , { t,0,2 Pi} , PlotPoints->31 ]

```

Printed Test Bank

EIGHTH EDITION ***CALCULUS***

Varberg / Purcell / Rigdon

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1 Preliminaries

1. The sum of a rational number and an irrational number must be _____.

Answer: irrational

Difficulty: 2 Section: 1

2. Give an example of irrational numbers whose product is irrational and an example whose product is rational.

Answer: Varies. For the first question, $\sqrt{2}$ and $\sqrt{3}$ work, and for the second question $\sqrt{2}$ and $\sqrt{8}$ work.

Difficulty: 2 Section: 1

3. Find a rational number between $\frac{3}{4}$ and $\frac{4}{5}$.

Answer: Varies. The average is $\frac{31}{40}$.

Difficulty: 1 Section: 2

4. Change the repeating decimal $1.163636363\dots$ to a ratio of two integers.

Answer: $\frac{64}{55}$.

Difficulty: 2 Section: 2

5. Find the best decimal approximation to $\frac{\sqrt{959} - 5^{2.3}}{5^4 - \sqrt{37.912}}$ that your calculator allows.

Answer: Depends on calculator. About -0.0154299 .

Difficulty: 2 Section: 2

6. Find the solution set for the inequality $2x - 11 < 4x - 1$.

Answer: $(-5, \infty)$.

Difficulty: 1 Section: 3

7. Find the solution set for the inequality $5x - 11 < 15x - 22$.

Answer: $\left(\frac{11}{10}, \infty\right)$.

Difficulty: 1 Section: 3

8. Find the solution set for the inequality $3x - 10 > 5x + 2$.

Answer: $(-\infty, -6)$

Difficulty: 1 Section: 3

9. Find the solution set for the inequality $x^2 - 9x + 20 > 0$.

Answer: $(-\infty, 4) \cup (5, \infty)$.

Difficulty: 2 Section: 3

10. Find the solution set for the inequality $(x - 1)^2 \geq 16$.

Answer: $(-\infty, -3] \cup [5, \infty)$.

Difficulty: 2 Section: 3

11. Find the solution set for the inequality $x^2 + x - 2 < 0$.

Answer: $(-2, 1)$.

Difficulty: 2 Section: 3

12. Find the solution set for the inequality $x^2 - x - 8 < 4$.

Answer: $(-3, 4)$.

Difficulty: 2 Section: 3

13. Find the solution set for the inequality $(x - 5)(x - 2) \geq -2$.

Answer: $(-\infty, 3] \cup [4, \infty)$.

Difficulty: 2 Section: 3

14. Find the solution set for the inequality $\frac{x+1}{x-4} \leq 2$.

Answer: $(-\infty, 4) \cup [9, \infty)$.

Difficulty: 2 Section: 3

15. Find the solution set for the inequality $\frac{x+1}{x-6} < 0$.

Answer: $(-1, 6)$.

Difficulty: 2 Section: 3

16. Find the solution set for the inequality $\frac{x-4}{x+5} \leq 0$.

Answer: $(-5, 4]$.

Difficulty: 2 Section: 3

17. Find the solution set for the inequality $\frac{x+1}{4x-3} > 0$.

Answer: $(-\infty, -1) \cup \left(\frac{3}{4}, \infty\right)$.

Difficulty: 2 Section: 3

18. Find the solution set for the inequality $\frac{3x-2}{x-2} \leq 1$.

Answer: $[0, 2)$.

Difficulty: 2 Section: 3

19. Find the solution set for the inequality $\frac{(x-3)^2}{2x+1} > 0$.

Answer: $\left(-\frac{1}{2}, \infty\right)$.

Difficulty: 2 Section: 3

20. Find the solution set for the inequality $|3x+1| \leq 7$.

Answer: $\left[-\frac{8}{3}, 2\right]$.

Difficulty: 2 Section: 4

21. Find the solution set for the inequality $|10x+17| < 45$.

Answer: $\left(-\frac{31}{5}, \frac{14}{5}\right)$.

Difficulty: 2 Section: 4

22. Find the solution set for the inequality $|3x+16| < 8$.

Answer: $\left(-8, -\frac{8}{3}\right)$.

Difficulty: 2 Section: 4

23. Find the solution set for the inequality $|5x+17| < 7$.

Answer: $\left(-\frac{24}{5}, -2\right)$.

Difficulty: 2 Section: 4

24. Describe the interval $[-5, 5]$ by means of an inequality involving absolute values.

Answer: $|x| \leq 5$.

Difficulty: 1 Section: 4

25. Describe the interval $(0, 6)$ by means of an inequality involving absolute values.

Answer: $|x - 3| < 3$.

Difficulty: 1 Section: 4

26. Describe the interval $(-2, 4)$ by means of an inequality involving absolute values.

Answer: $|x - 1| < 3$.

Difficulty: 1 Section: 4

27. Find the distance between the points $(-1, 1)$ and $(2, -1)$.

Answer: $\sqrt{13}$.

Difficulty: 1 Section: 5

28. Find the distance between the points $(-4, 6)$ and $(-3, -1)$.

Answer: $5\sqrt{2}$.

Difficulty: 1 Section: 5

29. Find the distance between the points $(-5, -1)$ and $(3, 1)$.

Answer: $2\sqrt{17}$.

Difficulty: 1 Section: 5

30. Find the distance between the points $(3, 6)$ and $(1, -5)$.

Answer: $5\sqrt{5}$.

Difficulty: 1 Section: 5

31. Find the distance between the points $(1, -1)$ and $(3, 1)$.

Answer: $2\sqrt{2}$.

Difficulty: 1 Section: 5

32. Write the equation of the circle with radius 7 and center $(-3, 7)$.

Answer: $(x + 3)^2 + (y - 7)^2 = 49$.

Difficulty: 1 Section: 5

33. Write the equation of the circle with radius 1 and center $(0, 1)$.

Answer: $x^2 + (y - 1)^2 = 1$.

Difficulty: 1 Section: 5

34. Write the equation of the circle with radius 2 and center $(2, -3)$.

Answer: $(x - 2)^2 + (y + 3)^2 = 4$.

Difficulty: 1 Section: 5

35. Write the equation of the circle with radius $\frac{3}{4}$ and center $(0, 0)$.

Answer: $x^2 + y^2 = \frac{9}{16}$.

Difficulty: 1 Section: 5

36. Write the equation of the circle with radius 3 and center $(-1, 3)$.

Answer: $(x + 1)^2 + (y - 3)^2 = 9$.

Difficulty: 1 Section: 5

37. Write the equation of the circle with diameter AB for $A(-4, 2)$ and $B(2, -2)$.

Answer: $(x + 1)^2 + y^2 = 13$.

Difficulty: 2 Section: 5

38. Give the center and radius of the circle $2x^2 + 2y^2 = 4y - 6x - 6$.

Answer: $\left(-\frac{3}{2}, 1\right), \frac{1}{2}$.

Difficulty: 2 Section: 5

39. Give the center and radius of the circle $3x^2 + 3y^2 - 12x = 6 - 8y$.

Answer: $\left(2, -\frac{4}{3}\right), \frac{\sqrt{70}}{3}$.

Difficulty: 2 Section: 5

40. Give the center and radius of the circle $3x^2 + 3y^2 - 12x + 78y + 492 = 0$.

Answer: $(2, -13), 3$.

Difficulty: 2 Section: 5

41. Show that the locus of $x^2 + y^2 - 2x + 2y + 2 = 0$ is a degenerate circle.

Answer: The equation is $(x - 1)^2 + (y + 1)^2 = 0$.

Difficulty: 2 Section: 5

42. Show that the locus of $x^2 + y^2 + 2x - 2y + 7 = 0$ is empty.

Answer: The equation is $(x + 1)^2 + (y - 1)^2 = -5$.

Difficulty: 2 Section: 5

43. Write the equation of the line through the points $(0, 1)$ and $(2, 3)$.

Answer: $y = x + 1$.

Difficulty: 1 Section: 6

44. Write the equation of the line through the points $(-3, 2)$ and $(0, 3)$.

Answer: $y = \frac{1}{3}x + 3$.

Difficulty: 1 Section: 6

45. Write the equation of the line through the points $(2, -3)$ and $(1, -2)$.

Answer: $y = -x - 1$.

Difficulty: 1 Section: 6

46. Write the equation of the line through the points $(0, 3)$ and $(1, 0)$.

Answer: $y = -3x + 3$.

Difficulty: 1 Section: 6

47. What is the slope of the line with equation $2x - 3y = 5$?

Answer: $m = \frac{2}{3}$.

Difficulty: 1 Section: 6

48. What is the slope of the line with equation $\frac{x}{5} + \frac{y}{4} = 5$?

Answer: $m = -\frac{4}{5}$.

Difficulty: 1 Section: 6

49. What is the slope of the line with equation $-3x + 4y = 7$?

Answer: $m = \frac{3}{4}$.

Difficulty: 1 Section: 6

50. What is the slope of the line with equation $\frac{x}{2} + \frac{y}{3} = 2$?

Answer: $m = -\frac{3}{2}$.

Difficulty: 1 Section: 6

51. Find the equation of the vertical line passing through $(3, 2)$.

Answer: $x = 3$.

Difficulty: 1 Section: 6

52. Find the equation of the vertical line passing through $(-1, 0)$.

Answer: $x = -1$.

Difficulty: 1 Section: 6

53. Find the equation of the vertical line passing through $(-6, 7)$.

Answer: $x = -6$.

Difficulty: 1 Section: 6

54. Write an equation of the line that is parallel to $-3x - 2y + 2 = 0$ and that passes through $(4, 3)$.

Answer: $y = -\frac{3}{2}x + 9$.

Difficulty: 2 Section: 6

55. Write an equation of the line that is parallel to $2y - 3x = 0$ and that passes through $(-4, -7)$.

Answer: $y = \frac{3}{2}x - 1$.

Difficulty: 2 Section: 6

56. Write an equation of the line that is parallel to $-3x + y = 0$ and that passes through $\left(\frac{1}{3}, -2\right)$.

Answer: $y = 3x - 3$.

Difficulty: 2 Section: 6

57. Write an equation of the line that is parallel to $x - 3y = 3$ and that passes through $(6, -5)$.

Answer: $y = \frac{1}{3}x - 7$.

Difficulty: 2 Section: 6

58. Write an equation of the line perpendicular to $5x + 2y = -4$ and passing through $(9, -4)$.

Answer: $y = \frac{2x}{5} - \frac{38}{5}$.

Difficulty: 2 Section: 6

59. Write an equation of the line perpendicular to $\frac{x}{8} + \frac{y}{2} = 1$ and passing through $(-5, 4)$.

Answer: $y = 4x + 24$.

Difficulty: 2 Section: 6

60. Write an equation of the line perpendicular to $2x + 3y = 7$ and passing through $(4, 7)$.

Answer: $y = \frac{3}{2}x + 1$.

Difficulty: 2 Section: 6

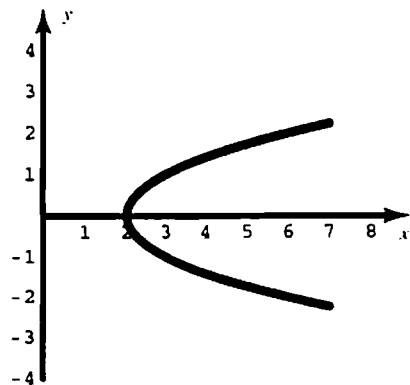
61. Write an equation of the line perpendicular to $-3x = 7y + 2$ and passing through $(3, 1)$.

Answer: $\frac{7}{3}x - 6$.

Difficulty: 2 Section: 6

62. Sketch the graph of $x = 2 + y^2$.

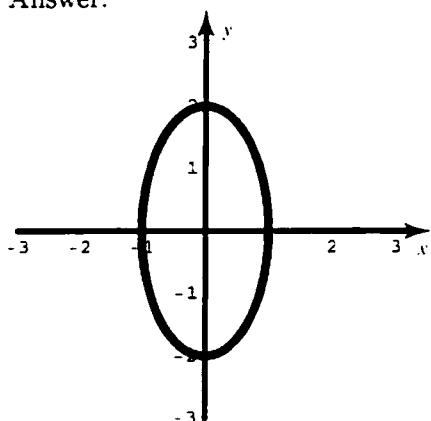
Answer:



Difficulty: 1 Section: 7

63. Sketch the graph of $4x^2 + y^2 = 4$.

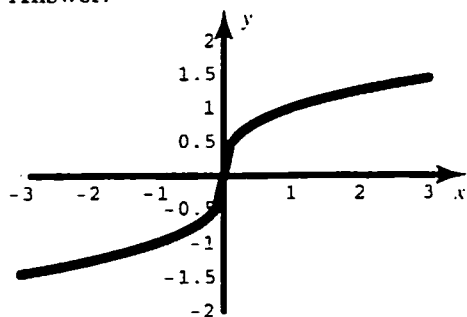
Answer:



Difficulty: 2 Section: 7

64. Sketch the graph of $y = \sqrt[3]{x}$.

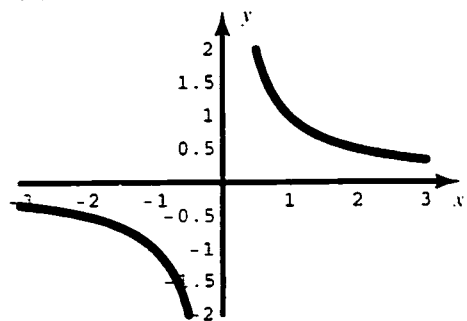
Answer:



Difficulty: 2 Section: 7

65. Sketch the graph of $y = \frac{1}{x}$.

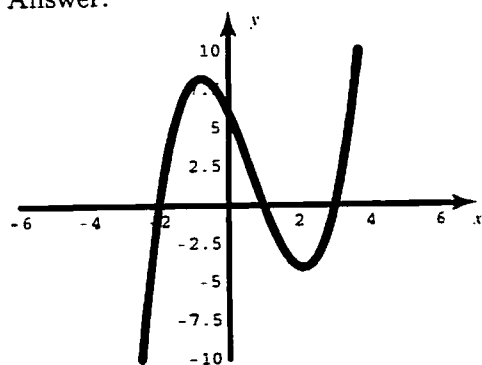
Answer:



Difficulty: 2 Section: 7

66. Sketch the graph of $y = (x - 1)(x + 2)(x - 3)$.

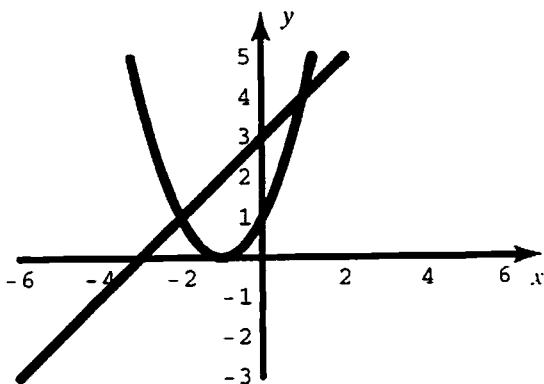
Answer:



Difficulty: 3 Section: 7

67. Sketch the graphs of $y = x^2 + 2x + 1$ and $y = x + 3$ on the same coordinate plane and give the points of intersection.

Answer: Points are $(-2, 1)$ and $(1, 4)$.



Difficulty: 2 Section: 7

2 Functions and Limits

1. If $f(x) = 2x^2 - 3x - 4$, find $\frac{f(5) - f(3)}{2}$.

Answer: 13

Difficulty: 1 Section: 1

2. If $h(x) = 3x^2 + 4x - 7$, find $\frac{h(6) - h(4)}{4}$.

Answer: 17

Difficulty: 1 Section: 1

3. If $f(x) = 5x^2 + 2x - 10$, find $\frac{f(6) - f(3)}{4}$.

Answer: 35.25

Difficulty: 1 Section: 1

4. Find the natural domain of the function $f(x) = \frac{3x+1}{2x-6}$.

Answer: All x except $x = 3$

Difficulty: 1 Section: 1

5. Find the natural domain of the function $f(x) = \frac{2x+1}{3x+2}$.

Answer: All x except $x = -\frac{2}{3}$

Difficulty: 1 Section: 1

6. Find the natural domain of the function $f(x) = \frac{x^2+5}{9-x^2}$.

Answer: All x except $x = \pm 3$

Difficulty: 1 Section: 1

7. Find the natural domain of the function $f(x) = \frac{x+5}{2x^2-8}$.

Answer: All x except $x = \pm 2$

Difficulty: 1 Section: 1

8. Find the natural domain of the function $g(x) = \sqrt{5-10x}$.

Answer: All x with $x \leq \frac{1}{2}$

Difficulty: 2 Section: 1

9. Find the natural domain of the function $I(x) = \frac{4x}{\sqrt{2x+8}}$.

Answer: All x with $x > -4$

Difficulty: 2 Section: 1

10. Find the natural domain of the function $f(x) = \frac{1}{\sqrt{x^2+1}}$.

Answer: All real numbers

Difficulty: 2 Section: 1

11. Find the range of the function $f(x) = \frac{1}{x-4}$.

Answer: All y with $y \neq 0$

Difficulty: 1 Section: 1

12. Find the range of the function $g(x) = \sqrt{2x-10}$.

Answer: All y with $y \geq 0$

Difficulty: 1 Section: 1

13. Find the range of the function $h(x) = 4 + (x-7)^2$.

Answer: All y with $y \geq 4$

Difficulty: 2 Section: 1

14. Find the range of the function $g(x) = \sqrt{3x+2}$.

Answer: All y with $y \geq 0$

Difficulty: 2 Section: 1

15. Find the domain and range of the function

$$f(t) = \begin{cases} -1 & \text{for } -3 < t < 0 \\ t & \text{for } t \geq 0 \end{cases}$$

Answer: The domain is $(-3, \infty)$ and the range is $\{-1\} \cup [0, \infty)$

Difficulty: 2 Section: 1

16. Find the domain and range of the function

$$f(\theta) = \begin{cases} 2 \sin \theta & \text{for } -\pi < \theta \leq 2\pi \\ \cos \theta & \text{for } 2\pi < \theta < 4\pi \end{cases}$$

Answer: The domain is $(-\pi, 4\pi)$ and the range is $[-2, 2]$.

Difficulty: 3 Section: 1

17. Find the domain and range of the function

$$g(s) = \begin{cases} \sqrt{s} & \text{for } s \geq 4 \\ 0 & \text{for } 0 \leq s < 4 \end{cases}$$

Answer: The domain is $[0, \infty)$ and the range is $\{0\} \cup [2, \infty)$.

Difficulty: 2 Section: 1

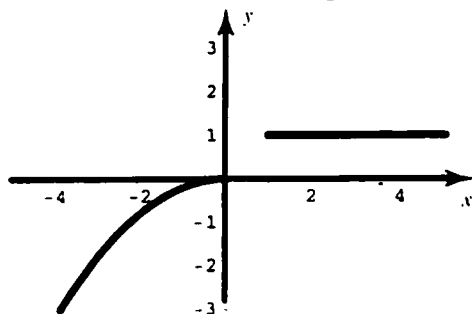
18. Find the domain and range of the function

$$h(x) = \begin{cases} 2 & \text{for } -1 \leq x < 0 \\ 1 & \text{for } 0 \leq x < 2 \\ 6 - x^2 & \text{for } x \geq 2 \end{cases}$$

Answer: The domain is $[-1, \infty)$ and the range is $(-\infty, 2]$.

Difficulty: 2 Section: 1

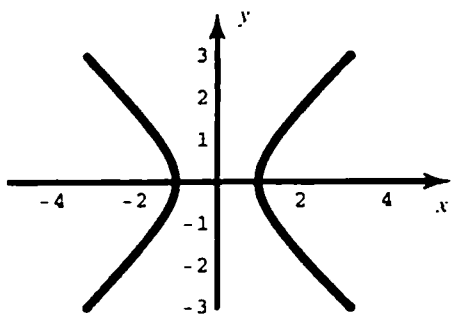
19. Determine whether the graph shown is the graph of a function.



Answer: Yes

Difficulty: 1 Section: 1

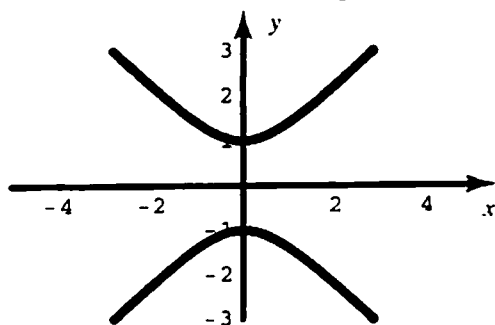
20. Determine whether the graph shown is the graph of a function.



Answer: No

Difficulty: 1 Section: 1

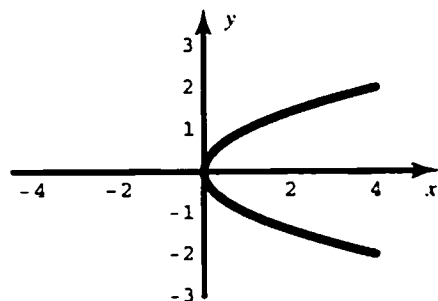
21. Determine whether the graph shown is the graph of a function.



Answer: No

Difficulty: 1 Section: 1

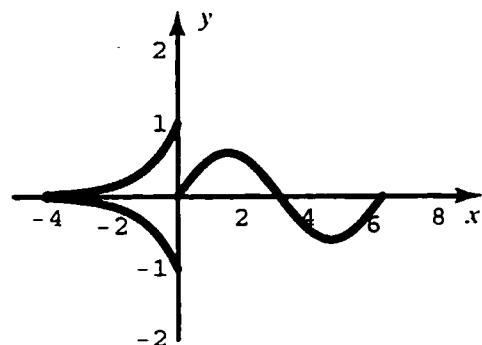
22. Determine whether the graph shown is the graph of a function.



Answer: No

Difficulty: 1 Section: 1

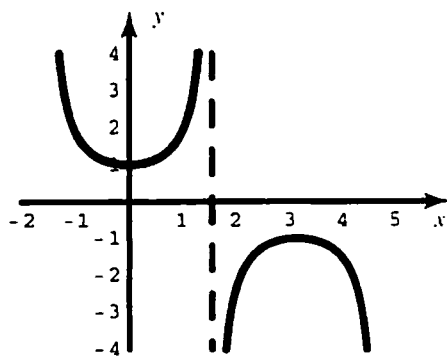
23. Determine whether the graph shown is the graph of a function.



Answer: No

Difficulty: 1 Section: 1

24. Determine whether the graph shown is the graph of a function.



Answer: Yes

Difficulty: 1 Section: 1

25. Determine whether the function $y = \sqrt[3]{x}$ is even, odd or neither.

Answer: Odd

Difficulty: 1 Section: 1

26. Determine whether the function $f(x) = \sqrt{x^2} + 9$ is even, odd, or neither.

Answer: Even

Difficulty: 2 Section: 1

27. Determine whether the function $f(x) = |2x - 5|$ is even, odd or neither.

Answer: Neither

Difficulty: 2 Section: 1

28. Let $f(x) = \frac{1}{x-1}$. Find and simplify $\frac{f(x+h) - f(x)}{h}$.

Answer: $\frac{-1}{(x+h-1)(x-1)}$

Difficulty: 2 Section: 1

29. If $F(x) = \sqrt{x-1}$ and $G(x) = 3x + 2$, give a formula for $(F \circ G)(x)$.

Answer: $\sqrt{3x+1}$

Difficulty: 2 Section: 2

30. If $F(x) = 2x + 5$ and $G(x) = x^2 - 3x$, give a formula for $(G \circ F)(x)$.

Answer: $4x^2 + 14x + 10$

Difficulty: 2 Section: 2

31. If $F(x) = \sqrt{x^2 + 1}$ and $G(x) = 3x - 2$, give a formula for $(F \circ G)(x)$.

Answer: $\sqrt{9x^2 - 12x + 5}$

Difficulty: 2 Section: 2

32. If $F(x) = |x^2 - x|$ and $G(x) = 3x - 1$, give a formula for $(F \circ G)(x)$.

Answer: $|9x^2 - 9x + 2|$

Difficulty: 2 Section: 2

33. If $F(x) = \frac{x}{x+2}$ and $G(x) = \frac{3x}{x+1}$, give a formula for $(F \circ G)(x)$.

Answer: $\frac{3x}{5x+2}$

Difficulty: 3 Section: 2

34. If $F(x) = \frac{x}{x-1}$ and $G(x) = 2x + 1$, give a formula for $(F \circ G)(x)$.

Answer: $\frac{2x+1}{2x}$

Difficulty: 2 Section: 2

35. If $F(x) = 3x + 5$ and $G(x) = \frac{x}{x+2}$, give a formula for $(F \circ G)(x)$.

Answer: $\frac{8x+10}{x+2}$

Difficulty: 2 Section: 2

36. If $F(x) = \frac{2x}{x+3}$ and $G(x) = \frac{x}{x-1}$, give a formula for $(F \circ G)(x)$.

Answer: $\frac{2x}{4x-3}$

Difficulty: 3 Section: 2

37. Let $f(x) = 2x - 3$ and $g(x) = x^2 - 3x - 28$. Write a formula for $(f + g)(x)$ and $(f \cdot g)(x)$ and state the domain of each.

Answer: $(f + g)(x) = x^2 - x - 31$ and $(f \cdot g)(x) = 2x^3 - 9x^2 - 47x + 84$.

Domain consists of all real numbers in both cases.

Difficulty: 2 Section: 2

38. Let $f(x) = \sqrt{2x-3}$ and $g(x) = \sqrt{3x+1}$. Write a formula for $(f+g)(x)$ and $(f \cdot g)(x)$ and state the domain of each.

Answer: $(f+g)(x) = \sqrt{2x-3} + \sqrt{3x+1}$ and $(f \cdot g)(x) = \sqrt{6x^2 - 7x - 3}$.

Domain consists of $\left\{x : x \geq \frac{3}{2}\right\}$ in both cases.

Difficulty: 2 Section: 2

39. Let $f(x) = x - 1$ and $g(x) = x^2 - 1$. Write a formula for $(f \cdot g)(x)$ and $(f/g)(x)$ and state the domain of each.

Answer: $(f \cdot g)(x) = x^3 - x^2 - x + 1$ and $(f/g)(x) = \frac{1}{x+1}$

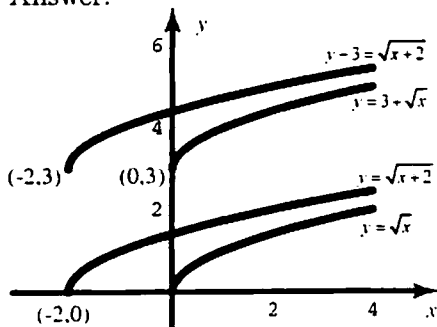
Domain consists of all reals in the first case and all reals except $x = \pm 1$ in the second case.

Difficulty: 2 Section: 2

40. Graph the following functions of the same coordinate axes:

$$y = \sqrt{x}, y = \sqrt{x+2}, y = 3 + \sqrt{x}, y - 3 = \sqrt{x+2}$$

Answer:

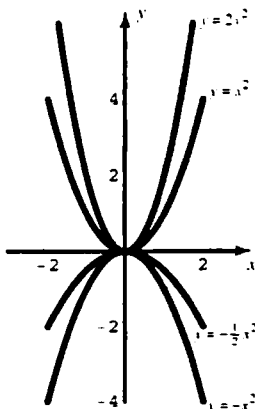


Difficulty: 2 Section: 2

41. Graph the following functions of the same coordinate axes:

$$y = x^2, y = 2x^2, y = -x^2, y = -\frac{1}{2}x^2$$

Answer:



Difficulty: 2 Section: 2

42. Convert 75° to radian measure.

Answer: $\frac{5\pi}{12}$

Difficulty: 1 Section: 3

43. Convert 20° to radian measure.

Answer: $\frac{\pi}{9}$

Difficulty: 1 Section: 3

44. Convert 40° to radian measure.

Answer: $\frac{2\pi}{9}$

Difficulty: 1 Section: 3

45. Convert 220° to radian measure.

Answer: $\frac{11\pi}{9}$

Difficulty: 1 Section: 3

46. Convert 320° to radian measure.

Answer: $\frac{16\pi}{9}$

Difficulty: 1 Section: 3

47. Convert 14° to radian measure.

Answer: $\frac{7\pi}{90}$

Difficulty: 1 Section: 3

48. Convert 100° to radian measure.

Answer: $\frac{5\pi}{9}$

Difficulty: 1 Section: 3

49. Convert $-\frac{5\pi}{3}$ to degree measure.

Answer: -300°

Difficulty: 1 Section: 3

50. Convert $\frac{5\pi}{2}$ to degree measure.

Answer: 450°

Difficulty: 1 Section: 3

51. Convert $-\frac{2\pi}{3}$ to degree measure.

Answer: -120°

Difficulty: 1 Section: 3

52. Convert $\frac{4\pi}{9}$ to degree measure.

Answer: 80°

Difficulty: 1 Section: 3

53. Convert $\frac{5\pi}{18}$ to degree measure.

Answer: 50°

Difficulty: 1 Section: 3

54. Convert $-\frac{11\pi}{9}$ to degree measure.

Answer: -220°

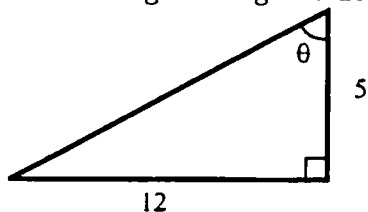
Difficulty: 1 Section: 3

55. Convert $-\frac{4\pi}{3}$ to degree measure.

Answer: -240°

Difficulty: 1 Section: 3

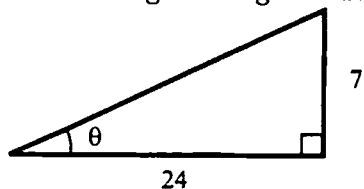
56. Use the right triangle to determine $\sin \theta$, $\cos \theta$, and $\tan \theta$.



Answer: $\sin \theta = \frac{12}{13}$, $\cos \theta = \frac{5}{13}$, $\tan \theta = \frac{12}{5}$

Difficulty: 2 Section: 3

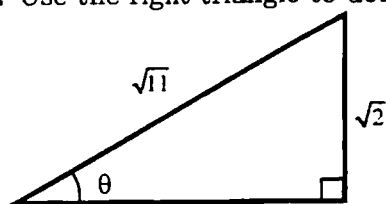
57. Use the right triangle to determine $\sin \theta$, $\cos \theta$, and $\tan \theta$.



Answer: $\sin \theta = \frac{7}{25}$, $\cos \theta = \frac{24}{25}$, $\tan \theta = \frac{7}{24}$

Difficulty: 2 Section: 3

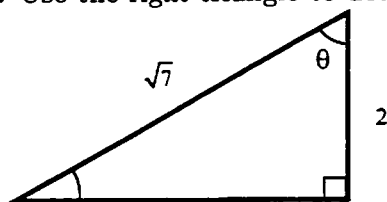
58. Use the right triangle to determine $\sin \theta$, $\cos \theta$, and $\tan \theta$.



Answer: $\sin \theta = \frac{\sqrt{2}}{\sqrt{11}}$, $\cos \theta = \frac{3}{\sqrt{11}}$, $\tan \theta = \frac{\sqrt{2}}{3}$

Difficulty: 2 Section: 3

59. Use the right triangle to determine $\sin \theta$, $\cos \theta$, and $\tan \theta$.



Answer: $\sin \theta = \frac{2}{\sqrt{7}}$, $\cos \theta = \frac{\sqrt{3}}{\sqrt{7}}$, $\tan \theta = \frac{2}{\sqrt{3}}$

Difficulty: 2 Section: 3

60. Find the value of $\sin^2\left(\frac{\pi}{5}\right) + \cos^2\left(\frac{\pi}{5}\right)$.

Answer: 1

Difficulty: 1 Section: 3

61. Evaluate $\sin t$, $\cos t$, and $\tan t$ for $t = \frac{\pi}{3}$.

Answer: $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$

Difficulty: 2 Section: 3

62. Evaluate $\sin t$, $\cos t$, and $\tan t$ for $t = \pi$.

Answer: 0, -1, 0

Difficulty: 2 Section: 3

63. Evaluate $\sin t$, $\cos t$, and $\tan t$ for $t = -\frac{\pi}{4}$.

Answer: $-\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}$, -1

Difficulty: 2 Section: 3

64. Find the value of $\csc t$ if $\sin t = -\frac{4}{5}$.

Answer: $-\frac{5}{4}$

Difficulty: 1 Section: 3

65. Verify that $\sin t = \sqrt{1 - \cos^2 t}$ is not an identity.

Answer: Varies. $t = \frac{3\pi}{2}$ is a counterexample.

Difficulty: 2 Section: 3

66. Find the signs of $\sin t$ and $\cos t$ if $t = 7$.

Answer: Both are positive.

Difficulty: 2 Section: 3

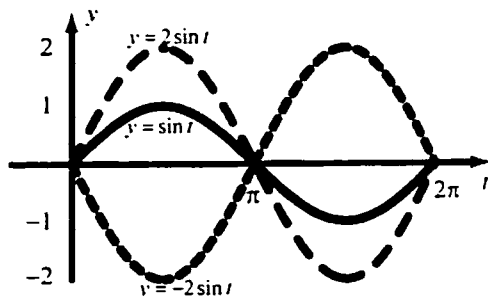
67. Find the signs of $\sin t$ and $\cos t$ if $t = -\frac{7\pi}{6}$.

Answer: $\sin t > 0$ and $\cos t < 0$.

Difficulty: 2 Section: 3

68. Sketch the graphs of $y = \sin t$, $y = 2 \sin t$, and $y = -2 \sin t$ on $[0, 2\pi]$

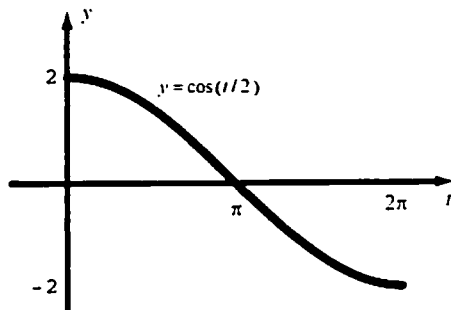
Answer:



Difficulty: 2 Section: 3

69. Sketch the graph of $2 \cos \frac{1}{2}t$ on $[0, 2\pi]$.

Answer:



Difficulty: 3 Section: 3

70. Let $f(x) = 3 - x^3$. Evaluate the limit $\lim_{x \rightarrow -2} f(x)$.

Answer: 11

Difficulty: 1 Section: 4

71. Let $g(x) = \frac{x-1}{x-5}$. Evaluate the limit $\lim_{x \rightarrow 5} g(x)$.

Answer: The limit does not exist.

Difficulty: 1 Section: 4

72. Let $s(x) = |x - 5|$. Evaluate the limit $\lim_{x \rightarrow -5} s(x)$.

Answer: 10

Difficulty: 1 Section: 4

73. Let $g(x) = \frac{x^2 - 4}{x + 2}$. Evaluate the limit $\lim_{x \rightarrow -2} g(x)$.

Answer: -4

Difficulty: 2 Section: 4

74. Let $t(x) = \cos 3x$. Evaluate the limit $\lim_{x \rightarrow \pi/3} t(x)$.

Answer: -1

Difficulty: 2 Section: 4

75. Let $f(x) = \sqrt{6 - x}$. Evaluate the limit $\lim_{x \rightarrow 3} f(x)$.

Answer: $\sqrt{3}$

Difficulty: 1 Section: 4

76. Let $f(x) = \frac{2+x}{\cos x}$. Evaluate the limit $\lim_{x \rightarrow \pi/2} f(x)$.

Answer: The limit does not exist.

Difficulty: 2 Section: 4

77. Let $g(x) = \frac{1}{|x-1|+1}$. Evaluate the limit $\lim_{x \rightarrow -1} g(x)$.

Answer: $\frac{1}{3}$

Difficulty: 2 Section: 4

78. Let $f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ \sqrt{x} & \text{for } x > 1 \end{cases}$. Evaluate the limit $\lim_{x \rightarrow 1} f(x)$.

Answer: 1

Difficulty: 2 Section: 4

79. Let $f(x) = \begin{cases} \sqrt{x-1} & \text{for } x \geq 1 \\ x & \text{for } x < 1 \end{cases}$. Evaluate the limit $\lim_{x \rightarrow 1} f(x)$.

Answer: The limit does not exist.

Difficulty: 2 Section: 4

80. Let $g(x) = \frac{|x|}{x}$. Evaluate the limit $\lim_{x \rightarrow -3} g(x)$.

Answer: -1

Difficulty: 2 Section: 4

81. Let $g(x) = \frac{|x|}{x}$. Evaluate the limit $\lim_{x \rightarrow 0} g(x)$.

Answer: The limit does not exist.

Difficulty: 2 Section: 4

82. Evaluate $\lim_{x \rightarrow -5^+} \sqrt{x+5}$.

Answer: 0

Difficulty: 2 Section: 4

83. Evaluate $\lim_{x \rightarrow -5^-} \sqrt{x+5}$.

Answer: The left limit does not exist.

Difficulty: 2 Section: 4

84. Evaluate $\lim_{x \rightarrow 0^+} \frac{|x|}{x}$.

Answer: 1

Difficulty: 2 Section: 4

85. Evaluate $\lim_{x \rightarrow 0^-} \frac{|x|}{x}$.

Answer: -1

Difficulty: 2 Section: 4

86. Evaluate $\lim_{x \rightarrow 0^+} \frac{1 - \sqrt{x}}{1 - x}$.

Answer: 1

Difficulty: 2 Section: 4

87. Evaluate $\lim_{x \rightarrow 0^-} \frac{1 - \sqrt{x}}{1 - x}$.

Answer: The left limit does not exist.

Difficulty: 2 Section: 4

88. Use a calculator to evaluate $\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x}$.

Answer: $\frac{2}{3}$

Difficulty: 2 Section: 4

89. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . Evaluate $\lim_{x \rightarrow -2^+} \lfloor x \rfloor$.

Answer: -2

Difficulty: 2 Section: 4

90. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . Evaluate $\lim_{x \rightarrow -2^-} \lfloor x \rfloor$.

Answer: -3

Difficulty: 2 Section: 4

91. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . Evaluate $\lim_{x \rightarrow -3^-} \frac{\lfloor x \rfloor}{x}$.

Answer: $\frac{4}{3}$

Difficulty: 2 Section: 4

92. Give an ε, δ proof that $\lim_{x \rightarrow 3} x^2 = 9$.

Answer: Choose $\delta = \min\left(1, \frac{\varepsilon}{7}\right)$. If $0 < |x - 3| < \delta$ then
 $|x^2 - 9| = |x + 3| |x - 3| < 7 |x - 3| < 7 \left(\frac{\varepsilon}{7}\right) = \varepsilon.$

Difficulty: 2 Section: 5

93. Give an ε, δ proof that $\lim_{x \rightarrow 3} x^3 = 27$.

Answer: Choose $\delta = \min\left(1, \frac{\varepsilon}{37}\right)$. If $0 < |x - 3| < \delta$ then
 $|x^3 - 27| = |x^2 + 3x + 9| |x - 3| < 37 |x - 3| < 37 \left(\frac{\varepsilon}{37}\right) = \varepsilon.$

Difficulty: 2 Section: 5

94. Give an ε, δ proof that $\lim_{x \rightarrow 2} (3x + 2) = 8$.

Answer: Choose $\delta = \frac{\varepsilon}{3}$.

If $0 < |x - 2| < \delta$ then $|(3x + 2) - 8| = |3x - 6| = 3|x - 2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$.

Difficulty: 2 Section: 5

95. Give an ε, δ proof that $\lim_{x \rightarrow 3} (2 - 5x) = -13$.

Answer: Choose $\delta = \frac{\varepsilon}{5}$. If $0 < |x - 3| < \delta$ then

$|(2 - 5x) - (-13)| = |15 - 5x| = 5|3 - x| = 5|x - 3| < 5\left(\frac{\varepsilon}{5}\right) = \varepsilon$.

Difficulty: 2 Section: 5

96. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . Give an ε, δ argument that $\lim_{x \rightarrow 0} \lfloor x \rfloor$ does not exist.

Answer: Suppose that $\lim_{x \rightarrow 0} \lfloor x \rfloor = L$. Let $\varepsilon = \frac{1}{2}$. Then there is a $\delta > 0$ such that $|\lfloor x \rfloor - L| < \frac{1}{2}$ whenever $0 < |x| < \delta$. If $L \geq 0$, $x = -\frac{\delta}{2}$ violates this inequality and if $L < 0$, $x = \frac{\delta}{2}$ violates the inequality.

Difficulty: 3 Section: 5

97. Let $\varepsilon > 0$ be given. Find the largest δ that will “work” for $\lim_{x \rightarrow 4} 3x = 12$.

Answer: $\frac{\varepsilon}{3}$

Difficulty: 1 Section: 5

98. Let $\varepsilon > 0$ be given. Find the largest δ that will “work” for $\lim_{x \rightarrow 6} \frac{1}{3}x = 2$.

Answer: 3ε

Difficulty: 1 Section: 5

99. Evaluate $\lim_{x \rightarrow -1} \frac{81 + x^4}{9 + x}$.

Answer: $\frac{41}{4}$

Difficulty: 2 Section: 6

100. Evaluate $\lim_{x \rightarrow -9} \frac{81 + x^4}{9 + x}$.

Answer: The limit does not exist.

Difficulty: 2 Section: 6

101. Let $f(x) = \begin{cases} 2x + c & \text{for } x < 2 \\ (x - 5)^2 & \text{for } x \geq 2 \end{cases}$.
What is the value of c if $\lim_{x \rightarrow 2} f(x) = 9$?

Answer: 5

Difficulty: 2 Section: 6

102. Let $f(x) = 2x - 3$. Determine $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$.

Answer: 2

Difficulty: 3 Section: 6

103. Let $f(x) = \sqrt{x}$. Find $\lim_{h \rightarrow 0} \frac{f(9+h) - f(9)}{h}$.

Answer: $\frac{1}{6}$

Difficulty: 3 Section: 6

104. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . Evaluate $\lim_{x \rightarrow -1/2} \lfloor x \rfloor$.

Answer: -1

Difficulty: 2 Section: 6

105. If $\lim_{x \rightarrow a} f(x) = M$ and $\lim_{x \rightarrow a} g(x) = N$, find

$$\lim_{x \rightarrow a} \frac{2[f(x)]^2 - g(x)}{1 + f(x)g(x)} \text{ if } MN \neq -1.$$

Answer: $\frac{2M^2 - N}{1 + MN}$

Difficulty: 2 Section: 6

106. Let $f(x) = \begin{cases} x^3 - 1 & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$. Evaluate $\lim_{x \rightarrow 1^-} f(x)$.

Answer: 0

Difficulty: 2 Section: 6

107. Evaluate $\lim_{x \rightarrow 0^-} (2 - x) \cos x$.

Answer: 2

Difficulty: 2 Section: 6

108. Evaluate: $\lim_{x \rightarrow 4^+} \sqrt{x^2 - 3x}$.

Answer: 2

Difficulty: 2 Section: 6

109. Evaluate $\lim_{x \rightarrow 3^-} \sqrt{x^2 - 3x}$.

Answer: The left limit does not exist.

Difficulty: 2 Section: 6

110. Find the value of k if $\lim_{x \rightarrow 0^-} \left(k + \frac{|x|}{x} \right) = 7$.

Answer: 8

Difficulty: 2 Section: 6

111. Evaluate $\lim_{x \rightarrow -1} \frac{x^4 + x^3 + x^2 + x + 1}{(1 - x)^2}$.

Answer: $\frac{1}{4}$

Difficulty: 2 Section: 6

112. Evaluate $\lim_{x \rightarrow 0} x \cot x$.

Answer: 1

Difficulty: 2 Section: 7

113. Evaluate $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 3x}$.

Answer: $\frac{4}{3}$

Difficulty: 2 Section: 7

114. Evaluate $\lim_{x \rightarrow \pi/6} \frac{\sin 3x}{\cos 2x}$.

Answer: 2

Difficulty: 2 Section: 7

115. Evaluate $\lim_{x \rightarrow 0} \frac{x + \sin(x^2)}{x}$.

Answer: 1

Difficulty: 2 Section: 7

116. Evaluate $\lim_{x \rightarrow 0} \frac{x}{1 - \cos x}$.

Answer: The limit does not exist.

Difficulty: 3 Section: 7

117. Evaluate $\lim_{x \rightarrow 0} \frac{x}{\tan 3x}$.

Answer: $\frac{1}{3}$

Difficulty: 2 Section: 7

118. Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos x}{\pi x}$.

Answer: 0

Difficulty: 2 Section: 7

119. Evaluate $\lim_{x \rightarrow 0^+} \frac{x^2 - 5x}{\tan x}$.

Answer: -5

Difficulty: 2 Section: 7

120. Let $f(x) = \begin{cases} \frac{\sin x}{2x} & \text{for } x > 0 \\ \cos 2x & \text{for } x \leq 0 \end{cases}$. Evaluate $\lim_{x \rightarrow 0^+} f(x)$.

Answer: $\frac{1}{2}$

Difficulty: 2 Section: 7

121. Let $f(x) = \begin{cases} \frac{\sin x}{2x} & \text{for } x > 0 \\ \cos 2x & \text{for } x \leq 0 \end{cases}$. Evaluate $\lim_{x \rightarrow 0^-} f(x)$.

Answer: 1

Difficulty: 2 Section: 7

122. Evaluate $\lim_{x \rightarrow \pi} [\tan x - \pi]$.

Answer: $-\pi$

Difficulty: 2 Section: 7

123. Evaluate $\lim_{x \rightarrow -\pi/4} \tan\left(x - \frac{\pi}{2}\right)$.

Answer: 1

Difficulty: 2 Section: 7

124. Evaluate $\lim_{x \rightarrow \pi/2} \frac{\sin x}{x}$

Answer: $\frac{2}{\pi}$

Difficulty: 1 Section: 7

125. Evaluate $\lim_{x \rightarrow \infty} \frac{7x^3 - 4x + 15}{3x^3 - 14x}$.

Answer: $\frac{7}{3}$

Difficulty: 1 Section: 8

127. Evaluate $\lim_{x \rightarrow \infty} \frac{2x + 3}{2 - 3x}$.

Answer: $-\frac{2}{3}$

Difficulty: 1 Section: 8

128. Evaluate $\lim_{x \rightarrow -\infty} \frac{4x^2 - 7x - 5}{13x + 6}$.

Answer: $-\infty$

Difficulty: 2 Section: 8

129. Evaluate $\lim_{x \rightarrow 4^+} \left(\frac{-2}{x - 4} + 2 \right)$.

Answer: $-\infty$

Difficulty: 2 Section: 8

130. Evaluate $\lim_{x \rightarrow 4^-} \left(\frac{-2}{x - 4} + 2 \right)$.

Answer: ∞

Difficulty: 2 Section: 8

131. Evaluate $\lim_{x \rightarrow 1^-} \left[\frac{x - 4}{-2(x^2 - 8x + 7)} \right]$.

Answer: ∞

Difficulty: 2 Section: 8

132. Evaluate $\lim_{x \rightarrow 7^+} \left[\frac{x - 4}{-2(x^2 - 8x + 7)} \right]$.

Answer: $-\infty$

Difficulty: 2 Section: 8

133. Determine all points at which $f(x) = \frac{1}{x^2 + 1}$ is continuous.

Answer: All real numbers

Difficulty: 2 Section: 9

134. Determine all points at which $f(x) = \begin{cases} \frac{\sin 2x}{2x} & \text{for } x \neq 0 \\ 2 & \text{for } x = 0 \end{cases}$ is continuous.

Answer: All real numbers except 0

Difficulty: 3 Section: 9

135. Determine all points at which $f(x) = \sqrt{x - 5}$ is continuous.

Answer: $x \geq 5$

Difficulty: 2 Section: 9

136. Determine all points at which $f(x) = \frac{x - 2}{|x - 2|}$ is continuous.

Answer: Continuous at every point in the domain. Note that 2 is not in the domain of f .

Difficulty: 2 Section: 9

137. Determine all points at which $f(x) = \begin{cases} \sqrt{x} & \text{for } x \geq 0 \\ x^2 - x & \text{for } x < 0 \end{cases}$ is continuous.

Answer: All real numbers

Difficulty: 2 Section: 9

138. Determine all points at which $f(x) = \begin{cases} \frac{2x^2 + 5x}{x} & \text{for } x \neq 0 \\ 5 & \text{for } x = 0 \end{cases}$ is continuous.

Answer: All real numbers

Difficulty: 2 Section: 9

139. Determine all points at which $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{for } x \neq 3 \\ 3 & \text{for } x = 3 \end{cases}$ is continuous.

Answer: All real numbers except 3.

Difficulty: 2 Section: 9

140. Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$. Determine all points at which f is continuous.

Answer: Continuous at all real numbers that are not integers.

Difficulty: 2 Section: 9

141. The function $f(x) = \frac{x^3 - 1}{x - 1}$ is continuous at every point of the domain. Define $f(1)$ so that f is continuous at every real number.

Answer: $f(1) = 3$

Difficulty: 2 Section: 9

142. The function $g(x) = \frac{\sin x}{2x}$ is continuous at every point of its domain. Define $g(0)$ so that g is continuous at every real number.

Answer: $g(0) = \frac{1}{2}$

Difficulty: 3 Section: 9

143. Let $f(x) = x^3 + x - 1$. Show that $f(x) = 0$ for some x in the interval $[0, 1]$.

Answer: $f(0) = -1$, $f(1) = 1$. Use the Intermediate Value Theorem.

Difficulty: 2 Section: 9

144. Let $f(x) = 2 \sin(x - 1) + 1$. Show that $f(x) = 0$ for some x in the interval $\left[0, \frac{\pi}{2}\right]$.

Answer: $f(0) < 0$, $f\left(\frac{\pi}{2}\right) > 0$. Use the Intermediate Value Theorem.

Difficulty: 2 Section: 9

3 The Derivative

1. For the position function $s = t^2 - 1$, calculate the instantaneous velocity at time $t_0 = 4$.

Answer: 8

Difficulty: 1 Section: 1

2. For the position function $s = 7t - 2$, calculate the instantaneous velocity at time $t_0 = 9$.

Answer: 7

Difficulty: 1 Section: 1

3. Find the equation of the tangent line to the graph of $y = \frac{3}{x}$ at the point where $x = -2$.

$$\text{Answer: } y = -\frac{3}{4}(x + 2) - \frac{3}{2}$$

Difficulty: 2 Section: 1

4. Find the equation of the tangent line to the graph of $y = \sqrt{x}$ at the point where $x = 9$.

$$\text{Answer: } y = \frac{1}{6}(x - 9) + 3$$

Difficulty: 2 Section: 1

5. Let $f(x) = -5x^2$. Use the limit definition of derivative to compute $f'(x)$ and then find $f'(-5)$.

$$\text{Answer: } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = -10x; 50.$$

Difficulty: 2 Section: 2

6. Let $f(x) = -\frac{1}{x}$. Use the limit definition of derivative to compute $f'(x)$ and then find $f'(-7)$.

$$\text{Answer: } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{x^2}; \frac{1}{49}$$

Difficulty: 2 Section: 2

7. Use the limit definition of derivative to find $p'(q)$ if $p = 1 - q^2$.

$$\text{Answer: } p'(q) = \lim_{h \rightarrow 0} \frac{[1 - (q+h)^2] - [1 - q^2]}{h} = -2q.$$

Difficulty: 2 Section: 2

8. Use the limit definition of derivative to find $v'(t)$ if $v = t - t^3$.

$$\text{Answer: } v'(t) = \lim_{h \rightarrow 0} \frac{[(t+h) - (t+h)^3] - [t - t^3]}{h} = 1 - 3t^2.$$

Difficulty: 2 Section: 2

9. Let $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$. Prove that $f(x)$ is not differentiable at $x = 0$.

$$\text{Answer: } \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 2; \quad \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = 0.$$

Difficulty: 2 Section: 2

10. Examine the graph of $f(x) = |5x + 12|$ and determine the points (if any) where $f'(x)$ does not exist.

$$\text{Answer: } -\frac{12}{5}$$

Difficulty: 2 Section: 2

11. Examine the graph of $f(x) = [x]$, where $[x] = \text{greatest integer } \leq x$, and determine the points (if any) where $f'(x)$ does not exist.

Answer: The derivative does not exist at any integer value of x .

Difficulty: 2 Section: 2

12. Give an example of a function $f(x)$, defined for all real numbers, such that $f'(x) = 0$ for all x .

Answer: $f(x) = c$ for any constant c .

Difficulty: 2 Section: 2

13. Find $D_x y$ if $y = 5x^5 - 2x^{-1}$.

$$\text{Answer: } 25x^4 + 2x^{-2}$$

Difficulty: 1 Section: 3

14. Find $D_x y$ if $y = x^4 - x - x^{-2}$.

$$\text{Answer: } 4x^3 - 1 + 2x^{-3}$$

Difficulty: 1 Section: 3

15. Find $D_x y$ if $y = \sqrt[3]{x} + x - 10$.

$$\text{Answer: } \frac{1}{3}x^{-2/3} + 1$$

Difficulty: 2 Section: 3

16. Let $f(t) = (2t^5 - t)(t^3 + 1)$. Compute $f'(t)$ and simplify.

Answer: $16t^7 + 10t^4 - 4t^3 - 1$

Difficulty: 2 Section: 3

17. Let $y = (x^2 + 2x - 5)(x^3 - 1)$. Compute $D_x y$ using the product rule and simplify.

Answer: $5x^4 + 8x^3 - 15x^2 - 2x - 2$

Difficulty: 2 Section: 3

18. Let $f(x) = \frac{x^2 - 1}{x^2 + 1}$. Find $f'(x)$.

Answer: $\frac{4x}{(x^2 + 1)^2}$

Difficulty: 2 Section: 3

19. Let $y = \frac{1+x}{1-x} - 4x^3$. Find $D_x y$.

Answer: $\frac{2}{(1-x)^2} - 12x^2$

Difficulty: 2 Section: 3

20. Let $f(x) = \frac{x^2 + 1}{x - 3}$. Compute $f'(x)$ and simplify.

Answer: $\frac{x^2 - 6x - 1}{(x - 3)^2}$

Difficulty: 2 Section: 3

21. Let $g(x) = \frac{x+1}{2-x^2}$. Compute $g'(x)$ and simplify.

Answer: $\frac{x^2 + 2x + 2}{(2 - x^2)^2}$

Difficulty: 2 Section: 3

22. Let $f(t) = \frac{t^3 - 1}{t^2 + 3t - 2}$. Find $f'(t)$ and simplify.

Answer: $\frac{t^4 + 6t^3 - 6t^2 + 2t + 3}{(t^2 + 3t - 2)^2}$

Difficulty: 2 Section: 3

23. Find an equation of the tangent line to the graph of the function $y = 1 + x^{-1} + x^{-1} + x^{-2} + x^{-3}$ at the point $(1, 4)$.

Answer: $y + 6x - 10 = 0$

Difficulty: 2 Section: 3

24. Find an equation of the tangent line to the graph of the function $f(x) = -3 + \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}$ at the point $(1, -2)$.

Answer: $y + 2x = 0$

Difficulty: 2 Section: 3

25. Find an equation of the tangent line to $f(x) = \sqrt{x} + 5$ at the point $(9, 8)$.

Answer: $x - 6y + 39 = 0$

Difficulty: 2 Section: 3

26. Find an equation of the tangent line to $f(x) = \frac{x^2 + 3}{1 - x}$ at the point $(0, 3)$.

Answer: $3x - y + 3 = 0$

Difficulty: 2 Section: 3

27. What is the point on the graph of $y = -2x^2 - 9x - 5$ at which the tangent line is perpendicular to the line $y = x - 10$?

Answer: $(-2, 5)$

Difficulty: 3 Section: 3

28. If $f(x) = \frac{1}{x} + \cos x$, find the value of $f'(x)$.

Answer: $-\frac{1}{x^2} - \sin(x)$

Difficulty: 1 Section: 4

29. Let $s = -5t \csc(t)$. Find $D_t s$.

Answer: $5 \csc(t) [t \cot(t) - 1]$

Difficulty: 2 Section: 4

30. Let $f(x) = 3 \tan(x) - 2 \sec(x) + 3x^2$. Find $f'(x)$.

Answer: $3 \sec^2(x) - 2 \sec(x) \tan(x) + 6x$

Difficulty: 2 Section: 4

31. Let $y = x^2 \cos(x)$. Find $D_x y$.

Answer: $2x \cos x - x^2 \sin(x)$

Difficulty: 2 Section: 4

32. Let $g(k) = (k^2 + 2k)[1 - \cos(k)]$. Find $g'(k)$.

Answer: $(k^2 + 2k)[\sin(k)] + 2[1 - \cos(k)](k + 1)$

Difficulty: 2 Section: 4

33. Let $f(x) = \frac{\sin(x) + \cos(x)}{x^2}$. Find $f'(x)$ and simplify.

Answer: $\frac{x \cos(x) - x \sin(x) - 2 \sin(x) - 2 \cos(x)}{x^3}$

Difficulty: 2 Section: 4

34. Find the equation of the tangent line to the curve $y = \cot(x)$ at the point where $x = \frac{\pi}{4}$.

Answer: $y + 2x - 1 - \frac{\pi}{2} = 0$

Difficulty: 2 Section: 4

35. Find the equation of the tangent line to the graph of $y = 2 \tan(x)$ at the point where $x = \frac{\pi}{3}$.

Answer: $y - 2\sqrt{3} = 8\left(x - \frac{\pi}{3}\right)$

Difficulty: 2 Section: 4

36. Find the equation of the tangent line to the graph of $y = \sqrt{3} \sec(x) - 2$ at the point where $x = \frac{\pi}{6}$.

Answer: $y = \frac{2\sqrt{3}}{3}\left(x - \frac{\pi}{6}\right)$

Difficulty: 2 Section: 4

37. Let $y = (x + 1)^5$. Find $D_x y$.

Answer: $5(x + 1)^4$

Difficulty: 1 Section: 5

38. Let $f(x) = (x^2 - x + 5)^4$. Find $f'(x)$.

Answer: $4(2x - 1)(x^2 - x + 5)^3$

Difficulty: 2 Section: 5

39. Let $g(x) = (1 - 3x)^{-2}$. Find $g'(x)$.

Answer: $6(1 - 3x)^{-3}$

Difficulty: 2 Section: 5

40. Let $g(x) = \left(x^2 + \frac{1}{x^2}\right)^3$. Find $g'(x)$.

Answer: $6\left(x^2 + \frac{1}{x^2}\right)^2 \left(x - \frac{1}{x^3}\right)$

Difficulty: 2 Section: 5

41. Let $f(x) = \left(\frac{x}{x^2 + 3}\right)^5$. Find $f'(x)$ and simplify.

Answer: $\frac{5x^4(3 - x^2)}{(x^2 + 3)^6}$

Difficulty: 2 Section: 5

42. Let $f(x) = \left(\frac{1+x}{1-x}\right)^4$. Find $f'(x)$ and simplify.

Answer: $\frac{8(1+x)^3}{(1-x)^5}$

Difficulty: 2 Section: 5

43. Let $f(x) = \frac{1}{x} + \cos 2x$. Find $f'(x)$.

Answer: $-\left(\frac{1}{x^2} + 2 \sin 2x\right)$

Difficulty: 2 Section: 5

44. Let $x = t^4 - \sec^2(t)$. Find $D_t x$.

Answer: $4t^3 - 2 \sec^2 t \tan t$

Difficulty: 2 Section: 5

45. Let $y = \cos(\sin 2x)$. Find $D_x y$.

Answer: $-2 \sin(\sin 2x) \cos 2x$

Difficulty: 2 Section: 5

46. Let $y = x \cos(\sin x)$. Find $D_x y$.

Answer: $\cos(\sin x) - x \sin(\sin x) \cos x$

Difficulty: 2 Section: 5

47. If the equation of motion of a particle is $x = (2t^2 - 7t)^3$, what is the velocity of the particle when $t = 1$?

Answer: -225

Difficulty: 2 Section: 5

48. Find an equation of the tangent line to the graph of $y = (3x^2 + 5x - 1)^7$ at the point where $x = 0$.

Answer: $y - 35x + 1 = 0$

Difficulty: 2 Section: 5

49. Find the slope of the line tangent to the graph of $y = x^2 \cos 2x$ at the point where $x = \frac{\pi}{2}$.

Answer: $-\pi$

Difficulty: 2 Section: 5

50. Making the usual assumptions, write a formula for the derivative of the function $\sin^4 [f(x)]$.

Answer: $4 \sin^3 [f(x)] \cos [f(x)] f'(x)$

Difficulty: 3 Section: 5

51. Making the usual assumptions, write a formula for the derivative of the function $f[\tan 2x]$.

Answer: $2 \sec^2(2x) f'[\tan 2x]$

Difficulty: 3 Section: 5

52. Let $y = (x^2 + 1)^5$. Find $\frac{dy}{dx}$.

Answer: $10x(x^2 + 1)^4$

Difficulty: 1 Section: 6

53. Let $y = (x^3 - 2)^5$. Find $\frac{dy}{dx}$.

Answer: $15x^2(x^3 - 2)^4$

Difficulty: 2 Section: 6

54. Let $y = (7 - x^3)^3$. Find $\frac{dy}{dx}$.

Answer: $9x^2(7 - x^3)^{-4}$

Difficulty: 2 Section: 6

55. Let $y = \left(\frac{x^2 + 1}{x}\right)^4$. Compute $\frac{dy}{dx}$ and simplify.

Answer: $\frac{4(x^2 + 1)^3(x^2 - 1)}{x^5}$

Difficulty: 2 Section: 6

56. Let $y = \cos(x^3)$. Find $\frac{dy}{dx}$.

Answer: $-3x^2 \sin(x^3)$

Difficulty: 2 Section: 6

57. Let $y = \cos^3(x)$. Find $\frac{dy}{dx}$.

Answer: $-3 \cos^2(x) \sin(x)$

Difficulty: 2 Section: 6

58. Let $y = \sec^2(5x)$. Find $\frac{dy}{dx}$.

Answer: $10 \sec^2(5x) \tan(5x)$

Difficulty: 2 Section: 6

59. Let $p = \tan(u^2)$. Find $\frac{dp}{du}$.

Answer: $2u \sec^2(u^2)$

Difficulty: 2 Section: 6

60. Let $y = x^2 \cos 2x$. Find $\frac{dy}{dx}$.

Answer: $2x(\cos 2x - x \sin 2x)$

Difficulty: 2 Section: 6

61. The radius of a circle at time t is given by $r(t) = (t^2 + t)^2$. Find the rate of change of the area of the circle with respect to time.

Answer: $4\pi(t^2 + t)^3(2t + 1)$

Difficulty: 2 Section: 6

62. Let $f(x) = \frac{1+x}{x}$. Find $f'(x)$ and $f''(x)$.

Answer: $-\frac{1}{x^2}; \frac{2}{x^3}$

Difficulty: 1 Section: 7

63. Let $y = x^6 - 3x^4 + 2x^2 - 3x + 10$. Find $\frac{d^2y}{dx^2}$.

Answer: $30x^4 - 36x^2 + 4$

Difficulty: 1 Section: 7

64. Let $p = q^{-2} + q^{-1}$. Find $\frac{d^3p}{d^3q}$.

Answer: $-24q^{-5} - 6q^{-4}$

Difficulty: 2 Section: 7

65. If $f(x) = x^7 + 3x^5 + x$, find the value of $f^{(8)}(x)$.

Answer: 0

Difficulty: 2 Section: 7

66. At time t , the position of a body moving in a line is given by $s = 3 + 5t - t^2$. What is the velocity at $t = 3$?

Answer: -1

Difficulty: 1 Section: 7

67. At time t , the position of a body moving in a line is given by $s = 3t^8 - 2t^5$. Find the velocity and acceleration as functions of time.

Answer: $v(t) = 24t^7 - 10t^4$, $a(t) = 168t^6 - 40t^3$

Difficulty: 1 Section: 7

68. Let $y = 2 \sin(x) - \frac{1}{x^2}$. Find $\frac{d^2y}{dx^2}$.

Answer: $-2 \sin(x) - 3x^{-4}$.

Difficulty: 2 Section: 7

69. Find $f''(x)$ if $f(x) = \sec(x)$.

Answer: $\sec^3(x) + \sec(x) \tan^2(x)$

Difficulty: 2 Section: 7

70. The function $s = 3 + t^2 - 2 \cos(t)$ specifies the position of a particle at time t . Find the acceleration at time $t = \frac{\pi}{6}$.

Answer: $2 + \sqrt{3}$

Difficulty: 2 Section: 7

71. Let $y = (1 - x)^4$. Find $\frac{d^2y}{dx^2}$.

Answer: $12(1 - x)^2$

Difficulty: 1 Section: 7

72. Let $y = \frac{x}{x+x}$. Find $\frac{d^2y}{dx^2}$.

Answer: $\frac{-2}{(1+x)^3}$

Difficulty: 2 Section: 7

73. Let $g(x) = (x-1)^5 + (x+1)^2 + 2$. Find $g^{(4)}(x)$.

Answer: 240

Difficulty: 2 Section: 7

74. Let $f(x) = \frac{3}{20}(2-5x)^{-4/3}$. Find $f'(x)$ and $f''(x)$.

Answer: $f'(x) = (2-5x)^{-7/3}$ and $f''(x) = \frac{35}{3}(2-5x)^{-10/3}$

Difficulty: 2 Section: 7

75. The function $s = \sin(2x^2) - \pi$ gives the position at time t of a particle. Find the acceleration at any time t .

Answer: $4\cos(2x^2) - 16x^2\sin(2x^2)$

Difficulty: 2 Section: 7

76. Let $f(x) = x\sqrt{1+x^3}$. Find the value of $f'(x)$ when $x = 2$.

Answer: 7

Difficulty: 2 Section: 8

77. Find the equation of the tangent line to the curve $y = \sqrt{25-x^2}$ at the point $(3, 4)$.

Answer: $3x + 4y = 25$

Difficulty: 2 Section: 8

78. Let $f(x) = \sqrt[3]{5x-8}$. Where is the equation of the tangent line to the graph of f at the point where $x = 7$?

Answer: $5x - 27y + 46 = 0$

Difficulty: 2 Section: 8

79. Use implicit differentiation to find $\frac{dy}{dx}$ if $x^2y + xy^2 = 5y$.

Answer: $\frac{2xy + y^2}{5 - x^2 - 2xy}$

Difficulty: 2 Section: 8

80. Use implicit differentiation to find $\frac{dy}{dx}$ if $x^2 - xy + y^2 = 3$.

Answer: $\frac{y - 2x}{2y - x}$

Difficulty: 2 Section: 8

81. Use implicit differentiation to find $\frac{dy}{dx}$ if $x^2 + xy + y^5 = 2$.

Answer: $-\frac{2x + y}{x + 5y^4}$

Difficulty: 2 Section: 8

82. What is the slope of the tangent line to the curve $4x^2 - 9y^2 = 36$ at the point $(6, 2\sqrt{3})$?

Answer: $\frac{4\sqrt{3}}{9}$

Difficulty: 2 Section: 8

83. Find the slope of the tangent line to the curve $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$.

Answer: 1

Difficulty: 2 Section: 8

84. Find the equation of the tangent line to the curve $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$?

Answer: $x - y + 2 = 0$

Difficulty: 2 Section: 8

85. Find the slope of the tangent line to the curve $xy + y^3 = 12$ at the point $(2, 2)$?

Answer: $\frac{1}{14}$

Difficulty: 2 Section: 8

86. Find an equation for the tangent line to the curve $x^3 - x + xy + y^3 + 24 = 0$ at the point $(-1, -3)$.

Answer: $y + 3 = \frac{1}{26}(x + 1)$ or $x - 26y - 77 = 0$

Difficulty: 2 Section: 8

87. Use implicit differentiation to find $\frac{du}{dw}$ if $w^3 - 2 + wu + u^3 + 24 = 0$.

Answer: $\frac{1 - 3w^2 - u}{w + 3u^2}$

Difficulty: 2 Section: 8

88. Use implicit differentiation to find $\frac{ds}{dt}$ if $s^2 - ts = 10$.

Answer: $\frac{s}{2s - t}$

Difficulty: 2 Section: 8

89. Use implicit differentiation to find $\frac{dr}{dw}$ if $5w^3 + 2wr^2 + r^3 = 10$.

Answer: $-\frac{15w^2 + 2r^2}{4wr + 3r^2}$

Difficulty: 2 Section: 8

90. Use implicit differentiation to find $\frac{du}{dw}$ if $w^2 - wu + u^2 = 3$.

Answer: $\frac{u - 2w}{2u - w}$

Difficulty: 2 Section: 8

91. Use implicit differentiation to find $\frac{ds}{dt}$ if $5t^2 + 2t^2s + s^2 = 1$.

Answer: $-\frac{5t + 2ts}{t^2 + s}$

Difficulty: 2 Section: 8

92. Use implicit differentiation to find $\frac{dy}{dx}$ if $y = x \cos y$.

Answer: $\frac{\cos y}{1 + x \sin y}$

Difficulty: 2 Section: 8

93. Use implicit differentiation to find $\frac{dy}{dx}$ if $y = x \sin y$.

Answer: $\frac{\sin y}{1 - x \cos y}$

Difficulty: 2 Section: 8

94. Use implicit differentiation to find $\frac{dy}{dx}$ if $x + \cos y - y^2 = 17$.

Answer: $\frac{1}{2y + \sin y}$

Difficulty: 2 Section: 8

95. Use implicit differentiation to find $\frac{dy}{dx}$ if $\tan y - y^3 + x^2 + 20 = 0$.

Answer: $\frac{2x}{3y^2 - \sec^2 y}$

Difficulty: 2 Section: 8

96. Use implicit differentiation to find $\frac{dy}{dx}$ if $y^2 - x^2 - \cos(xy) = 0$.

Answer: $\frac{2x - y \sin(xy)}{2y + x \sin(xy)}$

Difficulty: 2 Section: 8

97. Use implicit differentiation to find $\frac{dy}{dx}$ if $\sin^2(2y) - y^2 = 3x^2$.

Answer: $\frac{3x^2}{4 \sin(2y) - 2y}$

Difficulty: 2 Section: 8

98. Use implicit differentiation to find $\frac{dy}{dx}$ if $\cos(yx) = x$.

Answer: $\frac{y \sin(yx) + 1}{x \sin(yx)}$

Difficulty: 2 Section: 8

99. Use implicit differentiation to find $\frac{dy}{dx}$ if $\sin x + \sin y = xy$.

Answer: $\frac{y - \cos x}{\cos y - x}$

Difficulty: 2 Section: 8

100. Use implicit differentiation to find $\frac{dy}{dx}$ if $\cos 3y = \tan 2x$.

Answer: $-\frac{2 \sec^2(2x)}{3 \sin 3y}$

Difficulty: 2 Section: 8

101. Use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x^2y^3 = 1$.

Answer: $\frac{dy}{dx} = -\frac{2y}{3x}, \frac{d^2y}{dx^2} = \frac{10y}{9x^2}$

Difficulty: 3 Section: 8

102. Use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $y^3 = x^2 - 17$.

Answer: $\frac{dy}{dx} = \frac{2x}{3y^2}$ and $\frac{d^2y}{dx^2} = \frac{6y^3 - 8x^2}{9y^5}$

Difficulty: 3 Section: 8

103. Use implicit differentiation to find $\frac{dy}{dx}$ if $x^{2/3} + y^{2/3} = 4$.

Answer: $-\left(\frac{y}{x}\right)^{1/3}$

Difficulty: 2 Section: 8

104. Use implicit differentiation to find $\frac{dy}{dx}$ if $3x^{4/3} + xy + 3y^{4/3} = 2$.

Answer: $\frac{4x^{1/3} + y}{x + 4y^{1/3}}$

Difficulty: 2 Section: 8

105. Use implicit differentiation to find $\frac{dy}{dx}$ if $\sqrt{x+y} = x$.

Answer: $2\sqrt{x+y} - 1$

Difficulty: 2 Section: 8

106. Use implicit differentiation to find $\frac{dy}{dx}$ if $2(xy)^{3/2} + y^2 = x^2 = 0$.

Answer: $\frac{2x - 3y\sqrt{xy}}{3x\sqrt{xy} + 2y}$

Difficulty: 3 Section: 8

107. The radius of a circle is increasing at the rate of 3 ft/sec. Find the rate of change of the area of the circle at the time that the radius is 15 ft.

Answer: 90π ft²/sec.

Difficulty: 1 Section: 9

108. A spherical balloon is inflated at a rate of 4 ft³/min. How fast is the radius increasing when the radius is 2 ft?

Answer: $\frac{1}{4\pi}$ ft/min

Difficulty: 1 Section: 9

109. A ladder 20 feet long leans against a vertical building. If the bottom of the ladder slides away from the building horizontally at a rate of 2 ft/sec, how fast is the ladder sliding down the building when the top of the ladder is 12 feet above the ground?

Answer: $\frac{8}{3}$ ft/sec

Difficulty: 2 Section: 9

110. A 16ft ladder is leaning against a wall. Suppose the bottom of the ladder is pulled along the level sidewalk, directly away from the wall, at 2 ft/sec. How fast is the height of the midpoint of the ladder decreasing when the ladder is 4 ft from the wall?

Answer: $\frac{\sqrt{15}}{15}$ ft/sec

Difficulty: 2 Section: 9

111. A two-piece extension ladder leaning against a wall is collapsing at a rate of 2 ft/sec while the foot of the ladder remains a constant 5 ft from the wall. How fast is the ladder moving down the wall when the ladder is 13 ft long?

Answer: $\frac{13}{6}$ ft/sec

Difficulty: 2 Section: 9

112. A balloon is rising vertically at a rate of 2 ft/sec. An observer is located 300 ft from a point on the ground directly below the balloon. At what rate is the distance between the balloon and the observer changing when the height of the balloon is 400 ft?

Answer: 1.6 ft/sec

Difficulty: 2 Section: 9

113. A car leaves Albuquerque and travels east at 36 mph. One hour later, a second car traveling southwards at a constant speed is 48 miles south of Albuquerque and the distance between them is increasing at a rate of $\frac{592}{15}$ mph. Determine the speed of the second car.

Answer: $\frac{67}{3}$ mph

Difficulty: 2 Section: 9

114. A conical tank has radius 3 ft and depth 10 ft. If water is poured into the tank at the rate of 2 ft³/min, how fast is the water level rising when the water in the tank is 5 ft deep?

Answer: $\frac{8}{9\pi}$ ft/min

Difficulty: 2 Section: 9

115. Sand is pouring from a pipe at the rate of 12ft³/sec. If the falling sand forms a conical pile on the ground whose height is always $\frac{1}{3}$ the diameter of the base, how fast is the height increasing when the pile is 4 ft high? The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the base and h is the height.

Answer: $\frac{1}{3\pi}$ ft/sec

Difficulty: 2 Section: 9

116. A baseball player runs from homeplate to first base at a rate of 25 ft/sec. At what rate is his distance from third base changing when he is 40 ft from first base? A baseball diamond is 90 ft by 90 ft square.

Answer: $\frac{125}{\sqrt{106}}$ ft/sec

Difficulty: 2 Section: 9

117. The top of a tower has the shape of a hemisphere of radius 5 ft. The top has a uniform coating of ice which is 3 inches thick. The thickness of the ice is decreasing at the rate of $\frac{1}{3}$ in/hr. How fast is the volume of the ice changing? The volume of a sphere is $\frac{4}{3}\pi r^3$ where r is the radius of the sphere.

Answer: 2646π in³/hr

Difficulty: 2 Section: 9

118. A trough 10 ft long has a cross section that is an isosceles triangle having a base of 4 ft and height 3 ft. If the water pours into the trough at 6 ft³/min, how fast is the depth of the water changing when the depth is 1 ft?

Answer: 0.45 ft/min

Difficulty: 2 Section: 9

119. A beacon 0.5 miles (perpendicular distance) from a straight shore revolves at 2 revolutions per minute (4π radians per minute). At what speed is the beam moving along the shore when it hits the shore 1 mile from the lighthouse?

Answer: 8π miles/minute

Difficulty: 2 Section: 9

120. In a football game, a receiver runs north at a rate of 30 ft/sec while the quarterback runs east at a rate of 20 ft/sec. How fast are the two players moving away from each other when the receiver is 80 ft downfield and the quarterback is 60 ft east of center?

Answer: 36 ft/sec

Difficulty: 2 Section: 9

121. Let $f(x) = \frac{1}{x^2}$. Find a general expression for dy . Evaluate dy if $x = -2$ and $dx = 0.5$.

Answer: $dy = -\frac{2}{x^3}, dx = 0.125$

Difficulty: 1 Section: 10

122. Let $f(x) = x \cos(x)$. Find a general expression for dy . Evaluate dy if $x = \pi$ and $dx = 0.15$.

Answer: $[\cos(x) - x \sin(x)] dx, 0.15$

Difficulty: 1 Section: 10

123. Let $f(x) = \sec(x)$. Find a general expression for dy . Evaluate dy if $x = \frac{\pi}{3}$ and $dx = -0.5$.
 Answer: $[\sec(x) \tan(x)] dx, -\sqrt{3}$
 Difficulty: 1 Section: 10
124. Let $f(x) = \sin(x) - 4x$. Find a general expression for dy . Evaluate dy if $x = 0$, and $dx = 0.3$.
 Answer: $[\cos(x) - 4] dx, -0.9$
 Difficulty: 1 Section: 10
125. Let $y = 3x^2 - 5$. Compute dy and Δy if x changes from 2 to 2.1.
 Answer: $dy = 1.2, \Delta y = 1.23$
 Difficulty: 2 Section: 10
126. Let $y = \frac{1}{2 - x^3}$. Use dy to approximate the change in y if x changes from 1 to 1.02.
 Answer: $\Delta y \approx dy = 0.06$
 Difficulty: 1 Section: 6
127. Approximate $\sqrt{98}$ by means of the differential.
 Answer: 9.9
 Difficulty: 2 Section: 10
128. Approximate $\sqrt[3]{26}$ by means of the differential. Round off answer to three decimal places.
 Answer: 2.963
 Difficulty: 2 Section: 10
129. The radius of a circle is measured as 17 cm with an error of ± 0.4 cm. Estimate the error in calculating the circumference of the circle
 Answer: $\pm 0.8\pi$ cm
 Difficulty: 2 Section: 10
130. The side of a cube is measured at 10 cm with an error of ± 0.2 cm. Estimate the error in calculating the surface area of the cube.
 Answer: ± 24 square cm
 Difficulty: 2 Section: 10
131. Estimate the error in computing $(x^3 - 2x)^3$ at $x = 2$ if the error in x is estimated at ± 0.02 .
 Answer: ± 9.6
 Difficulty: 2 Section: 10
132. Estimate the error in computing $2 + \sin^3 x$ at $x = \frac{\pi}{4}$ if the error in x is estimated at ± 0.2 .
 Answer: $\pm (0.15) \sqrt{2}$
 Difficulty: 2 Section: 10

4 Applications of the Derivative

1. Determine the maximum and the minimum of the function $f(x) = x^3 - 3x^2 + 2$ on the interval $[-1, 1]$.

Answer:

maximum: 2

minimum: -2

Difficulty: 1 Section: 1

2. Determine the maximum and the minimum of the function $f(x) = x^3 - 6x^2 + 9x + 3$ on the interval $[1, 5]$.

Answer:

maximum: 23

minimum: 3

Difficulty: 1 Section: 1

3. Determine the maximum and the minimum of the function $f(x) = \frac{8x}{x^2 + 4}$ on the interval $[0, 3]$.

Answer:

maximum: 2

minimum: 0

Difficulty: 2 Section: 1

4. Determine the maximum and the minimum of the function $f(x) = 4x^3 - x^4$ on the interval $[-1, 2]$.

Answer:

maximum: 16

minimum: -5

Difficulty: 2 Section: 1

5. Determine the maximum and the minimum of the function $f(x) = x^4 - 4x^3 + 10$ on the interval $[0, 4]$.

Answer:

maximum: 10

minimum: -17

Difficulty: 2 Section: 1

6. Determine the maximum and the minimum of the function $f(x) = x^3 - 12x$ on the interval $[0, 3]$.

Answer:

maximum: 0

minimum: -16

Difficulty: 2 Section: 1

7. Determine the maximum and the minimum of the function $f(x) = x^3 - 3x + 2$ on the interval $[0, 2]$.

Answer:

maximum: 4

minimum: 0

Difficulty: 1 Section: 1

8. Determine the maximum and the minimum of the function $f(x) = \frac{x}{x^2 + 1}$ on the interval $[-1, 0]$.

Answer:

maximum: 0

minimum: $-\frac{1}{2}$

Difficulty: 2 Section: 1

9. Determine the maximum and the minimum of the function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$ on the interval $[-4, 0]$.

Answer:

maximum: $\frac{43}{2}$

minimum: 8

Difficulty: 2 Section: 1

10. Determine the maximum and the minimum of the function $f(x) = x^{2/3}$ on the interval $[-8, 1]$.

Answer:

maximum: 4

minimum: 0

Difficulty: 1 Section: 1

11. Determine the maximum and the minimum of the function $f(x) = -\frac{4}{x^2}$ on the interval $[-3, -1]$.

Answer:

maximum: -3

minimum: -5

Difficulty: 2 Section: 1

12. Determine the maximum and the minimum of the function $f(x) = x(1-x)^{2/5}$ on the interval $\left[\frac{1}{2}, 2\right]$.

Answer:

maximum: 2

minimum: 0

Difficulty: 2 Section: 1

13. Determine the maximum and the minimum of the function $f(x) = x^3 - 3x^2$ on the interval $[1, 2]$.

Answer:

maximum: -2

minimum: -4

Difficulty: 2 Section: 1

14. Find two positive numbers whose sum is 17 and whose product is a maximum.

Answer: $\frac{17}{2}, \frac{17}{2}$

Difficulty: 2 Section: 1

15. A rectangular field is to be enclosed by a fence and divided into 3 lots by fences parallel to one of its sides. Find the dimensions of the largest field that can be enclosed with a total of 800 meters of fencing.

Answer: 100 m by 200 m

Difficulty: 2 Section: 1

16. A farmer wishes to enclose a 4000 square meter field and subdivide the field into four rectangular plots with fences parallel to one of the sides. What should the dimensions of the field be in order to minimize the amount of fencing required?

Answer: 100 m by 40 m

Difficulty: 2 Section: 1

17. Determine the intervals where $f(x) = \frac{1}{3}x^3 + x^2 - 3x$ is increasing.

Answer: $(-\infty, -3], [1, \infty)$

Difficulty: 2 Section: 2

18. Determine the intervals where $f(x) = \frac{1}{3}x^3 + x^2 - 3x$ is decreasing.

Answer: $[-3, 1]$

Difficulty: 2 Section: 2

19. Determine the intervals where $f(x) = \frac{1}{1+x^2}$ is increasing.

Answer: $(-\infty, 0]$

Difficulty: 2 Section: 2

20. Determine the intervals where $f(x) = x^4 - 4x^3 + 10$ is increasing.

Answer: $[3, 0]$

Difficulty: 2 Section: 2

21. Determine the intervals where $f(x) = \frac{x}{x^2 + 1}$ is increasing.

Answer: $[-1, 1]$

Difficulty: 2 Section: 2

22. Determine the intervals where $f(x) = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$ is increasing.

Answer: $(-\infty, 1], [3, \infty)$

Difficulty: 2 Section: 2

23. Determine the intervals where $f(x) = \frac{8x}{x^2 + 4}$ is decreasing.

Answer: $(-\infty, -2], [2, \infty)$

Difficulty: 2 Section: 2

24. Determine the intervals where $f(x) = x^3 - 3x^2 + 2$ is decreasing.

Answer: $[0, 2]$

Difficulty: 1 Section: 2

25. Determine the intervals where $f(x) = 4x^3 - x^4$ is increasing.

Answer: $(-\infty, 3]$

Difficulty: 2 Section: 2

26. Determine the intervals where $f(x) = x^3 - 6x^2 + 9x + 3$ is increasing.

Answer: $(-\infty, 1], [3, \infty)$

Difficulty: 2 Section: 2

27. Determine the intervals where $f(x) = x^2 - 4x + 5$ is decreasing.

Answer: $(-\infty, 2]$

Difficulty: 1 Section: 2

28. Determine the intervals where $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x - 4$ is decreasing.

Answer: $[-2, 3]$

Difficulty: 2 Section: 2

29. Determine the intervals where $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$ is decreasing.

Answer: $[-3, 2]$

Difficulty: 2 Section: 2

30. Determine the intervals where $f(x) = x^2 - 7x + 12$ is increasing.

Answer: $[\frac{7}{2}, \infty)$

Difficulty: 1 Section: 2

31. Let $f(x) = x^3 - 3x^2 + 2$. Determine the intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer:

concave up: $(1, \infty)$

concave down: $(-\infty, 1)$

points of inflection: $(1, 0)$

Difficulty: 1 Section: 2

32. Let $f(x) = x^3 - 12x$. Determine the intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer:

concave up: $(0, \infty)$

concave down: $(-\infty, 0)$

points of inflection: $(0, 0)$

Difficulty: 1 Section: 2

33. Let $f(x) = x - 3 + \frac{2}{x+1}$. Determine the intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer:

concave up: $(-1, \infty)$

concave down: $(-\infty, -1)$

points of inflection: none

Difficulty: 2 Section: 2

34. Let $f(x) = \frac{x}{x^2+1}$. Determine the intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer:

concave up: $(-\sqrt{3}, 0), (\sqrt{3}, \infty)$

concave down: $(-\infty, -\sqrt{3}), (0, \sqrt{3})$

points of inflection: $(-\sqrt{3}, -\sqrt{3}/4), (0, 0), (\sqrt{3}, \sqrt{3}/4)$

Difficulty: 2 Section: 2

35. Let $f(x) = 4x^3 - x^4$. Determine the intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer: concave up: $(0, 2)$
 concave down: $(-\infty, 0), (2, \infty)$
 points of inflection: $(0, 0), (2, 16)$

Difficulty: 1 Section: 2

36. Let $f(x) = \frac{x^3}{(x-2)^3}$. Determine the intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer: concave up: $(-2, 0), (2, \infty)$
 concave down: $(-\infty, -2), (0, 2)$
 points of inflection: $(0, 0), (-2, 1/8)$

Difficulty: 2 Section: 2

37. Let $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x - 4$. Determine the intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer: concave up: $(\frac{1}{2}, \infty)$
 concave down: $(-\infty, \frac{1}{2})$
 points of inflection: $(\frac{1}{2}, -\frac{85}{12})$

Difficulty: 2 Section: 2

38. Let $f(x) = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$. Determine the intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer: concave up: $(2, \infty)$
 concave down: $(-\infty, 2)$
 points of inflection: $(2, 5/6)$

Difficulty: 2 Section: 2

39. Let $f(x) = x^3 - 6x^2 + 9x + 3$. Determine the intervals where $f(x)$ is increasing, intervals where $f(x)$ is decreasing, intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer: increasing: $(-\infty, 1], [3, \infty)$
 decreasing: $[1, 3]$
 concave up: $(2, \infty)$
 concave down: $(-\infty, 2)$
 points of inflection: $(2, 5)$

Difficulty: 2 Section: 2

40. Let $f(x) = x^3 - 3x^2 + 1$. Determine the intervals where $f(x)$ is increasing, intervals where $f(x)$ is decreasing, intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer:

increasing: $(-\infty, 0] , [2, \infty)$

decreasing: $[0, 2]$

concave up: $(1, \infty)$

concave down: $(-\infty, 1)$

points of inflection: $(1, -1)$

Difficulty: 2 Section: 2

41. Let $f(x) = x^4 - 4x^3 + 10$. Determine the intervals where $f(x)$ is increasing, intervals where $f(x)$ is decreasing, intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer:

increasing: $[3, \infty)$

decreasing: $(-\infty, 3]$

concave up: $(-\infty, 0) , (2, \infty)$

concave down: $(0, 2)$

points of inflection: $(0, 10) , (2, -6)$

Difficulty: 2 Section: 2

42. Let $f(x) = \frac{8x}{x^2 + 4}$. Determine the intervals where $f(x)$ is increasing, intervals where $f(x)$ is decreasing, intervals where $f(x)$ is concave up, intervals where $f(x)$ is concave down, and find all points of inflection.

Answer:

increasing: $[-2, 2]$

decreasing: $(-\infty, -2] , [2, \infty)$

concave up: $(-2\sqrt{3}, 0) , (2\sqrt{3}, \infty)$

concave down: $(-\infty, -2\sqrt{3}) , (0, 2\sqrt{3})$

points of inflection: $(-2\sqrt{3}, -\sqrt{3}) , (0, 0) , (2\sqrt{3}, \sqrt{3})$

Difficulty: 2 Section: 2

43. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = x^3 - 6x^2 + 9x + 3$.

Answer:

local maximum: at $x = 1$

local minimum: at $x = 3$

Difficulty: 2 Section: 3

44. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = x^3 - 3x^2 + 2$.

Answer:

local maximum: at $x = 0$

local minimum: at $x = 2$

Difficulty: 1 Section: 3

45. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = x^4 - 4x^3 + 10$.

Answer:

local maximum: none

local minimum: at $x = 3$

Difficulty: 2 Section: 3

46. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = x^3 - 12x$.

Answer:

local maximum: at $x = -2$

local minimum: at $x = 2$

Difficulty: 2 Section: 3

47. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = x - 3 + \frac{2}{x+1}$

Answer:

local maximum: at $x = -1 - \sqrt{2}$

local minimum: at $x = -1 + \sqrt{2}$

Difficulty: 2 Section: 3

48. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = \frac{8x}{x^2 + 4}$

Answer:

local maximum: at $x = 2$

local minimum: at $x = -2$

Difficulty: 2 Section: 3

49. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = x^3 - 3x^2 + 1$.

Answer:

local maximum: at $x = 0$

local minimum: at $x = 2$.

Difficulty: 1 Section: 3

50. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = \frac{1}{1+x^2}$

Answer:

local maximum: at $x = 0$

local minimum: none

Difficulty: 2 Section: 3

51. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = \frac{x}{1+x^2}$.

Answer:

local maximum: at $x = 1$

local minimum: at $x = -1$

Difficulty: 2 Section: 3

52. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$.

Answer:

local maximum: at $x = 1$

local minimum: at $x = 3$

Difficulty: 2 Section: 3

53. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = 4x^3 - x^4$.

Answer:

local maximum: at $x = 3$

local minimum: none

Difficulty: 2 Section: 3

54. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = \frac{x^3}{(x-2)^2}$.

Answer:

local maximum: none

local minimum: none

Difficulty: 2 Section: 3

55. Find all points which give a local maximum and all points which give a local minimum value for the function $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x - 4$.

Answer:

local maximum: at $x = -2$

local minimum: at $x = 3$

Difficulty: 2 Section: 3

56. An open box is formed from a square sheet of cardboard by cutting equal squares from each corner and folding up the edges. If the dimensions of the cardboard are 18 cm by 18 cm, what should be the dimensions of the box so as to maximize the volume, and what is the maximum volume?

Answer: 3 cm by 12 cm by 12 cm. The maximum volume is 432 cm^3 .

Difficulty: 2 Section: 4

57. An open box is formed from a rectangular sheet of cardboard by cutting equal squares from each corner and folding up the edges. If the dimensions of the cardboard are 15 in by 24 in, what size squares should be cut to obtain a box of maximum volume?

Answer: 3 in by 3 in

Difficulty: 2 Section: 4

58. A box is made with a square bottom and open top to hold a volume of 8 ft^3 . The material for the sides costs \$4 per square foot and the material for the bottom costs \$1 per square foot. What dimensions will minimize the cost of the box?

Answer: Bottom 4 ft by 4 ft and height 0.5 ft

Difficulty: 2 Section: 4

59. A car rental agency has 24 identical cars. The owner of the agency finds that at the price of \$10 per day, all the cars can be rented. However, for each \$1 increase in rental price, one of the cars is not rented. How much should be charged to maximize the income of the agency?

Answer: \$17

Difficulty: 2 Section: 4

60. Suppose a rectangle has its lower base on the x -axis and upper vertices on the graph of the function $y = 8 - x^2$. Find the area of the largest such rectangle.

Answer: $\frac{64\sqrt{6}}{9}$

Difficulty: 2 Section: 4

61. A right triangle of hypotenuse 10 units is rotated about one of its legs to generate a right circular cone. What is the greatest possible volume of the cone?

Answer: $\frac{2000\pi}{9\sqrt{3}} \approx 403 \text{ units}^3$

Difficulty: 2 Section: 4

62. Find the base radius r and the height h of the right circular cone of maximum volume which will fit inside a sphere of radius 3 units.

Answer: $r = 2\sqrt{2}$, $h = 4$

Difficulty: 2 Section: 4

63. A motorist is in a desert in a jeep. The closest point on a straight road is town A and it is $4\sqrt{2}$ miles from the motorist. He wishes to reach town B, 10 miles from A on the road, in the shortest time. If he can drive 15 mi/hr on the desert and 45 mi/hr on the road, where should he intersect the road?

Answer: 2 mi from A

Difficulty: 2 Section: 4

64. The total cost of producing and selling x units of a certain commodity per month is $C(x) = 4x^2 + 100x + 4000$. If the production level is 2000 units per month, find the average cost, $\frac{C(x)}{x}$, of each unit and the marginal cost.

Answer: \$8102, \$16100

Difficulty: 1 Section: 5

65. The total cost of producing x units is given by $C(x) = 3000 - 20x + 0.03x^2$. If the total revenue is $R(x) = 50x - 0.01x^2$, find the marginal profit when $x = 100$ units.

Answer: \$62

Difficulty: 1 Section: 5

66. A manufacturer has total cost $C(x) = 2000 + 196x + 3x^2$ and total revenue is $R(x) = 500x - x^2$ when x units are manufactured per week. What is the marginal profit when $x = 10$? How many units should be manufactured for maximum profit? What is the maximum profit?

Answer: \$224/unit, 38, \$3776

Difficulty: 2 Section: 5

67. If x is the number of items produced in one week, the selling price per item is $\$(100 - 0.02x)$. The total cost for x items per week is $\$(40x + 15000)$. How many items should be produced and sold per week for maximum profit? What is the maximum profit?

Answer: 1500, \$30000

Difficulty: 2 Section: 5

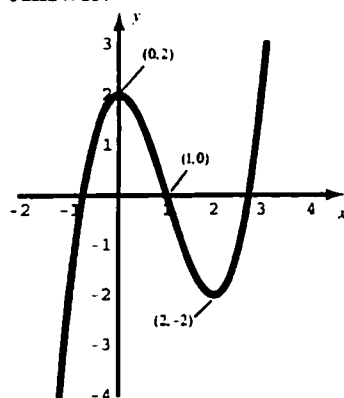
68. A shopping club charges its members \$200 per year. However, for each new member in excess of 60, the charge for every member is reduced by \$2. What number of members leads to a maximum revenue? What is the maximum revenue?

Answer: 80, \$12800

Difficulty: 2 Section: 5

69. Sketch the graph of the function $f(x) = x^3 - 3x^2 + 2$.

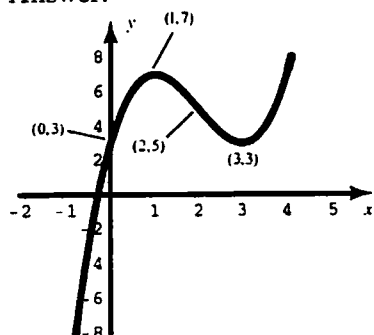
Answer:



Difficulty: 1 Section: 6

70. Sketch the graph of the function $f(x) = x^3 - 6x^2 + 9x + 3$.

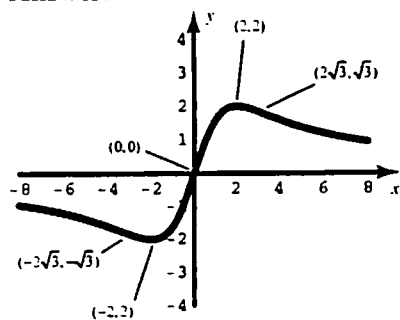
Answer:



Difficulty: 1 Section: 6

71. Sketch the graph of the function $f(x) = \frac{8x}{x^2 + 4}$.

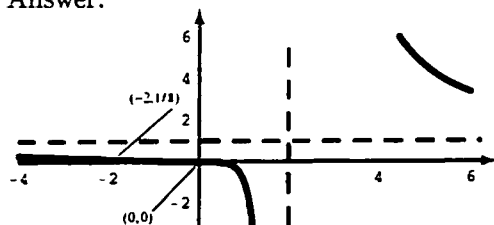
Answer:



Difficulty: 2 Section: 6

72. Sketch the graph of the function $f(x) = \frac{x^3}{(x-2)^2}$.

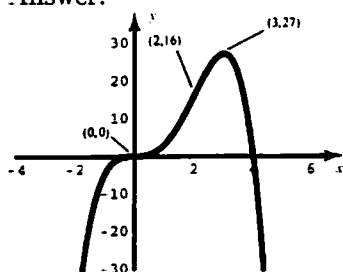
Answer:



Difficulty: 2 Section: 6

73. Sketch the graph of the function $f(x) = 4x^3 - x^4$.

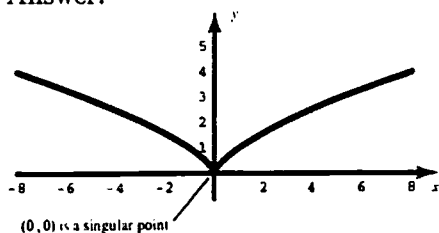
Answer:



Difficulty: 2 Section: 6

74. Sketch the graph of the function $f(x) = x^{2/3}$. Label the important features.

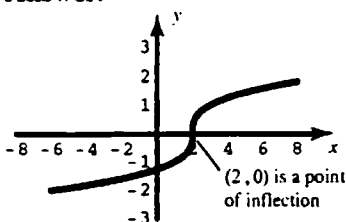
Answer:



Difficulty: 2 Section: 6

75. Sketch the graph of the function $f(x) = (x-2)^{1/3}$. Label the important features.

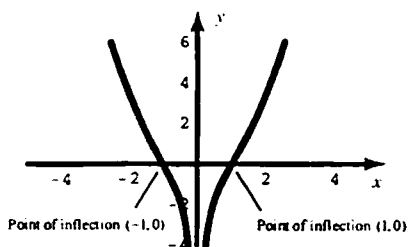
Answer:



Difficulty: 2 Section: 6

76. Sketch the graph of the function $f(x) = x^2 - \frac{1}{|x|}$. Label the important features.

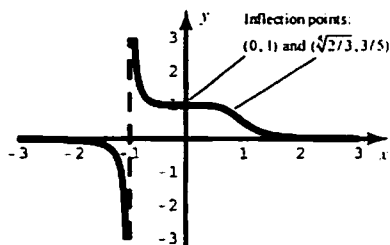
Answer:



Difficulty: 2 Section: 6

77. Sketch the graph of the function $f(x) = (1 + x^5)^{-1}$. Label the important features.

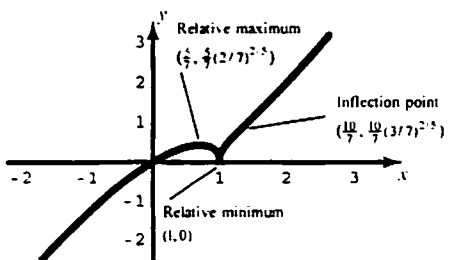
Answer:



Difficulty: 3 Section: 6

78. Sketch the graph of the function $f(x) = x(1 - x)^{2/5}$. Label the important features.

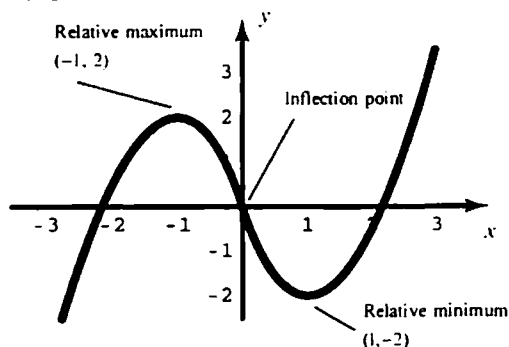
Answer:



Difficulty: 3 Section: 6

79. Sketch the graph of the function $f(x) = 3x^{5/3} - 5x$. Label the important features.

Answer:



Difficulty: 2 Section: 6

80. Let $f(x) = x^3$ on the interval $[0, 6]$. Find all numbers c , $0 < c < 6$, which satisfy the conclusion of the Mean Value Theorem.

Answer: $c = 2\sqrt{3}$

Difficulty: 1 Section: 7

81. Let $f(x) = \sqrt{2 - x^2}$ on the interval $[-\sqrt{2}, 0]$. Find all numbers c , $-\sqrt{2} < c < 0$, which satisfy the conclusion of the Mean Value Theorem.

Answer: $c = -1$

Difficulty: 2 Section: 7

82. Let $f(x) = \sqrt{2x - x^2}$ on the interval $[0, 2]$. Find all numbers c , $0 < c < 2$, which satisfy the conclusion of the Mean Value Theorem.

Answer: $c = 1$

Difficulty: 2 Section: 7

5 The Integral

1. Find $\int 2x^2 dx$.

Answer: $\frac{2}{3}x^3 + C$

Difficulty: 1 Section: 1

2. Find $\int (-3x^{-4}) dx$.

Answer: $x^{-3} + C$

Difficulty: 1 Section: 1

3. Find $\int \frac{x^2 + 2x}{x} dx$

Answer: $\frac{1}{2}x^2 + 2x + C$

Difficulty: 1 Section: 1

4. Find $\int \frac{1}{\sqrt{x}} dx$.

Answer: $2\sqrt{x} + C$

Difficulty: 1 Section: 1

5. Find $\int (5x^4 + 3) dx$.

Answer: $x^5 + 3x + C$

Difficulty: 1 Section: 1

6. Find $\int \left(x^2 - \frac{1}{x^2}\right) dx$.

Answer: $\frac{1}{3}x^3 + \frac{1}{x} + C$

Difficulty: 1 Section: 1

7. Find $\int (7x^4 - x^{1/2} - 3) dx$.

Answer: $\frac{7}{5}x^5 - \frac{2}{3}x^{3/2} - 3x + C$

Difficulty: 1 Section: 1

8. Find $\int (9 - y) dy$.

Answer: $9y - \frac{1}{2}y^2 + C$

Difficulty: 1 Section: 1

9. Find $\int \left(x - \frac{1}{x}\right)^2 dx$.

Answer: $\frac{1}{3}x^3 - 2x - x^{-1} + C$

Difficulty: 2 Section: 1

10. Find $\int (t^2 - 1)^2 dt$.

Answer: $\frac{1}{5}t^5 - \frac{2}{3}t^3 + t + C$

Difficulty: 1 Section: 1

11. Find $\int (\sin u + \cos u - u) du$.

Answer: $-\cos u + \sin u - \frac{1}{2}u^2 + C$

Difficulty: 2 Section: 1

12. Prove that $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$.

Answer:

Show that the derivative of the right side of the equation is equal to $\sin^2 x$. A trigonometric identity will be needed.

Difficulty: 3 Section: 1

13. Prove that $\int \cos^3 x dx = \sin x - \frac{\sin^3 x}{3} + C$.

Answer:

Show that the derivative of the right side of the equation is equal to $\cos^3 x$. A trigonometric identity will be needed.

Difficulty: 3 Section: 1

14. Prove that $\int u(2u^2 + 3)^3 du = \frac{1}{16}(2u^2 + 3)^4 + C$.

Answer:

Show that the derivative of the right side of the equation is $u(2u^2 + 3)^3$.

Difficulty: 2 Section: 1

15. Find $\int (x^3 + 2)^6 x^2 dx$.

Answer: $\frac{1}{21} (x^3 + 2)^7 + C$

Difficulty: 2 Section: 1

16. Find $\int (z^3 + 1)^2 dz$.

Answer: $\frac{1}{5} z^5 + \frac{2}{3} z^3 + z + C$

Difficulty: 2 Section: 1

17. Find $\int x^2 (2x^3 + 1)^7 dx$.

Answer: $\frac{1}{48} (2x^3 + 1)^8 + C$

Difficulty: 2 Section: 1

18. Find $\int 2x (4 - 7x^2)^{-7} dx$.

Answer: $\frac{1}{42} (4 - 7x^2)^{-6} + C$

Difficulty: 2 Section: 1

19. Find $\int \frac{1}{(3x + 2)^2} dx$.

Answer: $-\frac{1}{3(3x + 2)} + C$

Difficulty: 2 Section: 1

20. Find $\int \frac{1}{\sqrt{3x + 2}} dx$.

Answer: $\frac{2}{3} \sqrt{3x + 2} + C$

Difficulty: 2 Section: 1

21. Find $\int \frac{4y}{\sqrt{25 - 4y^2}} dy$.

Answer: $\sqrt{25 - 4y^2} + C$

Difficulty: 2 Section: 1

22. Find $\int x \sqrt[4]{1-x^2} dx$.

Answer: $-\frac{2}{5} (1-x^2)^{5/4} + C$

Difficulty: 2 Section: 1

23. Find $\int \sqrt{5-x} dx$.

Answer: $-\frac{2}{3} (5-x)^{3/2} + C$

Difficulty: 2 Section: 1

24. Find $\int x \cos(2x^2 - 1) dx$.

Answer: $\frac{1}{4} \sin(2x^2 - 1) + C$

Difficulty: 2 Section: 1

25. Find $\int x^3 \sin(4x^4 - \pi) dx$.

Answer: $-\frac{1}{16} \cos(4x^4 - \pi) + C$

Difficulty: 2 Section: 1

26. Find the solution to the differential equation $\frac{dy}{dx} = 2x + 3x^2$ satisfying $y = 5$ when $x = 1$.

Answer: $y = x^2 + x^3 + 3$

Difficulty: 1 Section: 2

27. Find the solution to the differential equation $\frac{dy}{dx} = 5x^4 + 3x^2 - 1$ satisfying $y = -7$ when $x = 1$.

Answer: $y = x^5 + x^3 - x - 8$.

Difficulty: 1 Section: 2

28. Find the solution to the differential equation $\frac{dy}{dx} = (x-2)(x+3)$ satisfying $y = -5$ when $x = 0$.

Answer: $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x - 5$

Difficulty: 2 Section: 2

29. Find the solution to the differential equation $\frac{dy}{dp} = \sqrt{p}$ satisfying $y = 0$ when $p = 4$.

Answer: $y = \frac{2}{3}p^{3/2} - \frac{16}{3}$

Difficulty: 2 Section: 2

30. Find the solution to the differential equation $\frac{dy}{dt} = ty^2$ satisfying $y = 7$ at $t = 1$.

Answer: $y = -\frac{2}{t^2} + 9$

Difficulty: 2 Section: 2

31. Find the solution to the differential equation $\frac{dy}{dx} = \frac{4x}{y}$ satisfying $y = 4$ at $x = 1$.

Answer: $y = 2\sqrt{x^2 + 3}$

Difficulty: 2 Section: 2

32. Find the solution to the differential equation $\frac{ds}{dt} = (t^2 - 7)s^2$ satisfying $s = \frac{1}{6}$ at $t = 2$.

Answer: $s = \frac{1}{7t - t^3}$.

Difficulty: 2 Section: 2

33. Find the solution to the differential equation $\frac{dy}{dx} = y^2$ satisfying $y = 1$ at $x = 0$.

Answer: $y = -\frac{1}{x - 1}$

Difficulty: 2 Section: 2

34. Find $g(t)$ if $g'(t) = \cos t$ and $g\left(\frac{\pi}{6}\right) = \frac{11}{2}$.

Answer: $\sin t + 5$

Difficulty: 2 Section: 2

35. Find $f(x)$ if $f'(x) = 2x - \frac{1}{x^2}$ and $f(2) = 4$.

Answer: $x^2 + \frac{1}{x} - \frac{1}{2}$

Difficulty: 2 Section: 2

36. A particle is moving along a line with velocity $v(t) = 3t^2 + t$. Find the position function $s(t)$ if $s(0) = 6$.

Answer: $t^3 + \frac{1}{2}t^2 + 6$

Difficulty: 1 Section: 2

37. A particle is moving along a line with velocity $v(t) = \sqrt{t} - 5$. Find the position function $s(t)$ if $s(0) = -7$.

Answer: $\frac{2}{3}t^{3/2} - 5t - 7$

Difficulty: 1 Section: 2

38. A particle is moving along a line with acceleration $a(t) = 3t^2 - 5$ feet/sec². Find the velocity and position functions if $v(0) = -5$ and $s(0) = 0$.

Answer: $v(t) = t^3 - 5t - 5$ ft/sec and $s(t) = \frac{1}{4}t^4 - \frac{5}{2}t^2 - 5t$ ft.

Difficulty: 2 Section: 2

39. A particle is moving along a line with acceleration $a(t) = \frac{1}{t^3} - t$ feet/sec². Find the velocity and position functions if $v(1) = 0$ and $s(1) = 0$.

Answer: $v(t) = -\frac{1}{2t^2} - \frac{1}{2}t^2 + 1$ ft/sec and $s(t) = \frac{1}{2t} - \frac{1}{6}t^3 + t - \frac{4}{3}$ ft.

Difficulty: 2 Section: 2

40. A ball is thrown straight downward from a height of 30 meters, with an initial velocity of 3 meters per second. When will the ball strike the ground?

Answer: $\frac{-3 + \sqrt{597}}{9.8} \approx 2.187$ seconds after it is thrown

Difficulty: 2 Section: 2

41. A ball is thrown straight downward from a height of 40 meters, with an initial velocity of 2 meters per second. When will the ball strike the ground?

Answer: $\frac{-2 + \sqrt{788}}{9.8} \approx 2.66$ seconds after it is thrown

Difficulty: 2 Section: 2

42. A bullet is shot straight upward from a point 5 feet above the ground. The muzzle velocity is 300 feet per second. At what time will the bullet reach its maximum height?

Answer: $\frac{75}{8} = 9.375$ seconds after firing.

Difficulty: 2 Section: 2

43. A bullet is shot straight upward from a point 5 feet above the ground. The muzzle velocity is 300 feet per second. What is the maximum height of the bullet?

Answer: $\frac{5645}{4} = 1411.25$ feet above the ground

Difficulty: 2 Section: 2

44. A bullet is shot straight upward from a point 5 feet above the ground. The muzzle velocity is 300 feet per second. At what time does the bullet strike the ground?

Answer: $\frac{300 + \sqrt{90320}}{32} \approx 18.77$ seconds after firing.

Difficulty: 2 Section: 2

45. A bullet is shot straight upward from ground level with a muzzle velocity of 256 feet per second. At what time will the bullet impact the ground?

Answer: 16 seconds after firing

Difficulty: 2 Section: 2

46. Compute the sum $\sum_{k=1}^6 (3k - 2)$.

Answer: 51

Difficulty: 1 Section: 3

47. Compute the sum $\sum_{k=0}^6 (3k - 10)$.

Answer: -7

Difficulty: 1 Section: 3

48. Compute the sum $\sum_{t=1}^5 4^t$.

Answer: 1364

Difficulty: 1 Section: 3

49. Compute the sum $\sum_{t=2}^7 (t^2 - t)$.

Answer: 112

Difficulty: 1 Section: 3

50. Compute the sum $\sum_{j=0}^6 (-1)^{j+1} j^2$.

Answer: -21

Difficulty: 1 Section: 3

51. Compute the sum $\sum_{i=0}^2 \left(\frac{i+2}{i+1} \right)^2$

Answer: $\frac{289}{36}$.

Difficulty: 1 Section: 3

52. Compute the sum $\sum_{t=0}^4 \left(\frac{t-t^2}{1+t} \right)$.

Answer: $-\frac{137}{30}$

Difficulty: 1 Section: 3

53. Compute the sum $\sum_{q=1}^5 \left(\frac{3-q}{q} \right)^2$.

Answer: $\frac{1789}{400}$

Difficulty: 1 Section: 3

54. Compute the sum $\sum_{j=0}^3 \left(\frac{j-1}{j+1} \right)^2$.

Answer: $\frac{49}{36}$

Difficulty: 1 Section: 3

55. Write the sum $9 + 7 + 5 + 3 + 1 - 1 - 3$ in sigma notation.

Answer: $\sum_{i=1}^7 (11 - 2i)$

Difficulty: 2 Section: 3

56. Write the sum $81 - 27 + 9 - 3 + 1$ in sigma notation.

Answer: $\sum_{k=0}^4 (-1)^k 3^{4-k}$

Difficulty: 2 Section: 3

57. Write the sum $-1^2 - 2^2 - 3^2 - 4^2 - 5^2 - 6^2$ in sigma notation.

Answer: $\sum_{m=1}^6 (-m^2)$

Difficulty: 2 Section: 3

58. Find the value of $\sum_{k=1}^{50} [(k+1)^2 - k^2]$.

Answer: 2600

Difficulty: 2 Section: 3

59. Evaluate the sum $\sum_{k=1}^n (6k^2 + 3)$.

Answer: $2n^3 + 3n^2 + 4n$

Difficulty: 2 Section: 3

60. Evaluate the sum $\sum_{j=1}^n (3j^2 - 2j + 1)$.

Answer: $\frac{1}{2} (2n^3 + n^2 + n)$

Difficulty: 2 Section: 3

61. Evaluate the sum $\sum_{t=1}^n (t^2 - 2t)$.

Answer: $\frac{1}{6} (2n^3 - 3n^2 - 5n)$

Difficulty: 2 Section: 3

62. Evaluate the sum $\sum_{k=2}^n (2k - 3)$.

Answer: $n^2 - 2n + 1$

Difficulty: 2 Section: 3

63. Find the area under the curve $y = 4x - 3$ over the interval $[1, 4]$. To do this, divide the interval into n equal subintervals, calculate the area of the corresponding circumscribed polygon and then let $n \rightarrow \infty$.

Answer: 21 square units

Difficulty: 1 Section: 4

64. Find the area under the curve $y = 2x + 5$ over the interval $[1, 5]$. To do this, divide the interval into n equal subintervals, calculate the area of the corresponding inscribed polygon and then let $n \rightarrow \infty$.

Answer: 44 square units

Difficulty: 1 Section: 4

65. Find the area under the curve $y = x^2 + 2$ over the interval $[0, 3]$. To do this, divide the interval into n equal subintervals, calculate the area of the corresponding inscribed polygon and then let $n \rightarrow \infty$.

Answer: 15 square units

Difficulty: 1 Section: 4

66. Find the area under the curve $y = x^3$ over the interval $[0, 2]$. To do this, divide the interval into n equal subintervals, calculate the area of the corresponding circumscribed polygon and then let $n \rightarrow \infty$.

Answer: 4 square units

Difficulty: 1 Section: 4

67. Find the area under the curve $y = 3 + x^3$ over the interval $[0, 2]$. To do this, divide the interval into n equal subintervals, calculate the area of the corresponding circumscribed polygon and then let $n \rightarrow \infty$.

Answer: 10 square units

Difficulty: 2 Section: 4

68. Use geometry to calculate $\int_1^3 (2x + 3) \, dx$.

Answer: 14

Difficulty: 1 Section: 5

69. Use geometry to calculate $\int_1^2 (2x - 1) \, dx$.

Answer: 2

Difficulty: 1 Section: 5

70. Use geometry to calculate $\int_{-1}^2 (2 + 4x) \, dx$.

Answer: 12

Difficulty: 1 Section: 5

71. Evaluate $\int_1^2 (3 - x) \, dx$ using the definition of the definite integral.

Answer: $\frac{3}{2}$

Difficulty: 2 Section: 5

72. Evaluate $\int_2^3 (x^2 + 6) \, dx$ using the definition of the definite integral.

Answer: $\frac{37}{3}$

Difficulty: 2 Section: 5

73. Evaluate $\int_0^3 (x^2 + 2) \, dx$ using the definition of the definite integral.

Answer: 15

Difficulty: 2 Section: 5

74. Evaluate $\int_3^3 (\cos^2 2x + \sqrt{x}) \, dx$.

Answer: 0

Difficulty: 1 Section: 5

75. Evaluate $\int_4^1 x \, dx$.

Answer: $-\frac{15}{2}$

Difficulty: 1 Section: 5

76. If $\int_{-2}^1 f(x) \, dx = 4$ and $\int_0^1 f(x) \, dx = 2$, find $\int_{-2}^0 f(x) \, dx$.

Answer: 2

Difficulty: 1 Section: 5

77. Find $D_x \left[\int_0^x \sin(5w) \, dw \right]$.

Answer: $\sin 5x$

Difficulty: 1 Section: 6

78. Find $D_z \left[\int_{-1}^z (2x - 5)^5 \, dx \right]$.

Answer: $(2z - 5)^5$

Difficulty: 1 Section: 6

79. Find $D_x \left[\int_x^0 (2p^3 - 7) \, dp \right]$.

Answer: $7 - 2p^3$

Difficulty: 1 Section: 6

80. Find $\frac{d}{dy} \left[\int_y^{-1} (\cos 5x - 3x^2) \, dx \right]$.

Answer: $3y^2 - \cos 5y$

Difficulty: 1 Section: 6

81. Find $\frac{d}{dx} \left[\int_y^{x^3} \cos(3w) \, dw \right]$.

Answer: $3x^2 \cos(3x^3)$

Difficulty: 2 Section: 6

82. Find $\frac{d}{dx} \left[\int_0^{\sin x} (2t^2 + t) dt \right]$.

Answer: $(2 \sin^2 x + \sin x) \cdot \cos x$

Difficulty: 2 Section: 6

83. Find $\frac{d}{dy} \left[\int_{-1}^{3y} (1 - 3t^2) dt \right]$.

Answer: $3(1 - 27y^2)$

Difficulty: 2 Section: 6

84. If $\int_0^2 f(x) dx = 3$ and $\int_{-3}^0 f(x) dx = -3$, find $\int_{-3}^2 f(x) dx$.

Answer: 0

Difficulty: 1 Section: 7

85. Use the Second Fundamental Theorem of Calculus to evaluate $\int_1^2 (2t - 1) dt$.

Answer: 2

Difficulty: 1 Section: 7

86. Use the Second Fundamental Theorem of Calculus to evaluate $\int_1^2 (3 - x) dx$.

Answer: $\frac{3}{2}$

Difficulty: 1 Section: 7

87. Use the Second Fundamental Theorem of Calculus to evaluate $\int_1^4 \pi x dx$.

Answer: $\frac{15\pi}{2}$

Difficulty: 1 Section: 7

88. Evaluate $\int_3^{-1} 4x^3 dx$.

Answer: -80

Difficulty: 1 Section: 7

89. Use the Second Fundamental Theorem of Calculus to evaluate $\int_{-1}^2 (2 + 4t) dt$.

Answer: 12

Difficulty: 1 Section: 7

90. Use the Second Fundamental Theorem of Calculus to evaluate $\int_1^9 \sqrt{x} dx$.

Answer: $\frac{52}{3}$

Difficulty: 1 Section: 7

91. Use the Second Fundamental Theorem of Calculus to evaluate $\int_2^3 \frac{4}{x^3} dx$.

Answer: $\frac{5}{18}$

Difficulty: 1 Section: 7

92. Use the Second Fundamental Theorem of Calculus to evaluate $\int_1^2 \frac{x^3 + 1}{x^2} dx$.

Answer: 2

Difficulty: 2 Section: 7

93. Use the Second Fundamental Theorem of Calculus to evaluate $\int_{-2}^2 (4 - x^2) dx$.

Answer: $\frac{32}{3}$

Difficulty: 2 Section: 7

94. Use the Second Fundamental Theorem of Calculus to evaluate $\int_{-1}^1 (x^2 - 3x) dx$.

Answer: $\frac{2}{3}$

Difficulty: 2 Section: 7

95. Use the Second Fundamental Theorem of Calculus to evaluate $\int_{-1}^3 (6x^2 - 1) dx$.

Answer: 52

Difficulty: 2 Section: 7

96. Use the Second Fundamental Theorem of Calculus to evaluate $\int_2^3 (x^2 + 6) dx$.

Answer: $\frac{37}{3}$

Difficulty: 2 Section: 7

97. Use the Second Fundamental Theorem of Calculus to evaluate $\int_{-1}^1 (x^2 - 1)^2 dx$.

Answer: $\frac{16}{15}$

Difficulty: 2 Section: 7

98. Use the Second Fundamental Theorem of Calculus to evaluate $\int_0^1 (z^2 + 1)^2 dz$.

Answer: $\frac{28}{15}$

Difficulty: 2 Section: 7

99. Use the Second Fundamental Theorem of Calculus to evaluate $\int_{-1}^2 (w + 2 - w^2) dw$.

Answer: $\frac{9}{2}$

Difficulty: 2 Section: 7

100. Use the Second Fundamental Theorem of Calculus to evaluate $\int_0^{\pi/6} \cos 2x dx$.

Answer: $\frac{\sqrt{3}}{4}$

Difficulty: 2 Section: 7

101. Evaluate $\int_1^2 2\sqrt{2-x} dx$.

Answer: $\frac{4}{3}$

Difficulty: 2 Section: 7

102. Evaluate $\int_0^2 x^2 \sqrt{x^3 + 1} dx$.

Answer: $\frac{52}{9}$

Difficulty: 2 Section: 8

103. Evaluate $\int_2^3 x \sqrt{x^2 - 4} dx$.

Answer: $\frac{5\sqrt{5}}{3}$

Difficulty: 2 Section: 8

104. Evaluate $\int_0^2 x^2 \sqrt{x^3 + 1} \, dx$.

Answer: $\frac{52}{9}$

Difficulty: 2 Section: 8

105. Find $\int \left(1 - \frac{3}{x}\right)^5 \frac{1}{x^2} \, dx$.

Answer: $\frac{1}{18} \left(1 - \frac{3}{x}\right)^6 + C$

Difficulty: 2 Section: 8

106. Find $\int \frac{5}{(7x - 2)^2} \, dx$.

Answer: $-\frac{5}{7(7x - 2)} + C$

Difficulty: 1 Section: 8

107. Find $\int \frac{x}{\sqrt[3]{1 + x^2}} \, dx$.

Answer: $\frac{3}{4} (1 + x^2)^{2/3} + C$

Difficulty: 2 Section: 8

108. Find $\int \frac{x}{\sqrt[5]{1 - x^2}} \, dx$.

Answer: $-\frac{5}{8} (1 - x^2)^{4/5} + C$

Difficulty: 2 Section: 8

109. Find $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} \, dx$.

Answer: $-\frac{2}{1 + \sqrt{x}} + C$

Difficulty: 2 Section: 8

110. Find $\int \frac{x - 2}{(x^2 - 4x + 3)^3} \, dx$.

Answer: $-\frac{1}{4(x^2 - 4x + 3)^2} + C$

Difficulty: 2 Section: 8

111. Find $\int (3x + 2) \sqrt{3x^2 + 4x + 9} dx$.

Answer: $\frac{1}{3} (3x^2 + 4x + 9)^{3/2} + C$

Difficulty: 2 Section: 8

112. Find $\int \frac{\sqrt[4]{1-x^{-1}}}{x^2} dx$.

Answer: $\frac{4}{5} (1 - x^{-1})^{5/4} + C$

Difficulty: 2 Section: 8

113. Find $\int \cos^2 x \sin x dx$.

Answer: $-\frac{1}{3} \cos^3 x + C$

Difficulty: 2 Section: 8

114. Find $\int 3t [\sec^2 (2t^2)] dt$.

Answer: $\frac{3}{4} \tan (2t^2) + C$

Difficulty: 2 Section: 8

115. Find $\int \sec 3x \tan 3x dx$.

Answer: $\frac{1}{3} \sec 3x + C$

Difficulty: 2 Section: 8

116. Find $\int [\sec^2 2x + x \sec (x^2) \tan (x^2)] dx$.

Answer: $\frac{1}{2} [\tan 2x + \sec (x^2)] + C$

Difficulty: 2 Section: 8

117. Find $\int \frac{1}{2} \sec^2 \left(\frac{3x}{2} \right) dx$.

Answer: $\frac{1}{3} \tan \left(\frac{3x}{2} \right) + C$

Difficulty: 2 Section: 8

118. Find $\int \frac{\cos x}{\sqrt{\sin x}} dx$.

Answer: $2\sqrt{\sin x} + C$

Difficulty: 2 Section: 8

119. Find $\int \frac{\cos x}{(1 - \sin x)^2} dx$.

Answer: $\frac{1}{1 - \sin x} + C$

Difficulty: 2 Section: 8

120. Find $\int \cos^2 2x \sin 2x dx$.

Answer: $-\frac{1}{6} \cos^3 2x + C$

Difficulty: 2 Section: 8

121. Find $\int \sqrt{\frac{\sin x}{\cos^5 x}} dx$.

Answer: $\frac{2}{3} \tan^{3/2} x + C$

Difficulty: 2 Section: 8

122. Find $\int \frac{\cos 3\sqrt{x}}{\sqrt{x}} dx$.

Answer: $\frac{2}{3} \sin 3\sqrt{x} + C$

Difficulty: 2 Section: 8

123. Find $\int \sin^2 x dx$ given that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

Answer: $\frac{1}{2}x - \frac{1}{4} \sin 2x + C$

Difficulty: 2 Section: 8

124. Let $f(x) = \cos 2x$. Find an antiderivative whose graph passes through $(\frac{\pi}{6}, 1)$.

Answer: $\frac{1}{2} \sin 2x + \frac{4 - \sqrt{3}}{4}$

Difficulty: 1 Section: 8

125. Let $f(x) = \cos 3x - \sin 3x$. Find an antiderivative whose graph passes through $(\pi, \frac{5}{3})$.

Answer: $\frac{1}{3}(\sin 3x + \cos 3x) + 2$

Difficulty: 1 Section: 8

126. Let $f(x) = \sec^2 \frac{x}{2}$. Find an antiderivative whose graph passes through $(\frac{\pi}{2}, 0)$.

Answer: $2 \tan \frac{x}{2} - 2$

Difficulty: 1 Section: 8

127. Let $f(x) = x^2 \sqrt{4 + 5x^3}$. Find an antiderivative whose graph passes through $(0, 1)$.

Answer: $\frac{2}{45} (4 + 5x^3)^{3/2} + \frac{29}{45}$

Difficulty: 2 Section: 8

128. Let $f(x) = (\cos x) \sqrt{4 + 5 \sin x}$. Find an antiderivative whose graph passes through $(\frac{\pi}{2}, \frac{18}{5})$.

Answer: $\frac{2}{15} (4 + 5 \sin x)^{3/2}$

Difficulty: 2 Section: 8

129. Let $f(x) = x^3 \sqrt{1 - x^2}$. Find an antiderivative whose graph passes through $(0, 5)$.

Answer: $-\frac{3}{8} (1 - x^2)^{4/3} + \frac{43}{8}$

Difficulty: 2 Section: 8

130. A particle is moving along a line with acceleration $a(t) = (2t + 1)^6$. Find the velocity function if $v(0) = 0$.

Answer: $\frac{1}{14} (2t + 1)^7 - \frac{1}{14}$

Difficulty: 2 Section: 8

131. A particle is moving along a line with velocity $v(t) = t \cos(t^2)$. Find the position function if $s(\sqrt{\pi}) = \pi$.

Answer: $\frac{1}{2} \sin(t^2) + \pi$

Difficulty: 2 Section: 8

132. A particle is moving along a line with velocity $v(t) = \sin 2t + \cos 2t$. Find the position function $s(t)$ if $s(\frac{\pi}{4}) = 2$.

Answer: $\frac{1}{2} (\sin 2t - \cos 2t) + \frac{3}{2}$

Difficulty: 2 Section: 8

133. Evaluate $\int_0^{4/3} \sqrt{1 + \frac{9}{4}x} dx$.

Answer: $\frac{56}{27}$

Difficulty: 1 Section: 8

134. Evaluate $\int_0^1 x^2 \sqrt{4 + 5x^3} dx$.

Answer: $\frac{38}{45}$

Difficulty: 1 Section: 8

135. Evaluate $\int_0^2 \frac{1}{(3x + 2)^2} dx$.

Answer: $\frac{1}{8}$

Difficulty: 2 Section: 8

136. Evaluate $\int_4^8 \frac{x}{\sqrt{x^2 - 15}} dx$.

Answer: 6

Difficulty: 2 Section: 8

137. Evaluate $\int_0^2 \frac{4y}{\sqrt{25 - 4y^2}} dy$.

Answer: 2

Difficulty: 2 Section: 8

138. Evaluate $\int_0^1 18x \sqrt{3x^2 + 1} dx$.

Answer: 14

Difficulty: 2 Section: 8

139. Evaluate $\int_0^{\sqrt{2}/4} \frac{t}{\sqrt{1 - 4t^2}} dt$.

Answer: $\frac{1}{4} \left(1 - \frac{\sqrt{2}}{2} \right)$

Difficulty: 2 Section: 8

140. Evaluate $\int_4^8 \frac{3t}{\sqrt{t^2 - 15}} dt$.

Answer: 18

Difficulty: 2 Section: 8

141. Evaluate $\int_0^1 (3x + 2) (3x^2 + 4x + 9)^{1/2} dx$.

Answer: $\frac{37}{3}$

Difficulty: 2 Section: 8

142. Evaluate $\int_{\pi/18}^{\pi/9} \sin 3w dw$.

Answer: $\frac{\sqrt{3} - 1}{6}$

Difficulty: 1 Section: 8

143. Evaluate $\int_0^{\pi/2} \cos(6p) dp$.

Answer: 0

Difficulty: 1 Section: 8

144. Evaluate $\int_0^{\pi/2} \sec^2 \frac{x}{2} dx$.

Answer: 2

Difficulty: 1 Section: 8

145. Evaluate $\int_{\pi/6}^{\pi/2} \cos^2 x \sin x dx$.

Answer: $\frac{\sqrt{3}}{8}$

Difficulty: 1 Section: 8

146. Evaluate $\int_0^{\pi/4} \cos^2 2x \sin 2x dx$.

Answer: $\frac{1}{6}$

Difficulty: 1 Section: 8

147. Evaluate $\int_{\pi/2}^{\pi} \sin^3 \theta \cos \theta d\theta$.

Answer: $-\frac{1}{4}$

Difficulty: 1 Section: 8

148. Evaluate $\int_{\pi/4}^{\pi/3} (1 + \cos t) \sin t \, dt$.

Answer: $\frac{1}{8} (4\sqrt{2} - 3)$

Difficulty: 1 Section: 8

149. Evaluate $\int_0^{\pi/2} \frac{\cos x}{(2 - \sin x)^{1/3}} \, dx$.

Answer: $\frac{3}{2} (\sqrt[3]{4} - 1)$

Difficulty: 2 Section: 8

6 Applications of the Integral

1. Find the area of the region bounded by the x -axis, the graph of $f(x) = x^2$, $x = 0$, and $x = 1$.

Answer: $\frac{1}{3}$

Difficulty: 1 Section: 1

2. Find the area of the region bounded by the x -axis, the graph of $f(x) = 4x - 3$, $x = 1$, and $x = 9$.

Answer: 21

Difficulty: 1 Section: 1

3. Find the area of the region bounded by the x -axis, the graph of $f(x) = x^3 + 3$, $x = 0$, and $x = 2$.

Answer: 10

Difficulty: 1 Section: 1

4. Find the area of the region bounded by the x -axis, the graph of $f(x) = x^2 - 4x + 5$, $x = 1$, and $x = 3$.

Answer: $\frac{8}{3}$

Difficulty: 1 Section: 1

5. Find the area of the region bounded by the x -axis, the graph of $f(x) = -x^2 + x + 1$, $x = 0$, and $x = 3$.

Answer: $\frac{7}{6}$

Difficulty: 1 Section: 1

6. Find the area of the region bounded by the x -axis, the graph of $f(x) = \sin x$, $x = 0$, and $x = \pi$.

Answer: 2

Difficulty: 1 Section: 1

7. Find the area between the curves $y = x^2$ and $y = 2x^2 - 4$.

Answer: $\frac{32}{3}$

Difficulty: 1 Section: 1

8. Find the area between the curves $y = x^2$ and $y = 2 - x^2$.

Answer: $\frac{8}{3}$

Difficulty: 1 Section: 1

9. Find the area between the curve $x = \cos^2 3y$ and the y -axis for $0 \leq y \leq \frac{\pi}{4}$.

Answer: $\frac{\pi}{8} - \frac{1}{12}$

Difficulty: 1 Section: 1

10. Find the area bounded by $y = x^2$ and $y = x + 2$.

Answer: $\frac{9}{2}$

Difficulty: 1 Section: 1

11. Find the area bounded by $y = x^2$ and $y = 4$.

Answer: $\frac{32}{3}$

Difficulty: 1 Section: 1

12. Find the area bounded by $y = x^2$ and $y = 2x - x^2$.

Answer: $\frac{1}{3}$

Difficulty: 1 Section: 1

13. Find the area bounded by $y = x^3 - x$ and the x -axis.

Answer: $\frac{1}{2}$

Difficulty: 1 Section: 1

14. Find the area bounded by $y = x^3 - 4x$ and the x -axis.

Answer: 8

Difficulty: 1 Section: 1

15. Find the area bounded by $x = y^2$ and $x = 2y - y^2$.

Answer: $\frac{1}{3}$

Difficulty: 2 Section: 1

16. Find the area bounded by $x = y - y^3$ and the y -axis.

Answer: $\frac{1}{2}$

Difficulty: 2 Section: 1

17. Interpret $\int_{-2}^2 \sqrt{4-x^2} dx$ as an area and evaluate the definite integral without finding an antiderivative.

Answer: The integral gives the area of a semicircle of radius 2 and the value is 2π .

Difficulty: 2 Section: 1

18. Interpret $\int_{-3}^3 \sqrt{9-x^2} dx$ as an area and evaluate the definite integral without finding an antiderivative.

Answer: The integral gives the area of a semicircle of radius 3 and the value is $\frac{9}{2}\pi$.

Difficulty: 2 Section: 1

19. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Answer: πab

Difficulty: 3 Section: 1

20. The base of a solid is the region bounded by $y = x^2$ and $y = x$. A cross section by a plane perpendicular to the x -axis is a square, one side of which is in the base. Find the volume.

Answer: $\frac{1}{30}$

Difficulty: 1 Section: 2

21. The base of a solid is the region bounded by $y = x^2$ and $y = 2x$. A cross section by a plane perpendicular to the x -axis is a square, one side of which is in the base. Find the volume.

Answer: $\frac{16}{15}$

Difficulty: 1 Section: 2

22. The base of a solid is the region bounded by $y = x^2$ and $y = 3x$. A cross section by a plane perpendicular to the x -axis is a square, one side of which is in the base. Find the volume.

Answer: $\frac{81}{10}$

Difficulty: 1 Section: 2

23. The base of a solid is the region bounded by the ellipse $\frac{x^2}{4} + y^2 = 1$. A cross section by a plane perpendicular to the x -axis is a circle with diameter in the base. Find the volume.

Answer: $\frac{8\pi}{3}$

Difficulty: 1 Section: 2

24. The base of a solid is the region bounded by the ellipse $x^2 + \frac{y^2}{4} = 1$. A cross section by a plane perpendicular to the x -axis is a circle with diameter in the base. Find the volume.

Answer: $\frac{16\pi}{3}$

Difficulty: 1 Section: 2

25. The base of a solid is the region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. A cross section by a plane perpendicular to the x -axis is a circle with diameter in the base. Find the volume.

Answer: 16π

Difficulty: 1 Section: 2

26. The base of a solid is the region bounded by $y = x$ and $y = x^2$. A cross section by a plane perpendicular to the x -axis is a circle with diameter in the base. Find the volume.

Answer: $\frac{\pi}{120}$

Difficulty: 1 Section: 2

27. The base of a solid is the region bounded by $y = 2x$ and $y = x^2$. A cross section by a plane perpendicular to the x -axis is a circle with diameter in the base. Find the volume.

Answer: $\frac{4\pi}{15}$

Difficulty: 1 Section: 2

28. The base of a solid is the region bounded by $y = 3x$ and $y = x^2$. A cross section by a plane perpendicular to the x -axis is a circle with diameter in the base. Find the volume.

Answer: $\frac{81\pi}{40}$

Difficulty: 1 Section: 2

29. The base of a solid is the region inside the ellipse $x^2 + \frac{y^2}{4} = 1$. A cross section by a plane perpendicular to the x -axis is a square with one side in the base. Find the volume.

Answer: $\frac{64}{3}$

Difficulty: 1 Section: 2

30. The base of a solid is the region inside the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. A cross section by a plane perpendicular to the x -axis is a square with one side in the base. Find the volume.

Answer: 64

Difficulty: 1 Section: 2

31. The region bounded by $y = \sqrt{1+x^2}$, the x -axis, the line $x = 1$ and the line $x = 2$ is rotated about the x -axis to form a solid. Find the volume.

Answer: $\frac{10\pi}{3}$

Difficulty: 1 Section: 2

32. The region bounded by $y = \sqrt{1+x^2}$, the x -axis, the line $x = 1$ and the line $x = 3$ is rotated about the x -axis to form a solid. Find the volume.

Answer: $\frac{32\pi}{3}$

Difficulty: 1 Section: 2

33. The region bounded by $y = \sqrt{1+x^2}$, the x -axis, the line $x = 1$ and the line $x = 2$ is rotated about the y -axis to form a solid. Find the volume.

Answer: $\frac{2\pi}{3} (5^{3/2} - 2^{3/2})$

Difficulty: 1 Section: 2

34. The region bounded by $y = \sqrt{2+x^2}$, the x -axis, the line $x = 1$ and the line $x = 2$ is rotated about the x -axis to form a solid. Find the volume.

Answer: $\frac{13\pi}{3}$

Difficulty: 1 Section: 2

35. The region bounded by $y = \sqrt{2+x^2}$, the x -axis, the line $x = 1$ and the line $x = 2$ is rotated about the y -axis to form a solid. Find the volume.

Answer: $\frac{2\pi}{3} (6^{3/2} - 3^{3/2})$

Difficulty: 1 Section: 2

36. Find the volume of the solid formed by revolving the region bounded by $y = x^2$, $x = 3$, and $y = 0$ about the y -axis.

Answer: $\frac{81\pi}{2}$

Difficulty: 1 Section: 2

37. Find the volume of the solid formed by revolving the region bounded by $y = x^2 - 2x$ and the x -axis about the line $y = 3$.

Answer: $\frac{136\pi}{15}$

Difficulty: 1 Section: 2

38. Find the volume of the solid formed by revolving the region bounded by $\frac{x^2}{9} + \frac{y^2}{4} = 1$ about the x -axis.

Answer: 16π

Difficulty: 1 Section: 2

39. Find the volume of the solid formed by revolving the region bounded by $x^2 + \frac{y^2}{4} = 1$ about the y -axis.

Answer: $\frac{8\pi}{3}$

Difficulty: 1 Section: 2

40. Find the volume of the solid formed by revolving the region bounded by $y = \sqrt{x}$, $x = 9$, and the x -axis about the y -axis.

Answer: $\frac{972\pi}{5}$

Difficulty: 1 Section: 2

41. Find the volume of the solid formed by revolving the region bounded by $y = \sqrt{x}$, $x = 9$, and the x -axis about the x -axis.

Answer: $\frac{81\pi}{2}$

Difficulty: 1 Section: 2

42. Find the volume of the solid formed by revolving the region bounded by $x^2 + \frac{y^2}{4} = 1$ about the x -axis.

Answer: $\frac{16\pi}{3}$

Difficulty: 1 Section: 2

43. The region bounded by $y = x$, $y = x^2$, $x = 1$, and $x = 2$ is rotated about the y -axis to form a solid. Use shells to find the volume.

Answer: $\frac{17\pi}{6}$

Difficulty: 1 Section: 3

44. The region bounded by $y = x$, $y = x^2$, $x = 1$, and $x = 2$ is rotated about the line $x = 2$. Use shells to find the volume.

Answer: $\frac{\pi}{2}$

Difficulty: 1 Section: 3

45. The region bounded by $y = x$, $y = x^2$, $x = 1$, and $x = 2$ is rotated about the line $x = 1$. Use shells to find the volume.
 Answer: $\frac{7\pi}{6}$
 Difficulty: 1 Section: 3
46. The region bounded by $y = \sqrt{1 + x^2}$, the x -axis, the line $x = 1$, and the line $x = 2$ is revolved about the y -axis to form a solid. Use shells to find the volume.
 Answer: $\frac{2\pi}{3} (5^{3/2} - 2^{3/2})$
 Difficulty: 1 Section: 3
47. The region bounded by $y = \sqrt{2 + x^2}$, the x -axis, the line $x = 1$, and the line $x = 2$ is revolved about the y -axis to form a solid. Use shells to find the volume.
 Answer: $\frac{2\pi}{3} (6^{3/2} - 3^{3/2})$
 Difficulty: 1 Section: 3
48. The region bounded by $y = x^2 - 2x$ and the x -axis is revolved about the line $x = 2$. Use shells to find the volume.
 Answer: $\frac{8\pi}{3}$
 Difficulty: 1 Section: 3
49. The region bounded by $y = x^2 - 2x$ and the x -axis is revolved about the line $x = 3$. Use shells to find the volume.
 Answer: $\frac{16\pi}{3}$
 Difficulty: 1 Section: 3
50. The region bounded by $y = x^2$ and $y = x + 2$ is revolved about the line $x = 2$. Use shells to find the volume.
 Answer: $\frac{27\pi}{2}$
 Difficulty: 1 Section: 3
51. The region bounded by $y = x^2$ and $y = x + 2$ is revolved about the line $x = -1$. Use shells to find the volume.
 Answer: $\frac{27\pi}{2}$
 Difficulty: 1 Section: 3
52. The region bounded by $y = 2x^2$ and the line $y = 2$ is revolved about the line $x = 1$. Use shells to find the volume.
 Answer: $\frac{16\pi}{3}$
 Difficulty: 1 Section: 3

53. The region bounded by $y = 2x^2$ and the line $y = 2$ is revolved about the line $x = 2$. Use shells to find the volume.

Answer: $\frac{32\pi}{3}$

Difficulty: 1 Section: 3

54. Write the integral whose value is the length of the curve $y = x^2$ from $x = 1$ to $x = 4$.

Answer: $\int_1^4 \sqrt{1 + 4x^2} dx$

Difficulty: 1 Section: 4

55. Write the integral whose value is the length of the curve $y = x^3$ from $x = 0$ to $x = 4$.

Answer: $\int_0^4 \sqrt{1 + 9x^4} dx$

Difficulty: 1 Section: 4

56. Write the integral whose value is the length of the curve $y = \sec(\pi x)$ from $x = 0$ to $x = \frac{1}{4}$.

Answer: $\int_0^{1/4} \sqrt{1 + \pi^2 \sec^2(\pi x) \tan^2(\pi x)} dx$

Difficulty: 1 Section: 4

57. Write the integral whose value is the length of the curve $y = x - x^2$ from $x = 0$ to $x = 2$.

Answer: $\int_0^2 \sqrt{1 + (1 - 2x)^2} dx$

Difficulty: 1 Section: 4

58. Write the integral whose value is the length of the curve $x = 2y - y^3$ from $y = 1$ to $y = 6$.

Answer: $\int_1^6 \sqrt{1 + (2 - 3y^2)^2} dy$

Difficulty: 1 Section: 4

59. The curve $y = x^2$ from $x = 1$ to $x = 4$ is rotated about the x -axis to form a surface. Write the integral whose value is the area of that surface.

Answer: $2\pi \int_1^4 x^2 \sqrt{1 + 4x^2} dx$

Difficulty: 1 Section: 4

60. The curve $y = x^3$ from $x = 0$ to $x = 4$ is rotated about the x -axis to form a surface. Write the integral whose value is the area of that surface.

Answer: $2\pi \int_0^4 x^3 \sqrt{1 + 9x^4} dx$

Difficulty: 1 Section: 4

61. The curve $y = \sec x$ from $x = 0$ to $x = \frac{\pi}{4}$ is rotated about the x -axis to form a surface. Write the integral whose value is the area of that surface.

Answer: $2\pi \int_0^{\pi/4} \sec x \sqrt{1 + \sec^2 x \tan^2 x} dx$

Difficulty: 1 Section: 4

62. The curve $y = \sqrt{x}$ from $x = 0$ to $x = 9$ is rotated about the y -axis to form a surface. Write the integral whose value is the area of that surface.

Answer: $2\pi \int_0^3 y^2 \sqrt{1 + 4y^2} dy$

Difficulty: 1 Section: 4

63. The curve $y = \sqrt[3]{x}$ from $x = 0$ to $x = 8$ is rotated about the y -axis to form a surface. Write the integral whose value is the area of that surface.

Answer: $2\pi \int_0^2 y^3 \sqrt{1 + 9y^4} dy$

Difficulty: 1 Section: 4

64. The curve $y = \sqrt[3]{x^2}$ from $x = 1$ to $x = 8$ is rotated about the y -axis to form a surface. Write the integral whose value is the area of that surface.

Answer: $2\pi \int_1^4 y^{3/2} \sqrt{1 + \frac{9}{4}y^{1/4}} dy$

Difficulty: 1 Section: 4

65. Find the area of the surface generated by rotating the curve $x = t$, $y = t^3$ from $t = 1$ to $t = 2$ about the x -axis.

Answer: $\frac{\pi}{27} (145^{3/2} - 10^{3/2})$

Difficulty: 2 Section: 4

66. Find the area of the surface generated by rotating the curve $x = 1 + t^2$ and $y = t$ from $t = 0$ to $t = 4$ about the x -axis.

Answer: $\frac{\pi}{6} (65^{3/2} - 1)$

Difficulty: 2 Section: 4

67. Find the length of the curve given by $y = t^3$ and $x = t^2$ from $t = 0$ to $t = 1$.

Answer: $\frac{1}{27} (13^{3/2} - 8)$

Difficulty: 2 Section: 4

68. A cylindrical water tank 6 feet in diameter and 8 feet long is lying on its side. If the tank is full of water, how much work is done in pumping the water to a height one foot above the top of the tank? Use 62.5 as the density factor.

Answer: 288π (62.5) foot-pounds

Difficulty: 2 Section: 5

69. A cylindrical water tank 6 feet in diameter and 8 feet long is lying on its side. If the tank is half full of water, how much work is done in pumping the water over the top of the tank? . Use 62.5 as the density factor.

Answer: 216π (62.5) foot-pounds

Difficulty: 2 Section: 5

70. A water trough is 6 feet long and 2 feet wide at the top. The vertical cross section is in the shape of the parabola $y = x^2$ with the vertex down. If the trough is full, how much work is done in pumping the water over the top of the tank? Use 62.5 as the density factor.

Answer: $\frac{16}{5}$ (62.5) foot-pounds

Difficulty: 2 Section: 5

71. The ends of a water trough 6 feet long have the shape of isosceles trapezoids of lower base 2 feet, upper base 3 feet and altitude 2 feet. If the trapezoid is full, find the work done in pumping the water over the top of the tank. Use 62.5 as the density factor.

Answer: 28 (62.5) foot-pounds

Difficulty: 2 Section: 5

72. A spring has spring constant $k = 3.5$. Find the work done in stretching the spring 4 inches from its natural length.

Answer: 28 inch-pounds

Difficulty: 2 Section: 5

73. A spring has natural length of 10 inches. It takes a force of 4 pounds to stretch it to a length of 12 inches. How much work is done in stretching the spring to a length of 14 inches?

Answer: 16 inch-pounds

Difficulty: 2 Section: 5

74. Find the centroid of the region bounded by $y = x + 2$ and $y = x^2$.

Answer: $\left(\frac{1}{2}, \frac{8}{5}\right)$

Difficulty: 1 Section: 6

75. Find the centroid of the upper half of the ellipse $x^2 + 4y^2 = 4$.

Answer: $\left(0, \frac{4}{3\pi}\right)$

Difficulty: 1 Section: 6

76. The centroid of the upper half of the ellipse $x^2 + 4y^2 = 4$ is $\left(0, \frac{4}{3\pi}\right)$. Use Pappus' Theorem to find the volume of the solid formed by rotating the region about the x -axis.

Answer: $\frac{8\pi}{3}$

Difficulty: 2 Section: 6

77. Use Pappus' Theorem to find the centroid of the upper half of the circle $x^2 + y^2 = 9$. (Note: Rotating the region about the x -axis gives the sphere a radius of 3.)

Answer: $\frac{4}{\pi}$

Difficulty: 2 Section: 6

78. Find the centroid of the region bounded by $y = x^2$ and $y = 1$.

Answer: $\left(0, \frac{3}{5}\right)$

Difficulty: 1 Section: 6

79. Find the centroid of the triangle with vertices $(0, 0)$, $(0, 2)$, and $(3, 0)$.

Answer: $\left(1, \frac{2}{3}\right)$

Difficulty: 1 Section: 6

80. An ellipse with major axis 4 and minor axis 2 is revolved about a line through a vertex on the major axis, perpendicular to the major axis. Find the volume of the solid thus formed.

Answer: $\frac{8\pi}{3}$

Difficulty: 2 Section: 6

7 Transcendental Functions

1. Use properties of $\ln(x)$ to simplify the function $\ln \left[(3x^3 + \sin 2x)^{1/3} \right]$.

Answer: $\frac{1}{3} \ln (3x^3 + \sin 2x)$

Difficulty: 1 Section: 1

2. Use properties of $\ln(x)$ to simplify the function $\ln \left[\frac{\sqrt{x} \cdot \cos 2x \cdot \tan x}{x^8 (1 + x^2)} \right]$.

Answer: $\ln(\cos 2x) + \ln(\tan x) - \frac{15}{2} \ln(x) - \ln(1 + x^2)$

Difficulty: 1 Section: 1

3. Let $f(x) = (x^3 - 1) \ln|x|$. Find $f'(x)$.

Answer: $\frac{x^3 - 1}{x} + 3x^2 \ln|x|$

Difficulty: 2 Section: 1

4. Let $f(x) = \ln|6x - 7|$. Find $f'(x)$.

Answer: $\frac{6}{6x - 7}$

Difficulty: 1 Section: 1

5. Let $y = \ln(1 + x^2)$. Find $\frac{dy}{dx}$.

Answer: $\frac{2x}{1 + x^2}$

Difficulty: 1 Section: 1

6. Let $f(x) = \ln \left| \frac{2x}{3x + 4} \right|$. Find $f'(x)$.

Answer: $\frac{1}{x} - \frac{3}{3x + 4}$

Difficulty: 2 Section: 1

7. Find the derivative of $[\ln|x|]^4$.

Answer: $\frac{4}{x} [\ln|x|]^3$

Difficulty: 2 Section: 1

8. Let $y = \ln|\sin x|$. Find $\frac{dy}{dx}$.

Answer: $\cot x$

Difficulty: 1 Section: 1

9. Let $f(x) = \ln(\cos^2 3x)$. Find $f'(x)$.

Answer: $-6 \tan 3x$

Difficulty: 2 Section: 1

10. Evaluate $\int_1^4 \left(\sqrt{u} + \frac{1}{u} \right) du$.

Answer: $\frac{14}{3} + \ln 4$

Difficulty: 1 Section: 1

11. Evaluate $\int_0^1 \frac{2}{2-x^3} dx$.

Answer: $\frac{\ln 2}{3}$

Difficulty: 1 Section: 1

12. Evaluate $\int_0^{\pi/4} \tan x dx$.

Answer: $\ln(2\sqrt{2})$

Difficulty: 2 Section: 1

13. Find $\int \frac{6x+1}{3x^2+x-19} dx$.

Answer: $\ln|3x^2+x-19| + C$

Difficulty: 2 Section: 1

14. Evaluate $\int \frac{1}{x[\ln(x)]^2} dx$.

Answer: $-\frac{1}{\ln(x)} + C$

Difficulty: 1 Section: 1

15. Let $f(x) = \frac{\sqrt{x^2+1}}{(9x-4)^2}$. Use logarithmic differentiation to find $f'(x)$.

Answer: $\frac{\sqrt{x^2+1}}{(9x-4)^2} \left(\frac{x}{x^2+1} - \frac{18}{9x-4} \right)$

Difficulty: 2 Section: 1

16. Let $f(x) = \sqrt{4x+7}(x-5)^3$. Use logarithmic differentiation to find $f'(x)$, then simplify.

Answer: $\frac{(14x+11)(x-5)^2}{\sqrt{4x+7}}$

Difficulty: 2 Section: 1

17. What is the slope of the tangent line to the graph of $3y - x^2 + \ln|xy| = 2$ at the point $(1, 1)$?

Answer: $\frac{1}{4}$

Difficulty: 2 Section: 1

18. Use implicit differentiation to find $\frac{dy}{dx}$ if $3y - x^2 + \ln|xy| + 5 = 0$.

Answer: $\frac{y(2x^2 - 1)}{x(3y + 1)}$

Difficulty: 2 Section: 1

19. Write the equation of the tangent line to the graph of $y^3 - 4y = x \ln x$ at the point $(1, 2)$.

Answer: $y - 2 = \frac{1}{8}(x - 1)$

Difficulty: 2 Section: 1

20. Find the area of the region in the first quadrant that is bounded by the curves $y = x$, $y = \frac{2}{x} - 1$, and $y = 0$.

Answer: $2 \ln(2) - \frac{1}{2}$ square units

Difficulty: 2 Section: 1

21. Let $f(x) = \frac{1}{3} - \frac{2x}{5}$. Find a formula for $f^{-1}(x)$.

Answer: $f^{-1}(x) = \frac{5}{6} - \frac{5x}{2}$

Difficulty: 1 Section: 2

22. Let $f(x) = 1 + \frac{x}{3}$. Find a formula for $f^{-1}(x)$.

Answer: $f^{-1}(x) = 3x - 3$

Difficulty: 1 Section: 2

23. Find the inverse function of $f(x) = 3(x+2)^2 - 4$, $x \geq -2$.

Answer: $f^{-1}(x) = \sqrt{\frac{x+4}{3}} - 2$, $x \geq -4$

Difficulty: 2 Section: 2

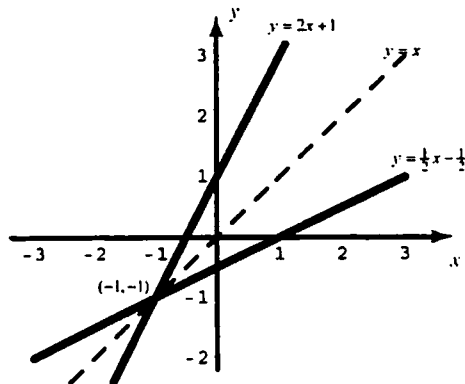
24. Find the inverse function of $f(x) = 2(x-1)^2 - 1$, $x \geq 1$.

Answer: $f^{-1}(x) = \sqrt{\frac{x+1}{2}} + 1$, $x \geq -1$

Difficulty: 2 Section: 1

25. Graph both the function $y = 2x + 1$ and its inverse on the same coordinate axes.

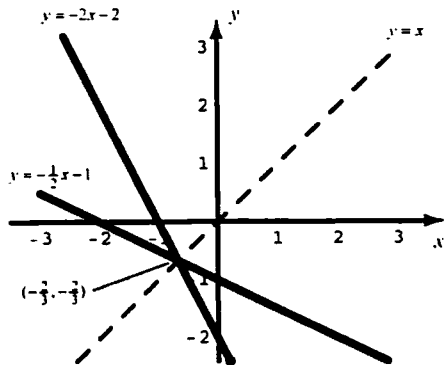
Answer:



Difficulty: 2 Section: 2

26. Graph both the function $y = -2x - 2$ and its inverse on the same coordinate axes.

Answer:



Difficulty: 2 Section: 2

27. Let $f(x) = \frac{x}{x-2}$ and $F(x) = \frac{2x}{x-1}$. Apply the definition to show that $F(x) = f^{-1}(x)$.

Answer: $F[f(x)] = F\left(\frac{x}{x-2}\right) = \frac{2\left(\frac{x}{x-2}\right)}{\left(\frac{x}{x-2}\right) - 1} = \frac{\frac{2x}{x-2}}{\frac{2}{x-2}} = x$

Difficulty: 2 Section: 2

28. Let $f(x) = \frac{1}{x^3+1}$ and $F(x) = \sqrt[3]{\frac{1}{x}} + 1$. Determine whether or not $F(x) = f^{-1}(x)$.

Answer: It does not

Difficulty: 2 Section: 2

29. Let $f(x) = \frac{x+2}{x-5}$ and $F(x) = \frac{5x-2}{x+1}$. Determine whether or not $F(x) = f^{-1}(x)$.

Answer: It does not

Difficulty: 2 Section: 2

30. Let $f(x) = \frac{x-1}{2x+1}$ and $F(x) = -\frac{x+1}{2x-1}$. Determine whether or not $F(x) = f^{-1}(x)$.

Answer: In this case, $F(x) = f^{-1}(x)$

Difficulty: 2 Section: 2

31. Let $f(x) = 4x + 5$. Evaluate $(f^{-1})'(x)$ at $x = -10$.

Answer: $\frac{1}{4}$

Difficulty: 1 Section: 2

32. Let $f(x) = 2 - x^2$ for $x \geq 0$. Evaluate $(f^{-1})'(x)$ at $x = 1$.

Answer: $-\frac{1}{2}$

Difficulty: 1 Section: 2

33. Find the value of the expression $\ln \frac{1}{\sqrt[3]{e}}$.

Answer: $-\frac{1}{3}$

Difficulty: 1 Section: 3

34. Find the value of the expression $e^{\ln(1/2)}$.

Answer: $\frac{1}{2}$

Difficulty: 1 Section: 3

35. Find the value of the expression $\left(\frac{1}{e}\right)^{-\ln 10^2}$

Answer: 100

Difficulty: 1 Section: 3

36. Find the value of the expression $\frac{e^{\ln 7}}{e^{\ln 5}}$.

Answer: $\frac{7}{5}$

Difficulty: 1 Section: 3

37. Let $f(x) = \frac{e^x}{x^2 - 1}$. Find $f'(x)$.

Answer: $\frac{e^x(x^2 - 2x - 1)}{(x^2 - 1)^2}$

Difficulty: 1 Section: 3

38. Solve the equation $\ln|6 - x| = 2$.

Answer: $x = 6 - e^2$ or $x = 6 + e^2$

Difficulty: 2 Section: 3

39. Let $y = 2^\pi e^x$. Find $\frac{dy}{dx}$.

Answer: $2^\pi e^x$

Difficulty: 1 Section: 3

40. Let $f(x) = e^{-\sqrt{x}}$. Find $f'(x)$.

Answer: $-\frac{e^{-\sqrt{x}}}{2\sqrt{x}}$

Difficulty: 2 Section: 3

41. Let $y = \sqrt{e^x} + e^{-x}$. Find $\frac{dy}{dx}$.

Answer: $\frac{1}{2}\sqrt{e^x} - e^{-x}$

Difficulty: 2 Section: 3

42. Find $\frac{dy}{dx}$ and simplify if $y = \frac{\ln x}{e^x}$.

Answer: $\frac{1 - x \ln x}{xe^x}$

Difficulty: 2 Section: 3

43. Let $f(x) = xe^{\sin 3x}$. Find $f'(x)$.

Answer: $e^{\sin 3x} (3x \cos 3x + 1)$

Difficulty: 2 Section: 3

44. Let $f(x) = e^{x \ln x}$. Find $f'(x)$.

Answer: $e^{x \ln x} (1 + \ln x)$

Difficulty: 2 Section: 3

45. Let $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Find $\frac{dy}{dx}$.

Answer: $\frac{4}{(e^x + e^{-x})^2}$

Difficulty: 2 Section: 3

46. Let $f(x) = e^x \sin x + e^x \cos x$. Find $f'(x)$.

Answer: $2e^x \cos x$

Difficulty: 2 Section: 3

47. Let $f(x) = \frac{1}{\sqrt{3e}} e^{\sin 3x}$. Find the slope of the tangent line to the graph of $f(x)$ at the point where $x = \frac{\pi}{18}$.

Answer: $\frac{3}{2}$

Difficulty: 2 Section: 3

48. Use implicit differentiation to find $\frac{dy}{dx}$ if $ye^x - \ln xy = 4$.

Answer: $\frac{y - xy^2 e^x}{xye^x - x}$

Difficulty: 2 Section: 3

49. Find $\int \frac{e^{4x}}{\sqrt{1 - e^{4x}}} dx$.

Answer: $-\frac{1}{2} \sqrt{1 - e^{4x}} + C$

Difficulty: 2 Section: 3

50. Find $\int \frac{e^{4x}}{1 + e^{4x}} dx$.

Answer: $\frac{1}{4} \ln(1 + e^{4x}) + C$

Difficulty: 2 Section: 3

51. Find $\int \frac{(e^x + 2)^2}{e^x} dx$.

Answer: $e^x + 2x - e^{-x} + C$

Difficulty: 2 Section: 3

52. Find $\int xe^{x^2} dx$.

Answer: $\frac{1}{2}e^{x^2} + C$

Difficulty: 2 Section: 3

53. Find $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$.

Answer: $2e^{\sqrt{x}} + C$

Difficulty: 1 Section: 3

54. Evaluate $\int_0^2 (e^x - 3x^2) dx$.

Answer: $e^2 - 9$

Difficulty: 1 Section: 3

55. Evaluate $\int_1^3 x(e^{x^2} - 1) dx$.

Answer: $\frac{1}{2}(e^8 - 1)$

Difficulty: 2 Section: 3

56. Evaluate $\int_1^e \frac{1}{x} \ln x dx$.

Answer: $\frac{1}{2}$

Difficulty: 2 Section: 3

57. Find the area bounded by the curves $y = e^{x/2}$, the y -axis, the x -axis, and the line $x = 2$.

Answer: $2(e - 1)$ square units

Difficulty: 2 Section: 3

58. Find the volume of the solid generated by revolving about the x -axis, the region bounded by $y = e^{-x}$, the x -axis, and the lines $x = 0$ and $x = 3$.

Answer: $\frac{\pi}{2}(1 - e^{-6})$ cubic units.

Difficulty: 2 Section: 3

59. Let $f(x) = 9^x$. Find $f'(x)$.

Answer: $(\ln 9)(9^x)$

Difficulty: 1 Section: 4

60. Let $f(x) = \cos(x \cdot 3^x)$. Find $f'(x)$.

Answer: $-3^x \sin(x \cdot 3^x)(x \ln 3 + 1)$

Difficulty: 2 Section: 4

61. Let $y = 3 \ln(9^x)$. Find $\frac{dy}{dx}$.

Answer: $3 \ln 9$

Difficulty: 1 Section: 4

62. Let $f(x) = 3^{2-x}$. Find $f'(x)$.

Answer: $-2(\ln 3)3^{2-x}$

Difficulty: 2 Section: 4

63. Find $\frac{dy}{dx}$ for $y = 2^{\sin x}$.

Answer: $(\ln 2)(2^{\sin x})(\cos x)$

Difficulty: 1 Section: 4

64. Find $\frac{dy}{dx}$ for $y = x^{\sin x}$.

Answer: $(x^{\sin x})\left(\frac{\sin x}{x} + (\cos x)(\ln x)\right)$

Difficulty: 2 Section: 4

65. Find $\frac{dy}{dx}$ for $y = (\ln x)^{\ln x}$.

Answer: $((\ln x)^{\ln x})\left(\frac{1 + \ln(\ln x)}{x}\right)$

Difficulty: 2 Section: 4

66. Find $\int 3^{2x} dx$.

Answer: $\left(\frac{1}{2 \ln 3}\right)3^{2x} + C$

Difficulty: 1 Section: 4

67. Find $\int x^2 3^{x^3} dx$.

Answer: $\frac{3^{x^3}}{3 \ln 3} + C$

Difficulty: 2 Section: 4

68. Find $\int 10^{1-x} dx$.

Answer: $-\frac{10^{1-x}}{\ln 10} + C$

Difficulty: 2 Section: 4

69. After 2 years at 9% compounded monthly what is the value of an initial investment of \$7000, rounded to the nearest dollar?

Answer: \$8375

Difficulty: 1 Section: 5

70. What is the value of an investment of \$1 after 1 year at 100% annual interest compounded daily? (rounded to the nearest cent).

Answer: \$2.71

Difficulty: 1 Section: 5

71. If an investment of \$3000 is compounded continuously for 2 years at 10% what will it be worth? (round to the nearest dollar).

Answer: \$3664

Difficulty: 1 Section: 5

72. The population of a certain town is approximated by the function $N(t) = 2500e^{0.05t}$, where t is measured in years after 2000. What will be the percent of increase in population from 2001 to 2010? (round to the nearest percent).

Answer: 57%

Difficulty: 2 Section: 5

73. Find the doubling time for a population given by the function $P(t) = 450,000e^{0.02t}$.

Answer: $50 \ln 2 \approx 34.7$ years

Difficulty: 2 Section: 5

74. A bone is found to contain 30% of the carbon-14 that it contained when it was part of a living organism. How long ago did the organism die? (The half life of carbon-14 is 5568 years).

Answer: $\frac{5568 [\ln 0.3]}{\ln 0.50} \approx 9670$ years ago

Difficulty: 2 Section: 5

75. The population P of a certain country is given by the function $P(t) = 450,000e^{kt}$, where t is the time in years from 1990 and k is constant. If the population was 495,000 in 1995, what will the population be in 2010 (answer to the nearest thousandth)?

Answer: 659,000

Difficulty: 2 Section: 5

76. Solve the differential equation $x \frac{dy}{dx} + y = x^3$.

Answer: $y = \frac{x^3}{4} + \frac{C}{x}$

Difficulty: 1 Section: 6

77. Solve the differential equation $x \frac{dy}{dx} + y = x^3$ if $y = 2$ when $x = 2$.

Answer: $y = \frac{x^3}{4}$

Difficulty: 1 Section: 6

78. Solve the differential equation $x \frac{dy}{dx} - y = x^3$.

Answer: $y = \frac{x^3}{2} + Cx$

Difficulty: 1 Section: 6

79. Solve the differential equation $x \frac{dy}{dx} - y = x^3$ if $y = 2$ when $x = 2$.

Answer: $y = \frac{x^3}{2} - x$

Difficulty: 1 Section: 6

80. Solve the differential equation $x \frac{dy}{dx} + 2y = \sin x$.

Answer: $y = \frac{\sin x - x \cos x + C}{x^2}$

Difficulty: 1 Section: 6

81. Solve the differential equation $x \frac{dy}{dx} + 2y = \sin x$ if $y = 0$ when $x = \pi$.

Answer: $y = \frac{\sin x - x \cos x - \pi}{x^2}$

Difficulty: 2 Section: 6

82. Solve the differential equation $x \frac{dy}{dx} - 2y = x^3$.

Answer: $y = x^3 + Cx^2$

Difficulty: 1 Section: 6

83. Solve the differential equation $x \frac{dy}{dx} - 2y = x^3$ if $y = 1$ when $x = 2$.

Answer: $y = x^3 - \frac{7}{4}x^2$

Difficulty: 1 Section: 6

84. Solve the differential equation $x \frac{dy}{dx} + 3y = x^3 + x$.

Answer: $y = \frac{x^3}{6} + \frac{x}{4} + \frac{C}{x^3}$

Difficulty: 1 Section: 6

85. Solve the differential equation $x \frac{dy}{dx} - 3y = x^3 + 2x$.

Answer: $y = x^3 \ln x - x + Cx^3$

Difficulty: 2 Section: 6

86. Determine $\sin^{-1}(-1)$.

Answer: $-\frac{\pi}{2}$

Difficulty: 1 Section: 7

87. Determine $\cos^{-1}\left(-\frac{1}{2}\right)$.

Answer: $\frac{2\pi}{3}$

Difficulty: 1 Section: 7

88. Determine $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$.

Answer: $-\frac{\pi}{4}$

Difficulty: 1 Section: 7

89. Determine $\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$.

Answer: $\frac{5\pi}{6}$

Difficulty: 2 Section: 7

90. Determine $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \tan^{-1}(-1)$.

Answer: $-\frac{7\pi}{12}$

Difficulty: 2 Section: 7

91. Determine $\sin^{-1}(0) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

Answer: $-\frac{\pi}{6}$

Difficulty: 2 Section: 7

92. Determine $\cos^{-1}\left(\sin \frac{\pi}{12}\right)$.

Answer: $\frac{5\pi}{12}$

Difficulty: 2 Section: 7

93. Determine $\tan\left[\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)\right]$.

Answer: $\frac{2}{3}$

Difficulty: 2 Section: 7

94. Determine $\tan(\sin^{-1} x)$ in terms of x .

Answer: $\frac{x}{\sqrt{1-x^2}}$

Difficulty: 2 Section: 7

95. Determine $\sin\left(\tan^{-1} \frac{5}{2x}\right)$ in terms of x .

Answer: $\frac{5}{\sqrt{4x^2+25}}$

Difficulty: 2 Section: 7

96. Determine $\sin\left[\tan^{-1}\left(\frac{x-2}{4}\right)\right]$ in terms of x .

Answer: $\frac{x-2}{\sqrt{x^2-4x+20}}$

Difficulty: 2 Section: 7

97. Determine $\tan\left(\sin^{-1}\sqrt{1-x^2}\right)$ in terms of x .

Answer: $\frac{\sqrt{1-x^2}}{|x|}$

Difficulty: 2 Section: 7

98. Determine $\sin\left(\cos^{-1}\frac{\sqrt{9-4x^2}}{3}\right)$ in terms of x .

Answer: 2

Difficulty: 2 Section: 7

99. Let $f(x) = \tan^{-1}\left(\frac{x-3}{1+3x}\right)$. Find $f'(x)$ and simplify.

Answer: $\frac{1}{x^2+1}$

Difficulty: 2 Section: 7

100. Let $f(x) = x \tan^{-1}\left(\frac{x}{2}\right)$. Find $f'(x)$.

Answer: $\tan^{-1}\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}$

Difficulty: 2 Section: 7

101. Let $y = \tan^{-1}\left(\sqrt{\ln(x^2)}\right)$. Find $\frac{dy}{dx}$.

Answer: $\frac{1}{x[1+\ln(x^2)]\sqrt{\ln(x^2)}}$

Difficulty: 2 Section: 7

102. Find $\frac{dy}{dx}$ if $y = \tan^{-1}(\sin 2x)$.

Answer: $\frac{2 \cos 2x}{1 + \sin^2 2x}$

Difficulty: 2 Section: 7

103. Let $f(x) = x^2 \sec^{-1}(e^x)$. Find $f'(x)$.

Answer: $\frac{x^2}{\sqrt{e^{2x}-1}} + 2x \sec^{-1}(e^x)$

Difficulty: 2 Section: 7

104. Let $y = \sin^{-1}(1-2x)$. Find $\frac{dy}{dx}$.

Answer: $-\frac{1}{\sqrt{x-x^2}}$

Difficulty: 2 Section: 7

105. Let $y = \tan^{-1}(\tan^{-1} x)$. Find $\frac{dy}{dx}$.

Answer: $\frac{1}{(1+x^2)[1+(\tan^{-1} x)^2]}$

Difficulty: 2 Section: 7

106. Evaluate $\int_{3/4}^{3/2} \frac{2}{\sqrt{9-4x^2}} dx$.

Answer: $\frac{\pi}{3}$

Difficulty: 2 Section: 7

107. Evaluate $2 \int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^4}} dx$.

Answer: $\frac{\pi}{6}$

Difficulty: 2 Section: 7

108. Evaluate $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx$.

Answer: $\frac{\pi}{4}$

Difficulty: 2 Section: 7

109. Find $\int \frac{1}{\sqrt{9-16x^2}} dx$.

Answer: $\frac{1}{2} \sin^{-1}\left(\frac{4x}{3}\right) + C$

Difficulty: 2 Section: 7

110. Find $\int \frac{e^{2x}}{1+e^{4x}} dx$.

Answer: $\frac{1}{2} \tan^{-1}(e^{2x}) + C$

Difficulty: 2 Section: 7

111. Find $\int \frac{3}{x\sqrt{x^4-1}} dx$.

Answer: $\frac{3}{2} \sec^{-1}(x^2) + C$ or $-\frac{3}{2} \tan^{-1} \frac{1}{\sqrt{(x^4-1)}}$

Difficulty: 1 Section: 7

112. Find $\int \frac{1}{\sqrt{\frac{25}{4} - (x-2)^2}} dx$.

Answer: $\sin^{-1} \left(\frac{2x-4}{5} \right) + C$

Difficulty: 1 Section: 7

113. Find $\int \frac{x^2}{7+x^6} dx$.

Answer: $\frac{1}{3\sqrt{7}} \tan^{-1} \left(\frac{x^3}{\sqrt{7}} \right) + C$

Difficulty: 1 Section: 7

114. Find $\int \frac{e^{2x} + e^{4x}}{\sqrt{1-e^{4x}}} dx$.

Answer: $\frac{1}{2} \sin^{-1}(e^{2x}) - \frac{1}{2} \sqrt{1-e^{4x}} + C$

Difficulty: 2 Section: 7

115. Find $\int \frac{1}{x\sqrt{1-(\ln x)^2}} dx$.

Answer: $\sin^{-1}(\ln x) + C$

Difficulty: 2 Section: 7

116. Let $f(x) = \cosh^2 3x$. Find $f'(x)$.

Answer: $6 \sinh 3x \cosh 3x = 3 \sinh 6x$

Difficulty: 1 Section: 8

117. Let $f(x) = \tanh(x^2 + 1)$. Find $f'(x)$.

Answer: $2x \operatorname{sech}^2(x^2 + 1)$

Difficulty: 1 Section: 8

118. Let $f(x) = 3e^{\sinh(3x^2)}$. Find $f'(x)$.

Answer: $18x \cosh(3x^2) e^{\sinh(3x^2)}$

Difficulty: 2 Section: 8

119. Let $f(x) = \frac{\sinh x}{1 + \cosh x}$. Find $f'(x)$.

Answer: $\frac{1}{1 + \cosh x}$

Difficulty: 2 Section: 8

120. Let $y = \ln |1 - \cosh 3x|$. Find $\frac{dy}{dx}$.

Answer: $\frac{-3 \sinh 3x}{1 - \cosh 3x}$

Difficulty: 2 Section: 8

121. Let $f(x) = \tanh(\ln |x|)$. Find $f'(x)$.

Answer: $\frac{1}{x} [\operatorname{sech}^2(\ln |x|)]$

Difficulty: 2 Section: 8

122. Find $\int x^2 \sinh(3x^3) dx$.

Answer: $\frac{1}{9} \cosh(3x^3) + C$

Difficulty: 1 Section: 8

123. Find $\int \operatorname{sech}^2 x \tanh x dx$.

Answer: $\frac{1}{2} \tanh^2 x + C$

Difficulty: 1 Section: 8

124. Find $\int \cosh 2x \sqrt{\sinh 2x} dx$.

Answer: $\frac{1}{3} (\sinh 2x)^{3/2} + C$

Difficulty: 1 Section: 8

125. Find $\int x \cosh(x^2) dx$.

Answer: $\frac{1}{2} \sinh(x^2) + C$

Difficulty: 1 Section: 8

126. Find $\int \operatorname{csch} 4x \coth 4x dx$.

Answer: $-\frac{1}{4} \operatorname{csch} 4x + C$

Difficulty: 1 Section: 8

127. Find $\int \frac{\sinh(\ln x)}{3x} dx$.

Answer: $\frac{1}{3} \cosh(\ln x) + C$

Difficulty: 1 Section: 8

128. Let $f(x) = 4 \sinh^{-1}(4x)$. Find $f'(x)$.

Answer: $\frac{16}{\sqrt{1+16x^2}}$

Difficulty: 1 Section: 8

129. Let $f(x) = \cosh^{-1}(2 \cos x)$. Find $f'(x)$.

Answer: $-\frac{2 \sin x}{\sqrt{4 \cos^2 x - 1}}$

Difficulty: 2 Section: 8

130. Let $f(x) = \tanh^{-1}(\sin x)$. Find $f'(x)$.

Answer: $\sec x$

Difficulty: 2 Section: 8

131. Find $\int \frac{e^x}{\sqrt{e^{2x} - 1}} dx$.

Answer: $\cosh^{-1}(e^x) + C$

Difficulty: 2 Section: 8

132. Find $\int \frac{1}{\sqrt{25x^2 + 4}} dx$.

Answer: $\frac{1}{5} \sinh^{-1}\left(\frac{5x}{2}\right) + C$

Difficulty: 2 Section: 8

8 Techniques of Integration

1. Show that $\int \frac{1}{x^2 + 2x + 5} dx = \frac{1}{2} \tan^{-1} \left[\frac{1}{2} (x + 1) \right] + C.$

Answer: The derivative of the right hand side of the equation is equal to the function being integrated.

Difficulty: 1 Section: 1

2. Show that $\int x \ln x dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C.$

Answer: The derivative of the right hand side of the equation is equal to the function being integrated.

Difficulty: 1 Section: 1

3. Show that $\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + C.$

Answer: The derivative of the right hand side of the equation is equal to the function being integrated.

Difficulty: 1 Section: 1

4. Show that $\int \cos^3 x \sin^4 x dx = \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C.$

Answer: The derivative of the right hand side of the equation is equal to the function being integrated.

Difficulty: 1 Section: 1

5. Show that $\int \frac{1}{\sqrt{4 + x^2}} dx = \ln (x + \sqrt{4 + x^2}) + C.$

Answer: The derivative of the right hand side of the equation is equal to the function being integrated.

Difficulty: 1 Section: 1

6. Find $\int \frac{1}{x^2 + 4x + 5} dx.$

Answer: $\tan^{-1} (x + 2) + C$

Difficulty: 2 Section: 1

7. Find $\int \frac{1}{6 + 3x^2} dx.$

Answer: $\frac{\sqrt{2}}{6} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + C$

Difficulty: 1 Section: 1

8. Find $\int \frac{1}{\sqrt{x^2 + 8x + 25}} dx$.

Answer: $\sinh^{-1} \left(\frac{x+4}{3} \right) + C$

Difficulty: 1 Section: 1

9. Find $\int \frac{3x+3}{\sqrt{x^2+2x-5}} dx$.

Answer: $3\sqrt{x^2+2x-5} + C$

Difficulty: 1 Section: 1

10. Find $\int \frac{2}{\sqrt{2x-x^2}} dx$.

Answer: $2\sin^{-1}(x-1) + C$

Difficulty: 1 Section: 1

11. Find $\int \frac{x-5}{x^2+9} dx$.

Answer: $\frac{1}{2} \ln(x^2+9) - \frac{5}{3} \tan^{-1} \left(\frac{x}{3} \right) + C$

Difficulty: 2 Section: 1

12. Find $\int \frac{2x-1}{x^2-6x+13} dx$.

Answer: $\ln|x^2-6x+13| + \frac{5}{2} \tan^{-1} \left(\frac{x-3}{2} \right) + C$

Difficulty: 2 Section: 1

13. Find $\int \frac{2x-1}{x^2+4x+8} dx$.

Answer: $\ln|x^2+4x+8| - \frac{5}{2} \tan^{-1} \left(\frac{x+2}{2} \right) + C$

Difficulty: 2 Section: 1

14. Find $\int \frac{1}{\sqrt{9+16x-4x^2}} dx$.

Answer: $\frac{1}{2} \sin^{-1} \left(\frac{2x-4}{5} \right) + C$

Difficulty: 2 Section: 1

15. Evaluate $\int_0^{2\sqrt{3}-2} \frac{1}{x^2 + 4x + 8} dx$.

Answer: $\frac{\pi}{24}$

Difficulty: 1 Section: 1

16. Evaluate $\int_{3/4}^{3/2} \frac{1}{\sqrt{9-4x^2}} dx$.

Answer: $\frac{\pi}{6}$

Difficulty: 2 Section: 1

17. Evaluate $\int_0^{\sqrt{3}-1} \frac{1}{\sqrt{3-2x-x^2}} dx$.

Answer: $\frac{\pi}{6}$

Difficulty: 2 Section: 1

18. Find $\int \sin^5 7x dx$.

Answer: $-\frac{1}{7} \left(\cos 7x - \frac{2 \cos^3 7x}{3} + \frac{\cos^5 7x}{5} \right) + C$

Difficulty: 2 Section: 2

19. Find $\int \cos^3 4x dx$.

Answer: $\frac{1}{4} \sin 4x - \frac{\sin^3 4x}{12} + C$

Difficulty: 2 Section: 2

20. Find $\int \cos^2 5x dx$.

Answer: $\frac{x}{2} + \frac{\sin 10x}{20} + C$

Difficulty: 2 Section: 2

21. Find $\int \sin^3 \left(\frac{x}{2} \right) dx$.

Answer: $-2 \cos \left(\frac{x}{2} \right) + \frac{2}{3} \cos^3 \left(\frac{x}{2} \right) + C$

Difficulty: 2 Section: 2

22. Find $\int \sin x (1 + \cos^2 x) dx$.

Answer: $-\cos x - \frac{1}{3} \cos^3 x + C$

Difficulty: 1 Section: 2

23. Find $\int \cos^2 x \sin^4 x dx$.

Answer: $\frac{1}{8} \left[\frac{x}{2} - \frac{\sin 4x}{8} - \frac{\sin^3 2x}{6} \right] + C$

Difficulty: 2 Section: 2

24. Find $\int \cos^3 x \sin^4 x dx$.

Answer: $\frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C$

Difficulty: 2 Section: 2

25. Find $\int \sin^3 x \cos^2 x dx$.

Answer: $-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

Difficulty: 2 Section: 2

26. Find $\int \sin^2 x \cos^3 x dx$.

Answer: $\frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$

Difficulty: 2 Section: 2

27. Find $\int \frac{\cos^3 x}{\sqrt{\sin x}} dx$.

Answer: $2\sqrt{\sin x} - \frac{2}{5} \sqrt{\sin^5 x} + C$

Difficulty: 2 Section: 2

28. Evaluate $\int_0^{\pi/4} \cos^2 x \sin^4 x dx$.

Answer: $\frac{1}{16} \left(\frac{\pi}{4} - \frac{1}{3} \right)$

Difficulty: 2 Section: 2

29. Evaluate $\int_{\pi/3}^{\pi} \frac{\cos^2 x}{\csc x} dx$.

Answer: $\frac{3}{8}$

Difficulty: 2 Section: 2

30. Evaluate $\int_0^{\pi/2} \cos^3\left(\frac{x}{2}\right) \sin^4\left(\frac{x}{2}\right) dx$.

Answer: $\sqrt{2} \left(\frac{1}{20} - \frac{1}{56} \right)$

Difficulty: 2 Section: 2

31. Evaluate $\int_0^{\pi/40} \sin^2 5x \cos^2 5x dx$.

Answer: $\frac{\pi - 2}{320}$

Difficulty: 2 Section: 2

32. Find $\int \sin 3x \sin 2x dx$.

Answer: $\frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C$

Difficulty: 1 Section: 2

33. Find $\int \cos x \cos 5x dx$.

Answer: $\frac{1}{2} \left(\frac{\sin 6x}{6} + \frac{\sin 4x}{4} \right) + C$

Difficulty: 1 Section: 2

34. Find $\int \tan^2 x \sec^2 x dx$.

Answer: $\frac{1}{3} \tan^3 x + C$

Difficulty: 1 Section: 2

35. Find $\int \tan^3 x \sec^3 x dx$.

Answer: $\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

Difficulty: 1 Section: 2

36. Find $\int \tan^6\left(\frac{x}{3}\right) \sec^2\left(\frac{x}{3}\right) dx$.

Answer: $\frac{3 \tan^7\left(\frac{x}{3}\right)}{7} + C$

Difficulty: 1 Section: 2

37. Find $\int \tan^5 x \sec^3 x dx$.

Answer: $\frac{\sec^7 x}{7} - \frac{2 \sec^5 x}{5} + \frac{\sec^3 x}{3} + C$

Difficulty: 2 Section: 2

38. Find $\int \tan^5 x \sec^4 x dx$.

Answer: $\frac{\tan^8 x}{8} + \frac{\tan^6 x}{6} + C$

Difficulty: 2 Section: 2

39. Find $\int \tan^3 x dx$.

Answer: $\frac{\tan^2 x}{2} + \ln |\cos x| + C$

Difficulty: 2 Section: 2

40. Find $\int \tan^5 3x dx$.

Answer: $\frac{\tan^4 3x}{12} - \frac{\tan^2 3x}{6} - \frac{1}{3} \ln |\cos 3x| + C$

Difficulty: 2 Section: 2

41. Find $\int \sec 5x \tan^3 5x dx$.

Answer: $\frac{\sec^3 5x}{15} - \frac{\sec 5x}{5} + C$

Difficulty: 2 Section: 2

42. Find $\int \frac{\cot \sqrt[3]{x}}{\sqrt[3]{x^2}} dx$.

Answer: $3 \ln |\sin \sqrt[3]{x}| + C$

Difficulty: 1 Section: 2

43. Find $\int (5 - \cot 3x)^2 dx$.

Answer: $24x - \frac{10}{3} \ln |\sin 3x| - \frac{1}{3} \cot 3x + C$

Difficulty: 2 Section: 2

44. Find $\int \frac{1}{4} \cot^4 \csc^2 x \, dx$.

Answer: $-\frac{1}{20} \cot^5 x + C$

Difficulty: 2 Section: 2

45. Find $\int (\csc x - 1)^2 \, dx$.

Answer: $-\cot x + 2 \ln |\csc x + \cot x| + x + C$

Difficulty: 2 Section: 2

46. Find $\int (\cot x + \csc x) \, dx$.

Answer: $\ln |\sin x| - \ln |\csc x + \cot x| + C$

Difficulty: 3 Section: 3

47. Find $\int \frac{x}{\sqrt{x^2 + 2x - 3}} \, dx$.

Answer: $\sqrt{x^2 + 2x - 3} - \ln |x + 1 + \sqrt{x^2 + 2x - 3}| + C$

Difficulty: 2 Section: 3

48. Find $\int \frac{1}{1 + \sqrt{x}} \, dx$.

Answer: $2(\sqrt{x} - \ln |\sqrt{x} + 1|) + C$

Difficulty: 2 Section: 3

49. Find $\int \frac{x^{1/2}}{4(x^{3/4} + 1)} \, dx$.

Answer: $\frac{1}{3} x^{3/4} - \frac{1}{3} \ln |x^{3/4} + 1| + C$

Difficulty: 2 Section: 3

50. Find $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}} \, dx$.

Answer: $2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6 \ln |x^{1/6} + 1| + C$

Difficulty: 2 Section: 3

51. Find $\int \frac{\cot \sqrt[3]{x}}{\sqrt[3]{x^2}} \, dx$.

Answer: $3 \ln |\sin \sqrt{x}| + C$

Difficulty: 1 Section: 3

52. Find $\int x\sqrt{x+4} \, dx$.

Answer: $\frac{2}{5}(x+4)^{5/2} - \frac{8}{3}(x+4)^{3/2} + C$

Difficulty: 2 Section: 3

53. Find $\int \frac{x^3}{\sqrt[3]{x^2+4}} \, dx$.

Answer: $\frac{3}{10}(x^2+4)^{5/3} - 3(x^2+4)^{2/3} + C$

Difficulty: 2 Section: 3

54. Find $\int \sqrt{1-x^2} \, dx$.

Answer: $\frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + C$

Difficulty: 1 Section: 3

55. Find $\int \sqrt{9-4x^2} \, dx$.

Answer: $\frac{9}{4}\left[\sin^{-1}\left(\frac{2x}{3}\right) + \frac{2x\sqrt{9-4x^2}}{9}\right] + C$

Difficulty: 2 Section: 3

56. Find $\int \frac{x^2}{\sqrt{9-x^2}} \, dx$.

Answer: $\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$

Difficulty: 2 Section: 3

57. Find $\int \frac{\sqrt{4-x^2}}{x^2} \, dx$.

Answer: $-\left[\frac{\sqrt{4-x^2}}{x} + \sin^{-1}\left(\frac{x}{2}\right)\right] + C$

Difficulty: 2 Section: 3

58. Find $\int \frac{1}{x^2\sqrt{4-x^2}} \, dx$.

Answer: $-\frac{\sqrt{4-x^2}}{4x} + C$

Difficulty: 2 Section: 3

59. Find $\int \frac{1}{\sqrt{9+4x^2}} dx$.

Answer: $\frac{1}{2} \ln \left| \sqrt{9+4x^2} + 2x \right| + C$

Difficulty: 2 Section: 3

60. Find $\int \frac{\sqrt{x^2-16}}{x} dx$.

Answer: $\sqrt{x^2-16} - 4 \tan^{-1} \left(\frac{\sqrt{x^2-16}}{4} \right) + C$

Difficulty: 2 Section: 3

61. Find $\int \frac{x^3}{\sqrt{25-x^2}} dx$.

Answer: $\frac{1}{3} (25-x^2)^{3/2} - 25 (25-x^2)^{1/2} + C$

Difficulty: 2 Section: 3

62. Find $\int \frac{1}{\sqrt{9+16x-4x^2}} dx$.

Answer: $\frac{1}{2} \sin^{-1} \left(\frac{2x-4}{5} \right) + C$

Difficulty: 2 Section: 3

63. Find $\int \frac{1}{\sqrt{x^2+8x+25}} dx$.

Answer: $\ln \left| \sqrt{x^2+8x+25} + (x+4) \right| + C$

Difficulty: 2 Section: 3

64. Find $\int \frac{x}{\sqrt{x^2+2x-3}} dx$.

Answer: $\sqrt{x^2+2x-3} - \ln \left| x+1 + \sqrt{x^2+2x-3} \right| + C$

Difficulty: 2 Section: 3

65. Find $\int \frac{(4-x^2)^{3/2}}{x^6} dx$.

Answer: $-\frac{1}{20} \left(\frac{\sqrt{4-x^2}}{x} \right)^5 + C$

Difficulty: 2 Section: 3

66. Find $\int_3^6 \frac{\sqrt{x^2 - 9}}{x} dx$.

Answer: $3\sqrt{3} - \pi$

Difficulty: 2 Section: 3

67. Find $\int x^2 e^{2x} dx$.

Answer: $\frac{1}{2} e^{2x} \left(x^2 - x + \frac{1}{2} \right) + C$

Difficulty: 1 Section: 4

68. Find $\int x e^{3x} dx$.

Answer: $\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C$

Difficulty: 1 Section: 4

69. Find $\int \cos \sqrt{x} dx$.

Answer: $2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C$

Difficulty: 2 Section: 4

70. Find $\int \sin^{-1} x dx$.

Answer: $x \sin^{-1} x + \sqrt{1 - x^2} + C$

Difficulty: 2 Section: 4

71. Find $\int x \tan^{-1} x dx$.

Answer: $\frac{1}{2} [x^2 \tan^{-1} x + \tan^{-1} x - x] + C$

Difficulty: 2 Section: 4

72. Find $\int x^2 \tan^{-1} x dx$.

Answer: $\frac{1}{3} x^3 \tan^{-1} x - \frac{1}{6} x^2 + \frac{1}{6} \ln |x^2 + 1| + C$

Difficulty: 2 Section: 4

73. Find $\int x \ln x \, dx$.

Answer: $\frac{1}{2}x^2 \left(\ln x - \frac{1}{2} \right) + C$

Difficulty: 2 Section: 4

74. Find $\int \ln(x^2 + 1) \, dx$.

Answer: $x \ln|x^2 + 1| - 2x + 2 \tan^{-1} x + C$

Difficulty: 2 Section: 4

75. Find $\int \ln(x^2 + 2) \, dx$.

Answer: $x \ln(x^2 + 2) - 2x + 2\sqrt{2} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + C$

Difficulty: 2 Section: 4

76. Find $\int (\ln x)^2 \, dx$.

Answer: $x (\ln x)^2 - 2x \ln x + 2x + C$

Difficulty: 2 Section: 4

77. Find $\int \sqrt{x} \ln x \, dx$.

Answer: $\frac{2}{3}x^{3/2} \ln x - \frac{4}{9}x^{3/2} + C$

Difficulty: 2 Section: 4

78. Evaluate $\int_1^4 \sqrt{x} \ln x \, dx$.

Answer: $\frac{16}{3} \ln 4 - \frac{28}{9}$

Difficulty: 2 Section: 4

79. Evaluate $\int_1^e e \ln(x^2) \, dx$.

Answer: $\frac{e^2 + 1}{2}$

Difficulty: 2 Section: 4

80. Find $\int x \sin^{-1} \left(\frac{3}{x} \right) dx$ (for $x > 0$).

Answer: $\frac{x^2}{2} \sin^{-1} \left(\frac{3}{x} \right) + \frac{3}{2} \sqrt{x^2 - 9} + C$

Difficulty: 2 Section: 4

81. Find $\int x^3 \sqrt{1 - x^2} dx$.

Answer: $\frac{1}{5} (1 - x^2)^{5/2} - \frac{1}{3} (1 - x^2)^{3/2} + C$

Difficulty: 2 Section: 4

82. Use partial fractions to find $\int \frac{x+1}{x^2(x-1)} dx$.

Answer: $\frac{1}{x} + 2 \ln |x-1| - 2 \ln |x| + C$

Difficulty: 1 Section: 5

83. Use partial fractions to find $\int \frac{x+1}{x^3+x^2-6x} dx$.

Answer: $\frac{3}{10} \ln |x-2| - \frac{1}{6} \ln |x| - \frac{2}{15} \ln |x+3| + C$

Difficulty: 1 Section: 5

84. Use partial fractions to find $\int \frac{x+1}{x^2(x-1)(x^2+1)} dx$.

Answer: $-2 \ln |x| + \frac{1}{x} + \ln |x-1| + \frac{1}{2} \ln |x^2+1| + C$

Difficulty: 2 Section: 5

85. Use partial fractions to evaluate $\int_0^4 \frac{1}{x^2+3x+2} dx$.

Answer: $\ln \frac{5}{3}$

Difficulty: 1 Section: 5

86. Use partial fractions to find $\int \frac{3}{x^3+x} dx$.

Answer: $3 \ln |x| - \frac{3}{2} \ln |x^2+1| + C$

Difficulty: 1 Section: 5

87. Use partial fractions to find $\int \frac{3x^2 + 3x + 2}{x^3 + 2x^2 + x} dx$.

Answer: $2 \ln |x| + \ln |x + 1| + \frac{2}{x + 1} + C$

Difficulty: 2 Section: 5

88. Use partial fractions to find $\int \frac{x^2 - 13x + 30}{(x - 1)(x - 4)^2} dx$.

Answer: $2 \ln |x - 1| - \ln |x - 4| + \frac{2}{x - 4} + C$

Difficulty: 2 Section: 5

89. Use partial fractions to find $\int \frac{1}{x(x + 1)^2} dx$.

Answer: $\ln |x| - \ln |x + 1| + \frac{1}{x + 1} + C$

Difficulty: 1 Section: 5

90. Use partial fractions to find $\int \frac{2x^2 - 12x + 4}{x^3 - 4x^2} dx$.

Answer: $\frac{11}{4} \ln |x| + \frac{1}{x} - \frac{3}{4} \ln |x - 4| + C$

Difficulty: 2 Section: 5

91. Use partial fractions to find $\int \frac{x^3 - 4x - 1}{x(x - 1)^3} dx$.

Answer: $\ln |x| - \frac{3}{x - 1} + \frac{2}{(x - 1)^2} + C$

Difficulty: 2 Section: 5

92. Use partial fractions to find $\int \frac{5x^2 + 30x + 43}{(x + 3)^3} dx$.

Answer: $5 \ln |x + 3| + \frac{1}{(x + 3)^2} + C$

Difficulty: 2 Section: 5

93. Use partial fractions to find $\int \frac{-2x}{(x + 1)(x^2 + 1)} dx$.

Answer: $\ln |x + 1| - \frac{1}{2} \ln |x^2 + 1| - \tan^{-1} x + C$

Difficulty: 2 Section: 5

94. Use partial fractions to find $\int \frac{5x^2 + 11x + 17}{(x^2 + 4)(x + 5)} dx$.

Answer: $\ln|x^2 + 4| + \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + 3 \ln|x + 5| + C$

Difficulty: 2 Section: 5

95. Use partial fractions to find $\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} dx$.

Answer: $2 \ln|x| + \frac{1}{x} + \frac{3}{2} \ln|x^2 + 1| - 2 \tan^{-1} x + C$

Difficulty: 2 Section: 5

96. Use partial fractions to find $\int \frac{x^2 - x - 21}{(x^2 + 4)(2x - 1)} dx$.

Answer: $\frac{3}{2} \ln|x^2 + 4| + \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) - \frac{5}{2} \ln|2x - 1| + C$

Difficulty: 2 Section: 5

9 Indeterminate Forms and Improper Integrals

1. Find the limit if it exists: $\lim_{x \rightarrow 0} \frac{(\sin x)^2}{\sin x^2}$.

Answer: 1

Difficulty: 1 Section: 1

2. Find the limit if it exists: $\lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{1 - \sin x}$.

Answer: 2

Difficulty: 1 Section: 1

3. Find the limit if it exists: $\lim_{x \rightarrow 0} \frac{\sinh 3x}{x^3}$.

Answer: limit does not exist

Difficulty: 1 Section: 1

4. Find the limit if it exists: $\lim_{x \rightarrow 0} \frac{\tan^{-1} x - x}{x^3}$.

Answer: $-\frac{1}{3}$

Difficulty: 1 Section: 1

5. Find the limit if it exists: $\lim_{x \rightarrow 2} \frac{(x - 2) - \ln(x - 1)}{(x - 2)^2}$.

Answer: $\frac{1}{2}$

Difficulty: 1 Section: 1

6. Find the limit if it exists: $\lim_{x \rightarrow 1} \frac{\tan \frac{\pi}{2} x}{x - 1}$.

Answer: $-\infty$

Difficulty: 1 Section: 1

7. Find the limit if it exists: $\lim_{x \rightarrow \infty} \frac{x^3 - 4x^2 + 7x - 9}{x^2 + 7x - 5}$.

Answer: ∞

Difficulty: 1 Section: 1

8. Find the limit if it exists: $\lim_{x \rightarrow \infty} \frac{4x^4 - 3x^3 + 2x^2 + 1009}{x^5 - 8x + 2,347}$.

Answer: zero

Difficulty: 1 Section: 1

9. Find the limit if it exists: $\lim_{x \rightarrow 0} \frac{\tan^{-1} 2x}{\sin^{-1} x}$.

Answer: 2

Difficulty: 1 Section: 1

10. Find the limit if it exists: $\lim_{x \rightarrow \infty} \frac{x \sin x}{\sqrt{x}}$.

Answer: limit does not exist

Difficulty: 1 Section: 1

11. Find the limit if it exists: $\lim_{x \rightarrow \infty} \frac{2^x + x^2}{3^x}$.

Answer: zero

Difficulty: 1 Section: 1

12. Find the limit if it exists: $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$.

Answer: 2

Difficulty: 1 Section: 1

13. Find the limit if it exists: $\lim_{x \rightarrow 0} x^2 \ln x$.

Answer: 0

Difficulty: 1 Section: 2

14. Find the limit if it exists: $\lim_{x \rightarrow \infty} x \left(\tan^{-1} x - \frac{\pi}{2} \right)$.

Answer: -1

Difficulty: 1 Section: 2

15. Find the limit if it exists: $\lim_{x \rightarrow \infty} x^2 e^{-2x}$.

Answer: zero

Difficulty: 1 Section: 2

16. Find the limit if it exists: $\lim_{x \rightarrow \infty} \csc x [\ln(1 + \sin 2x)]$.

Answer: 2

Difficulty: 2 Section: 2

17. Find the limit if it exists: $\lim_{x \rightarrow 0} (x - \ln x)$. (Hint: Write the function as a quotient with denominator $\frac{1}{x}$.)

Answer: ∞

Difficulty: 2 Section: 2

18. Find the limit if it exists: $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 3} - \sqrt{x^2 + 6x})$.

Answer: -3

Difficulty: 1 Section: 1

19. Find the limit if it exists: $\lim_{x \rightarrow 0^+} x^{3/\ln x}$.

Answer: e^3

Difficulty: 1 Section: 2

20. Find the limit if it exists: $\lim_{x \rightarrow \infty} x^{3/\ln x}$.

Answer: e^3

Difficulty: 1 Section: 2+

21. Find the limit if it exists: $\lim_{t \rightarrow \infty} \left(\frac{t-1}{t+1} \right)^t$.

Answer: e^2

Difficulty: 1 Section: 2

22. Find the limit if it exists: $\lim_{x \rightarrow 0} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$.

Answer: $-\frac{1}{2}$

Difficulty: 1 Section: 2

23. Find the limit if it exists: $\lim_{x \rightarrow \infty} (1 - 3x)^{1/x}$.

Answer: e^{-3}

Difficulty: 1 Section: 2

24. Find the limit if it exists: $\lim_{x \rightarrow \infty} \left(\frac{x+2}{x-1} \right)^x$.

Answer: e^3

Difficulty: 1 Section: 2

25. Find the limit if it exists: $\lim_{x \rightarrow 0^+} x \csc x$.

Answer: 1

Difficulty: 1 Section: 2

26. Find the limit if it exists: $\lim_{x \rightarrow \infty} x (e^{2/x} - 1)$.

Answer: 2

Difficulty: 1 Section: 2

27. Find the limit if it exists: $\lim_{x \rightarrow 1} \left(\frac{\ln x}{(x-1)^2} - \frac{1}{x-1} \right)$.

Answer: $-\frac{1}{2}$

Difficulty: 1 Section: 2

28. Determine if the integral converges or diverges and find the value if it converges: $\int_0^{\infty} \frac{1}{(x+1)^2} dx$.

Answer: converges to 1

Difficulty: 1 Section: 3

29. Determine if the integral converges or diverges and find the value if it converges: $\int_{1/2}^{\infty} \frac{1}{1+4x^2} dx$.

Answer: converges to $\frac{\pi}{8}$

Difficulty: 1 Section: 3

30. Determine if the integral converges or diverges and find the value if it converges: $\int_0^{\infty} \frac{1}{\sqrt{x+1}} dx$.

Answer: diverges

Difficulty: 1 Section: 3

31. Determine if the integral converges or diverges and find the value if it converges: $\int_2^{\infty} \frac{2}{x^2-1} dx$.

Answer: converges to $\ln 3$

Difficulty: 2 Section: 3

32. Does the integral: $\int_{-\infty}^{\infty} \frac{1}{1+4x^2} dx$ converge or diverge? If it converges, find the value.

Answer: converges to $\frac{\pi}{2}$

Difficulty: 1 Section: 3

33. Does the integral: $\int_1^{\infty} \frac{1}{1+x^3} dx$ converge or diverge?

Answer: converges by comparison to the integral $\int_1^{\infty} \frac{1}{x^3} dx$

Difficulty: 2 Section: 4

34. Does the integral: $\int_{-1}^1 \frac{1}{x^{1/3}} dx$ converge or diverge? If it converges, find the value.

Answer: converges to zero

Difficulty: 1 Section: 4

35. Does the integral: $\int_{3/2}^3 \frac{1}{\sqrt{9-x^2}} dx$ converge or diverge? If it converges, find the value.

Answer: converges to $\frac{3\sqrt{3}}{2}$

Difficulty: 1 Section: 4

36. Does the integral: $\int_0^2 \frac{x}{(x-1)^3} dx$ converge or diverge? If it converges, find the value.

Answer: diverges

Difficulty: 2 Section: 4

37. Does the integral: $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$ converge or diverge? If it converges, find the value.

Answer: converges to $\frac{\pi}{2}$

Difficulty: 1 Section: 4

38. Does the integral: $\int_{-\infty}^{\infty} \frac{1}{x^2} dx$ converge or diverge? If it converges, find the value.

Answer: diverges

Difficulty: 2 Section: 4

10 Infinite Series

1. Let $s_n = \tan^{-1} n$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: $\frac{\pi}{2}$

Difficulty: 1 Section: 1

2. Let $s_n = \frac{3^n + n}{4^n}$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: 0

Difficulty: 1 Section: 1

3. Let $s_n = \left(-\frac{1}{n}\right)^n$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: 0

Difficulty: 2 Section: 1

4. Let $s_n = 2^{1/n}$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: 1

Difficulty: 1 Section: 1

5. Let $s_n = \left(1 + \frac{3}{n}\right)^n$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: e^3

Difficulty: 1 Section: 1

6. Let $s_n = \frac{n^3 + 3n - 2}{4n^3 + n^2 - 19n}$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: $\frac{1}{4}$

Difficulty: 1 Section: 1

7. Let $s_n = \frac{n!}{x^3}$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: ∞

Difficulty: 2 Section: 1

8. Let $s_n = \frac{|\cos n|}{\sqrt{n}}$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: 0

Difficulty: 2 Section: 1

9. Define $s_n = 1 + \cos n\pi$ for all n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: the limit does not exist

Difficulty: 2 Section: 1

10. Define $s_n = (1.001)^n$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: ∞

Difficulty: 1 Section: 1

11. Define $s_n = (-.999)^n$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: 0

Difficulty: 1 Section: 1

12. Define $s_n = \sqrt[3]{3n}$ for each n . Find $\lim_{n \rightarrow \infty} s_n$ if it exists.

Answer: 1

Difficulty: 2 Section: 1

13. Define $s_n = \frac{3^n}{n!}$ for each n . Show that the sequence is eventually decreasing and bounded from below.

Answer: $\frac{s_{n+1}}{s_n} = \frac{3}{n+1} \leq 1$ if $n \geq 2$. Also, $s_n \geq 0$ for all n .

Difficulty: 2 Section: 1

14. Define $s_n = \frac{x+1}{n+2}$ for each n . Show that the sequence is eventually decreasing and bounded from below.

Answer: Let $f(x) = \frac{x+1}{x+2}$, then $f'(x) = (x+2)^{-2} \geq 0$ for $x \geq 1$ and $\frac{n+1}{n+2} \leq 1$ for all n .

Difficulty: 2 Section: 1

15. Define $s_n = \frac{n+2}{n+1}$ for each n . Show that the sequence is eventually decreasing and bounded from below.

Answer: Let $f(x) = \frac{x+2}{x+1}$, then $f'(x) = -(x+1)^{-2} \leq 0$ for $x \geq 1$. Also, $\frac{n+1}{n+1} \geq 1$ for all n .

Difficulty: 2 Section: 1

16. Define $s_n = \frac{n}{2^n}$ for each n . Show that the sequence is eventually decreasing and bounded from below.

Answer: $\frac{s_n}{s_{n+1}} = \frac{n+1}{2n} \leq 1$ for $n \geq 2$ and $\frac{n}{2^n} \geq 0$ for all n .

Difficulty: 2 Section: 1

17. Define $s_n = 1 + \frac{(-1)^n}{n}$ for each n . Show that the sequence is monotone.

Answer: For n even, $s_n \geq 1$ and for n odd, $s_n \leq 1$.

Difficulty: 2 Section: 1

18. Define $s_n = \frac{2^n}{n}$ for each n . Show that the sequence is increasing but not convergent.

Answer: $\frac{s_{n+1}}{s_n} = \frac{n+1}{2n} \leq 1$ for $n \geq 2$. But the sequence is unbounded, hence it doesn't converge

Difficulty: 2 Section: 1

19. Find the sum of the infinite series $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$.

Answer: 2

Difficulty: 1 Section: 2

20. Find the sum of the infinite series $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}$.

Answer: 4

Difficulty: 1 Section: 2

21. Find the sum of the infinite series $\sum_{n=0}^{\infty} \frac{3^n}{5^n}$.

Answer: $\frac{5}{2}$

Difficulty: 1 Section: 2

22. Find the sum of the infinite series $\sum_{n=0}^{\infty} (0.6)^n$.

Answer: 2.5

Difficulty: 1 Section: 2

23. Find the sum of the infinite series $\sum_{n=0}^{\infty} (0.75)^n$.

Answer: 4

Difficulty: 1 Section: 2

24. Use partial fractions to find a formula for s_n , the n th partial sum for $\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$. Does the series converge?

Answer: $s_n = \frac{1}{2} - \frac{1}{n+2}$. The series converges to $\frac{1}{2}$.

Difficulty: 2 Section: 2

25. Use partial fractions to find a formula for s_n , the n th partial sum for $\sum_{n=3}^{\infty} \frac{1}{n^2 - 3n + 2}$. Does the series converge?

Answer: $s_n = 1 - \frac{1}{n-1}$. The series converges to 1.

Difficulty: 2 Section: 2

26. Write the repeating decimal $0.\overline{7}$ (in other words, $0.777 \dots$) as a geometric series and write the sum as a fraction.

Answer: $0.\overline{7} = \sum_{n=1}^{\infty} 7(0.1)^n = \frac{7}{9}$

Difficulty: 2 Section: 2

27. Write the repeating decimal $0.\overline{5}$ (in other words, $0.555 \dots$) as a geometric series and write the sum as a fraction.

Answer: $0.\overline{5} = \sum_{n=1}^{\infty} 5(0.1)^n = \frac{5}{9}$

Difficulty: 2 Section: 2

28. Write the repeating decimal $0.\overline{15}$ (in other words, $0.151515 \dots$) as a geometric series and write the sum as a fraction.

Answer: $0.\overline{15} = \sum_{n=1}^{\infty} 15(0.01)^n = \frac{5}{33}$

Difficulty: 2 Section: 2

29. Write the repeating decimal $0.\overline{51}$ (in other words, $0.515151 \dots$) as a geometric series and write the sum as a fraction.

Answer: $0.\overline{51} = \sum_{n=1}^{\infty} 51(0.01)^n = \frac{17}{33}$

Difficulty: 2 Section: 2

30. Does the series $\sum_{n=1}^{\infty} \frac{n}{8n+3}$ converge? If so, find the sum.

Answer: The series diverges because $\lim_{n \rightarrow \infty} \frac{n}{8n+3} \neq 0$.

Difficulty: 1 Section: 2

31. Does the series $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$ converge? If so, find the sum.

Answer: The series diverges because $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \neq 0$.

Difficulty: 1 Section: 2

32. Does the series $\sum_{n=1}^{\infty} \frac{5}{n}$ converge? If so, find the sum.

Answer: The series diverges, it is a multiple of the harmonic series.

Difficulty: 1 Section: 2

33. Does the series $\sum_{n=1}^{\infty} \left(\frac{5}{2} - \frac{1}{2^n} \right)$ converge? If so, find the sum.

Answer: The series diverges because $\sum_{n=1}^{\infty} \frac{5}{2}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

Difficulty: 1 Section: 2

34. Does the series $\sum_{n=1}^{\infty} \left(\frac{5}{2^n} + \frac{2}{5^n} \right)$ converge? If so, find the sum.

Answer: The series converges to $\frac{11}{2}$, it is the sum of two geometric series.

Difficulty: 1 Section: 2

35. Does the series $\sum_{n=1}^{\infty} \left(\sin n + \frac{1}{3^n} \right)$ converge? If so, find the sum.

Answer: The series diverges since $\lim_{n \rightarrow \infty} \left(\sin n + \frac{1}{3^n} \right) \neq 0$.

Difficulty: 1 Section: 2

36. Find the sum of the series $\sum_{n=3}^{\infty} 2^{-n}$.

Answer: $\frac{1}{4}$

Difficulty: 1 Section: 2

37. Find the sum of the series $\sum_{n=3}^{\infty} 4^{-n}$.

Answer: $\frac{1}{48}$

Difficulty: 1 Section: 2

38. Find the sum of the series $\sum_{n=1}^{\infty} (-1)^n 2^{-n}$.

Answer: This is a geometric series with $r = -\frac{1}{2}$, the sum is $-\frac{1}{3}$

Difficulty: 1 Section: 2

39. Find the sum of the series $\sum_{n=2}^{\infty} (-1)^n 3^{-n}$.

Answer: This is a geometric series with $r = -\frac{1}{3}$, the sum is $\frac{1}{12}$.

Difficulty: 1 Section: 2

40. Does the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$ converge? If so, find the sum.

Answer: The series diverges because $\lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} \neq 0$.

Difficulty: 2 Section: 2

41. Use the integral test to test the series $\sum_{n=1}^{\infty} \frac{1}{1+4n^2}$ for convergence. If the series converges, find an upper bound for the sum.

Answer: The series converges and the sum is less than or equal to $\frac{1}{5} + \frac{1}{2} \left(\frac{\pi}{2} - \tan^{-1} 2 \right)$.

Difficulty: 1 Section: 3

42. Use the integral test to test the series $\sum_{n=1}^{\infty} (n+1)^{-1/2}$ for convergence. If the series converges, find an upper bound for the sum.

Answer: The series diverges.

Difficulty: 1 Section: 3

43. Use the integral test to test the series $\sum_{n=1}^{\infty} \frac{n^2}{1+n^3}$ for convergence. If the series converges, find an upper bound for the sum.

Answer: The series diverges

Difficulty: 1 Section: 3

44. Use the integral test to test the series $\sum_{n=1}^{\infty} n e^{-2n}$ for convergence. If the series converges, find an upper bound for the sum.

Answer: The series converges; an upper bound for the sum is $\frac{3}{2} e^{-1}$

Difficulty: 1 Section: 3

45. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ converge? Give a reason for your conclusion.

Answer: The series converges; it is a p-series with $p = 1.5 > 1$.

Difficulty: 1 Section: 3

46. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$ converge? Give a reason for your conclusion.

Answer: The series diverges; it is a p-series with $p = 0.9 < 1$.

Difficulty: 1 Section: 3

47. Can the integral test be used for the series $\sum_{n=2}^{\infty} \frac{n}{\ln n}$?

Answer: No, $f(x) = \frac{x}{\ln x}$ is not a decreasing function.

Difficulty: 2 Section: 3

48. Does the series $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$ converge? Try the integral test.

Answer: yes

Difficulty: 2 Section: 3

49. Does the series $\sum_{n=1}^{\infty} n^{-1.1}$ converge? Give a reason for your conclusion.

Answer: Yes; it is a p-series with $p = 1.1 > 1$.

Difficulty: 1 Section: 3

50. Use the integral test to show that $\sum_{n=1}^{\infty} n^{-4}$ converges and find an upper bound for the sum.

Answer: $\int_1^{\infty} x^{-4} dx = \frac{1}{3}$, an upper bound is $\frac{7}{12}$

Difficulty: 1 Section: 3

51. Does the series $\sum_{n=1}^{\infty} 4n^{-2}$ converge? Give reasons.

Answer: Yes, it is a multiple of a convergent p-series, $p = 2$.

Difficulty: 1 Section: 3

52. Does the series $\sum_{n=1}^{\infty} [2^{-n} + n^{-2}]$ converge? Give reasons.

Answer: Yes, it is the sum of a convergent geometric series and a convergent p-series.

Difficulty: 2 Section: 3

53. Does the series $\sum_{n=1}^{\infty} \frac{7}{3n^2 + 5}$ converge?

Answer: Yes, compare with the p-series with $p = 2$.

Difficulty: 1 Section: 4

54. Does the series $\sum_{n=1}^{\infty} \frac{7}{3n+5}$ converge?.

Answer: No, compare with the harmonic series

Difficulty: 1 Section: 4

55. Does the series $\sum_{n=1}^{\infty} \frac{5}{2^n + n^2}$ converge? Give reasons.

Answer: Yes, compare with the geometric series with $r = \frac{1}{2}$.

Difficulty: 2 Section: 4

56. Does the series $\sum_{n=1}^{\infty} \frac{5n}{2^n}$ converge?

Answer: Yes, use whether the ratio or root test

Difficulty: 1 Section: 4

57. Does the series $\sum_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^n}{3^n}$ converge? Give reasons.

Answer: Yes, use the root test

Difficulty: 2 Section: 4

58. Does the series $\sum_{n=1}^{\infty} \frac{2n-7}{3n^2+5n-1}$ converge?

Answer: No, compare with the harmonic series.

Difficulty: 1 Section: 4

59. Does the series $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!}$ converge?

Answer: Yes, use the ratio test.

Difficulty: 1 Section: 4

60. Does the series $\sum_{n=1}^{\infty} \frac{(n+1)!}{2n!}$ converge?

Answer: No, the limit of the n th term is not zero

Difficulty: 2 Section: 4

61. Does the series $\sum_{n=1}^{\infty} \frac{2^n}{n^3+1}$ converge?

Answer: No, use the ratio test

Difficulty: 1 Section: 4

62. Does the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converge?

Answer: No, use the ratio test

Difficulty: 1 Section: 4

63. Does the series $\sum_{n=1}^{\infty} 4n^{-2}$ converge? Give reasons.

Answer: Yes, it is a multiple of a convergent p-series, $p = 2$.

Difficulty: 1 Section: 3

64. Does the series $\sum_{n=1}^{\infty} \frac{n^2 - 7n + 5}{n^n}$ converge?

Answer: Yes, use the ratio or root test.

Difficulty: 2 Section: 4

65. Does the series $\sum_{n=1}^{\infty} e^{1/n}$ converge?

Answer: No, the limit of the n th term is not zero.

Difficulty: 1 Section: 4

66. Does the series $\sum_{n=1}^{\infty} \left(\frac{2n}{3n+1} \right)^n$ converge?

Answer: Yes, use the root test

Difficulty: 1 Section: 4

67. Does the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ converge? Give reasons.

Answer: Yes, $\frac{\cos^2 n}{n^2} \leq \frac{1}{n^2}$ for all n .

Difficulty: 1 Section: 3

68. Does the series $\sum_{n=1}^{\infty} \frac{2^n + 1}{2^{n+1}}$ converge?

Answer: No, the limit of the n th term is not 0.

Difficulty: 1 Section: 4

69. Does the series $\sum_{n=1}^{\infty} \pi^{-n} e^n$ converge?

Answer: Yes, it is a geometric series with $r = \frac{e}{\pi} < 1$.

Difficulty: 1 Section: 4

70. Does the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ converge?

Answer: No, compare with the harmonic series

Difficulty: 1 Section: 3

71. Does the series $\sum_{n=1}^{\infty} (-1)^n \frac{2^n - 1}{n + 1}$ converge?

Answer: No, the limit of the n th term is not zero.

Difficulty: 1 Section: 5

72. Does the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$ converge?

Answer: Yes, by the alternating series test

Difficulty: 1 Section: 5

73. Does the series $\sum_{n=1}^{\infty} (-1)^n n^{-1/3}$ converge? Give reasons.

Answer: Yes, by the alternating series test

Difficulty: 1 Section: 5

74. Does the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$ converge?

Answer: Yes, $\cos n\pi = (-1)^n$, so the alternating series test applies.

Difficulty: 1 Section: 5

75. Does the series $\sum_{n=1}^{\infty} (-1)^n \left(\frac{3}{2}\right)^n$ converge?

Answer: No, the limit of the n th term is not zero.

Difficulty: 1 Section: 5

76. How many terms of the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{2n}$ must be summed to estimate the sum of the series with error less than .001?

Answer: 500

Difficulty: 2 Section: 5

77. How many terms of the series $\sum_{n=1}^{\infty} (-1)^n n^{-1/2}$ must be summed to estimate the sum of the series with error less than .01?

Answer: 10,000

Difficulty: 2 Section: 5

78. How many terms of the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!}$ must be summed to estimate the sum of the series with error less than .01?

Answer: 4

Difficulty: 2 Section: 5

79. Estimate the sum of the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ with error less than 0.1.

Answer: $-\frac{31}{36}$

Difficulty: 2 Section: 5

80. Estimate the sum of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$ with error less than 0.1.

Answer: $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9}$

Difficulty: 2 Section: 5

81. Classify the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{3n+7}$ as absolutely convergent, conditionally convergent, or divergent.

Answer: Divergent, the limit of the n th term is not zero.

Difficulty: 1 Section: 5

82. Classify the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n^2+5}$ as absolutely convergent, conditionally convergent, or divergent.

Answer: Absolutely convergent, compare the series of absolute values to a p -series with $p = 2$.

Difficulty: 2 Section: 5

83. Classify the series $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{3}{5}\right)^n$ as absolutely convergent, conditionally convergent, or divergent.

Answer: Absolutely convergent, the series of absolute values is a geometric series with $r = \frac{3}{5}$.

Difficulty: 1 Section: 5

84. Classify the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$ as absolutely convergent, conditionally convergent, or divergent.

Answer: Conditionally convergent by the alternating series test, the series of absolute values is the harmonic series which diverges.

Difficulty: 1 Section: 5

85. Classify the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n^2 - 1}$ as absolutely convergent, conditionally convergent, or divergent.
 Answer: Absolutely convergent, compare the series of absolute values to a p-series with $p = 2$.
 Difficulty: 2 Section: 5
86. Classify the series $\sum_{n=1}^{\infty} (-1)^{2n+1} \frac{1}{\sqrt{n+5}}$ as absolutely convergent, conditionally convergent, or divergent.
 Answer: Divergent, all terms are negative; compare to a p-series with $p = \frac{1}{2}$.
 Difficulty: 1 Section: 5
87. Classify the series $\sum_{n=1}^{\infty} (-1)^{n+3} (0.6)^n$ as absolutely convergent, conditionally convergent, or divergent.
 Answer: Absolutely convergent, the series of absolute values is a geometric series with $r = 0.6$.
 Difficulty: 1 Section: 5
88. Classify the series $\sum_{n=1}^{\infty} (-1)^{n+2}$ as absolutely convergent, conditionally convergent, or divergent.
 Answer: Divergent, the limit of the n th term is not zero.
 Difficulty: 1 Section: 5
89. Classify the series $\sum_{n=1}^{\infty} (-1)^n \sin n$ as absolutely convergent, conditionally convergent, or divergent.
 Answer: Divergent, the limit of the n th term is not zero.
 Difficulty: 2 Section: 5
90. Classify the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt[n]{n}}$ as absolutely convergent, conditionally convergent, or divergent.
 Answer: Divergent, the limit of the n th term is not zero.
 Difficulty: 2 Section: 5
91. Find the convergence set of $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.
 Answer: $-1 \leq x \leq 1$.
 Difficulty: 1 Section: 6
92. Find the convergence set of $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.
 Answer: x can be any real number.
 Difficulty: 1 Section: 6

93. Find the convergence set of $\sum_{n=1}^{\infty} \frac{x^n}{3n+1}$.

Answer: $-1 \leq x < 1$

Difficulty: 1 Section: 6

94. Find the convergence set of $\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$.

Answer: $-\frac{1}{2} \leq x < \frac{1}{2}$

Difficulty: 1 Section: 6

95. Find the convergence set of $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$.

Answer: $-3 < x < 3$

Difficulty: 1 Section: 6

96. Find the convergence set of $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

Answer: $-1 \leq x < 1$, the convergence at -1 is conditional.

Difficulty: 1 Section: 6

97. Find the convergence set of $\sum_{n=1}^{\infty} \frac{x^n}{x^3}$.

Answer: $-1 \leq x \leq 1$

Difficulty: 1 Section: 6

98. Find the convergence set of $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n n}$.

Answer: $-2 < x \leq 2$

Difficulty: 1 Section: 6

99. Find the convergence set of $\sum_{n=0}^{\infty} \frac{2^n x^{2n}}{n!}$.

Answer: all real numbers

Difficulty: 1 Section: 6

100. Find the convergence set of $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$.

Answer: $-1 < x < 5$

Difficulty: 1 Section: 6

101. Find the convergence set of $\sum_{n=0}^{\infty} \frac{8^{3n} x^{2n}}{n!}$.

Answer: all real numbers

Difficulty: 1 Section: 6

102. Find the convergence set of $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n}$.

Answer: $-\sqrt{2} < x < \sqrt{2}$

Difficulty: 1 Section: 6

103. Find the convergence set of $\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^n} x^n$.

Answer: $-\frac{2}{3} < x < \frac{2}{3}$

Difficulty: 1 Section: 6

104. Find the convergence set of $\sum_{n=0}^{\infty} \frac{n}{4^{n+1}} (x-1)^n$.

Answer: $-3 < x < 5$

Difficulty: 1 Section: 6

105. Find the convergence set of $\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n n}$.

Answer: $-5 \leq x < -1$

Difficulty: 1 Section: 6

106. Find the convergence set of $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^{2n-1}}{(2n-1)!}$.

Answer: all real numbers

Difficulty: 1 Section: 6

107. Find the convergence set of $\sum_{n=0}^{\infty} \frac{n(x-1)^n}{(3n-1)3^n}$.

Answer: $-2 < x < 4$

Difficulty: 1 Section: 6

108. Find the convergence set of $\sum_{n=1}^{\infty} \frac{(x+3)^{2n}}{\sqrt{3n}}$.

Answer: $-4 < x < -2$

Difficulty: 1 Section: 6

109. Find the convergence set of $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

Answer: $-1 \leq x \leq 1$

Difficulty: 1 Section: 6

110. Find the convergence set of $\sum_{n=1}^{\infty} \frac{(x+4)^n}{3^n n^2}$.

Answer: $-7 \leq x \leq -1$

Difficulty: 1 Section: 6

111. Find the convergence set of $\sum_{n=1}^{\infty} \frac{2^n (x+1)^n}{n^2}$.

Answer: $-\frac{3}{2} \leq x \leq -\frac{1}{2}$

Difficulty: 1 Section: 6

112. Write a power series for $f(x) = \frac{x}{(1-x)^2}$. Give the convergence set.

Answer: $\sum_{n=1}^{\infty} n x^n, -1 < x < 1$

Difficulty: 2 Section: 7

113. Write a power series for $f(x) = \frac{x}{(1+x)^2}$. Give the convergence set.

Answer: $\sum_{n=1}^{\infty} (-1)^n n x^n, -1 < x < 1$

Difficulty: 1 Section: 7

114. Write a power series for $f(x) = \frac{x^2}{1-x}$. Give the convergence set.

Answer: $\sum_{n=0}^{\infty} x^{n+2}, -1 < x < 1$

Difficulty: 1 Section: 7

115. Write a power series for $f(x) = \frac{1}{2+3x}$. Give the convergence set.

Answer: $\sum_{n=0}^{\infty} \frac{3^n (-1)^n}{2^{n+1}} x^n, -\frac{2}{3} < x < \frac{2}{3}$

Difficulty: 2 Section: 7

116. Find the power series for $f(x) = \frac{x}{3+2x}$. Give the convergence set.

Answer: $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{n+1}}{3^{n+1}} x^n, -\frac{3}{2} < x < \frac{3}{2}$

Difficulty: 2 Section: 7

117. Find the power series for $f(x) = \frac{x^2}{1-3x}$. Give the convergence set.

Answer: $\sum_{n=0}^{\infty} 3^n x^{n+2}, -\frac{1}{3} < x < \frac{1}{3}$

Difficulty: 2 Section: 7

118. Find the power series for $f(x) = \tan^{-1}\left(\frac{x}{2}\right)$. Give the convergence set.

Answer: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1) 2^{2n+1}}, -2 \leq x \leq 2$

Difficulty: 1 Section: 7

119. Find the power series for $f(x) = \tan^{-1}(3x)$. Give the convergence set.

Answer: $\sum_{n=0}^{\infty} \frac{(-1)^n - 3^{2n+1} x^{2n+1}}{2n+1}, -\frac{1}{3} \leq x \leq \frac{1}{3}$

Difficulty: 1 Section: 7

120. Find the power series for $f(x) = \ln(1+3x)$. Give the convergence set.

Answer: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n x^n}{n}, -\frac{1}{3} < x \leq \frac{1}{3}$

Difficulty: 1 Section: 7

121. Find the power series for $f(x) = x \ln(1-2x)$. Give the convergence set.

Answer: $\sum_{n=1}^{\infty} (-1) \frac{2^n x^{n+1}}{n}, -\frac{1}{2} \leq x < \frac{1}{2}$

Difficulty: 1 Section: 7

122. Find the power series for $f(x) = x^2 \tan^{-1}\left(\frac{x}{2}\right)$. Give the convergence set.

Answer: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1) 2^{2n+1}}, -2 \leq x \leq 2$

Difficulty: 1 Section: 7

123. Find the power series for $f(x) = \frac{1}{1-x^2}$. Give the convergence set.

Answer: $\sum_{n=0}^{\infty} x^{2n}, -1 < x < 1$

Difficulty: 1 Section: 7

124. Find the Taylor series for $f(x) = x^4 - x^3 + 4x$ in powers of $x + 2$.

Answer: $16 - 40(x + 2) + 30(x + 2)^2 - 9(x + 2)^3 + (x + 2)^4$

Difficulty: 2 Section: 8

125. Find the Maclaurin series for $f(x) = \frac{1}{3+x}$.

Answer: $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{3^{n+1}}$

Difficulty: 1 Section: 8

126. Find the Maclaurin series for $f(x) = \cos x^2$.

Answer: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$

Difficulty: 1 Section: 8

127. Write the Taylor series for $f(x) = \ln x^2$ in powers of $(x - 1)$.

Answer: $\sum_{n=0}^{\infty} \frac{2(-1)^n (x - 1)^{n+1}}{n + 1}$

Difficulty: 2 Section: 8

128. Write the Taylor series for $f(x) = e^{-x}$ in powers of $(x - 1)$.

Answer: $\sum_{n=0}^{\infty} \frac{e^{-1} (-1)^n (x - 1)^n}{n!}$

Difficulty: 2 Section: 8

129. Write the terms through x^4 in the Maclaurin series for $f(x) = \sec 2x$ about $c = 0$.

Answer: $1 + 2x^2 + \frac{10}{3}x^4$

Difficulty: 1 Section: 8

130. Write the Taylor series in $(x - 8)$ through the term in $(x - 8)^2$ for $f(x) = x^{2/3}$ about $c = 8$.

Answer: $4 + \frac{1}{3}(x - 8) = \frac{1}{144}(x - 8)^2$

Difficulty: 1 Section: 8

131. Write the Taylor series in $(x - 1)$ for $f(x) = x^3 - 3x^2 + 5x - 2$.

Answer: $1 + 2(x - 1) + (x - 1)^3$

Difficulty: 1 Section: 8

132. Write the Taylor series for $f(x) = \frac{1}{x}$ in powers of $(x - 2)$

Answer: $\sum_{n=0}^{\infty} \frac{(-1)^n (x - 2)^n}{2^{n+1}}$

Difficulty: 2 Section: 8

133. Write the Taylor series for $f(x) = x \cos x$.

Answer: $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$

Difficulty: 1 Section: 8

134. Write the first two nonzero terms of the Maclaurin series for $f(x) = \tan 2x$.

Answer: $2x + \frac{8}{3}x^3$

Difficulty: 1 Section: 8

135. Write the first two nonzero terms of the Maclaurin series for $f(x) = \tanh 2x$

Answer: $2x - \frac{8}{3}x^3$

Difficulty: 1 Section: 8

136. Write the first four terms of the binomial series for $f(x) = (1 - x)^{1/2}$.

Answer: $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

Difficulty: 1 Section: 8

137. Write the first four terms of the binomial series for $f(x) = (1 + x)^{1/3}$.

Answer: $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{1}{81}x^3$

Difficulty: 1 Section: 8

138. Write the first four terms of the binomial series for $f(x) = (1 - x)^{-1/2}$.

Answer: $1 + \frac{1}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3$

Difficulty: 1 Section: 8

139. Write the first four terms of the binomial series for $f(x) = (1 + x)^{-3}$.

Answer: $1 - 3x + 6x^2 - 10x^3$

Difficulty: 1 Section: 8

140. Find a polynomial approximation $f(x) = e^x$ accurate to within 0.01 for $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

Answer: $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

Difficulty: 2 Section: 8

141. Find a polynomial approximation $f(x) = \sin x$ accurate to within 0.01 for $-1 \leq x \leq 1$.

Answer: $x - \frac{x^3}{6}$

Difficulty: 2 Section: 8

142. Find a polynomial approximation $f(x) = \cos x$ accurate to within 0.01 for $-1 \leq x \leq 1$.

Answer: $1 - \frac{x^2}{2} + \frac{x^4}{24}$

Difficulty: 2 Section: 8

143. Write the first four terms of the binomial series for $f(x) = (4+x)^{1/2}$.

Answer: $2 + \frac{x}{4} - \frac{1}{64}x^2 + \frac{1}{512}x^3$

Difficulty: 2 Section: 8

11 Numerical Methods, Approximations

1. Write the Taylor polynomial of order 3 based at 1 for $f(x) = e^x$.

$$\text{Answer: } e \left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} \right)$$

Difficulty: 1 Section: 1

2. Write the Taylor polynomial of order 3 based at 2 for $f(x) = e^x$.

$$\text{Answer: } e^2 \left(1 + (x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{6} \right)$$

Difficulty: 1 Section: 1

3. Write the Taylor polynomial of order 3 based at -1 for $f(x) = e^x$.

$$\text{Answer: } e^{-1} \left(1 + (x+1) + \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} \right)$$

Difficulty: 1 Section: 1

4. Write the Taylor polynomial of order 3 based at -2 for $f(x) = e^x$.

$$\text{Answer: } e^{-2} \left(1 + (x+2) + \frac{(x+2)^2}{2} + \frac{(x+2)^3}{2} \right)$$

Difficulty: 1 Section: 1

5. Write the Maclaurin polynomial of order 4 for $f(x) = \sec 2x$.

$$\text{Answer: } 1 + 2x^2 + \frac{10}{3}x^4$$

Difficulty: 1 Section: 1

6. Write the Maclaurin polynomial of order 4 for $f(x) = \sec 3x$.

$$\text{Answer: } 1 + \frac{9}{2}x^2 + \frac{135}{8}x^4$$

Difficulty: 1 Section: 1

7. Write the Taylor polynomial of order 3 based at 1 for $f(x) = x^3 - 3x^2 + 5x - 2$.

$$\text{Answer: } 1 + 2(x-1) + (x-1)^3$$

Difficulty: 1 Section: 1

8. Write the Taylor polynomial of order 3 based at 2 for $f(x) = x^3 - 3x^2 + 5x - 2$.

$$\text{Answer: } 4 + 5(x-2) + 3(x-2)^2 + (x-2)^3$$

Difficulty: 1 Section: 1

9. Write the Maclaurin polynomial of order 3 for $\tan^{-1} \frac{x}{3}$.

Answer: $\frac{x}{3} - \frac{1}{81}x^3$

Difficulty: 1 Section: 1

10. Write the Taylor polynomial of order 2 based at 8 for $f(x) = x^{2/3}$.

Answer: $4 + \frac{1}{3}(x-8) - \frac{1}{144}(x-8)^2$

Difficulty: 1 Section: 1

11. Determine the order of the Maclaurin polynomial required to approximate e^x correct to 2 decimal places for $|x| \leq \frac{1}{2}$.

Answer: order 4 using 1.7 as an upper bound for $e^{1/2}$

Difficulty: 2 Section: 1

12. Determine the order of the Maclaurin polynomial required to approximate e^x correct to 3 decimal places for $|x| \leq \frac{1}{2}$.

Answer: order 5 using 1.7 as an upper bound for $e^{1/2}$

Difficulty: 2 Section: 1

13. Determine the order of the Maclaurin polynomial required to approximate e^x correct to 4 decimal places for $|x| \leq \frac{1}{2}$.

Answer: order 6 using 1.7 as an upper bound for $e^{1/2}$

Difficulty: 2 Section: 1

14. Find the third order of Maclaurin polynomial for $(1+x)^{1/2}$ and bound the error $R_3(x)$ for $0 \leq x \leq \frac{1}{2}$.

Answer: $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3, |R_3(x)| \leq .019$

Difficulty: 2 Section: 1

15. Find the fourth order of Maclaurin polynomial for $(1+x)^{1/2}$ and bound the error $R_4(x)$ for $0 \leq x \leq \frac{1}{2}$.

Answer: $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4, |R_4(x)| \leq 1.008$

Difficulty: 2 Section: 1

16. Find $\lim_{x \rightarrow 0} \frac{\ln(1+x) - x + \frac{x^2}{2}}{x^3}$.

Answer: $\frac{1}{3}$

Difficulty: 2 Section: 1

17. Find $\lim_{x \rightarrow 0} \frac{\tan^{-1} x - x + \frac{x^3}{3}}{2x^5}$.

Answer: $\frac{1}{10}$

Difficulty: 2 Section: 1

18. Find $\lim_{x \rightarrow 0} \frac{2x^5 - 7x^6}{\tan x - x - \frac{x^3}{3}}$.

Answer: 15

Difficulty: 2 Section: 1

19. Find $\lim_{x \rightarrow 0} \frac{e^{-2x} - 1 + x^2}{3x^4 + 5x^7 - x^9}$.

Answer: $\frac{1}{6}$

Difficulty: 2 Section: 1

20. Use the Trapezoid Rule, with $n = 4$, to approximate $\int_0^1 \frac{1}{(x+1)^2} dx$. Round your answer to two decimal places.

Answer: 0.51

Difficulty: 1 Section: 2

21. What value of n should be used with the Trapezoid Rule to approximate $\int_1^4 \sqrt{x} dx$ to within 0.01?

Answer: $n \geq 8$

Difficulty: 2 Section: 2

22. What value of n should be used with the Trapezoid Rule to approximate $\int_1^2 \frac{1}{t} dt$ to within 0.0001?

Answer: $n \geq 41$

Difficulty: 2 Section: 2

23. Use Simpson's Rule, with $n = 8$, to approximate $\int_0^1 \frac{1}{1+x^2} dx$. Round your answer to 4 decimal places.
Answer: 0.7854
Difficulty: 1 Section: 2
24. Use Simpson's Rule, with $n = 4$, to approximate $\int_0^1 \frac{1}{\sqrt{1+x^2}} dx$. Round your answer to three decimal places.
Answer: 0.881
Difficulty: 1 Section: 2
25. What value of n should be used in Simpson's Rule to approximate $\int_1^4 \sqrt{x} dx$ to within 0.01?
Answer: $n \geq 4$
Difficulty: 2 Section: 2
26. What value of n should be used in Simpson's Rule to approximate $\int_1^2 \frac{1}{x} dx$ to within 0.0001?
Answer: $n \geq 8$
Difficulty: 2 Section: 2
27. Use Newton's method to find an approximate solution of $x^3 + 3x - 20 = 0$ between 2 and 3. Round to two decimal places.
Answer: 2.35
Difficulty: 2 Section: 3
28. Use Newton's method to find an approximate solution of $3x^3 + 14x^2 + 13x - 2 = 0$ between 0 and 1. Round to two decimal places.
Answer: 0.13
Difficulty: 2 Section: 3
29. Use Newton's method to find an approximate solution of $x^4 - 2x - 100 = 0$ between 3 and 4. Round to two decimal places.
Answer: 3.21
Difficulty: 2 Section: 3
30. Use Newton's method to find a positive approximate solution of $x^3 - 144x + 665 = 0$. Round to two decimal places.
Answer: There are two, 7 and 6.86.
Difficulty: 2 Section: 3

31. In Newton's method to estimate the root of the equation $x^3 - x - 1 = 0$, take the first guess $x_0 = 1$. What is the second approximation x_1 ?

Answer: 1.5

Difficulty: 2 Section: 3

32. In Newton's method to estimate the root of the equation $x^2 - 5 = 0$, take the first guess $x_0 = 2$. What is the second approximation x_1 ?

Answer: 2.25

Difficulty: 2 Section: 3

33. Use the fixed-point method to solve the equation $x^3 + 3x - 20 = 0$ with initial guess of 2 and 4 iterations.

Answer: the process diverges

Difficulty: 2 Section: 4

34. Use the fixed-point method to solve the equation $3x^3 + 14x^2 + 13x - 2 = 0$ with initial guess of 0.5 and 4 iterations. Round off to two decimal places.

Answer: 0.13

Difficulty: 1 Section: 4

35. Use the fixed-point method to solve the equation $x^3 - 144x + 665 = 0$ with initial guess of 6.5 and 4 iterations. Round off to two decimal places.

Answer: 6.58

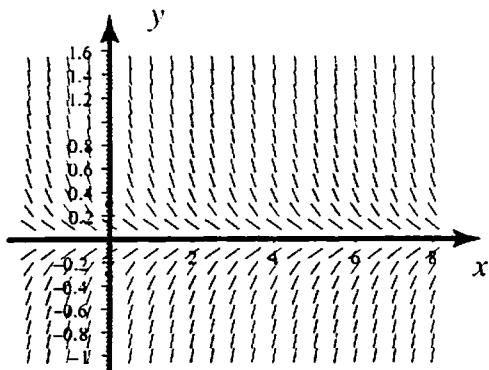
Difficulty: 1 Section: 4

36. Use the fixed-point method to solve the equation $x^3 - x - 1 = 0$ with initial guess of 2 and 4 iterations. Round off to two decimal places

Answer: the process diverges

Difficulty: 2 Section: 4

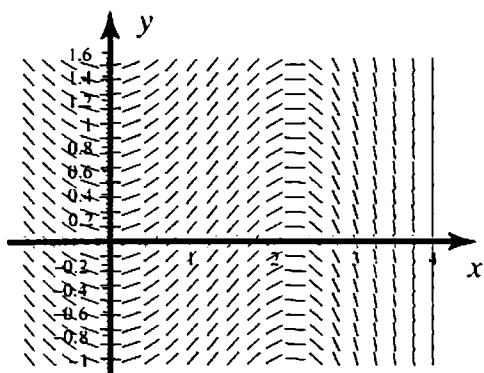
37. The figure below contains a slope field for the differential equation $y' = f(x, y)$. Sketch the solution to this differential equation that satisfies $y(0) = 0.5$ and approximate $y(3)$.



Answer: $y(3) \approx 0$

Difficulty: 1: Section: 5

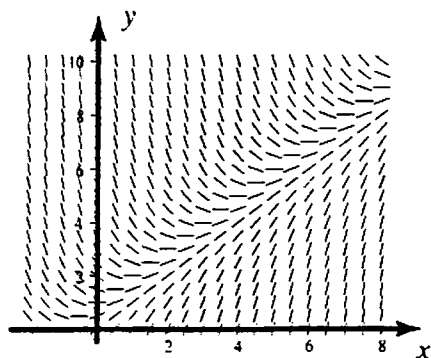
38. The figure below contains a slope field for the differential equation $y' = f(x, y)$. Sketch the solution to this differential equation that satisfies $y(0) = 0.2$ and approximate $y(2)$.



Answer: $y(2) \approx 1.5$

Difficulty: 1: Section: 5

39. The figure below contains a slope field for the differential equation $y' = f(x, y)$. Sketch the solution to this differential equation that satisfies $y(0) = 4$ and approximate $y(8)$.



Answer: $y(8) \approx 8$

Difficulty: 1: Section: 5

40. Use Euler's Method with $h = 0.2$ to approximate $y(1)$ where y is the solution of $y' = 3y$ that satisfies $y(0) = 1$.

Answer: 10.48576

Difficulty: 2: Section 5

41. Use Euler's Method with $h = 0.2$ to approximate $y(2)$ where y is the solution of $y' = xy$ that satisfies $y(1) = 2$.

Answer: 6.838419

Difficulty: 2: Section 5

42. Use the Improved Euler Method with $h = 0.2$ to approximate $y(1)$ where y is the solution of $y' = 3y$ that satisfies $y(0) = 1$.

Answer: 17.86899

Difficulty: 2: Section 5

43. Use the Improved Euler Method with $h = 0.2$ to approximate $y(2)$ where y is the solution of $y' = xy$ that satisfies $y(1) = 2$.

Answer: 8.783297

Difficulty: 2: Section 5

12 Conics and Polar Coordinates

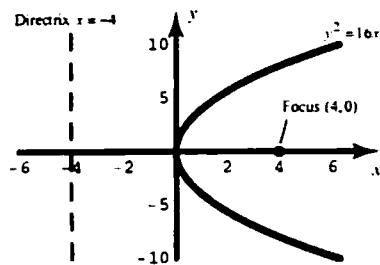
1. Find the equation of the parabola with focus $(2, 0)$ and directrix $x = -2$.

Answer: $y^2 = 8x$

Difficulty: 1 Section: 1

2. Sketch the parabola $y^2 = 16x$ showing vertex, focus, and directrix on the graph.

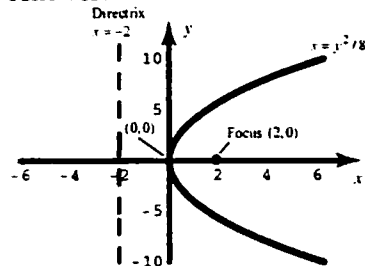
Answer:



Difficulty: 1 Section: 1

3. Sketch the parabola $x = \frac{1}{8}y^2$ showing vertex, focus, and directrix on the graph.

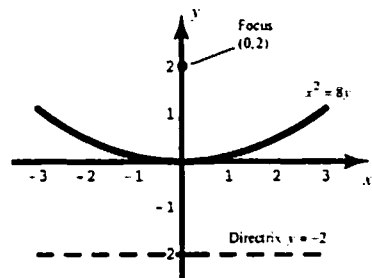
Answer:



Difficulty: 1 Section: 1

4. Sketch the parabola $x^2 = 8y$ showing vertex, focus, and directrix on the graph.

Answer:



Difficulty: 1 Section: 1

5. Find the vertices on the major axis of the ellipse $3x^2 + 2y^2 = 16$.

Answer: $(0, 2\sqrt{2})$ and $(0, -2\sqrt{2})$

Difficulty: 1 Section: 2

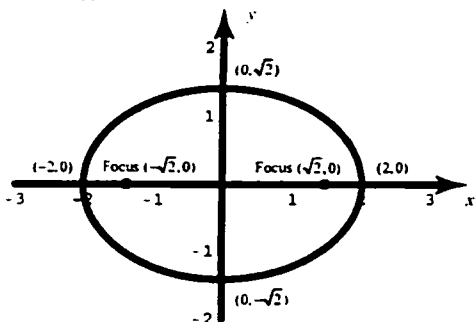
6. Find the foci of the ellipse $4y^2 + x^2 = 36$.

Answer: $(-\sqrt{27}, 0)$ and $(\sqrt{27}, 0)$

Difficulty: 1 Section: 2

7. Sketch the ellipse $2x^2 + 4y^2 = 8$ showing center, vertices, and foci on the graph.

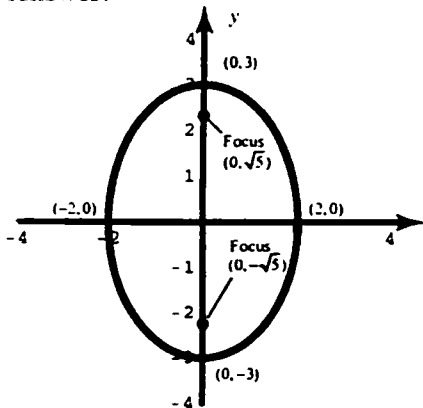
Answer:



Difficulty: 1 Section: 2

8. Sketch the ellipse $9x^2 + 4y^2 = 36$ showing center, vertices, and foci on the graph.

Answer:



Difficulty: 1 Section: 2

9. Find the foci and vertices of the hyperbola $9x^2 - 4y^2 = 36$.

Answer: foci $(\sqrt{13}, 0)$, $(-\sqrt{13}, 0)$ and vertices $(-2, 0)$, $(2, 0)$

Difficulty: 1 Section: 2

10. Find the foci and vertices of the hyperbola $9x^2 - 16y^2 = -144$.

Answer: foci $(0, 5)$, $(0, -5)$ and vertices $(0, 3)$, $(0, -3)$

Difficulty: 1 Section: 2

11. Find the equation of the hyperbola with foci at $(5, 0)$, $(-5, 0)$ and vertices at $(4, 0)$, $(-4, 0)$.

Answer: $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Difficulty: 1 Section: 2

12. Find the equation of the hyperbola with foci at $(0, 6)$, $(0, -6)$ and vertices at $(0, 2)$, $(0, -2)$.

Answer: $8y^2 - x^2 = 32$

Difficulty: 1 Section: 2

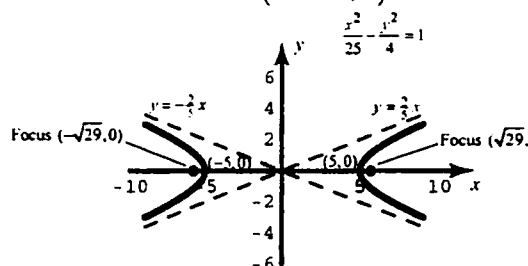
13. Find the equation of the hyperbola with foci at $(0, 2)$, $(0, -2)$ and an asymptote $y = 2x$.

Answer: $\frac{5y^2}{16} - \frac{5x^2}{4} = 1$

Difficulty: 1 Section: 2

14. Sketch the hyperbola $\frac{x^2}{25} - \frac{y^2}{4} = 1$ showing vertices, foci, and asymptotes on the graph.

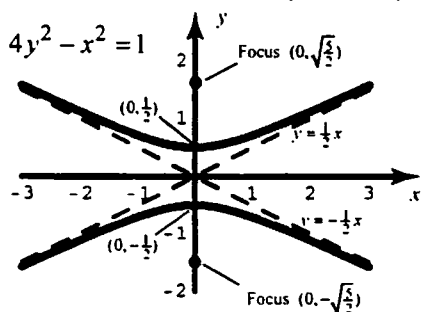
Answer: Foci are at $(\pm\sqrt{29}, 0)$.



Difficulty: 2 Section: 2

15. Sketch the hyperbola $4y^2 - x^2 = 1$ showing vertices, foci, and asymptotes on the graph.

Answer: Foci are at $(0, \pm\sqrt{\frac{5}{2}})$.



Difficulty: 2 Section: 2

16. Find the equation of the hyperbola with foci at $(5, 0)$ and $(-5, 0)$ and a fixed difference of distances equal to 6.

Answer: $16x^2 - 9y^2 = 144$

Difficulty: 1 Section: 3

17. Find the equation of the hyperbola with foci at $(0, 8)$ and $(0, -8)$ and a fixed difference of distances equal to 10.

Answer: $\frac{y^2}{25} - \frac{x^2}{39} = 1$

Difficulty: 1 Section: 3

18. Find the equation of the ellipse with foci at $(-3, 0)$ and $(3, 0)$ and a fixed sum of distances equal to 12.

Answer: $\frac{x^2}{36} + \frac{y^2}{27} = 1$

Difficulty: 1 Section: 3

19. Find the equation of the ellipse with foci at $(0, 1)$ and $(0, -1)$ and a fixed sum of distances equal to 5.

Answer: $\frac{4x^2}{21} + \frac{4y^2}{25} = 1$

Difficulty: 1 Section: 3

20. Find the slope of the tangent line to the curve $x^2 - 4y^2 = 9$ at the point $(5, 2)$.

Answer: $\frac{5}{8}$

Difficulty: 1 Section: 8

21. Find the equation of the tangent line to the curve $9x^2 + 4y^2 = 72$ at the point $(-2, 3)$.

Answer: $y - 3 = \frac{3}{2}(x + 2)$

Difficulty: 2 Section: 3

22. Find the equation of the tangent line to the curve $4x^2 + 9y^2 = 40$ at the point $(1, 2)$.

Answer: $y - 2 = -\frac{2}{9}(x - 1)$

Difficulty: 2 Section: 3

23. Find the equation of the tangent line to the curve $x^2 - y^2 = 16$ at the point $(5, 3)$.

Answer: $y - 3 = \frac{5}{3}(x - 5)$

Difficulty: 2 Section: 3

24. Find the total area enclosed by the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

Answer: 15π square units

Difficulty: 2 Section: 3

25. Find the equation of the parabola with focus $(2, 9)$ and vertex $(2, 6)$.

Answer: $(x - 2)^2 = 12(y - 6)$

Difficulty: 1 Section: 4

26. Find the equation of the parabola with focus $(4, 6)$ and vertex $(1, 6)$.

Answer: $(y - 6)^2 = 12(x - 1)$

Difficulty: 1 Section: 4

27. Find the equation of the parabola with focus $(3, -1)$ and vertex $(-2, -1)$.

Answer: $(y + 1)^2 = 20(x + 2)$

Difficulty: 1 Section: 4

28. Find the equation of the parabola with focus $(-2, 3)$ and directrix $y = 5$.

Answer: $(x + 2)^2 = -4(y - 4)$

Difficulty: 1 Section: 4

29. Find the equation of the parabola with focus $(-3, 2)$ and directrix $x = 5$.

Answer: $(y - 2)^2 = -16(x - 1)$

Difficulty: 1 Section: 4

30. Find the vertex, focus, and directrix of the parabola $y^2 - 2y + 9 = 8x$.

Answer: $(1, 1)$, $(3, 1)$, $x = -1$

Difficulty: 2 Section: 4

31. Find the vertex, focus, and directrix of the parabola $x^2 + 2x + 8y = 15$.

Answer: $(-1, 2)$, $(-1, 0)$, $y = 4$

Difficulty: 2 Section: 4

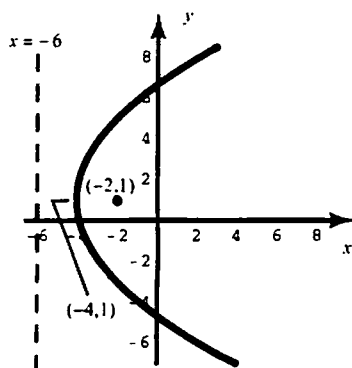
32. Find the vertex, focus, and directrix of the parabola $x^2 - 2x + 4y = 11$.

Answer: $(1, 3)$, $(1, 2)$, $y = 4$

Difficulty: 2 Section: 4

33. Sketch the parabola $y^2 - 2y - 31 = 8x$ showing vertex, focus, and directrix on the graph.

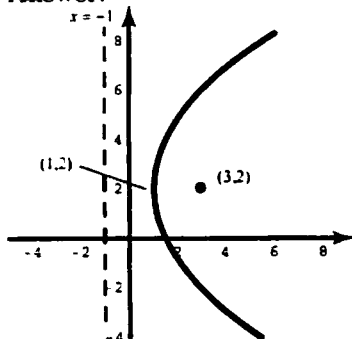
Answer:



Difficulty: 2 Section: 4

34. Sketch the parabola $y^2 - 4y + 12 = 8x$ showing vertex, focus, and directrix on the graph.

Answer:



Difficulty: 2 Section: 4

35. Find the center of the ellipse $4x^2 + y^2 + 16x - 4y = 80$.

Answer: $(-2, 2)$

Difficulty: 1 Section: 4

36. Find the foci of the ellipse $25x^2 + 4y^2 - 50x + 16y = 59$.

Answer: $(1, -2 + \sqrt{21})$ and $(1, -2 - \sqrt{21})$

Difficulty: 2 Section: 4

37. Find the foci of the ellipse $3x^2 - 6x + 2y^2 + 4y = 19$.

Answer: $(1, 1)$ and $(1, -3)$

Difficulty: 2 Section: 4

38. Find the equation of the ellipse with vertices at $(2, 4)$, $(2, -6)$, $(5, -1)$, and $(-1, -1)$.

Answer: $\frac{(x-2)^2}{9} + \frac{(y+1)^2}{25} = 1$

Difficulty: 2 Section: 4

39. Find the equation of the ellipse with vertices at $(3, 2)$, $(-7, 2)$, $(-2, 12)$, and $(-2, -8)$.

Answer: $\frac{(x+2)^2}{25} + \frac{(y-2)^2}{100} = 1$

Difficulty: 2 Section: 4

40. Find the equation of the ellipse with foci at $(-4, 2)$ and $(2, 2)$ and a fixed sum of distances equal to 12.

Answer: $\frac{(x+1)^2}{36} + \frac{(y-2)^2}{27} = 1$

Difficulty: 2 Section: 4

41. Find the equation of the ellipse with foci at $(4, -1)$ and $(-4, -1)$ and a fixed sum of distances equal to 10.

Answer: $\frac{x^2}{25} + \frac{(y+1)^2}{9} = 1$

Difficulty: 2 Section: 4

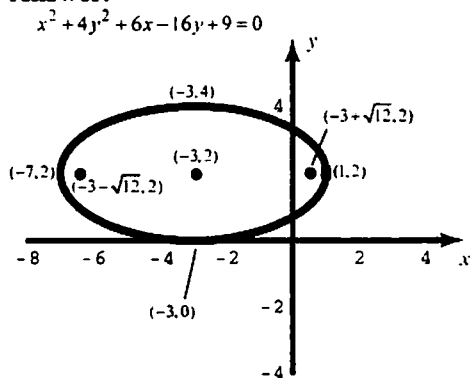
42. Find the equation of the ellipse with foci at $(1, 1)$ and $(-3, 1)$ and a fixed sum of distances equal to $2\sqrt{5}$.

Answer: $\frac{(x+1)^2}{5} + (y-1)^2 = 1$

Difficulty: 2 Section: 4

43. Sketch the ellipse $x^2 + 4y^2 + 6x - 16y + 9 = 0$ showing center, vertices, and foci on the graph.

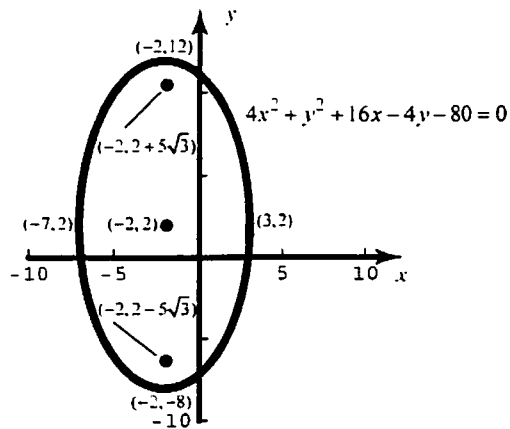
Answer:



Difficulty: 2 Section: 4

44. Sketch the ellipse $4x^2 + y^2 + 16x - 4y - 80 = 0$ showing center, vertices, and foci on the graph.

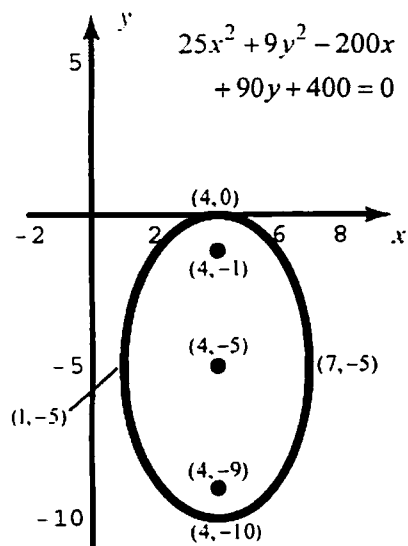
Answer:



Difficulty: 2 Section: 4

45. Sketch the ellipse $25x^2 + 9y^2 - 200x + 90y + 400 = 0$ showing center, vertices, and foci on the graph.

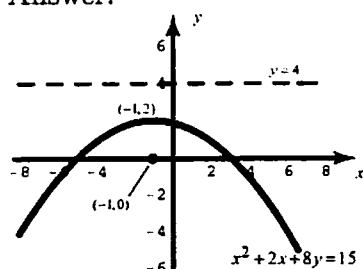
Answer:



Difficulty: 2 Section: 4

46. Sketch the parabola $x^2 + 2x + 8y = 15$ showing vertex, focus, and directrix on the graph.

Answer:



Difficulty: 2 Section: 4

47. Find the foci and vertices of the hyperbola $4x^2 - y^2 + 16x - 4y = 88$.

Answer: foci $(-2 + 5\sqrt{5}, -2)$, $(-2 - 5\sqrt{5}, -2)$ and vertices $(-7, -2)$, $(3, -2)$

Difficulty: 2 Section: 4

48. Find the foci and vertices of the hyperbola $25x^2 - 9y^2 - 50x - 18y = -241$.

Answer: foci $(1, -1 + \sqrt{34})$, $(-1, -1 - \sqrt{34})$ and vertices $(1, 4)$, $(1, -6)$

Difficulty: 2 Section: 4

49. Find the equation of the hyperbola with foci at $(5, -2)$, $(-5, 2)$ and vertices at $(4, -2)$, $(-4, -2)$.

Answer: $9x^2 - 16(y + 2)^2 = 144$

Difficulty: 1 Section: 4

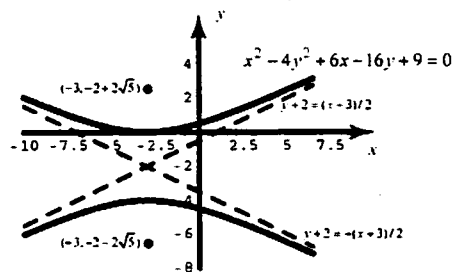
50. Find the equation of the hyperbola with vertices $(1, 2)$, $(1, 6)$ and an asymptote with slope $\frac{2}{3}$.

Answer: $\frac{(y - 4)^2}{4} - \frac{(x - 1)^2}{9} = 1$

Difficulty: 2 Section: 4

51. Sketch the hyperbola $x^2 - 4y^2 + 6x - 16y + 9 = 0$ showing vertices, foci, and asymptotes on the graph.

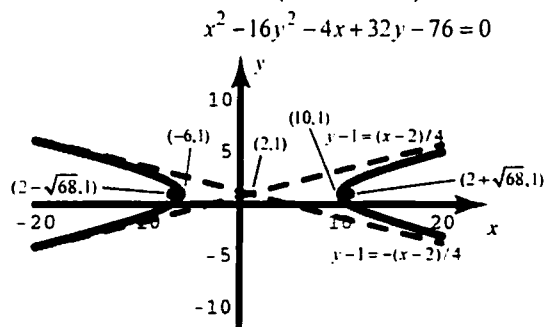
Answer: Foci are at $(-3, -3 \pm \sqrt{20})$.



Difficulty: 2 Section: 4

52. Sketch the hyperbola $x^2 - 16y^2 - 4x + 32y - 76 = 0$ showing vertices, foci, and asymptotes on the graph.

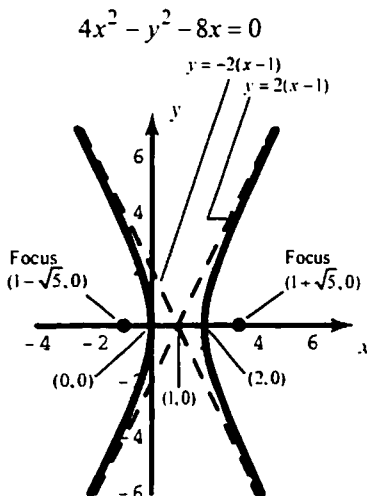
Answer: Foci are at $(2 \pm \sqrt{68}, 1)$.



Difficulty: 2 Section: 4

53. Sketch the hyperbola $4x^2 - y^2 - 8x = 0$ showing vertices, foci, and asymptotes on the graph.

Answer: Foci are at $(1 \pm \sqrt{5}, 0)$.



Difficulty: 2 Section: 4

54. Let θ be the angle of rotation to eliminate the xy term from the equation $4x^2 - 7xy = 4y^2 - 5x - 15$. Find the value of $\cot 2\theta$.

Answer: $-\frac{8}{7}$

Difficulty: 1 Section: 5

55. Find the angle θ so that the rotation through that angle will eliminate the xy term from the equation $x^2 - 2\sqrt{3}xy + 3y^2 - 16\sqrt{3}x - 16y = 0$.

Answer: $\frac{\pi}{6}$

Difficulty: 1 Section: 5

56. Let θ be the angle so that the rotation through θ will eliminate the xy term from the equation $4x^2 - 6xy - 4y^2 - 15 = 0$. Find $\sin \theta$ and $\cos \theta$.

Answer: $\sin \theta = \frac{3}{\sqrt{10}}, \cos \theta = \frac{1}{\sqrt{10}}$

Difficulty: 1 Section: 5

57. Write the equation of the curve $x^2 + y^2 - xy = 4$ in terms of u and v after rotating through the appropriate axes through the angle $\theta = \frac{\pi}{4}$.

Answer: $u^2 + 3v^2 = 8$

Difficulty: 1 Section: 5

58. Write the equation of the curve $8x^2 - 4xy + 5y^2 = 36$ in terms of u and v after rotating through the appropriate angle to eliminate the xy term.

Answer: $\frac{u^2}{9} + \frac{v^2}{4} = 1$

Difficulty: 1 Section: 5

59. Write the equation of the curve $16x^2 - 24xy + 9y^2 - 60x - 80y + 100$ in terms of u and v after rotating through the appropriate angle to eliminate the xy term.

Answer: $v^2 = 4(u - 1)$

Difficulty: 2 Section: 5

60. Write the equation of the curve $4x^2 - 6xy - 4y^2 - 15 = 0$ in terms of u and v after rotating through the appropriate angle to eliminate the xy term.

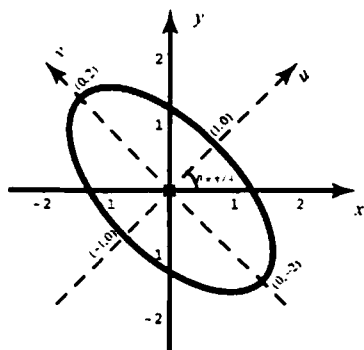
Answer: $\frac{v^2}{3} - \frac{u^2}{3} = 1$

Difficulty: 1 Section: 5

61. Use rotation of axes to eliminate the xy term and graph the conic $5x^2 + 6xy + 5y^2 - 8 = 0$. Identify the important parts.

Answer: $\frac{u^2}{1} + \frac{v^2}{4} = 1$

$5x^2 + 6xy + 5y^2 - 8 = 0$

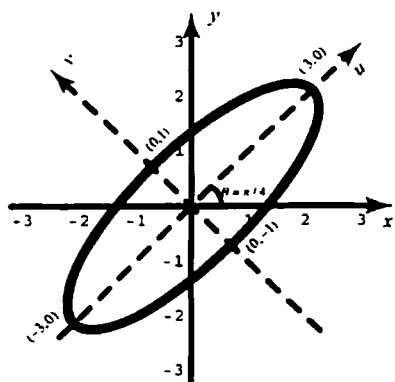


Difficulty: 2 Section: 5

62. Use rotation of axes to eliminate the xy term and graph the conic $5x^2 - 8xy + 5y^2 - 9 = 0$. Identify the important parts.

Answer: $\frac{u^2}{9} + \frac{v^2}{1} = 1$

$$5x^2 - 8xy + 5y^2 - 9 = 0$$

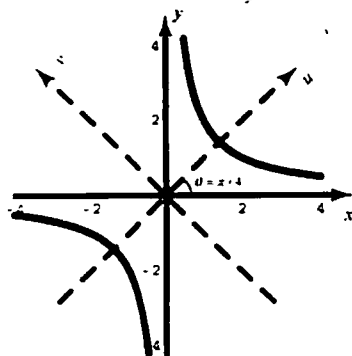


Difficulty: 2 Section: 5

63. Use rotation of axes to eliminate the xy term and graph the conic $2xy - 1 = 0$. Identify the important parts.

Answer: $u^2 - v^2 = 1$

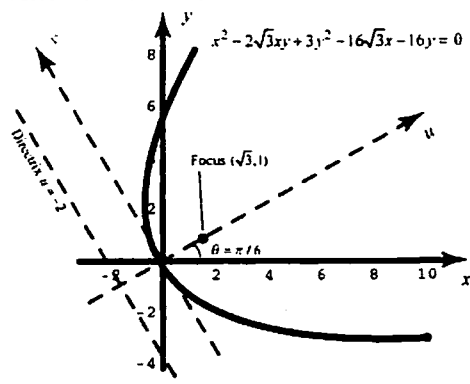
$$2xy = 1$$



Difficulty: 2 Section: 5

64. Use rotation of axes to eliminate the xy term and graph the conic $x^2 - 2\sqrt{3}xy + 3y^2 - 16\sqrt{3}x - 16y = 0$. Identify the important parts.

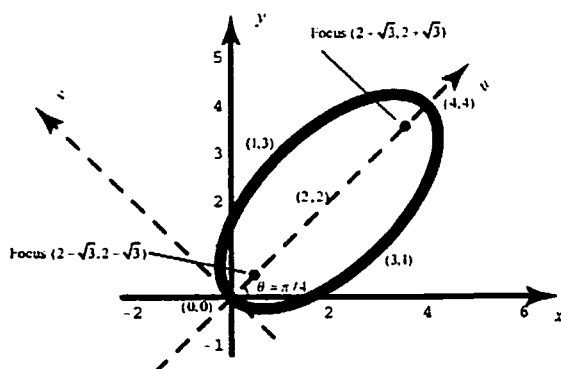
Answer: $v^2 = 8u$



Difficulty: 2 Section: 5

65. Use rotation of axes to eliminate the xy term and graph the conic $5x^2 - 6xy + 5y^2 - 8x - 8y = 0$. Identify the important parts.

Answer: $\frac{(u - 2\sqrt{2})^2}{8} + \frac{v^2}{2} = 1$
 $5x^2 - 6xy + 5y^2 - 8x - 8y = 0$



Difficulty: 3 Section: 5

66. The polar coordinates of P are $(4, \frac{\pi}{3})$. Give the rectangular coordinates of P.

Answer: $(2, 2\sqrt{3})$

Difficulty: 1 Section: 6

67. The polar coordinates of P are $(2, \frac{7\pi}{6})$. Give the rectangular coordinates of P.

Answer: $(-\sqrt{3}, -1)$

Difficulty: 1 Section: 6

68. The polar coordinates of P are $(4, \pi)$. Give the rectangular coordinates of P.

Answer: $(-4, 0)$

Difficulty: 1 Section: 6

69. The polar coordinates of P are $\left(3, -\frac{\pi}{4}\right)$. Give the rectangular coordinates of P.

Answer: $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

Difficulty: 1 Section: 6

70. The rectangular coordinates of P are $(2, 2)$. Give the polar coordinates of P.

Answer: $\left(2\sqrt{2}, \frac{\pi}{4} + 2n\pi\right)$ or $\left(-2\sqrt{2}, \frac{\pi}{4} + (2n + 1)\pi\right)$

Difficulty: 1 Section: 6

71. The rectangular coordinates of P are $(0, 3)$. Give the polar coordinates of P.

Answer: $\left(3, \frac{\pi}{2} + 2n\pi\right)$ or $\left(-3, \frac{3\pi}{2} + 2n\pi\right)$

Difficulty: 1 Section: 6

72. The rectangular coordinates of P are $(0, 3)$. Give the polar coordinates of P.

Answer: $\left(3, \frac{\pi}{2} + 2n\pi\right)$ or $\left(-3, \frac{3\pi}{2} + 2n\pi\right)$

Difficulty: 1 Section: 6

73. The polar coordinates of P are $\left(-3, \frac{\pi}{2}\right)$. Give the rectangular coordinates of P.

Answer: $(0, -3)$

Difficulty: 1 Section: 6

74. The polar coordinates of P are $\left(-2, \frac{\pi}{3}\right)$. Give the rectangular coordinates of P.

Answer: $(-1, \sqrt{3})$

Difficulty: 1 Section: 6

75. The rectangular coordinates of P are $(-2, -3)$. Give the polar coordinates of P.

Answer: $(\sqrt{13}, \tan^{-1}(1.5) + (2n + 1)\pi)$ or $(-\sqrt{13}, \tan^{-1}(1.5) + 2n\pi)$

Difficulty: 1 Section: 6

76. The rectangular coordinates of P are $(1, -4)$. Give the polar coordinates of P.

Answer: $(\sqrt{17}, \tan^{-1}(-4) + 2n\pi)$ or $(-\sqrt{17}, \tan^{-1}(-4) + (2n + 1)\pi)$

Difficulty: 1 Section: 6

77. The rectangular coordinates of P are $(-3, -3)$. Give the polar coordinates of P.

Answer: $(3\sqrt{2}, \frac{5\pi}{4} + 2n\pi)$ or $(-3\sqrt{2}, \frac{\pi}{4} + 2n\pi)$

Difficulty: 1 Section: 6

78. Find a polar equation for the graph of the Cartesian equation $x^2 + y^2 - 2x = 9$.

Answer: $r^2 = 2r(\cos \theta) + 9$

Difficulty: 1 Section: 6

79. Find a polar equation for the graph of the Cartesian equation $y^2 = 4x$

Answer: $r(\sin^2 \theta) = 4 \cos \theta$

Difficulty: 1 Section: 6

80. Find a Cartesian equation for the graph of the polar equation $r = 3 \cos \theta$.

Answer: $x^2 + y^2 = 3y$.

Difficulty: 1 Section: 6

81. Find a Cartesian equation for the graph of the polar equation $r^2 = 2r(\cos \theta) + r(\sin \theta)$.

Answer: $x^2 + y^2 = 2x + y$

Difficulty: 1 Section: 6

82. Name the curve that has the polar equation $r = 2 \cos \theta$.

Answer: circle

Difficulty: 1 Section: 6

83. Name the curve that has the polar equation $r = \frac{5}{3 + 2 \cos \theta}$ and give its eccentricity.

Answer: ellipse, $e = \frac{2}{3}$

Difficulty: 2 Section: 6

84. Name the curve that has the polar equation $f = \frac{5}{2 + 3 \cos \theta}$ and give its eccentricity.

Answer: hyperbola, $e = \frac{3}{2}$

Difficulty: 2 Section: 6

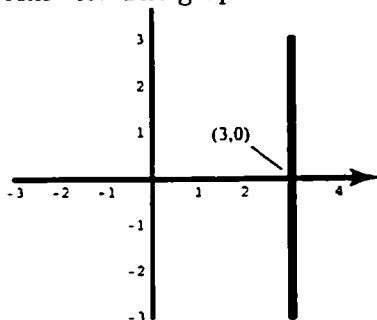
85. Name the curve that has the polar equation $r = \frac{5}{3 - 3 \cos \theta}$ and give its eccentricity.

Answer: parabola, $e = 1$

Difficulty: 2 Section: 6

86. Sketch the graph of $r = 3 \sec \theta$.

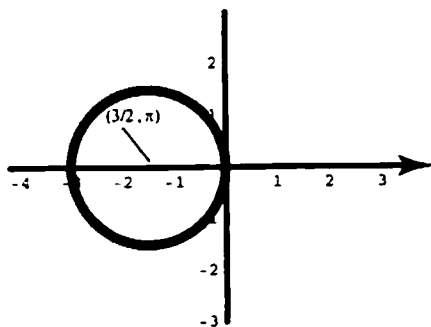
Answer: The graph is the line $x = 3$.



Difficulty: 1 Section: 7

87. Sketch the graph of $r = -3 \cos \theta$.

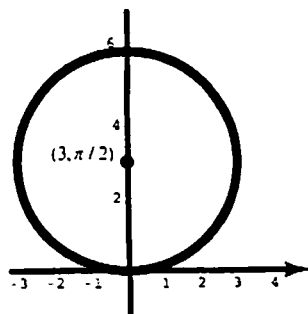
Answer: The graph is a circle of radius $\frac{3}{2}$ with center at $\left(\frac{3}{2}, \pi\right)$.



Difficulty: 1 Section: 7

88. Sketch the graph of $r = 6 \sin \theta$.

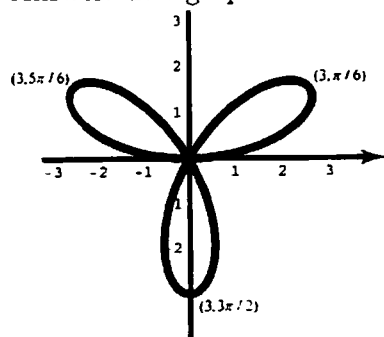
Answer: The graph is a circle of radius 3, with center at $\left(3, \frac{\pi}{2}\right)$.



Difficulty: 1 Section: 7

89. Sketch the graph of $r = 3 \sin 3\theta$.

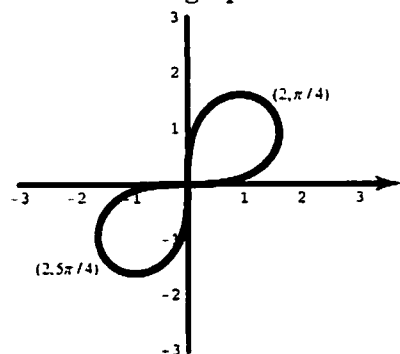
Answer: The graph is a 3-leaved rose.



Difficulty: 2 Section: 7

90. Sketch the graph of $r^2 = 4 \sin 2\theta$.

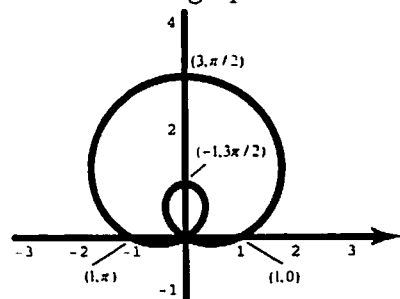
Answer: The graph is a lemniscate.



Difficulty: 2 Section: 7

91. Sketch the graph of $r = 1 + 2 \sin \theta$.

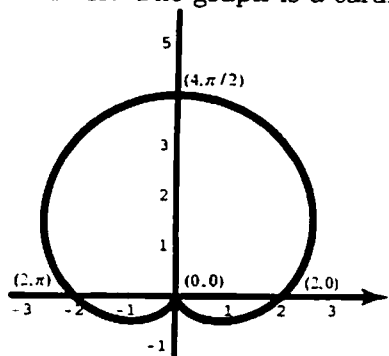
Answer: The graph is a limaçon.



Difficulty: 2 Section: 7

92. Sketch the graph of $r = 2 + 2 \sin \theta$.

Answer: The graph is a cardioid.



Difficulty: 2 Section: 7

93. Find the points of intersection of the curves $r = 2$ and $r = 4 \sin 2\theta$ that are in quadrant I.

Answer: $\left(2, \frac{\pi}{12}\right)$ and $\left(2, \frac{5\pi}{12}\right)$

Difficulty: 2 Section: 7

94. Find the points of intersection of the curves $r = 2 \sin \theta$ and $r = 2 \cos \theta$.

Answer: $\left(\sqrt{2}, \frac{\pi}{4}\right)$ and $(0, 0)$

Difficulty: 2 Section: 7

95. Find the area bounded by the curve $r^2 = 2 \cos 2\theta + 1$.

Answer: $\sqrt{3} + \frac{2\pi}{3}$

Difficulty: 1 Section: 8

96. Find the area bounded by the curve $r^2 = \frac{1}{2} \sin 2\theta$.

Answer: $\frac{1}{2}$

Difficulty: 1 Section: 8

97. Find the area bounded by the curve $r^2 = 2 \sin 3\theta$.

Answer: 4

Difficulty: 1 Section: 8

98. Find the area bounded by the curve $r^2 = 2 \cos 3\theta$.

Answer: 4

Difficulty: 1 Section: 8

99. Find the area inside the circle $r = 3 \cos \theta$ and outside the cardioid $r = 1 + \cos \theta$.

Answer: π

Difficulty: 2 Section: 8

100. Find the area inside the circle $r = 3 \sin \theta$ and outside the cardioid $r = 1 + \sin \theta$.

Answer: π

Difficulty: 2 Section: 8

101. Find the area inside the cardioid $r = 1 + \cos \theta$.

Answer: $\frac{3\pi}{2}$

Difficulty: 1 Section: 8

102. Find the area inside the cardioid $r = 1 + \sin \theta$.

Answer: $\frac{3\pi}{2}$

Difficulty: 2 Section: 8

103. Find the area inside the cardioid $r = 2 \cos 3\theta$.

Answer: $\frac{\pi}{3}$

Difficulty: 2 Section: 8

104. Find the area inside one petal of the rose $r = 2 \sin 3\theta$.

Answer: $\frac{\pi}{3}$

Difficulty: 2 Section: 8

105. Find the length of the cardioid $r = 1 + \cos \theta$.

Answer: 8

Difficulty: 2 Section: 8

106. Find the length of the cardioid $r = 1 + \sin 2\theta$.

Answer: 8

Difficulty: 2 Section: 8

107. Find the length of the curve $r = \sin^3 \left(\frac{\theta}{3} \right)$.

Answer: $\frac{3\pi}{2}$

Difficulty: 2 Section: 8

108. Find the equation of the tangent line to $r = 4 \cos 2\theta$ at $\theta = \frac{\pi}{2}$.

Answer: $y = -4$

Difficulty: 1 Section: 8

109. Find the slope of the tangent line to the circle $r = 2 \sin \theta$ at the point where $\theta = \frac{\pi}{3}$.

Answer: $-\sqrt{3}$

Difficulty: 1 Section: 8

110. Find the slope of the cardioid $r = 1 - \cos \theta$ at $\theta = \frac{\pi}{2}$.

Answer: -1

Difficulty: 1 Section: 8

111. Find all points where the cardioid $r = 1 + \sin \theta$ has a horizontal tangent line.

Answer: $\left(2, \frac{\pi}{2}\right), \left(\frac{1}{2}, \frac{7\pi}{6}\right), \left(\frac{1}{2}, \frac{11\pi}{6}\right)$

Difficulty: 2 Section: 8

112. Find all points where the cardioid $r = 1 + \sin \theta$ has a vertical tangent line.

Answer: $\left(0, \frac{3\pi}{2}\right), \left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right)$

Difficulty: 2 Section: 8

13 Geometry in the Plane, Vectors

1. Eliminate the parameter and identify the curve $x = \sqrt{1+t}$, $y = \sqrt{t}$, $t \geq 0$.

Answer: $x^2 - y^2 = 1$, $x \geq 1$, $y \geq 1$, part of the hyperbola.

Difficulty: 1 Section: 1

2. Eliminate the parameter and identify the curve $x = 3t^2$, $y = 2t - 1$, $-1 \leq t \leq 1$.

Answer: $4x = 3(y^2 + 2y + 1)$, part of the parabola with $0 \leq x \leq 3$.

Difficulty: 1 Section: 1

3. Eliminate the parameter and identify the curve $y = \sqrt{1 - \cos^2 t}$, $x = 1 - \sin^2 t$.

Answer: $y^2 = 1 - x$, part of the parabola with $0 \leq x$, $0 \leq y$.

Difficulty: 1 Section: 1

4. Eliminate the parameter and identify the curve $x = \sec t$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

Answer: $x^2 - y^2 = 1$, part of the hyperbola with $x \geq 0$.

Difficulty: 1 Section: 1

5. Eliminate the parameter and identify the curve $x = 2 \sin \theta$, $y = 3 \cos \theta$, $0 \leq \theta \leq \pi$.

Answer: $\frac{x^2}{4} + \frac{y^2}{9} = 1$, the right half of an ellipse.

Difficulty: 1 Section: 1

6. Find $\frac{dy}{dx}$ if $x = \sec t$ and $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

Answer: $\frac{\sec t}{\tan t}$ if $t \neq 0$.

Difficulty: 1 Section: 1

7. Find $\frac{dy}{dx}$ if $x = 2 \sin t$ and $y = 3 \cos t$.

Answer: $-\frac{3}{2} \tan t$

Difficulty: 1 Section: 1

8. Find $\frac{d^2y}{dx^2}$ if $x = 2 \sin t$ and $y = 3 \cos t$.

Answer: $-\frac{3}{4} \sec^3 t$

Difficulty: 1 Section: 1

9. Find $\frac{dy}{dx}$ if $x = 4t^2$ and $y = t^3 - 12t$.

Answer: $\frac{3t^2 - 12}{8t}$

Difficulty: 1 Section: 1

10. Find $\frac{d^2y}{dx^2}$ if $x = 4t^2$ and $y = t^3 - 12t$.

Answer: $\frac{3t^2 + 12}{64t^3}$

Difficulty: 1 Section: 1

11. An airplane flies east with an airspeed in still air of 200 mph. A wind is blowing E 60° N at 120 mph. What is the velocity of the airplane relative to the ground and what is its direction of travel?

Answer: 279.99 mph, E 21.78° N

Difficulty: 2 Section: 2

12. An airplane flies east with an airspeed in still air of 200 mph. A wind is blowing E 30° N at 120 mph. What is the velocity of the airplane relative to the ground and what is its direction of travel?

Answer: 309.77 mph, E 11.17° N

Difficulty: 2 Section: 2

13. A motorboat travels at a speed of 30 mph in still water. The boat starts upriver at an angle of 40° with the riverbank against a 5 mph current. At what angle with the riverbank is the boat moving and at what speed relative to the riverbank?

Answer: 47.77° , 26.4 mph

Difficulty: 2 Section: 2

14. A river flows south with a current of 5 mph. A motorboat travels at a speed of 25 mph in still water. In what direction must the boat be headed in order to move directly east across the river?

Answer: 78.5°

Difficulty: 2 Section: 2

15. An airplane is flying at a heading of 40° (40° east of north) with an airspeed of 600 mph. The wind is blowing from the south at 25 mph. What is the direction in which the plane flies and what is its ground speed?

Answer: A heading of 39° (39° east of north) with an airspeed of 620 mph

Difficulty: 2 Section: 2

16. If $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$, find $\mathbf{u} + \mathbf{v}$.

Answer: $\mathbf{i} + \mathbf{j}$

Difficulty: 1 Section: 3

17. If $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$, find $2\mathbf{u} - \mathbf{v}$.

Answer: $4\mathbf{i} + \mathbf{j}$

Difficulty: 1 Section: 3

18. If $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$, compute the length of $\mathbf{u}$.

Answer: 5

Difficulty: 1 Section: 3

19. If $\mathbf{u} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, compute the length of $\mathbf{u}$.

Answer: $\sqrt{10}$

Difficulty: 1 Section: 3

20. Find the vector extending from $(0, 2)$ to $(2, -4)$.

Answer: $2\mathbf{i} - 6\mathbf{j}$

Difficulty: 1 Section: 3

21. Find the vector extending from $(1, -2)$ to $(1, 2)$.

Answer: $4\mathbf{j}$

Difficulty: 1 Section: 3

22. Find the vector extending from $(-2, 3)$ to $(-1, 3)$.

Answer: $\mathbf{i}$

Difficulty: 1 Section: 3

23. Find a vector of length 4 in the direction $\mathbf{i} - \mathbf{j}$.

Answer: $\frac{4}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

Difficulty: 1 Section: 3

24. Find a vector of length 2 in the direction of $2\mathbf{i} - \mathbf{j}$.

Answer: $\frac{2}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$

Difficulty: 1 Section: 3

25. Find the vector of length 5 in the direction of $4\mathbf{i} - 3\mathbf{j}$.

Answer: $4\mathbf{i} - 3\mathbf{j}$

Difficulty: 1 Section: 3

26. Find a vector of length 5 in the direction of $3\mathbf{i} - 4\mathbf{j}$.

Answer: $3\mathbf{i} - 4\mathbf{j}$

Difficulty: 1 Section: 3

27. Find the angle between the vectors $\sqrt{3}\mathbf{i} - \mathbf{j}$ and $-\mathbf{i} + \sqrt{3}\mathbf{j}$.

Answer: $\frac{5\pi}{6}$

Difficulty: 1 Section: 3

28. Find the cosine of the angle between $\mathbf{i} - 4\mathbf{j}$ and $4\mathbf{i} + 2\mathbf{j}$.

Answer: $\frac{-2}{\sqrt{85}}$

Difficulty: 1 Section: 3

29. If $\mathbf{r}(t) = t^2\mathbf{i} + \frac{t^4 - 4}{t - 2}\mathbf{j}$, find $\lim_{t \rightarrow 2} \mathbf{r}(t)$.

Answer: $4\mathbf{i} + 4\mathbf{j}$

Difficulty: 1 Section: 4

30. If $\mathbf{r}(t) = 3t^2\mathbf{i} + \frac{t^2 - 9}{t - 3}\mathbf{j}$, find $\lim_{t \rightarrow 3} \mathbf{r}(t)$.

Answer: $27\mathbf{i} + 6\mathbf{j}$

Difficulty: 1 Section: 4

31. If $\mathbf{r}(t) = \ln t\mathbf{i} + 2t\mathbf{j}$, find $\lim_{t \rightarrow 0} \mathbf{r}(t)$ if it exists.

Answer: limit does not exist

Difficulty: 1 Section: 4

32. If $\mathbf{r}(t) = \frac{t+1}{t-1}\mathbf{i} + \ln(2t^3 + 1)\mathbf{j}$, find the velocity vector and acceleration vector when $t = 0$.

Answer: $\mathbf{v} = -2\mathbf{i}$, $\mathbf{a} = -4\mathbf{i}$

Difficulty: 1 Section: 4

33. If $\mathbf{r}(t) = \sec t \mathbf{i} + \tan t \mathbf{j}$, find the velocity vector and acceleration vector when $t = 0$.

Answer:

$$\mathbf{v} = (\sec t \tan t) \mathbf{i} + \sec^2 t \mathbf{j}$$
$$\mathbf{a} = (\sec t \tan^2 t + \sec^3 t) \mathbf{i} + (2 \sec^2 t \tan t) \mathbf{j}$$

Difficulty: 1 Section: 4

34. Find the position and velocity vectors if $\mathbf{a} = 2t\mathbf{i} - \mathbf{j}$, $\mathbf{v}(0) = \mathbf{i}$, and $\mathbf{r}(0) = 2\mathbf{i} - \mathbf{j}$.

Answer: $\mathbf{v}(t) = (t^2 + 1) \mathbf{i} - t \mathbf{j}$, $\mathbf{r}(t) = \left(\frac{t^3}{3} + t + 2\right) \mathbf{i} - \left(\frac{t^2}{2} + 1\right) \mathbf{j}$

Difficulty: 1 Section: 4

35. Find the position and velocity vectors if $\mathbf{a} = \cos t \mathbf{i} + \sin^2 t \mathbf{j}$, $\mathbf{v}\left(\frac{\pi}{2}\right) = \frac{3}{2} \mathbf{j}$, $\mathbf{r}\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \mathbf{j}$.

Answer:

$$\mathbf{v}(t) = (\sin t - 1) \mathbf{i} + \left(\frac{-\cos 2t}{2} + 1\right) \mathbf{j}$$
$$\mathbf{r}(t) = \left(-\cos t - t + \frac{\pi}{2}\right) \mathbf{i} + \left(\frac{-\sin 2t}{4} + t\right) \mathbf{j}$$

Difficulty: 1 Section: 4

36. Find the position and velocity vectors if $\mathbf{a} = 12t^2\mathbf{i} - (4 \cos t)\mathbf{j}$, $\mathbf{v}(0) = -\mathbf{i}$, and $\mathbf{r}(0) = 2\mathbf{i} + \mathbf{j}$.

Answer:

$$\mathbf{v}(t) = (4t^3 - 1) \mathbf{i} - (4 \sin t) \mathbf{j}$$
$$\mathbf{r}(t) = (t^4 - t + 2) \mathbf{i} + (4 \cos t - 3) \mathbf{j}$$

Difficulty: 1 Section: 4

37. A shell is fired from ground level, with muzzle velocity of 600 ft/sec., and an angle of elevation of 30 degrees. Find the parametric equations of the trajectory.

Answer: $x(t) = 300\sqrt{3}t$, $y = 300t - 16t^2$

Difficulty: 1 Section: 4

38. A shell is fired from ground level, with muzzle velocity of 600 ft/sec., and an angle of elevation of 60 degrees. Find the parametric equations of the trajectory.

Answer: $x(t) = 300t$, $y = 300\sqrt{3}t - 16t^2$

Difficulty: 1 Section: 4

39. Find the velocity and acceleration vectors for $x = \sin^2 t$, $y = \cos^2 t$.

Answer:

$$\mathbf{v} = (2 \sin t \cos t) \mathbf{i} - (2 \sin t \cos t) \mathbf{j}$$
$$\mathbf{a} = 2(\cos^2 t - \sin^2 t) \mathbf{i} + 2(\sin^2 t - \cos^2 t) \mathbf{j}$$

Difficulty: 1 Section: 4

40. Find the velocity and acceleration vectors for $\frac{t+1}{t-1}$ and $y = \ln(2t^3 + 1)$ when $t = 0$.

Answer:

$$\mathbf{v} = -2\mathbf{i}$$

$$\mathbf{a} = -4\mathbf{i}$$

Difficulty: 1 Section: 4

41. If $\mathbf{R}(t) = t^2\mathbf{i} + \sec t \mathbf{j}$, find the velocity and acceleration vectors.

Answer:

$$\mathbf{v} = 2t\mathbf{i} + \sec t \tan t \mathbf{j}$$

$$\mathbf{a} = 2\mathbf{i} + (\sec t \tan^2 t + \sec^3 t) \mathbf{j}$$

Difficulty: 1 Section: 4

42. Find the curvature of the curve given by $x = t$, $y = \ln t$.

$$\text{Answer: } \frac{t}{(1+t^2)^{3/2}}$$

Difficulty: 2 Section: 5

43. Find the curvature of the curve given by $x = \cos^3 t$, $y = \sin^3 t$.

$$\text{Answer: } \frac{1}{3 \cos t \sin t}$$

Difficulty: 2 Section: 5

44. Find the curvature of the curve given by $x = 3 \cos 2t$, $y = \sin 2t$ at $(0, 1)$.

$$\text{Answer: } \frac{1}{9}$$

Difficulty: 2 Section: 5

45. Find the curvature of the curve given by $x = 4 \cos 3t$, $y = \sin 3t$.

$$\text{Answer: } \frac{4}{(16 \sin^2 3t + \cos^2 3t)^{3/2}}$$

Difficulty: 2 Section: 5

46. Find the curvature of the curve given by $x = 2 \cos 5t$, $y = \sin 5t$.

$$\text{Answer: } \frac{2}{(4 \sin^2 5t + \cos^2 5t)}$$

Difficulty: 2 Section: 5

47. Find the curvature of the curve given in polar coordinates by $r = e^{2\theta}$.

$$\text{Answer: } \frac{1}{r\sqrt{5}}$$

Difficulty: 2 Section: 5

48. Find the curvature of the curve $y = \ln \sec x$.

Answer: $\cos x$

Difficulty: 2 Section: 5

49. Find the point on the parabola $y = x^2 + 2x$ where the curvature is a maximum.

Answer: at the vertex $(-1, -1)$

Difficulty: 2 Section: 5

50. Find the radius of curvature of the curve $y = x^3$ at $(1, 1)$.

Answer: $\frac{5\sqrt{10}}{3}$

Difficulty: 2 Section: 5

51. Find the radius of curvature of the curve $y = \frac{x^2}{2}$ at $\left(1, \frac{1}{2}\right)$.

Answer: $2\sqrt{2}$

Difficulty: 2 Section: 5

52. If $\mathbf{r}(t) = \sin^2 t \mathbf{i} + \cos^2 t \mathbf{j}$, find the velocity and acceleration vectors.

Answer:

$$\mathbf{v} = (2 \sin t \cos t) \mathbf{i} + (-2 \sin t \cos t) \mathbf{j} = \sin 2t \mathbf{i} - \sin 2t \mathbf{j}$$

$$\mathbf{a} = 2(\cos^2 t - \sin^2 t) \mathbf{i} + 2(\sin^2 t - \cos^2 t) \mathbf{j} = 2 \cos 2t \mathbf{i} - 2 \cos 2t \mathbf{j}$$

Difficulty: 1 Section: 5

53. If $\mathbf{r}(t) = \sin^2 t \mathbf{i} + \cos^2 t \mathbf{j}$, find the normal and tangential components of the acceleration.

Answer:

$$a_T = 2\sqrt{2} \cos 2t, a_N = 0$$

Difficulty: 1 Section: 5

54. Find the unit tangent and unit normal vectors to the curve given by $x = \ln t$, $y = 2t$.

$$\text{Answer: } \frac{\mathbf{i} + 2t \mathbf{j}}{\sqrt{1 + 4t^2}} = \mathbf{T}, \mathbf{N} = \frac{-2t \mathbf{i} + \mathbf{j}}{\sqrt{1 + 4t^2}}$$

Difficulty: 2 Section: 5

55. Find the unit tangent and unit normal vectors to the curve given by $x = 2t$, $y = \ln(\cos t)^2$.

$$\text{Answer: } \cos t \mathbf{i} - \sin t \mathbf{j} = \mathbf{T}, \mathbf{N} = \sin t \mathbf{i} + \cos t \mathbf{j}$$

Difficulty: 2 Section: 5

56. Find the unit tangent and unit normal vectors to the curve described by the function $\mathbf{r}(t) = t^2\mathbf{i} - 4t\mathbf{j}$.

$$\text{Answer: } \mathbf{T} = \frac{t\mathbf{i} - 2\mathbf{j}}{\sqrt{4 + t^2}}, \mathbf{N} = \frac{2\mathbf{i} + t\mathbf{j}}{\sqrt{4 + t^2}}$$

Difficulty: 2 Section: 5

57. Find the unit tangent and unit normal vectors to the curve described by the function $\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j}$.

$$\begin{aligned} \text{Answer: } \mathbf{T} &= \frac{\cos t + \sin t}{\sqrt{2}} \mathbf{i} + \frac{\cos t - \sin t}{\sqrt{2}} \mathbf{j} \\ \mathbf{N} &= \frac{\cos t + \sin t}{\sqrt{2}} \mathbf{i} + \frac{\cos t - \sin t}{\sqrt{2}} \mathbf{j} \end{aligned}$$

Difficulty: 2 Section: 5

58. Find the unit tangent and unit normal vectors to the curve described by the function $\mathbf{r}(t) = 3t \cos t \mathbf{i} + 3t \sin t \mathbf{j}$.

$$\begin{aligned} \text{Answer: } \mathbf{T} &= \frac{(\cos t - t \sin t) \mathbf{i} + (\sin t + t \cos t) \mathbf{j}}{\sqrt{1 + t^2}} \\ \mathbf{N} &= \frac{(-\sin t - t \cos t) \mathbf{i} + (\cos t - t \sin t) \mathbf{j}}{\sqrt{1 + t^2}} \end{aligned}$$

Difficulty: 2 Section: 5

14 Geometry in Space, Vectors

1. Find the distance from $(0, 1, 2)$ to $(1, -2, 3)$.

Answer: $\sqrt{35}$

Difficulty: 1 Section: 1

2. Find the distance from $(1, -4, 5)$ to $(2, 0, 3)$.

Answer: $\sqrt{21}$

Difficulty: 1 Section: 1

3. Find the distance from $(-4, 3, -2)$ to $(-2, 1, 0)$.

Answer: $\sqrt{12}$

Difficulty: 1 Section: 1

4. Find the midpoint of the line segment joining $(0, 1, -2)$ and $(1, -2, 3)$.

Answer: $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$

Difficulty: 1 Section: 1

5. Find the midpoint of the line segment joining $(1, -4, 5)$ and $(2, 0, 3)$.

Answer: $\left(\frac{3}{2}, -2, 4\right)$

Difficulty: 1 Section: 1

6. Find the midpoint of the line segment joining $(-4, 3, -2)$ and $(-2, 1, 0)$.

Answer: $\left(-3, \frac{3}{2}, -1\right)$

Difficulty: 1 Section: 1

7. Find the center and radius of the sphere with equation $x^2 - 2x + y^2 + 6y + z^2 = 6$.

Answer: center $(1, -3, 0)$, radius 4

Difficulty: 1 Section: 1

8. Find the equation of the sphere with center $(1, -3, 0)$ and radius 4.

Answer: $(x - 1)^2 + (y + 3)^2 + z^2 = 16$

Difficulty: 1 Section: 1

9. Find the center and radius of the sphere with equation $x^2 + 6x + y^2 - 4x + z^2 - 8z = -4$.

Answer: center $(-3, 2, 4)$, radius 5

Difficulty: 1 Section: 1

10. Find the equation of the sphere with center $(-3, 2, 4)$ and radius 5.

Answer: $(x + 3)^2 + (y - 2)^2 + (z - 4)^2 = 25$

Difficulty: 1 Section: 1

11. If $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$, find $\mathbf{u} + \mathbf{v}$.

Answer: $\mathbf{i} + \mathbf{j} - \mathbf{k}$

Difficulty: 1 Section: 2

12. If $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + 9\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, find $2\mathbf{u} - \mathbf{v}$.

Answer: $4\mathbf{i} + \mathbf{j} + 17\mathbf{k}$

Difficulty: 1 Section: 2

13. If $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j} - \mathbf{k}$, compute the length of $\mathbf{u}$.

Answer: $\sqrt{26}$

Difficulty: 1 Section: 2

14. If $\mathbf{u} = \mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, compute the length of $\mathbf{u}$.

Answer: $\sqrt{14}$

Difficulty: 1 Section: 2

15. Write the vector extending from $(0, 2, 3)$ to $(2, -4, -1)$.

Answer: $2\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$

Difficulty: 1 Section: 2

16. Write the vector extending from $(1, -2, 4)$ to $(1, 2, -1)$.

Answer: $4\mathbf{j} - 5\mathbf{k}$

Difficulty: 1 Section: 2

17. Write the vector extending from $(-2, 3, -1)$ to $(-1, 3, 4)$.

Answer: $\mathbf{i} + 5\mathbf{k}$

Difficulty: 1 Section: 2

18. Find a vector of length 4 in the direction of $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Answer: $\frac{4}{\sqrt{6}}(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

Difficulty: 1 Section: 2

19. Find a vector of length 2 in the direction of $2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$.

Answer: $\frac{2}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} - 3\mathbf{k})$

Difficulty: 1 Section: 2

20. Find a vector of length 5 in the direction of $4\mathbf{i} - 3\mathbf{j}$.

Answer: $4\mathbf{i} - 3\mathbf{j}$

Difficulty: 1 Section: 2

21. Find the vector of length 5 in the direction of $3\mathbf{j} - 4\mathbf{k}$.

Answer: $3\mathbf{j} - 4\mathbf{k}$

Difficulty: 1 Section: 2

22. If the length of $\mathbf{u}$ is 4 and the length of $\mathbf{v}$ is 3, what can be said about the length of $\mathbf{u} + \mathbf{v}$?

Answer: The length of $\mathbf{u} + \mathbf{v}$ is less than or equal to 7.

Difficulty: 1 Section: 2

23. If the length of $\mathbf{u}$ is 3 and the length of $\mathbf{v}$ is 5, what can be said about the length of $\mathbf{u} + \mathbf{v}$?

Answer: The length of $\mathbf{u} + \mathbf{v}$ is less than or equal to 8.

Difficulty: 1 Section: 2

24. If the length of $\mathbf{u}$ is 4, what is the length of $-2\mathbf{u}$?

Answer: 8

Difficulty: 1 Section: 2

25. If the length of $\mathbf{u}$ is $\sqrt{2}$, what is the length of $-\mathbf{u}$?

Answer: $\sqrt{2}$

Difficulty: 1 Section: 2

26. Find a unit vector in the direction of $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Answer: $\frac{1}{\sqrt{6}}(2\mathbf{i} - \mathbf{j} + \mathbf{k})$

Difficulty: 1 Section: 2

27. Find a unit vector in the direction opposite to that of $3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Answer: $\frac{1}{\sqrt{19}}(-3\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

Difficulty: 1 Section: 2

28. Find a unit vector in the direction opposite to that of $2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$.

Answer: $\frac{1}{\sqrt{22}}(-2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$

Difficulty: 1 Section: 2

29. Find a vector of length $\sqrt{2}$ in the direction of the vector extending from $(2, -1, 0)$ to $(-1, 2, 2)$.

Answer: $\frac{\sqrt{2}}{\sqrt{22}}(-3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$

Difficulty: 1 Section: 2

30. The vector $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ extends from $(1, -4, -5)$ to (a, b, c) . What are the coordinates of a , b , and c ?

Answer: $a = 4, b = -6, c = -1$

Difficulty: 2 Section: 2

31. Find the dot product of $2\mathbf{i} - \mathbf{k}$ and $3\mathbf{i} + 14\mathbf{j} - \mathbf{k}$.

Answer: 7

Difficulty: 1 Section: 2

32. Find the dot product of $4\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Answer: 2

Difficulty: 1 Section: 2

33. Find the dot product of $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Answer: 0

Difficulty: 1 Section: 2

34. Find the dot product of $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Answer: 0

Difficulty: 1 Section: 2

35. Find the dot product of $4\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

Answer: -2

Difficulty: 1 Section: 2

36. Find the dot product of $\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

Answer: -3

Difficulty: 1 Section: 2

37. Find the angle between the vectors $\sqrt{3}\mathbf{i} - \mathbf{j}$ and $-\mathbf{i} + \sqrt{3}\mathbf{j}$.

Answer: $\frac{5\pi}{6}$

Difficulty: 1 Section: 2

38. Find the cosine of the angle between $\mathbf{i} - 4\mathbf{j}$ and $4\mathbf{i} - 2\mathbf{j}$.

Answer: $\frac{-2}{\sqrt{85}}$

Difficulty: 1 Section: 2

39. Find the cosine of the angle between $4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Answer: $\frac{4}{3\sqrt{26}}$

Difficulty: 1 Section: 2

40. Find the cosine of the angle between $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Answer: $-\frac{9}{\sqrt{84}}$

Difficulty: 1 Section: 2

41. Find the equation of the plane containing $(2, -3, 1)$ and perpendicular to the vector $\mathbf{i} - \mathbf{j} + 9\mathbf{k}$.

Answer: $(x - 2) - (y + 3) + 9(z - 1) = 0$

Difficulty: 1 Section: 2

42. Find the equation of the plane containing $(1, -1, 9)$ and perpendicular to the vector $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Answer: $2(x - 1) - 3(y + 1) + (z - 9) = 0$

Difficulty: 1 Section: 2

43. Find the equation of the plane containing $(1, -1, 3)$ and perpendicular to the vector $3\mathbf{i} - \mathbf{j}$.

Answer: $3(x - 1) - (y + 1) = 0$

Difficulty: 1 Section: 2

44. Find the equation of the plane containing $(1, -1, 3)$ and perpendicular to the vector $3\mathbf{j} - \mathbf{k}$.

Answer: $3(y + 1) - (z - 3) = 0$

Difficulty: 1 Section: 2

45. Find the equation of the plane containing $(3, 1, 0)$ and perpendicular to the vector $4\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$.

Answer: $4(x - 3) - 5(y - 1) + 8z = 0$

Difficulty: 1 Section: 2

46. If $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, find $\text{pr}_{\mathbf{v}}\mathbf{u}$.

Answer: $\frac{-4}{19}(\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$

Difficulty: 2 Section: 2

47. If $\mathbf{u} = \mathbf{i} + \mathbf{j} - 8\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, find $\text{pr}_{\mathbf{v}}\mathbf{u}$.

Answer: $\frac{12}{9}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

Difficulty: 2 Section: 2

48. If $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} - 8\mathbf{k}$, find $\text{pr}_{\mathbf{v}}\mathbf{u}$.

Answer: $\frac{12}{66}(\mathbf{i} + \mathbf{j} - 8\mathbf{k})$

Difficulty: 2 Section: 2

49. If $\mathbf{u} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, find $\text{pr}_{\mathbf{v}}\mathbf{u}$.

Answer: $\frac{-4}{6}(2\mathbf{i} - \mathbf{j} - \mathbf{k})$

Difficulty: 2 Section: 2

50. If $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, find $\text{pr}_{\mathbf{v}}\mathbf{u}$.

Answer: the zero vector

Difficulty: 2 Section: 2

51. If $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j} - \mathbf{k}$, find $\text{pr}_{\mathbf{v}}\mathbf{u}$.

Answer: the zero vector

Difficulty: 2 Section: 2

52. If $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$.

Answer: $-31\mathbf{i} - 20\mathbf{j} + 7\mathbf{k}$

Difficulty: 1 Section: 3

53. If $\mathbf{u} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$.

Answer: $\mathbf{i} - \mathbf{j} + \mathbf{k}$

Difficulty: 1 Section: 3

54. If $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$.

Answer: $7\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

Difficulty: 1 Section: 3

55. If $\mathbf{u} = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$.

Answer: $22\mathbf{i} - 2\mathbf{j} + 17\mathbf{k}$

Difficulty: 1 Section: 3

56. If $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$.

Answer: $\mathbf{i} + 11\mathbf{j} + 3\mathbf{k}$

Difficulty: 1 Section: 3

57. Find the equation of the plane through $(1, 2, -3)$, $(3, 1, 0)$, and $(2, 3, -7)$.

Answer: $x + 11y + 3z - 14 = 0$

Difficulty: 2 Section: 3

58. Find the equation of the plane through $(1, 2, -1)$, $(3, 1, 2)$, and $(2, 3, -5)$.

Answer: $x + 11y + 3z - 20$

Difficulty: 2 Section: 3

59. Find the equation of the plane through $(1, 2, -1)$, $(3, -3, 7)$, and $(-2, 3, 2)$.

Answer: $31x + 20y - 7z = -16$

Difficulty: 2 Section: 3

60. Find the equation of the plane through $(6, 1, 0)$, $(5, -2, -1)$, and $(1, -3, 4)$.

Answer: $16x - 9y + 11z = 87$

Difficulty: 2 Section: 3

61. Find the equation of the plane through $(2, 3, 1)$, $(1, -1, 2)$, and $(-4, 3, 0)$.

Answer: $4x - 7y - 24z = -37$

Difficulty: 2 Section: 3

62. Find the equation of the plane through $(2, 3, 1)$, $(1, -1, 2)$, and $(-4, 3, 1)$.

Answer: $y + 4z = 7$

Difficulty: 2 Section: 3

63. Find a unit vector perpendicular to both $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Answer: $\frac{1}{\sqrt{77}}(4\mathbf{i} - 5\mathbf{j} - 6\mathbf{k})$

Difficulty: 2 Section: 3

64. Find a unit vector perpendicular to both $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Answer: $\frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

Difficulty: 2 Section: 3

65. Find a vector of length 2 perpendicular to both $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

Answer: $\frac{2}{\sqrt{83}}(7\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$

Difficulty: 2 Section: 3

66. Write the parametric equations of the line through $(2, -1, 0)$ and $(-1, 2, 2)$.

Answer: $x = 2 + 3t$; $y = -1 - 3t$; $z = -2t$

Difficulty: 1 Section: 4

67. Write the symmetric equations of the line through $(2, -1, 0)$ and $(-1, 2, 2)$.

Answer: $\frac{x-2}{3} = \frac{y+1}{-3} = \frac{z}{-2}$

Difficulty: 1 Section: 4

68. Write the parametric equations of the line through $(1, -4, -5)$ and $(4, -6, -1)$.

Answer: $x = 4 + 3t$, $y = -6 - 2t$, $z = -1 + 4t$

Difficulty: 1 Section: 4

69. Write the symmetric equations of the line through $(1, -4, -5)$ and $(4, -6, -1)$.

Answer: $\frac{x-4}{3} = \frac{y+6}{-2} = \frac{z+1}{4}$

Difficulty: 1 Section: 4

70. Write the symmetric equations of the line through $(0, 2, 3)$ and $(2, -4, -1)$.

Answer: $\frac{x-2}{1} = \frac{y+4}{-3} = \frac{z+1}{-2}$

Difficulty: 1 Section: 4

71. Write the parametric equations of the line through $(0, 2, 3)$ and $(2, -4, -1)$.

Answer: $x = 2 + t$, $y = -4 - 3t$, $z = -1 - 2t$

Difficulty: 1 Section: 4

72. Write the parametric equations of the line through $(2, -3, 1)$ parallel to $3\mathbf{i} - \mathbf{j}$.

Answer: $x = 2 + 3t$, $y = -3 - t$, $z = 1$

Difficulty: 1 Section: 4

73. Write the symmetric equations of the line through $(2, -3, 1)$ parallel to $3\mathbf{i} - \mathbf{j}$.

Answer: $\frac{x-2}{3} = \frac{y+3}{-1}, z = 1$

Difficulty: 1 Section: 4

74. Write the parametric equations of the line through $(-2, 3, -1)$ and $(-1, 3, 4)$.

Answer: $x = -1 + t, y = 3, z = 4 + 5t$

Difficulty: 1 Section: 4

75. Write the symmetric equations of the line through $(-2, 3, -1)$ and $(-1, 3, 4)$.

Answer: $\frac{x+1}{1} = \frac{z-4}{5}, y = 3$

Difficulty: 1 Section: 4

76. Find a vector parallel to the line given by the equations $x = -3 - t, y = 2 + 5t, z = 3t$.

Answer: any scalar multiple of $-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

Difficulty: 1 Section: 4

77. Find a vector parallel to the line given by the equations $\frac{x+3}{-1} = \frac{y-2}{5} = \frac{z}{3}$.

Answer: Any scalar multiple of $-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

Difficulty: 1 Section: 4

78. Find a vector parallel to the line given by the equations $x = 1, y = 9 - 2t, z = -5 - t$.

Answer: Any scalar multiple of $2\mathbf{j} + \mathbf{k}$

Difficulty: 1 Section: 4

79. Find a vector parallel to the line given by the equations $x = 1, \frac{y-9}{-2} = \frac{z+5}{-1}$.

Answer: Any scalar multiple of $2\mathbf{j} + \mathbf{k}$

Difficulty: 1 Section: 4

80. Find a point on the line given by the equations $\frac{x-3}{2} = \frac{y+1}{4} = \frac{z+2}{-1}$ such that the first coordinate is 4.

Answer: $\left(4, 1, -\frac{5}{2}\right)$

Difficulty: 2 Section: 4

81. Find a point on the line given by the equations $x = 3 + 2t, y = -1 + 4t, z = -2 - t$ such that the first coordinate is 4.

Answer: $\left(4, 1, -\frac{5}{2}\right)$

Difficulty: 2 Section: 4

82. Find an equation for the line tangent to the curve $x = \sin t$, $y = \cos t$, $z = 1$ at $(0, -1, 1)$.

Answer: $-\mathbf{i}$

Difficulty: 2 Section: 4

83. Find an equation for the line tangent to the curve $x = t^2$, $y = t^3$, $z = 1 - t$ at $(1, -1, 2)$.

Answer: $-2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

Difficulty: 2 Section: 4

84. Find a tangent vector for the curve $x = 3t \cos t$, $y = 3t \sin t$, $z = 4t$ in terms of t .

Answer: $(3 \cos t - 3t \sin t)\mathbf{i} + (3 \sin t + 3t \cos t)\mathbf{j} + 4\mathbf{k}$

Difficulty: 2 Section: 4

85. Find a tangent vector for the curve $x = \sec t$, $y = \tan t$, $z = 3$, at $(\sqrt{2}, 1, 3)$.

Answer: $\sqrt{2}\mathbf{i} + 2\mathbf{j}$

Difficulty: 2 Section: 4

86. Find a unit vector tangent to the curve $x = 2 \cos 6t$, $y = 2 \sin 6t$, $z = 5t$ at $(0, 2, \frac{5\pi}{12})$.

Answer: $\frac{1}{13}(-12\mathbf{i} + 5\mathbf{k})$

Difficulty: 2 Section: 4

87. Write the integral for finding the length of the curve given by the equations $x = 4$, $y = t$, $z = t^2$, $0 \leq t \leq 2$.

Answer: $\int_0^4 \sqrt{1 + 4t^2} dt$

Difficulty: 1 Section: 5

88. Write the integral for finding the length of the curve given by the equations $x = e^t \sin t$, $y = e^t \cos t$, $z = 1$, $0 \leq t \leq 2$.

Answer: $\int_0^2 \sqrt{2}e^t dt$

Difficulty: 1 Section: 5

89. Write the integral for finding the length of the curve given by the equations $x = 1 + t^3$, $y = t$, $z = 1$, $0 \leq t \leq 1$.

Answer: $\int_0^1 \sqrt{9t^4 + 1} dt$

Difficulty: 1 Section: 5

90. Find the velocity and acceleration vectors for $x = \sin^2 t$, $y = \cos^2 t$, $z = t$.

Answer:

$$\mathbf{v} = (2 \sin t \cos t) \mathbf{i} - (2 \sin t \cos t) \mathbf{j} + \mathbf{k}$$
$$\mathbf{a} = 2 (\cos^2 t - \sin^2 t) \mathbf{i} + 2 (\sin^2 t - \cos^2 t) \mathbf{j}$$

Difficulty: 1 Section: 5

91. Find the velocity and acceleration vectors for $x = \frac{t+1}{t-1}$, $y = \ln(2t^3 + 1)$ and $z = t^3 - 5t$ when $t = 0$.

Answer:

$$\mathbf{v} = -2\mathbf{i} - 5\mathbf{k}$$
$$\mathbf{a} = -4\mathbf{i}$$

Difficulty: 1 Section: 5

92. If $\mathbf{r}(t) = t^2 \mathbf{i} + \sec t \mathbf{j} + \tan t \mathbf{k}$, find the velocity and acceleration vectors.

Answer:

$$\mathbf{v} = 2t \mathbf{i} + \sec t \tan t \mathbf{j} + \sec^2 t \mathbf{k}$$
$$\mathbf{a} = 2\mathbf{i} + (\sec t \tan^2 t + \sec^3 t) \mathbf{j} + (2 \sec^2 t \tan t) \mathbf{k}$$

Difficulty: 1 Section: 5

93. Find the velocity and acceleration vectors if $\mathbf{r}(t) = (t^2 - t + 2) \mathbf{i} + (4 \cos t - 3) \mathbf{j} + (e^{2t} - t) \mathbf{k}$.

Answer:

$$\mathbf{v} = (2t - 1) \mathbf{i} + (-4 \sin t) \mathbf{j} + (2e^{2t} - 1) \mathbf{k}$$
$$\mathbf{a} = 2\mathbf{i} - 4 \cos t \mathbf{j} + 4e^{2t} \mathbf{k}$$

Difficulty: 1 Section: 5

94. Given that $\mathbf{r}(t) = (3t + 6) \mathbf{i} + (t^3) \mathbf{j} + \left(\frac{3\sqrt{2}}{2} t^2\right) \mathbf{k}$, find the curvature.

Answer: $\frac{\sqrt{2}}{3(1+t^2)^2}$

Difficulty: 2 Section: 5

95. Given that $\mathbf{r}(t) = (3t + 6) \mathbf{i} + (t^3) \mathbf{j} + \left(\frac{3\sqrt{2}}{2} t^2\right) \mathbf{k}$, find the normal and tangential components of the acceleration.

Answer: $a_T = 6t$, $a_N = 3\sqrt{2}$

Difficulty: 2 Section: 5

96. Given that $\mathbf{r}(t) = (2 \cos 6t)\mathbf{i} + (2 \sin 6t)\mathbf{j} + 3\mathbf{k}$, find the unit tangent vector and the principal normal vector.

Answer:

$$\mathbf{T} = \frac{1}{\sqrt{153}} (-12 \sin 6t \mathbf{i} + 12 \cos 6t \mathbf{j} + 3\mathbf{k})$$

$$\mathbf{N} = (-\cos 6t)\mathbf{i} + (-\sin 6t)\mathbf{j}$$

Difficulty: 2 Section: 5

97. Given that $\mathbf{r}(t) = (2 \cos 6t)\mathbf{i} + (2 \sin 6t)\mathbf{j} + 3t\mathbf{k}$, find the curvature.

Answer: $\frac{72}{153}$

Difficulty: 2 Section: 5

98. If $\mathbf{r}(t) = 2t^2\mathbf{i} + \frac{3}{2}t^2\mathbf{j} + 5t\mathbf{k}$, find the normal and tangential components of the acceleration.

Answer: $a_T = \frac{5t}{\sqrt{t^2 + 1}}$, $a_N = \frac{5}{\sqrt{t^2 + 1}}$

Difficulty: 2 Section: 5

99. If $\mathbf{r}(t) = 2t^2\mathbf{i} + \frac{3}{2}t^2\mathbf{j} + 5t\mathbf{k}$, find the curvature.

Answer: $\frac{1}{5(1+t^2)^{3/2}}$

Difficulty: 2 Section: 5

100. If $\mathbf{r}(t) = 2t^2\mathbf{i} + \frac{3}{2}t^2\mathbf{j} + 5t\mathbf{k}$, find the unit tangent vector and the unit normal vector.

Answer:

$$\mathbf{T} = \frac{1}{\sqrt{t^2 + 1}} (4t\mathbf{i} + 3t\mathbf{j} + 5\mathbf{k})$$

$$\mathbf{N} = \frac{1}{5\sqrt{t^2 + 1}} (4\mathbf{i} + 3\mathbf{j} - 5t\mathbf{k})$$

Difficulty: 2 Section: 5

101. Write the equation of the surface $x^2 + z^2 = x$ in cylindrical coordinates.

Answer: $r^2 \cos^2 \theta + z^2 = r \cos \theta$

Difficulty: 1 Section: 7

102. Write the equation of the surface $z = x^2 + y^2 - y$ in cylindrical coordinates.

Answer: $z = r(r - \sin \theta)$

Difficulty: 1 Section: 7

103. Write the equation of the surface $x^2 + y^2 = 3x$ in cylindrical coordinates.

Answer: $r = 3 \cos \theta$

Difficulty: 1 Section: 7

104. Write the equation of the surface $x^2 + y^2 = 4y$ in cylindrical coordinates.

Answer: $r = 4 \sin \theta$

Difficulty: 1 Section: 7

105. Write the equation $\rho = 2 \sec \rho$ in rectangular coordinates.

Answer: $z = 2$

Difficulty: 1 Section: 7

106. Write the equation $z = 5$ in spherical coordinates.

Answer: $\rho = 5 \sec \theta$

Difficulty: 1 Section: 7

107. Write the equation $x = 6$ in cylindrical coordinates.

Answer: $r = 6 \sec \theta$

Difficulty: 1 Section: 7

108. Write the equation $x^2 - y^2 = 1$ in cylindrical coordinates.

Answer: $r^2 \cos^2 \theta = 1$

Difficulty: 1 Section: 7

15 The Derivative in n -Space

1. What is the natural domain of the function $f(x, y) = \tan^{-1} \frac{y}{x}$?

Answer: $\{(x, y) : x \neq 0\}$

Difficulty: 1 Section: 1

2. What is the natural domain of the function $f(x, y) = \frac{2x - y}{x + 3y}$?

Answer: $\{(x, y) : x \neq -3y\}$

Difficulty: 1 Section: 1

3. What is the natural domain of the function $f(x, y) = \frac{2x - y}{x - 3y}$?

Answer: $\{(x, y) : x \neq 3y\}$

Difficulty: 1 Section: 1

4. If $f(x, y) = \tan^{-1} \frac{y}{x}$, what is $f(1, 1)$?

Answer: $\frac{\pi}{4}$

Difficulty: 1 Section: 1

5. If $f(x, y) = \tan^{-1} \frac{y}{x}$, what is $f(1, 0)$?

Answer: 0

Difficulty: 1 Section: 1

6. If $f(x, y) = \tan^{-1} \frac{y}{x}$, what is $f(\sqrt{3}, 1)$?

Answer: $\frac{\pi}{6}$

Difficulty: 1 Section: 1

7. If $f(x, y) = \tan^{-1} \frac{y}{x}$, what is $f(1, \sqrt{3})$?

Answer: $\frac{\pi}{3}$

Difficulty: 1 Section: 1

8. If $f(x, y) = \frac{2x - y}{x + 3y}$, what is $f(c, 0)$, $c \neq 0$?

Answer: 2

Difficulty: 1 Section: 1

9. If $f(x, y) = \frac{2x - y}{x + 3y}$, what is $f(0, c)$, $c \neq 0$?

Answer: $-\frac{1}{3}$

Difficulty: 1 Section: 1

10. If $f(x, y) = -\frac{2x - y}{x + 3y}$, what is $f(c, c)$, $c \neq 0$?

Answer: $\frac{1}{4}$

Difficulty: 1 Section: 1

11. If $f(x, y) = \ln(x^2 + y^2)^{1/2}$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Answer: $\frac{\partial f}{\partial x} = x(x^2 + y^2)^{-1}$, $\frac{\partial f}{\partial y} = y(x^2 + y^2)^{-1}$

Difficulty: 1 Section: 2

12. If $f(x, y) = \ln(x^2 + y^2)^{1/2}$, find the second partial derivatives.

Answer: $\frac{\partial^2 f}{\partial x^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$, $\frac{\partial^2 f}{\partial y^2} = -\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{2xy}{(x^2 + y^2)^2}$

Difficulty: 1 Section: 2

13. If $f(x, y) = \tan^{-1} \frac{x}{y}$, find f_x and f_y .

Answer: $f_x = \frac{y}{x^2 + y^2}$, $f_y = \frac{-x}{x^2 + y^2}$

Difficulty: 1 Section: 2

14. If $f(x, y) = \tan^{-1} \frac{x}{y}$, find f_{xx} , f_{yy} , f_{xy} , and f_{yx} .

Answer: $f_{xx} = \frac{-2xy}{(x^2 + y^2)^2}$, $f_{yy} = -f_{xx}$, $f_{xy} = \frac{x^2 - y^2}{(x^2 + y^2)^2} = f_{yx}$

Difficulty: 1 Section: 2

15. If $f(x, y) = \tan^{-1} \frac{ty}{x}$, find f_x and f_y .

Answer: $f_x = \frac{-y}{x^2 + y^2}$, $f_y = \frac{x}{x^2 + y^2}$

Difficulty: 1 Section: 2

16. If $f(x, y) = \tan^{-1} \frac{y}{x}$, find f_{xx} , f_{yy} , f_{xy} , and f_{yx} .

$$\text{Answer: } f_{xx} = \frac{2xy}{(x^2 + y^2)^2}, f_{yy} = -f_{xx}, f_{xy} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = f_{yx}$$

Difficulty: 1 Section: 2

17. If $f(x, y) = 2x^2y^3 + 17e^{xy}$, find f_x and f_y .

$$\text{Answer: } f_x = 4xy^3 + 17e^{xy}, f_y = 6x^2y^2 + 17xe^{xy}$$

Difficulty: 1 Section: 7

18. If $f(x, y) = 2x^2y^3 + 17e^{xy}$, find f_{xx} , f_{yy} , f_{xy} , and f_{yx} .

$$\text{Answer: } f_{xx} = 4y^3 + 17y^2e^{xy}, f_{yy} = 12x^2y + 17x^2e^{xy}, f_{xy} = 12xy^2 + 17e^{xy} + 17xye^{xy} = f_{yx}$$

Difficulty: 1 Section: 2

19. Let C be the curve formed by the intersection of the plane $y = -2$ and the surface $z = x^2 + 2xy - y^2$. Find the slope of the line tangent to C at $(1, -2, -7)$.

$$\text{Answer: } -2$$

Difficulty: 1 Section: 2

20. Let C be the curve formed by the intersection of the plane $x = 1$ and the surface $z = x^2 + 2xy - y^2$. Find the slope of the line tangent to C at $(1, -2, -7)$.

$$\text{Answer: } 6$$

Difficulty: 1 Section: 2

21. Let C be the curve formed by the intersection of the plane $x = 0$ and the surface $z = x^3 + y^3 - 9xy + 27$. Find the equations of the line tangent to C at $(0, -3, 0)$.

$$\text{Answer: } x = 0, z = 27t, y = -3 + t$$

Difficulty: 1 Section: 2

22. Let C be the curve formed by the intersection of the plane $y = -3$ and the surface $z = x^3 + y^3 - 9xy + 27$. Find the equations of the line tangent to C at $(0, -3, 0)$.

$$\text{Answer: } y = -3, z = 27t, x = t$$

Difficulty: 1 Section: 2

23. Find C so that $f(x, y) = x^3 + Cxy^2 - x^2 + y^2$ satisfies the equation $f_{xx} + f_{yy} = 0$.

$$\text{Answer: } C = 3$$

Difficulty: 2 Section: 2

24. Find C so that $f(x, y) = x^3 - 3xy^2 + Cx^2 + y^2$ satisfies the equation $f_{xx} + f_{yy} = 0$.

Answer: $C = -1$

Difficulty: 2 Section: 2.

25. Find C so that $f(x, y) = e^{-Cy} \sin 2x$ satisfies the equation $f_y = 4f_{xx}$.

Answer: $C = 16$

Difficulty: 2 Section: 2

26. Find C so that $f(x, y) = e^{-4y} \sin Cx$ satisfies the equation $f_y = f_{xx}$.

Answer: $C = 2$ or -2

Difficulty: 2 Section: 2

27. Find a so that $f(x, y) = \cos(ax + y)$ is a solution to the equation $f_{xx} = 9f_{yy}$.

Answer: $a = 3$ or -3

Difficulty: 2 Section: 2

28. Find a so that $f(x, y) = \cos(x + ay)$ is a solution to the equation $f_{yy} = 4f_{xx}$.

Answer: $a = 2$ or -2

Difficulty: 2 Section: 2

29. If $f(x, y, z) = x^3y - \frac{xy}{z} + \tan^{-1}xy$, find f_x , f_y , and f_z .

Answer:

$$f_x = 3x^2y - \frac{y}{z} + \frac{y}{1+x^2y^2}$$
$$f_y = x^3 - \frac{x}{z} + \frac{x}{1+x^2y^2}, f_z = \frac{xy}{z^2}$$

Difficulty: 1 Section: 2

30. If $f(x, y, z) = x^2yz - \sin xyz + xe^{yz}$, find f_x , f_y , and f_z .

Answer:

$$f_x = 2xyz - yz \sin xyz + e^{yz}$$
$$f_y = x^2z - xz \sin xyz + xz e^{yz}$$
$$f_z = x^2y - xy \sin xyz + xy e^{yz}$$

Difficulty: 1 Section: 2

31. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x-7y}{x+y}$ doesn't exist.

Answer: $f(0, y) = -1$ if $y \neq 0$ and $f(x, 0) = 1$ if $x \neq 0$.

Difficulty: 2 Section: 2

32. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{y}$ doesn't exist.

Answer: along $x = 2\sqrt{y}$, $f(x, y) = 4$, along $x = \sqrt{y}$, $f(x, y) = 1$

Difficulty: 2 Section: 3

33. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x - y^2}{x + y^2}$ doesn't exist.

Answer: along $x = 0$, $f(0, y) = -1$, along $y = 0$, $f(x, 0) = 1$ if $x \neq 0$.

Difficulty: 2 Section: 3

34. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{2x - y}{x + 3y}$ doesn't exist.

Answer: along $x = 0$, $f(0, y) = -\frac{1}{3}$ if $y \neq 0$, along $y = 0$, $f(x, 0) = 2$ if $x \neq 0$.

Difficulty: 2 Section: 3

35. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - 2y}{x + 3y}$ doesn't exist.

Answer: along $x = 0$, $f(0, y) = -\frac{2}{3}$ if $y \neq 0$ and if $y = 0$, $f(x, 0) = x$ if $x \neq 0$.

Difficulty: 2 Section: 3

36. Find the gradient ∇f for $f(x, y) = x^2 + 2xy - y^2$.

Answer: $\nabla f(x, y) = (2x + 2y)\mathbf{i} + (2x - 2y)\mathbf{j}$

Difficulty: 1 Section: 4

37. Find the gradient ∇f for $f(x, y) = x^3 + y^3 - 9xy + 27$.

Answer: $\nabla f(x, y) = (3x^2 - 9y)\mathbf{i} + (3y^3 - 9x)\mathbf{j}$

Difficulty: 1 Section: 4

38. Find the gradient ∇f for $f(x, y) = \ln(x^2 + y^2)^{1/2}$.

Answer: $\nabla f(x, y) = (x^2 + y^2)^{-1/2} (x\mathbf{i} + y\mathbf{j})$

Difficulty: 1 Section: 4

39. Find the gradient ∇f for $f(x, y) = \tan^{-1} \frac{y}{x}$.

Answer: $\nabla f(x, y) = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j}$

Difficulty: 1 Section: 4

40. Find the gradient ∇f for $f(x, y) = e^x \cos y$.

Answer: $\nabla f(x, y) = (e^x \cos y)\mathbf{i} + (-e^x \sin y)\mathbf{j}$

Difficulty: 1 Section: 4

41. Find the gradient ∇f for $f(x, y) = \sqrt{9 - x^2 - y^2}$.

Answer: $\nabla f(x, y) = (9 - x^2 - y^2)^{-1/2}(-x \mathbf{i} - y \mathbf{j})$

Difficulty: 1 Section: 4

42. Find the equation of the tangent plane for the surface $z = x^2 + 2xy - y^2$ at $(1, -2, -7)$.

Answer: $z + 7 = -2(x - 1) + 6(y + 2)$

Difficulty: 2 Section: 4

43. Find the equation of the tangent plane for the surface $z = x^3 + y^3 - 9xy + 27$ at $(0, -3, 0)$.

Answer: $z = 27x + 27(y + 3)$

Difficulty: 2 Section: 4

44. Find the equation of the tangent plane for the surface $z = \ln(x^2 + y^2)^{1/2}$ at $(1, 0, 0)$.

Answer: $z = x - 1$

Difficulty: 2 Section: 4

45. Find the equation of the tangent plane for the surface $z = \tan^{-1} \frac{y}{x}$ at $(1, 1, \frac{\pi}{4})$.

Answer: $z - \frac{\pi}{4} = -\frac{1}{2}(x - 1) + \frac{1}{2}(y - 1)$

Difficulty: 2 Section: 4

46. Find the equation of the tangent plane for the surface $z = \sqrt{9 - x^2 - y^2}$ at $(1, -2, 2)$.

Answer: $z - 2 = -\frac{1}{2}(x - 1) + (y + 2)$

Difficulty: 2 Section: 4

47. Find the equation of the tangent plane for the surface $z = e^x \cos y$ at $(1, 0, e)$.

Answer: $z - e = e(x - 1)$

Difficulty: 2 Section: 4

48. Find the directional derivative of $f(x, y) = x^2 - 2y^2$ at $(3, 3)$ in the direction of $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$.

Answer: $\frac{24}{\sqrt{5}}$

Difficulty: 2 Section: 4

49. Find the directional derivative of $f(x, y) = x^2 - 2y^2$ at $(3, 3)$ in the direction of $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$.

Answer: $-\frac{24}{\sqrt{13}}$

Difficulty: 2 Section: 5

50. Find the directional derivative of $f(x, y) = x^2 - 2y^2$ at $(3, 3)$ in the direction of $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$.

Answer: 0

Difficulty: 2 Section: 5

51. In what direction does $f(x, y) = x^2 - 2y^2$ increase most rapidly at $(3, 3)$? What is the maximum rate of increase?

Answer: $\frac{1}{\sqrt{5}}(\mathbf{i} - 2\mathbf{j})$; maximum rate is $\sqrt{180}$

Difficulty: 2 Section: 5

52. In what direction does $f(x, y) = x^3y - x^2y^3$ increase most rapidly at $(2, -1)$? What is the maximum rate of increase?

Answer: $\frac{-2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$; maximum rate is $\sqrt{80}$

Difficulty: 2 Section: 5

53. The function $f(x, y)$ at $(1, 2)$ has a directional derivative equal to 2 in the direction towards $(2, 2)$ and equal to -2 in the direction toward $(1, 1)$. What is the directional derivative in the direction toward $(2, 3)$?

Answer: $\frac{4}{\sqrt{2}}$

Difficulty: 3 Section: 5

54. The function $f(x, y)$ at $(1, 2)$ has a directional derivative equal to 3 in the direction towards $(2, 2)$ and equal to -1 in the direction toward $(1, 1)$. What is the directional derivative in the direction toward $(2, 3)$?

Answer: $\frac{4}{\sqrt{2}}$

Difficulty: 3 Section: 5

55. Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(2, -3) = 0$ where $f(x, y) = x^3y - x^2y$.

Answer: $\frac{\mathbf{i} + 6\mathbf{j}}{\sqrt{37}}$

Difficulty: 2 Section: 5

56. Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(2, -1) = 0$ where $f(x, y) = x^3y - x^2y^3$.

Answer: $\frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$

Difficulty: 2 Section: 5

57. Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(1, -2) = 0$ where $f(x, y) = x^2 + 2xy - y^2$.

Answer: $\frac{3\mathbf{i} + \mathbf{j}}{\sqrt{10}}$

Difficulty: 2 Section: 5

58. Find the directional derivative of $f(x, y, z) = xy + yz - xz$ at $(1, -1, 2)$ in the direction $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

Answer: $-\frac{1}{3}$

Difficulty: 1 Section: 5

59. Find the directional derivative of $f(x, y, z) = xy + yz + xz$ at $(1, -1, 2)$ in the direction $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

Answer: $-\frac{5}{3}$

Difficulty: 1 Section: 5

60. Find the directional derivative of $f(x, y, z) = x^2 - 2y^2 + z^2$ at $(3, 3, 1)$ in the direction $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Answer: $\frac{26}{\sqrt{6}}$

Difficulty: 1 Section: 5

61. In what direction does $f(x, y, z) = xy + yz + xz$ increase most rapidly at $(1, 2, 3)$? What is the rate of change in that direction?

Answer: $5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}, \sqrt{50}$

Difficulty: 2 Section: 5

62. Find the directional derivative of $f(x, y, z) = xy + 2x^2 - z^3$ at $(2, -3, -1)$ in the direction $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

Answer: $-\frac{5}{3}$

Difficulty: 2 Section: 5

63. In what direction does $f(x, y, z) = xy + 2x^2 - z^3$ increase most rapidly at $(2, -3, -1)$?

Answer: $\frac{5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{38}}$

Difficulty: 2 Section: 5

64. Use the chain rule to find $\frac{df}{dt}$ if $f(x, y) = x^2 - 2xy + y^2$, $x = 2t + 1$ and $y = 3t - 7$.

Answer: $2t - 16$

Difficulty: 2 Section: 6

65. Use the chain rule to find $\frac{df}{dt}$ if $f(x, y) = x^2 - 2xy + y^2$, $x = 4 - 3t$ and $y = 2t + 1$.

Answer: $50t - 30$

Difficulty: 2 Section: 6

66. Use the chain rule to find $\frac{df}{dt}$ if $f(x, y) = x \ln y$ where $x = t^2$ and $y = e^t$.

Answer: $3t^2$

Difficulty: 2 Section: 6

67. Use the chain rule to find f_u if $f(x, y) = x^2 + 2y^2$ and $x = \sin u \cos v$, $y = \sin u \sin v$.

Answer: $2 \cos u \sin u (\cos^2 v + 2 \sin^2 v)$

Difficulty: 2 Section: 6

68. If $f(x, y) = x^3 + y^3 - 2xy + 18$, find $\frac{df}{dt}$ at $(2, 1)$ if $\frac{dx}{dt} = -2$ and $\frac{dy}{dt} = 3$.

Answer: -23

Difficulty: 2 Section: 6

69. If $f(x, y) = x^3 + y^3 - 2xy + 18$, find $\frac{df}{dt}$ at $(2, 1)$ if $\frac{dx}{dt} = -\frac{1}{2}$ and $\frac{dy}{dt} = -2$.

Answer: -3

Difficulty: 2 Section: 6

70. If z is defined implicitly as a function of x and y by $x^2 + 2xy + y^2 + z^2 = 6$, find $\frac{\partial z}{\partial x}$.

Answer: $-\frac{x+y}{z}$

Difficulty: 2 Section: 6

71. If z is defined implicitly as a function of x and y by $2x^2 + 3y^2 + 4z^2 = 9$, find $\frac{\partial z}{\partial y}$.

Answer: $-\frac{3y}{4z}$

Difficulty: 2 Section: 6

72. If z is defined implicitly as a function of x and y by $x^2 + 3xyz + 2xy^3 - z^3 = -15$, find $\frac{\partial z}{\partial x}$.

Answer: $\frac{2x + 3yz}{3z^2 - 3xy}$

Difficulty: 2 Section: 6

73. Determine the tangent plane and normal line to the surface $x^2 + 2xy - y^2 + z^2 = 6$ at $(1, 1, -2)$.

Answer: $4(x - 1) - 4(z + 2) = 0$; $\frac{x-1}{4} = \frac{z+2}{-4}$, $y = 1$

Difficulty: 1 Section: 7

74. Determine the tangent plane and normal line to the surface $2x^2 + 3y^2 + 4z^2 = 9$ at $(1, 1, 1)$.

Answer: $4(x - 1) + 6(y - 1) + 8(z - 1) = 0$; $\frac{x - 1}{4} = \frac{y - 1}{6} = \frac{z - 1}{8}$

Difficulty: 1 Section: 7

75. Determine the tangent plane and normal line to the surface $x^3 + 3xyz + 2y^3 - z^3 = -15$ at $(1, -1, 2)$.

Answer: $-3(x - 1) + 12(x + 1) - 15(z - 2) = 0$; $\frac{x - 1}{-3} = \frac{y + 1}{12} = \frac{z - 2}{-15}$

Difficulty: 1 Section: 7

76. Determine the tangent plane and normal line to the surface $xy + yz + xz - 1 = 0$ at $(3, -1, 2)$.

Answer: $(x - 3) + 5(y + 1) + 2(z - 2) = 0$; $\frac{x - 3}{1} = \frac{y + 1}{5} = \frac{z - 2}{2}$

Difficulty: 1 Section: 7

77. Find the points on the surface $5x^2 - 6xy + 5y^2 - 8x + 8y + z^2 - 3z = 0$ which have a tangent plane that is parallel to the xy plane.

Answer: $\left(\frac{1}{2}, -\frac{1}{2}, 4\right), \left(\frac{1}{2}, -\frac{1}{2}, -1\right)$

Difficulty: 2 Section: 7

78. Find all relative maxima, relative minima and saddle points for $f(x, y) = x^3 + y^3 - 9xy + 27$.

Answer: $(0, 0)$ is a saddle point, a relative minimum occurs at $(3, 3)$

Difficulty: 1 Section: 8

79. Find all relative maxima, relative minima and saddle points for $f(x, y) = x^3 + y^3 + 3xy + 18$.

Answer: $(0, 0)$ is a saddle point, a relative maximum occurs at $(-1, -1)$

Difficulty: 1 Section: 8

80. Find all relative maxima, relative minima and saddle points for $f(x, y) = x^3 + y^3 - 3xy + 9$.

Answer: $(0, 0)$ is a saddle point, a relative minimum occurs at $(1, 1)$

Difficulty: 1 Section: 8

81. Find all relative maxima, relative minima and saddle points for $f(x, y) = x^2 - 4xy + y^3 + 4y$.

Answer: a relative minimum occurs at $(4, 2)$, $\left(\frac{4}{3}, \frac{2}{3}\right)$ is a saddle point

Difficulty: 1 Section: 8

82. Find all relative maxima, relative minima and saddle points for $f(x, y) = x^2 + 2y^2 - 4x + 4y - 3$.

Answer: a relative minimum occurs at $(2, -1)$

Difficulty: 1 Section: 8

83. Find all relative maxima, relative minima and saddle points for $f(x, y) = x^3 + y^2 - 6x^2 + y - 1$.

Answer: a relative minimum occurs at $\left(4, -\frac{1}{2}\right)$, $\left(0, -\frac{1}{2}\right)$ is a saddle point

Difficulty: 1 Section: 8

84. Find all relative maxima, relative minima and saddle points for $f(x, y) = \frac{1}{x} - \frac{27}{y} + xy$.

Answer: a relative maximum occurs at $\left(-\frac{1}{3}, 9\right)$

Difficulty: 1 Section: 8

85. Find all relative maxima, relative minima and saddle points for $f(x, y) = \frac{1}{x} - \frac{1}{y} + xy$.

Answer: a relative maximum occurs at $(-1, 1)$

Difficulty: 1 Section: 8

86. Find all relative maxima, relative minima and saddle points for $f(x, y) = x^2 - 3xy + 5x - 2y + 2y^2 + 8$.

Answer: $(14, 11)$ is a saddle point

Difficulty: 1 Section: 8

87. Find all relative maxima, relative minima and saddle points for $f(x, y) = x^2 - 3xy - x + 2y + 2y^2 + 9$.

Answer: $(2, 1)$ is a saddle point

Difficulty: 1 Section: 8

88. Find all relative maxima, relative minima and saddle points for $f(x, y) = x^2 - 3xy + 4x - 5y + 2y^2 + 10$.

Answer: $(1, 2)$ is a saddle point

Difficulty: 1 Section: 8

89. Find the maximum value of $f(x, y) = y^2 - 3x^2 + 2x$ if $x^2 + y^2 = 9$.

Answer: $9\frac{1}{4}$ at $x = \frac{1}{4}$, $y = \frac{\sqrt{143}}{4}$

Difficulty: 2 Section: 8

90. Find the distance from the point $(-6, 2, 3)$ to the plane $4x - 5y + 8z = 7$.

Answer: $\frac{17}{\sqrt{105}}$

Difficulty: 1 Section: 8

91. Find the distance from the point $(4, -3, 1)$ to the plane $2x + 3y - 6z = -14$.

Answer: $\frac{17}{\sqrt{105}}$

Difficulty: 1 Section: 8

92. Find the largest and smallest values of $f(x, y) = 2x^2 + y^2 - y$ on the circle $x^2 + y^2 = 1$.

Answer: $f(0, 1) = 0$ and $f\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{5}{4}$

Difficulty: 2 Section: 8

93. Find the largest and smallest value of $f(x, y) = x^2 + y^2$ on the parabola $x^2 + y = 1$.

Answer: $f\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right) = \frac{3}{4}$

Difficulty: 2 Section: 9

94. Find the largest and smallest value of $f(x, y) = x - x^2 - y^2$ if $x^2 + y^2 = 4$.

Answer: $f(2, 0) = -2$, $f(-2, 0) = -6$

Difficulty: 2 Section: 9

95. Find the largest and smallest value of $f(x, y) = x - x^2 - y^2$ if $x^2 + y^2 = 1$.

Answer: $f(1, 0) = 0$, $f(-1, 0) = -2$

Difficulty: 2 Section: 9

96. Find the largest and smallest value of $f(x, y) = x^2 + y^2$ on the parabola $x^2 + y = \frac{3}{2}$.

Answer: $f\left(\pm 1, \frac{1}{2}\right) = \frac{5}{4}$

Difficulty: 2 Section: 9

97. Find the largest and smallest value of $f(x, y) = x^2 + y$ if $x^2 + y^2 = 4$.

Answer: $f(0, -2) = -2$ and $f\left(\pm\frac{\sqrt{15}}{2}, \frac{1}{2}\right) = \frac{17}{4}$

Difficulty: 2 Section: 9

16 The Integral in n -Space

1. Compute $\iint_D 2 \, dA$ where D is given by $1 \leq x \leq 3, -1 \leq y \leq 2$.

Answer: 12

Difficulty: 1 Section: 1

2. Compute $\iint_D 3 \, dA$ where D is given by $1 \leq x \leq 3, 2 \leq y \leq 6$.

Answer: 24

Difficulty: 1 Section: 1

3. Compute $\iint_D 5 \, dA$ where D is bounded by $x^2 + y^2 = 2$.

Answer: 10π

Difficulty: 1 Section: 1

4. Compute $\iint_D 4 \, dA$ where D is bounded by $x^2 + y^2 = 1$.

Answer: 4π

Difficulty: 1 Section: 1

5. Compute $\iint_D 3 \, dA$ where D is bounded by $(x - 1)^2 + (y + 2)^2 = 3$.

Answer: 9π

Difficulty: 1 Section: 1

6. Compute $\iint_D 2 \, dA$ where D is bounded by $x^2 + (y - 7)^2 = 1$.

Answer: 2π

Difficulty: 1 Section: 1

7. Evaluate the integral $\int_0^1 \int_0^2 (x^2 + y^2) \, dy \, dx$.

Answer: $\frac{10}{3}$

Difficulty: 1 Section: 2

8. Evaluate the integral $\int_0^1 \int_0^3 (x^2 + y^2) dy dx$.

Answer: 10

Difficulty: 1 Section: 2

9. Evaluate the integral $\int_0^2 \int_0^1 x^2 y dx dy$.

Answer: $\frac{2}{3}$

Difficulty: 1 Section: 2

10. Evaluate the integral $\int_0^1 \int_0^2 x^2 y dx dy$.

Answer: $\frac{4}{3}$

Difficulty: 1 Section: 2

11. Evaluate the integral $\int_1^4 \int_0^2 (x + y^2) dx dy$.

Answer: 48

Difficulty: 1 Section: 2

12. Find the volume of the solid above the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 2$ and below the surface $z = x^2 + y^2$.

Answer: $\frac{10}{3}$

Difficulty: 1 Section: 2

13. Find the volume of the solid above the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 3$ and below the surface $z = x^2 + y^2$.

Answer: 10

Difficulty: 1 Section: 2

14. Find the volume of the solid above the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 2$ and below the surface $z = x^2 y$.

Answer: $\frac{2}{3}$

Difficulty: 1 Section: 2

15. Find the volume of the solid above the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$ and below the surface $z = x^2y$.

Answer: $\frac{4}{3}$

Difficulty: 1 Section: 2

16. Find the volume of the solid above the rectangle $0 \leq x \leq 2$, $1 \leq y \leq 4$ and below the surface $z = x + y^2$.

Answer: 48

Difficulty: 1 Section: 2

17. Compute $\iint_D 4 \, dA$ where D is bounded by $y = x^2$, the x -axis, and the line $x = 1$.

Answer: $\frac{4}{3}$

Difficulty: 1 Section: 3

18. Compute $\iint_D 3 \, dA$ where D is bounded by $y = x^2$, the x -axis, and the line $x = 2$.

Answer: 8

Difficulty: 1 Section: 3

19. Compute $\iint_D 3 \, dA$ where D is bounded by $y = 1 - x^2$ and the x -axis.

Answer: 4

Difficulty: 1 Section: 3

20. Write a double integral to express the volume of the solid with base D where D is the region bounded by $x^2 + 2y^2 = 4$ and the solid extends from each point (x, y) in the base to the surface $z = 2x^2 + y^2$.

Answer: $\iint_D (2x^2 + y^2) \, dA$

Difficulty: 1 Section: 3

21. Write a double integral to express the volume of the solid with base D where D is the region bounded by $x^2 + 5y^2 = 24$ and the solid extends from each point (x, y) in the base to the surface $z = 3x^2 + 5y^2$.

Answer: $\iint_D (3x^2 + 5y^2) \, dA$

Difficulty: 1 Section: 3

22. Write a double integral to express the volume of the solid with base D where D is the region bounded by x^2 , $y = 0$, $x = 1$, and the solid extends from each point (x, y) in the base to the surface $z = 4xy$.

Answer: $\iint_D 4xy \, dA$

Difficulty: 1 Section: 3

23. Evaluate the integral $\iint_D xy \, dA$ where D is the region bounded by $y = x^2$, the x -axis and the line $x = 2$.

Answer: $\frac{16}{3}$

Difficulty: 1 Section: 3

24. Evaluate $\iint_D \sin \pi x^2 \, dA$ where D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$. The order is important!

Answer: $\frac{2}{\pi}$

Difficulty: 2 Section: 3

25. Evaluate $\iint_D x \, dA$ where D is the region bounded by $y = x^2$ and $y = 2x$.

Answer: $\frac{4}{3}$

Difficulty: 1 Section: 3

26. Evaluate $\iint_D x \, dA$ where D is the region in the first quadrant bounded by $y = 4 - x^2$, $x = 0$ and $y = 0$.

Answer: 4

Difficulty: 1 Section: 3

27. Evaluate $\iint_D y \sin xy \, dA$ where D is the rectangle given by $1 \leq x \leq 2$, $0 \leq y \leq \pi$.

Answer: 0

Difficulty: 1 Section: 3

28. Evaluate $\iint_D (x^2 + 2xy) \, dA$ where D is the region between $y = x^2$ and $y = x^3$.

Answer: $\frac{3}{40}$

Difficulty: 1 Section: 3

29. Evaluate $\iint_D 3 \, dA$ where D is the region bounded by $x = 0$, $y = e^x$ and $y = 2$.

Answer: $3(\ln 4 - 1)$

Difficulty: 1 Section: 3

30. Find the volume of the solid above the region bounded by $y = x^2$, the x -axis and the line $x = 2$, and below the surface $z = xy$.

Answer: $\frac{16}{3}$

Difficulty: 1 Section: 3

31. Find the volume of the solid above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$ below the surface $z = \sin \pi x^2$.

Answer: $\frac{2}{\pi}$

Difficulty: 1 Section: 3

32. Find the volume of the solid above the region bounded by $y = x^2$ and $y = 2x$ below the surface $z = x$.

Answer: $\frac{4}{3}$

Difficulty: 1 Section: 3

33. Find the volume of the solid above the region in the first quadrant bounded by $y = 4 - x^2$, $x = 0$, and $y = 0$, below the surface $z = x$.

Answer: 4

Difficulty: 1 Section: 3

34. Find the volume of the solid if the first octant bounded by the coordinate planes and the plane $2x + 3y + 6z = 24$.

Answer: 64

Difficulty: 2 Section: 3

35. Find the volume of the solid in the first octant bounded by the coordinate planes and the plane $x + 2y + 3z = 8$.

Answer: $\frac{128}{9}$

Difficulty: 2 Section: 3

36. Find the volume of the solid above the region bounded by $y = x^2$ and $y = x^3$ and below the surface $z = x^2 + 2xy$.

Answer: $\frac{3}{40}$

Difficulty: 1 Section: 3

37. Reverse the order of integration: $\int_0^1 \int_0^{\sqrt{1-x}} f(x, y) dy dx$.

Answer: $\int_0^1 \int_0^{1-y^2} f(x, y) dx dy$

Difficulty: 2 Section: 3

38. Reverse the order of integration: $\int_0^1 \int_y^{\sqrt{y}} f(x, y) dx dy$.

Answer: $\int_0^1 \int_{x^2}^x f(x, y) dy dx$

Difficulty: 2 Section: 3

39. Reverse the order of integration: $\int_0^2 \int_0^{4-x^2} f(x, y) dy dx$.

Answer: $\int_0^4 \int_0^{\sqrt{4-y}} f(x, y) dx dy$

Difficulty: 2 Section: 3

40. Reverse the order of integration: $\int_0^{\sqrt{\pi/2}} \int_{y^2}^{\pi/2} f(x, y) dx dy$.

Answer: $\int_0^{\pi/2} \int_0^{\sqrt{x}} f(x, y) dy dx$

Difficulty: 2 Section: 3

41. Reverse the order of integration: $\int_0^4 \int_{y/2}^{\sqrt{y}} f(x, y) dx dy$.

Answer: $\int_0^2 \int_{x^2}^{2x} f(x, y) dy dx$

Difficulty: 2 Section: 3

42. Reverse the order of integration: $\int_1^2 \int_0^{\ln y} f(x, y) dx dy$.

Answer: $\int_0^{\ln 2} \int_{e^x}^2 f(x, y) dy dx$

Difficulty: 2 Section: 3

43. Reverse the order of integration: $\int_{-1}^2 \int_{x^2}^{x+2} f(x, y) dy dx$.

$$\text{Answer: } \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) dx dy + \int_1^4 \int_{y-2}^{\sqrt{y}} f(x, y) dx dy$$

Difficulty: 2 Section: 3

44. Reverse the order of integration: $\int_0^2 \int_{x^2}^{4x-x^2} f(x, y) dy dx$.

$$\text{Answer: } \int_0^4 \int_{2-\sqrt{4-y}}^{\sqrt{y}} f(x, y) dx dy$$

Difficulty: 2 Section: 3

45. Evaluate $\iint_R \sin \theta dA$ over the region R given by $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq r \leq \sqrt{\cos \theta}$.

$$\text{Answer: } \frac{1}{4}$$

Difficulty: 1 Section: 4

46. Find the volume of the solid above the region bounded by $r = 2 \cos \theta$ and below the surface $z = 1 - y$.

$$\text{Answer: } \pi$$

Difficulty: 1 Section: 4

47. Find the area inside the circle $r = 3 \cos \theta$ and outside the cardioid $r = 1 + \cos \theta$.

$$\text{Answer: } \pi$$

Difficulty: 1 Section: 4

48. Find the area inside the circle $r = 3 \sin \theta$ and outside the cardioid $r = 1 + \sin \theta$.

$$\text{Answer: } \pi$$

Difficulty: 1 Section: 4

49. Find the volume of the solid above the region bounded by $x^2 + y^2 = 2y$ and below the surface $z = x^2 + y^2$.

$$\text{Answer: } \frac{3\pi}{2}$$

Difficulty: 2 Section: 4

50. Use polar coordinates to evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)} dy dx$.

Answer: $\frac{\pi}{4} \left(1 - \frac{1}{e}\right)$

Difficulty: 2 Section: 4

51. Use polar coordinates to evaluate $\int_0^2 \int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2} dx dy$.

Answer: $\frac{\pi \ln 5}{8}$

Difficulty: 2 Section: 4

52. Find the volume of the solid above the region bounded by $x^2 + y^2 = 2x$ and below the surface $z = x^2 + y^2$.

Answer: $\frac{3\pi}{2}$

Difficulty: 2 Section: 4

53. Find the mass of a thin lamina bounded by $y = 0$, $y = \sin x$, $0 \leq x \leq \pi$ if the density at (x, y) is $2y$.

Answer: $\frac{\pi}{2}$

Difficulty: 2 Section: 5

54. Find the mass of a thin lamina bounded by $y = x^2$, $y = 0$, and $x = 1$ if the density at (x, y) is $x + y$.

Answer: $\frac{7}{20}$

Difficulty: 2 Section: 5

55. Find the mass of a thin lamina bounded by $x = 1 - y^2$, $x = 0$ and $y = 0$ if the density at (x, y) is x .

Answer: $\frac{4}{15}$

Difficulty: 2 Section: 5

56. Find the mass of a thin lamina bounded by $y = 0$, $y = x^2$, and $x = 2$ if the density at (x, y) is xy .

Answer: $\frac{16}{3}$

Difficulty: 2 Section: 5

57. Find the center of mass of the spherical shell between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, $z \geq 0$. Assume $\delta(x, y) = 1$.

Answer: $\left(0, 0, \frac{45}{56}\right)$

Difficulty: 2 Section: 5

58. Find the moment of inertia about the x -axis of the region bounded by $y = x^2$, $y = 0$, and $x = 2$. Assume density factor of 1.

Answer: $\frac{128}{21}$

Difficulty: 2 Section: 5

59. Find the surface area of the portion of the surface $z = \sqrt{16 - x^2}$ inside the cylinder $x^2 + y^2 = 16$.

Answer: 64

Difficulty: 2 Section: 6

60. Find the surface area of the region formed by the intersection of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

Answer: 16

Difficulty: 2 Section: 6

61. Find the surface area of the region formed by the intersection of the cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.

Answer: 64

Difficulty: 2 Section: 6

62. Find the surface area of the region formed by the intersection of the cylinders $9x^2 + 9y^2 = 1$ and $9x^2 + 9z^2 = 1$.

Answer: $\frac{16}{9}$

Difficulty: 2 Section: 6

63. Find the surface area of the portion of the surface of the sphere $x^2 + y^2 + z^2 = 4$ cut by the cylinder $x^2 + y^2 = 2y$.

Answer: $8(\pi - 2)$

Difficulty: 2 Section: 6

64. Find the surface area of the portion of the surface of the sphere $x^2 + y^2 + z^2 = 1$ cut by the cylinder $x^2 + y^2 = y$.

Answer: $2(\pi - 2)$

Difficulty: 2 Section: 6

65. Find the surface area of the portion of the cylinder $x^2 + y^2 = y$ inside the sphere $x^2 + y^2 + z^2 = 1$.

Answer: 4

Difficulty: 2 Section: 6

66. Find the surface area of the portion of the cylinder $x^2 + y^2 = 2y$ inside the sphere $x^2 + y^2 + z^2 = 4$.

Answer: 16

Difficulty: 2 Section: 6

67. Compute $\iiint_V (xz + 3z) dV$ where V is the region in the first octant bounded by $x^2 + z^2 = 9$ and $x + y = 3$.

Answer: $\frac{324}{5}$

Difficulty: 2 Section: 7

68. Compute $\iiint_V (xz + 2z) dV$ where V is the region in the first octant bounded by $x^2 + z^2 = 4$ and $x + y = 2$.

Answer: $\frac{128}{5}$

Difficulty: 2 Section: 7

69. Compute $\iiint_V (xz + z) dV$ where V is the region in the first octant bounded by $x^2 + z^2 = 1$ and $x + y = 1$.

Answer: $\frac{4}{15}$

Difficulty: 2 Section: 7

70. Compute $\iiint_V z^2 dV$ where V is the region in the first octant bounded by $z = 2$, $y = 2z$, and $x = y + z$.

Answer: $\frac{128}{5}$

Difficulty: 2 Section: 7

71. Compute $\int_0^1 \int_{2x}^{x+y} (x + y + z) dz dy dx$.

Answer: $\frac{13}{8}$

Difficulty: 2 Section: 7

72. Evaluate $\int_{-1}^1 \int_{3x^2}^{4-x^2} \int_0^{6-z} dy \, dz \, dx$.

Answer: $\frac{304}{15}$

Difficulty: 2 Section: 7

73. Evaluate $\int_0^1 \int_0^x \int_0^{x-y} x \, dz \, dy \, dx$.

Answer: $\frac{1}{8}$

Difficulty: 1 Section: 7

74. Find the volume of the region in the first octant bounded by $x^2 + y^2 = 4$, the plane $y + z = 2$ and the plane $z = 0$.

Answer: $\frac{2}{3}(3\pi - 4)$

Difficulty: 2 Section: 7

75. Find the volume of the region in the first octant bounded by $x^2 + y^2 = 1$, the plane $y + z = 1$ and the plane $z = 0$.

Answer: $\frac{1}{12}(3\pi - 4)$

Difficulty: 2 Section: 7

76. Find the volume of the region formed by the intersection of the cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.

Answer: $\frac{138}{3}$

Difficulty: 2 Section: 7

77. Find the volume of the region formed by the intersection of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

Answer: $\frac{16}{3}$

Difficulty: 2 Section: 7

78. Use cylindrical coordinates to find the volume of the region in the first octant bounded by the cylinder $x^2 + y^2 = 4$ and the plane $z = 1 + 2x + 3y$.

Answer: $\pi + \frac{40}{3}$

Difficulty: 2 Section: 8

79. Use cylindrical coordinates to find the volume of the region in the first octant bounded by the cylinder $x^2 + y^2 = 4$ and the plane $z = 1 + 3x + 3y$.

Answer: $\pi + 16$

Difficulty: 2 Section: 8

80. Use cylindrical coordinates to find the volume of the region in the first octant bounded by the cylinder $x^2 + y^2 = 4$ and the plane $z = 1 + x + 2y$.

Answer: $\pi + 8$

Difficulty: 2 Section: 8

81. Use cylindrical coordinates to find the volume of the region in the first octant bounded by the cylinder $x^2 + y^2 = 9$ and the plane $zx^2 + z^2 = 9$.

Answer: 18

Difficulty: 2 Section: 8

82. Use cylindrical coordinates to find the volume of the region in the first octant bounded by $x^2 + y^2 = 16$ and $x^2 + z^2 = 16$.

Answer: $\frac{128}{3}$

Difficulty: 2 Section: 8

83. Use cylindrical coordinates to find the volume of the region in the first octant bounded by the cone $z = r$ and the cylinder $r = 2 \sin \theta$.

Answer: $\frac{16}{9}$

Difficulty: 2 Section: 8

84. Use cylindrical coordinates to find the volume of the region in the first octant bounded by the cone $z = 2r$ and the cylinder $r = 3 \sin \theta$.

Answer: 12

Difficulty: 2 Section: 8

85. Use cylindrical coordinates to find the volume of the region inside the paraboloid $z = 9 - x^2 - y^2$, outside the cylinder $x^2 + y^2 = 4$, and above the plane $z = 0$.

Answer: $\frac{25\pi}{2}$

Difficulty: 2 Section: 8

86. Find the volume of the region within the cylinder $z = 2 \sin \theta$ bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$.

Answer: $\frac{3\pi}{2}$

Difficulty: 2 Section: 8

87. Find the volume of the solid bounded by the plane $z = 0$, the plane $z = 1 - y$, and the cylinder $r = 2 \cos \theta$.

Answer: π

Difficulty: 2 Section: 8

88. The plane $z = 1$ divides the ball $x^2 + y^2 + z^2 \leq 2$ into two parts. Find the volume of the smaller part.

Answer: $\frac{\pi}{3} (4\sqrt{2} - 5)$

Difficulty: 2 Section: 8

89. Evaluate $\iiint_V z^2 dV$ where V is the region described by $x^2 + y^2 + z^2 \leq 9$, $x \geq 0$, $y \geq 0$, and $z \geq 0$.

Answer: $\frac{81\pi}{10}$

Difficulty: 2 Section: 8

90. Find the volume of the region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Answer: $\frac{\pi}{8}$

Difficulty: 2 Section: 8

91. Find the volume of the region above the plane $z = 3$ and below the sphere $x^2 + y^2 + z^2 = 18$.

Answer: $9\pi (4\sqrt{2} - 5)$

Difficulty: 2 Section: 8

92. Find the volume of the region above the plane $z = 2$ and below the sphere $x^2 + y^2 + z^2 = 8$.

Answer: $\frac{8\pi}{3} (4\sqrt{2} - 5)$

Difficulty: 2 Section: 8

93. Find the volume of the region above the plane $z = 4$ and below the sphere $x^2 + y^2 + z^2 = 32$.

Answer: $\frac{64\pi}{3} (4\sqrt{2} - 5)$

Difficulty: 2 Section: 8

17 Vector Calculus

1. If $\mathbf{F} = x^2 \mathbf{i} + 2xy^2z \mathbf{j} + \cos z \mathbf{k}$, find $\nabla \cdot \mathbf{F}$.

Answer: $2x + 4xyz - \sin z$

Difficulty: 1 Section: 1

2. If $\mathbf{F} = x^2 \mathbf{i} + 2xy^2z \mathbf{j} + \cos z \mathbf{k}$, find $\nabla \times \mathbf{F}$.

Answer: $-2x^2y \mathbf{i} + 2y^2z \mathbf{k}$

Difficulty: 1 Section: 1

3. If $\mathbf{F} = (x^2 - y^2) \mathbf{i} + 2xyz \mathbf{j} + z^2 \mathbf{k}$, find $\nabla \cdot \mathbf{F}$.

Answer: $2x + 2xz + 2z$

Difficulty: 1 Section: 1

4. If $\mathbf{F} = (x^2 - y^2) \mathbf{i} + 2xyz \mathbf{j} + z^2 \mathbf{k}$, find $\nabla \times \mathbf{F}$.

Answer: $-2xy \mathbf{i} + (2yz + 2y) \mathbf{k}$

Difficulty: 1 Section: 1

5. If $\mathbf{F} = xyz^2 \mathbf{i} + xe^{yz} \mathbf{j} - y \cosh xz \mathbf{k}$, find $\nabla \cdot \mathbf{F}$.

Answer: $yz^2 + xze^{yz} - xy \sinh xz$

Difficulty: 1 Section: 1

6. If $\mathbf{F} = xyz^2 \mathbf{i} + xe^{yz} \mathbf{j} - y \cosh xz \mathbf{k}$, find $\nabla \times \mathbf{F}$.

Answer: $(\cosh xz - xye^{yz}) \mathbf{i} + (2xyz - yz \sinh xz) \mathbf{j} + (e^{yz} - xz^2) \mathbf{k}$

Difficulty: 1 Section: 1

7. If $\phi = 2x^2yz - z \cos x + e^{yz}$, find $\nabla \phi$.

Answer: $(4xyz + z \sin x) \mathbf{i} + (2x^2z + ze^{yz}) \mathbf{j} + (2x^2y - \cos x + ye^{yz}) \mathbf{k}$

Difficulty: 1 Section: 1

8. If $\phi = 2x^2yz - z \cos x + e^{yz}$, find $\nabla \times (\nabla \phi)$.

Answer: $\mathbf{0}$

Difficulty: 1 Section: 1

9. If $\mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} + xyz \mathbf{k}$, compute $\nabla \cdot (\nabla \times \mathbf{F})$.

Answer: 0

Difficulty: 1 Section: 1

10. If $\mathbf{F} = (2x - 5y) \mathbf{i} + (-5x + 3y^2) \mathbf{j} + \mathbf{k}$, find $\nabla \times \mathbf{F}$.

Answer: $\mathbf{0}$

Difficulty: 1 Section: 1

11. If $\mathbf{F} = (6xy^3 + 2z^2) \mathbf{i} + 9x^2y^2 \mathbf{j} + (4xz + 1) \mathbf{k}$, find $\nabla \cdot \mathbf{F}$.

Answer: 0

Difficulty: 1 Section: 1

12. Compute $\int_C yz \, dx + zx \, dy + xy \, dz$ where C is the line segment from $(0, 0, 1)$ to $(2, 3, 4)$.

Answer: 24

Difficulty: 1 Section: 2

13. Compute $\int_C (x^2 - y) \, dx + (y^2 + x) \, dy$ from $(0, 1)$ to $(1, 2)$ along the curve $C : y = x^2 + 1$.

Answer: 2

Difficulty: 1 Section: 2

14. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 3x^2 \mathbf{i} + (2xz - y) \mathbf{j} + z\mathbf{k}$ and C is the path given by $x = t^2$, $y = t$, $z = 2t^2 - t$, $0 \leq t \leq 1$.

Answer: $\frac{13}{10}$

Difficulty: 1 Section: 2

15. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 3x^2 \mathbf{i} + (2xz - y) \mathbf{j} + z\mathbf{k}$ and C is the straight line from $(0, 0, 0)$ to $(1, 1, 1)$.

Answer: $\frac{5}{3}$

Difficulty: 1 Section: 2

16. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (y^2 \cos x + z^3) \mathbf{i} + (2y \sin x - 4) \mathbf{j} + (3xz^2 + 2) \mathbf{k}$ and C is the straight line from $(0, 0, 0)$ to $\left(\frac{\pi}{2}, -\frac{1}{2}\right)$.

Answer: $9 + 4\pi$

Difficulty: 1 Section: 2

17. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (x^2 - 2xy) \mathbf{i} + (x^2y + 3) \mathbf{j}$ and C is the path $8y = x^2$ from $(-4, 2)$ to $(4, 2)$.

Answer: $\frac{128}{3}$

Difficulty: 1 Section: 2

18. Evaluate $\int_C (x^2 + y) ds$ where C is the path $x = 3t, y = 4t, 0 \leq t \leq 1$.

Answer: 25

Difficulty: 1 Section: 2

19. Evaluate $\int_C (x^2 + y^2) ds$ where C is the path $x = e^t \sin t, y = e^t \cos t, 0 \leq t \leq 2$.

Answer: $\frac{\sqrt{2}}{3} (e^6 - 1)$

Difficulty: 1 Section: 2

20. Is $\mathbf{F} = (2x - 5y) \mathbf{i} + (-5x + 3y^2) \mathbf{j}$ conservative? If so, find f such that $\mathbf{F} = \nabla f$.

Answer: Yes, a potential function is $x^2 - 5xy + x^3$

Difficulty: 1 Section: 3

21. Is $\mathbf{F} = (2x + 5y) \mathbf{i} + (2y - 4) \mathbf{j}$ conservative? If so, find f such that $\mathbf{F} = \nabla f$.

Answer: Yes, a potential function is $x^2 + 5x + y^2 - 4y$

Difficulty: 1 Section: 3

22. Is $\mathbf{F} = (4xy^3 + 17ye^{xy}) \mathbf{i} + (6x^2y^2 + 17xe^{xy}) \mathbf{j}$ conservative? If so, find f such that $\mathbf{F} = \nabla f$.

Answer: Yes, a potential function is $2x^2y^3 + 17e^{xy}$

Difficulty: 1 Section: 3

23. Is $\mathbf{F} = (3x^2 - 3y^2 - 2x) \mathbf{i} + (-6xy + 2y) \mathbf{j}$ conservative? If so, find f such that $\mathbf{F} = \nabla f$.

Answer: Yes, a potential function is $x^3 - 3xy^2 - x^2 + y^2$

Difficulty: 1 Section: 3

24. Find f so that $\mathbf{F} = \nabla f$ for $\mathbf{F} = (6x^3 + 2z^2) \mathbf{i} + (9x^2y^2) \mathbf{j} + (4xz + 1) \mathbf{k}$.

Answer: $3x^2y^3 + 2xz^2 + z$

Difficulty: 1 Section: 3

25. Find f so that $\mathbf{F} = \nabla f$ for $\mathbf{F} = (y^2 \cos x + z^3) \mathbf{i} + (2y \sin x - 4) \mathbf{j} + (3xz^2 + 2) \mathbf{k}$.
 Answer: $y^2 \sin x + xz^3 - 4y + 2x$
 Difficulty: 1 Section: 3
26. Is $\mathbf{F} = (2xy - y^3 + 2) \mathbf{i} + (3xy^2 - x^2 - 4) \mathbf{k}$ conservative? If so, find f such that $\mathbf{F} = \nabla f$.
 Answer: No
 Difficulty: 1 Section: 3
27. Is $\mathbf{F} = (3x - 4y) \mathbf{i} + (4x - 3y) \mathbf{j}$ conservative? If so, find f such that $\mathbf{F} = \nabla f$.
 Answer: No
 Difficulty: 1 Section: 3
28. If $\mathbf{F} = (2x + 5) \mathbf{i} + (2y - 4) \mathbf{j}$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is any piecewise smooth curve from $(0, 1)$ to $(1, 2)$.
 Answer: 5
 Difficulty: 1 Section: 3
29. If $\mathbf{F} = (2x - 5y) \mathbf{i} + (-5x + 3y^2) \mathbf{j}$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is any piecewise smooth curve from $(0, 1)$ to $(1, 2)$.
 Answer: -2
 Difficulty: 1 Section: 3
30. If $\mathbf{F} = (6xy^3 + 2z^2) \mathbf{i} + (9x^2y^2) \mathbf{j} + (4xz + 1) \mathbf{k}$ compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is any piecewise smooth curve from $(0, 1, 0)$ to $(1, -1, 1)$.
 Answer: 0
 Difficulty: 1 Section: 3
31. If $\mathbf{F} = 6xy \mathbf{i} + (3x^2 - 3y^2z^2) \mathbf{j} - 2y^3z \mathbf{k}$ compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is any piecewise smooth curve from $(1, 1, 2)$ to $(0, 1, 3)$.
 Answer: -8
 Difficulty: 1 Section: 3
32. Use Green's Theorem to evaluate $\oint_C (3x - 4y) dx + (4x - 2y) dy$ where C is the path counterclockwise around the ellipse $x^2 + 4y^2 = 16$ beginning and ending at $(4, 0)$.
 Answer: 64π
 Difficulty: 1 Section: 4

33. Use Green's Theorem to evaluate $\oint_C (5x - 2y) dx + (2x + 3y) dy$ where C is the path counter-clockwise around the ellipse $x^2 + 4y^2 = 16$ beginning and ending at $(4, 0)$.

Answer: 32π

Difficulty: 1 Section: 4

34. Use Green's Theorem to evaluate $\oint_C (5x - 2y) dx + (-2x + 3y) dy$ where C is the path counter-clockwise around the ellipse $x^2 + 4y^2 = 16$ beginning and ending at $(4, 0)$.

Answer: 0

Difficulty: 1 Section: 4

35. Use Green's Theorem to evaluate $\oint_C 2xy dx + (e^x + x^2) dy$ where C is the boundary of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$, oriented counter-clockwise.

Answer: 1

Difficulty: 1 Section: 4

36. Use Green's Theorem to evaluate $\oint_C x^2y dx + xy dy$ where C is the boundary of the rectangle with vertices $(1, 1)$, $(3, 1)$, $(1, 2)$, and $(3, 2)$, oriented counter-clockwise.

Answer: $-\frac{17}{3}$

Difficulty: 1 Section: 4

37. Use Green's Theorem to evaluate $\oint_C y dx + 2x dy$ where C is the boundary of the rectangle with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$, oriented counter-clockwise.

Answer: 1

Difficulty: 1 Section: 4

38. Use Green's Theorem to evaluate $\oint_C (2xy - 3x^2e^y) dx + (x^2 - x^3e^y) dy$ where C is the circle $x^2 + y^2 = 4$ oriented counter-clockwise.

Answer: 0

Difficulty: 1 Section: 4

39. Use Green's Theorem to evaluate $\oint_C x(x^2 + y^2)^{-1} dx + y(x^2 + y^2)^{-1} dy$ where C is the circle $(x - 5)^2 + (y - 5)^2 = 4$ oriented counter-clockwise.

Answer: 0

Difficulty: 1 Section: 4

40. Compute $\iint_G x dS$ where G is the surface of that portion of the cylinder $x^2 + z^2 = 16$ inside the cylinder $x^2 + y^2 = 16$ in the first octant.

Answer: 32

Difficulty: 2 Section: 5

41. Compute $\iint_G x dS$ where G is the surface of that portion of the cylinder $x^2 + z^2 = 4$ inside the cylinder $x^2 + y^2 = 4$ in the first octant.

Answer: 4

Difficulty: 2 Section: 5

42. Compute $\iint_G x dS$ where G is the surface of that portion of the cylinder $x^2 + z^2 = 1$ inside the cylinder $x^2 + y^2 = 1$ in the first octant.

Answer: $\frac{1}{2}$

Difficulty: 2 Section: 5

43. Compute $\iint_G \mathbf{F} \cdot \mathbf{n} dS$ where G is the portion of the surface $3x = 9 - z^2 - y^2$ above the plane $z = 0$, and $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$.

Answer: $\frac{135\pi}{2}$

Difficulty: 2 Section: 5

44. Compute $\iint_G \mathbf{F} \cdot \mathbf{n} dS$ where G is the portion of the plane $6z + 3y + 2x = 6$ in the first octant and $\mathbf{F} = 6z\mathbf{i} + (2x + y)\mathbf{j} - x\mathbf{k}$.

Answer: 3

Difficulty: 2 Section: 5

45. Use Gauss' Theorem to compute $\iint_G \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = (x - y^2) \mathbf{i} + (y + 3xz^2) \mathbf{j} + 5z \mathbf{k}$ and G is the surface consisting of the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, and the plane surface $z = 0$, $x^2 + y^2 = 4$.

Answer: $\frac{112\pi}{3}$

Difficulty: 2 Section: 6

46. Use Gauss' Theorem to compute $\iint_G \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = (x - 2y) \mathbf{i} + (2y - 3z) \mathbf{j} + (3z - x) \mathbf{k}$ and G is the surface of the cube bounded by $x = 0$, $y = 0$, $z = 0$, and $x = 1$, $y = 1$, $z = 1$.

Answer: 6

Difficulty: 2 Section: 6

47. Use Gauss' Theorem to compute $\iint_G \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = x^2 \mathbf{i} + 2xy \mathbf{j} + 2z \mathbf{k}$ and G is the surface of the region in the first octant bounded by $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Answer: $\frac{1}{2}$

Difficulty: 2 Section: 6

48. Use Gauss' Theorem to compute $\iint_G \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = (x + e^{-y} \sin z) \mathbf{i} + (x^2 + \tan^{-1} z) \mathbf{j} + (y \cos x) \mathbf{k}$ and G is the surface of the region bounded by the cylinder $z = 4 - x^2$, the plane $y + z = 5$, $z = 0$, and $y = 0$.

Answer: $\frac{544}{15}$

Difficulty: 2 Section: 6

49. Use Gauss' Theorem to compute $\iint_G \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = (xe^{-y} \sin z) \mathbf{i} + (x^2 + \tan^{-1} x) \mathbf{j} + (y \cos x) \mathbf{k}$ and G is the surface of the region above the plane $z = 2$ and below the hemisphere $x^2 + y^2 + z^2 = 8$.

Answer: $\frac{8\pi}{3} (4\sqrt{2} - 5)$

Difficulty: 2 Section: 6

50. Use Gauss' Theorem to compute $\iint_G \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = (xz - 3 \cos y) \mathbf{i} + (4x - 3z^2) \mathbf{j} + (5xy - \sin x) \mathbf{k}$ and G is the surface of the region between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$, $z \geq 0$.

Answer: 20π

Difficulty: 2 Section: 6

51. Use Gauss' Theorem to compute $\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS$ where

$\mathbf{F} = (4x - yz^2) \mathbf{i} + (5y + xz^3) \mathbf{j} + (4xy \cos y) \mathbf{k}$ and G is the surface of the region in the first octant bounded by the cylinder $x^2 + y^2 = 4$ and the plane $z = 1 + x + 2y$.

Answer: $9\pi + 72$

Difficulty: 2 Section: 6

52. Use Gauss' Theorem to compute $\iiint_G \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = 3xz \mathbf{i} + xyz \mathbf{j} + (x^2 + y^2) \mathbf{k}$ and G is

the surface of the region in the first octant bounded by $z^2 + x^2 = 9$ and $x + y = 3$.

Answer: $\frac{324}{5}$

Difficulty: 2 Section: 6

53. Use Stoke's Theorem to compute $\iint_G (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$ where $\mathbf{F} = \frac{y \mathbf{i} - x \mathbf{j}}{x^2 + y^2 + z^2}$

and G is the surface of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Answer: -2π

Difficulty: 2 Section: 7

54. Use Stoke's Theorem to compute $\oint_C \mathbf{F} \cdot \mathbf{T} ds$ where C is the curve bounding the portion

of the cylinder $x^2 + z^2 = 4$ with $0 \leq y \leq 2$ and $z \geq 0$, and $\mathbf{F} = (x^2 - yz) \mathbf{i} + y^2 \mathbf{j} + (z^2 + xy) \mathbf{k}$.

Answer: 8π

Difficulty: 2 Section: 7

55. Use Stoke's Theorem to compute $\oint_C \mathbf{F} \cdot \mathbf{T} ds$ where C is the curve bounding the portion of the

cylinder $x^2 + z^2 = 1$ with $0 \leq y \leq 2$ and $z \geq 0$, and $\mathbf{F} = (x^2 - yz) \mathbf{i} + y^2 \mathbf{j} + (z^2 + xy) \mathbf{k}$.

Answer: 2π

Difficulty: 2 Section: 7

56. Use Stoke's Theorem to compute $\oint_C \mathbf{F} \cdot \mathbf{T} ds$ where C is the curve bounding the portion of the

cylinder $x^2 + z^2 = 6$ with $0 \leq y \leq 2$ and $z \geq 0$, and $\mathbf{F} = (x^2 - yz) \mathbf{i} + y^2 \mathbf{j} + (z^2 + xy) \mathbf{k}$.

Answer: 12π

Difficulty: 2 Section: 7

57. Use Stoke's Theorem to compute $\oint_C \mathbf{F} \cdot \mathbf{T} \, ds$ where C is the curve of the intersection of the surfaces $z = 4xy$ and $x^2 + y^2 = 9$, and $\mathbf{F} = yz^2 \mathbf{i} + xz^2 \mathbf{j} - 4xyz \mathbf{k}$.

Answer: 0

Difficulty: 2 Section: 7

58. Use Stoke's Theorem to compute $\oint_C \mathbf{F} \cdot \mathbf{T} \, ds$ where C is the curve of the intersection of the surfaces $x^2 + y^2 = 1$ and $x + y + z = 1$, $\mathbf{F} = y^2 \mathbf{i} + x^2 \mathbf{j} + xy \mathbf{k}$.

Answer: 0

Difficulty: 2 Section: 7

18 Differential Equations

1. Solve the differential equation $y'' - 3y' - 4y = 0$.
Answer: $y = Ae^{4x} + Be^{-x}$
Difficulty: 1 Section: 1
2. Solve the differential equation $y'' + 3y' - 4y = 0$.
Answer: $y = Ae^{-4} + Be^x$
Difficulty: 1 Section: 1
3. Solve the differential equation $y'' + 2y' - 8y = 0$.
Answer: $y = Ae^{2x} + Be^{-4x}$
Difficulty: 1 Section: 1
4. Solve the differential equation $y'' + 2y' + y = 0$.
Answer: $y = Axe^{-x} + Be^{-x}$
Difficulty: 1 Section: 1
5. Solve the differential equation $y'' + 4y' + 4y = 0$.
Answer: $y = Axe^{-2x} + Be^{-2x}$
Difficulty: 1 Section: 1
6. Solve the differential equation $y'' - 4y' + 5y = 0$.
Answer: $y = Ae^{2x} \cos x + Be^{2x} \sin x$
Difficulty: 1 Section: 1
7. Solve the differential equation $y'' + 4y = 0$.
Answer: $y = A \cos 2x + B \sin 2x$
Difficulty: 1 Section: 1
8. Solve the differential equation $y'' - 4y = 0$.
Answer: $y = A \cosh 2x + B \sinh 2x$ or $y = Ae^{2x} + Be^{-2x}$
Difficulty: 1 Section: 1
9. Solve the differential equation $(D^3 - 4D^2 + 6D)y = 0$.
Answer: $y = A + e^{2x} (B \sin \sqrt{2}x + C \cos \sqrt{2}x)$
Difficulty: 1 Section: 1

10. Solve the differential equation $(D^2 + 4)(D - 2)(D - 1)y = 0$.

Answer: $y = e^{2x}(A \sin 2x + B \cos 2x) + e^x(C \sin 2x + D \cos 2x)$

Difficulty: 1 Section: 1

11. Solve the differential equation $y'' - 3y' - 4y = 4x^2 + 6x - 10$.

Answer: $y = Ae^{4x} + Be^{-x} + 2 - x^2$

Difficulty: 2 Section: 2

12. Solve the differential equation $y'' + 3y' - 4y = 4x^2 - 6x - 10$.

Answer: $y = Ae^{-4x} + Be^x + 2 - x^2$

Difficulty: 2 Section: 2

13. Solve the differential equation $y'' + 2y' - 8y = 4 \cos 2x - 12 \sin 2x$.

Answer: $y = Ae^{2x} + Be^{-4x} + \sin 2x$

Difficulty: 2 Section: 2

14. Solve the differential equation $y'' + 2y' + y = -25 \sin 2x$.

Answer: $y = Axe^{-x} + Be^{-x} + 4 \cos 2x + 3 \sin 2x$

Difficulty: 2 Section: 2

15. Solve the differential equation $y'' + 4y' + 4y = 2e^{-2x}$.

Answer: $y = Axe^{-2x} + Be^{-2x} + x^2e^{-2x}$

Difficulty: 2 Section: 2

16. Solve the differential equation $y'' - 4y' + 5y = \sin x - \cos x$.

Answer: $y = Ae^{2x} \cos x + Be^{2x} \sin x + \frac{1}{4} \sin x$

Difficulty: 2 Section: 2

17. Solve the differential equation $y'' + 4y = \cos 2x - \sin 2x$

Answer: $y = A \cos 2x + B \sin 2x + \frac{1}{4}x(\cos 2x + \sin 2x)$

Difficulty: 2 Section: 2

18. Solve the differential equation $y'' - 4y = 4e^{2x}$.

Answer: $y = A \cosh 2x + B \sinh 2x + xe^{2x}$ or $y = Ae^{2x} + Be^{-2x} + xe^{2x}$

Difficulty: 2 Section: 2

19. Solve the differential equation $y'' - 4y' + 5y = \sin x - \cos x$ if $y(0) = 1$ and $y'(0) = 0$.

Answer: $y = e^{2x} \cos x - \frac{9}{4}e^{2x} \sin x + \frac{1}{4} \sin x$

Difficulty: 2 Section: 2

20. Solve the differential equation $y'' + 4y = \cos 2x - \sin 2x$ if $y(0) = 0$ and $y'(0) = 1$.

Answer: $y = \frac{3}{8} \sin 2x + \frac{1}{4}x (\cos 2x + \sin 2x)$

Difficulty: 2 Section: 2

21. A spring with a spring constant k of 96 pounds per foot is loaded with a 12 pound weight and brought to equilibrium. It is then stretched 1 inch and released. Find the motion and the period. Neglect friction.

Answer: $\frac{1}{12} \cot 16t$, period $\frac{\pi}{8}$

Difficulty: 2 Section: 3

22. A spring with a spring constant k of 64 pounds per foot is loaded with a 16 pound weight and brought to equilibrium. It is then stretched 1 inch and released. Find the motion and the period. Neglect friction.

Answer: $\frac{1}{12} \cos 8\sqrt{2}t$, period $\frac{\sqrt{2}\pi}{8}$

Difficulty: 2 Section: 3

23. A spring with a spring constant k of 18 pounds per foot is loaded with a 12 pound weight and brought to equilibrium. It is then stretched 2 inches and released. Find the motion and the period. Neglect friction.

Answer: $\frac{1}{6} \cos 4\sqrt{3}t$, period $\frac{\sqrt{3}\pi}{6}$

Difficulty: 2 Section: 3

24. A 12 pound weight stretches a spring 2 inches. The weight is raised 2 inches and given an initial velocity of 2 feet per second upward. Find the equation of motion.

Answer: $y = -\frac{1}{6} \cos 8\sqrt{3}t - \frac{\sqrt{3}}{12} \sin 8\sqrt{3}t$

Difficulty: 2 Section: 3

25. A 12 pound weight stretches a spring 3 inches. The weight is raised 2 inches and given an initial velocity of 2 feet per second upward. Find the equation of motion.

Answer: $y = -\frac{1}{6} \cos 8\sqrt{2}t - \frac{\sqrt{2}}{8} \sin 8\sqrt{2}t$

Difficulty: 2 Section: 3

26. A 12 pound weight stretches a spring 4 inches. The weight is raised 2 inches and given an initial velocity of 2 feet per second downward. Find the equation of motion.

Answer: $y = -\frac{1}{6} \cos 4\sqrt{6}t + \frac{\sqrt{6}}{12} \sin 4\sqrt{6}t$

Difficulty: 2 Section: 3

27. A spring with constant k of 10 pounds per foot is loaded with a 32 pound weight and brought to equilibrium. It is then stretches 1 inch and released. Find the equation of motion if the damping force is proportional to twice the velocity.

Answer: $y = e^{-t} \left(\frac{1}{12} \cos 3t + \frac{1}{36} \sin 3t \right)$

Difficulty: 2 Section: 3

28. A spring with constant k of 10 pounds per foot is loaded with a 32 pound weight and brought to equilibrium. It is then stretches 2 inches and released. Find the equation of motion if the damping force is proportional to twice the velocity.

Answer: $y = e^{-t} \left(\frac{1}{6} \cos 3t + \frac{1}{18} \sin 3t \right)$

Difficulty: 2 Section: 3